Physics in Spacetime (F4051)

Lecture notes

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Spring 2025

Chapter 1

Space and Time

This course deals with the special theory of relativity introduced by Einstein in a famous 1905 paper. The traditional way of introducing special relativity is to derive it, in much the same way that Einstein did, from two basic principles:

1. **The principle of relativity**
2. **The constancy of the speed of light**

From these assumptions the notion of a spacetime with (inertial) observers being connected by Lorentz transformations follows. This is a natural way to proceed if one starts from a knowledge of classical mechanics and Maxwell's equations of electrodynamics. However, it is not the best way to understand the geometrical aspects of spacetime. This is part of the reason it took Einstein another ten years to formulate the general theory of relativity, describing gravity, where the geometry of spacetime is the key player.

In this course we will follow a different route[1] which leads more directly to a geometric picture. Rather than starting with the principles described above we will derive the same physics from what is known as

- **The principle of maximum proper time**

This approach is more in line with general relativity, and this course can be thought of as a first step towards studying general relativity.

1.1 What is space and time?

The notions of Space and Time are central to physics. In physics we are interested in answering questions like

Given some configuration of particles with given positions and momenta at some initial time t_i, what will the configuration look like at some later time t_f?

[1]The approach to relativity taken here is inspired by lecture notes and a book by B. Laurent (*Introduction To Spacetime*, World Scientific, 1994).

For such questions to make sense we must have a precise way of defining what we mean by time and also what we mean by a particle's position in space. So what are space and time? Rather than get into a philosophical discussion about the nature of space and time, a more useful approach when faced with such deep questions in physics is to try to replace it by a different, more down to earth, question. After all, in physics we deal only with things that can be measured, and therefore a better question to ask is

> *How do we* **measure** *space and time?*

We say that we define space and time *operationally*, by declaring how we measure them.

So how do we measure distances in space? The most basic way is to take a reference object, say stick of a certain length, and use it to measure the distance between two points. We will call such a reference object a *ruler*. Of course, a good ruler should not bend or change its length with temperature etc., so we will assume it is always possible to find a sufficiently good ruler (or equivalent) so that we can measure lengths to the precision we need. How do we measure time? To measure time we need a clock. It does not have to be what we normally think of as a clock, it can be any physical process which has a known time dependence, e.g. a periodic process with definite period like a pendulum or a non-periodic process like an atom in an excited state with known half-life. Again, we will assume that there exist such clocks with good enough precision for the time measurements we need to perform.

We allow each person, or *observer*, to measure time with their own clock and spatial distances with their own ruler. We will assume that these are small enough that the observer can carry them with her, i.e. they will be assumed to be in the same state of motion as the observer and experience the same forces she experiences. But if each observer makes their own measurements using their own clock and ruler, how do we relate the measurements of two *different* observers? Newton and his contemporaries assumed that there was an absolute notion of time, so that all observers clocks would tick at the same rate. In that case it is very easy to relate the measurements of two observers. We now know that this assumption was wrong. For example, taking two synchronized atomic clocks, putting one on a plane circling the earth and leaving one on the ground, one finds when comparing them at the end that they differ (by a few hundred nanoseconds). This observation is clearly inconsistent with the Newtonian idea of an absolute time.

1.2 The principle of maximum proper time

Experiments show that time runs differently for different observers. We must therefore assign each observer their own time, their *proper time*, which is the time measured on their clock. We can now state the key principle that will allow us to compare the measurements of different observers

The principle of maximum proper time:

> *If two observers are separated and then meet again, the one that does not experience any acceleration always measures the* **longest** *proper time.*

It says that proper time is maximized for *inertial*, i.e. unaccelerated, observers. There is plenty of experimental evidence to support this principle, such as the experiments with atomic clocks on planes, or the operation of GPS satellites which requires very precise time measurements. In this course we will take this principle as the starting point from which we will derive the theory of special relativity.

1.3 Spacetime

We are familiar with the fact that to specify the position of an object in our three dimensions we need to give three numbers – the coordinates with respect to some specified coordinate system. For positions on the earth we might for example give the longitude, the latitude and the height above sea level. To specify an *event* – something happening at a certain place at a certain instant of time – we must give one more number, namely the time on a clock associated to the coordinate system. In our example this could be the time GMT.

We have argued that we must allow each observer to measure distances and times using their own coordinate system defined by their ruler and clock. Each observer will therefore associate to a given event four numbers (t, x, y, z) – the *spacetime coordinates* relative to their coordinate system. Note that we are defining an event here in an idealized way as a single point in spacetime, i.e. something that happens at a point in space at a single instant of time. The set of all events make up the four-dimensional *spacetime*. Note that each observer will (in general) assign different coordinates to the same event because they are using different coordinate systems, there is no preferred coordinate system in spacetime. One of our first tasks will therefore be to understand how to relate the observations of different observers.

1.4 Worldlines

The trajectory of an object traces out a continuous path in spacetime – a *worldline* (really "worldtube" if the object is not point-like, but this distinction won't be very important to us). In ordinary Euclidean space we are familiar with the fact that there is a shortest path between any two points. This path is called a straight line. It is the path an object follows if it is not acted upon by any external forces, i.e. it is unaccelerated. Similarly, we will assume that there is precisely one straight line connecting any two events in spacetime and that any object not acted on by external forces, i.e. not experiencing any acceleration, follows such a straight worldline. To a worldline connecting two events in spacetime we can associate a number – the proper time along that worldline. Recall that this is the time an observer traveling along the worldline measures on her clock between the two events. The principle of maximum proper time says that a straight worldline corresponds to the *longest* proper time. Therefore the analog of shortest length in Euclidean space is longest proper time in spacetime and a clock can be thought of as measuring distances in spacetime. When we draw *spacetime diagrams* we will draw the worldlines of unaccelerated objects as straight lines. Curved lines will correspond to worldlines of accelerated objects (Figure 1.1).

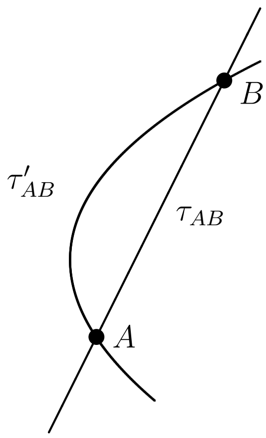


Figure 1.1: Spacetime diagram showing an accelerated (curved worldline) and an unaccelerated (straight worldline) observer meeting at events A and B. The proper time measured on their respective clocks between the meetings is τ'_{AB} and τ_{AB}. The principle of maximum proper time then says that $\tau_{AB} > \tau'_{AB}$.

An important notion in Euclidean geometry is the notion of two lines being parallel. In spacetime we can similarly have the notion of two observers being on the same course. How can two observers, e.g. two spaceships traveling in outer space, determine whether they are on the same course? One way to do this uses a construction from Euclidean space adapted to spacetime. Imagine that the two observers each send out a probe fitted with a clock, which travels freely until it is picked up by the other observer at some later time (Figure 1.2). If the two probes happen to meet halfway, i.e. after half of the proper time (from being emitted to being picked up) has elapsed on each clock, then we will say that the observers are on the same course, or that their worldlines are *parallel*. From the figure we see that this also implies that the lines AB and CD (not drawn) are parallel. Note that to carry out the experiment we really need to send clocks that also have a recording device that records the time they were sent and the time they met. We would also need to do the experiment several times to get them to meet halfway.

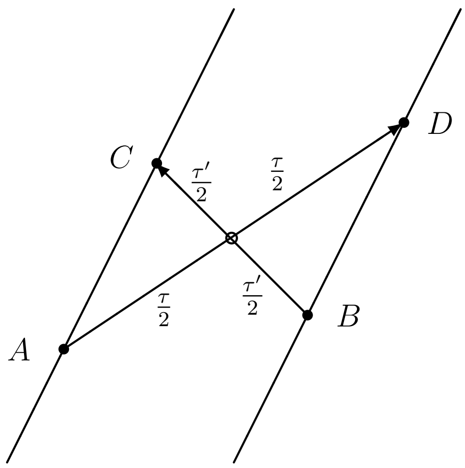


Figure 1.2: The worldlines of two observers are parallel if they can send out probes, at A and B, that meet halfway before encountering the other observer at D and C.

Notice that this construction does not refer to space or time separately, only to the full spacetime picture or the proper time measured by a particular clock. This is in sharp contrast to how we would describe such an experiment in Newtonian physics.

Just like in Euclidean space the line AC in Figure 1.2 defines a vector, which we can draw as an arrow starting at A and ending at C. We declare the length of the vector to be given by the proper time elapsed from A to C. The construction in the figure gives us a way to *parallel transport* vectors, i.e. moving a vector while keeping it parallel to itself. The vector AC can be parallel transported to the vector BD. Taking the two worldlines to approach each other we obtain the special case of parallel transport of the vector along the worldline. A general parallel transport is obtained by a sequence of such "elementary" parallel transports.

We will now make a very important assumption. We will assume that the vector one obtains by such a sequence of elementary parallel transports from a point A to a point A' in spacetime does not depend on how one chooses the sequence of parallel transports, i.e. it does not depend on the path taken. This assumption is actually *not true* close to gravitating bodies and in that case one must use the more advanced theory of *general relativity*. The assumption *is true* if gravity is very weak, which is the case we consider in this course. In this case we are working with *special relativity*. In fact, the change of a vector under parallel transport is directly related to the curvature of a space. In special relativity spacetime is *flat*, while in general relativity it can be curved.

Chapter 2

Spacetime vectors

In the last chapter we defined a vector on the straight worldline of an observer as an arrow from one event to a later event on the worldline, with length given by the proper time elapsed between the two events. The notion of vectors is familiar from Euclidean space and we will use the same notation $\bar{v}$ for a vector in spacetime. Such vectors are often called *four-vectors* since spacetime is four-dimensional. Just as any point in Euclidean space $\mathbb{R}^3$ can be associated with a vector going from the origin to that point, any event in spacetime can be associated to a spacetime vector from an origin (which we can choose as we please) to the point in question. We have also seen that we can move vectors around using parallel transport. Two vectors related by parallel transport will be considered the same vector (this is consistent since we are assuming that the vector obtained by parallel transport is independent of the path taken).

Spacetime vectors obey the usual axioms familiar from Euclidean space:

- Commutativity of addition: $\bar{u} + \bar{v} = \bar{v} + \bar{u}$
- Associativity of addition: $\bar{u} + (\bar{v} + \bar{w}) = (\bar{u} + \bar{v}) + \bar{w}$
- Identity element of addition: $\bar{v} + \bar{0} = \bar{v}$
- Inverse element of addition: Given $\bar{v}$ there exists a vector $-\bar{v}$ such that $\bar{v} + (-\bar{v}) = 0$
- Compatibility of scalar multiplication: $a(b\bar{v}) = (ab)\bar{v}$ for $a, b \in \mathbb{R}$
- Identity element of scalar multiplication: $1\bar{v} = \bar{v}$
- Distributivity of scalar multiplication with respect to vector addition: $a(\bar{u} + \bar{v}) = a\bar{u} + a\bar{v}$
- Distributivity of scalar multiplication with respect to addition: $(a + b)\bar{v} = a\bar{v} + b\bar{v}$

Addition of spacetime vectors can be done by the geometric construction familiar from Euclidean space, which is illustrated in Figure 2.1. Recall that a *basis* for a vector space

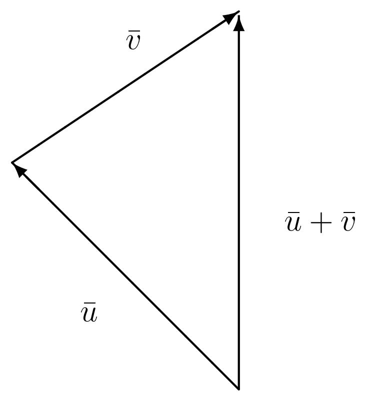


Figure 2.1: Geometric addition of the vectors $\bar{u}$ and $\bar{v}$ producing a third vector $\bar{u}+\bar{v}$.

is a set of linearly independent vectors $\bar{v}_i$ with $i = 1, \ldots, n$ which span the space, so that any vector is expressed uniquely as a linear combination

$$a_1\bar{v}_1 + a_2\bar{v}_2 + \ldots + a_n\bar{v}_n\,, \tag{2.1}$$

for some numbers $a_i \in \mathbb{R}$ with $i = 1, \ldots, n$. The vector can be denoted in this basis as $(a_1, a_2, \ldots, a_n)$ and n is called the dimension of the vector space. A basis of spacetime vectors consists of four linearly independent spacetime vectors.

2.1 Inner product

An important notion in linear algebra is that of the *inner product* between two vectors. Given two vectors their inner product is a real number. We will denote the inner product with a dot, e.g. $\bar{u}\cdot\bar{v}$ denotes the inner product between vectors $\bar{u}$ and $\bar{v}$. The inner product satisfies the following standard axioms

- Symmetry: $\bar{u}\cdot\bar{v} = \bar{v}\cdot\bar{u}$
- Linearity: $(a\bar{u})\cdot\bar{v} = a(\bar{u}\cdot\bar{v})$ and $(\bar{u}+\bar{v})\cdot\bar{w} = \bar{u}\cdot\bar{w} + \bar{v}\cdot\bar{w}$
- Non-degeneracy: If $\bar{u}\cdot\bar{v} = 0$ for all vectors $\bar{v}$ then $\bar{u} = \bar{0}$

Often the inner product is required to be positive definite, so that $\bar{u}^2 = \bar{u}\cdot\bar{u} \geq 0$, which is a stronger requirement than being non-degenerate. This is the case in Euclidean space where we are used to identifying $\bar{u}^2$ with the length-squared of a vector, which is obviously positive. We will see below that it is not possible to require this for spacetime vectors. Instead, for a spacetime vector that goes along the straight worldline of an object from point A to point B we will take

$$\bar{u}^2 = -\tau^2 \,, \tag{2.2}$$

where τ is the proper time along the worldline from A to B. The minus sign seems strange at this point but we will see shortly that it is needed if we want vectors representing lengths in space to have positive square. All the differences between Euclidean space and spacetime are due to the fact that the inner product in spacetime is not positive definite. As we will see this is what makes it possible to separate the time-direction from the spatial directions.

The assumption that $\bar{u}^2 = -\tau^2$ for vectors corresponding to a segment of a straight worldline determines also the inner product $\bar{u} \cdot \bar{v}$ of two straight worldline vectors $\bar{u}$, $\bar{v}$. To see this consider three such worldline vectors related by

$$a\bar{u} = \bar{v} + \bar{w} \,, \tag{2.3}$$

for some $a \in \mathbb{R}$. Writing this as $\bar{w} = a\bar{u} - \bar{v}$ and squaring both sides we get

$$\bar{w}^2 = (a\bar{u} - \bar{v}) \cdot (a\bar{u} - \bar{v}) = a^2\bar{u}^2 - 2a\bar{u} \cdot \bar{v} + \bar{v}^2 \,. \tag{2.4}$$

Rearranging this we have

$$\bar{u} \cdot \bar{v} = \frac{1}{2a} \left(a^2\bar{u}^2 + \bar{v}^2 - \bar{w}^2 \right) \,. \tag{2.5}$$

The right-hand-side involves only squares of vectors, which are expressed in terms of the corresponding proper times. Therefore we see that the inner product $\bar{u} \cdot \bar{v}$ is also determined in terms of the proper times corresponding to the lengths of the vectors $\bar{u}$, $\bar{v}$, $\bar{w}$.

It is important to understand that the assumption that there exists an inner product for spacetime vectors satisfying the above axioms is not a trivial statement. The mere existence of this inner product has physical consequences. To see this consider the identity

$$(\bar{u} + \bar{v})^2 + (\bar{u} - \bar{v})^2 = 2\bar{u}^2 + 2\bar{v}^2 \,. \tag{2.6}$$

Let's assume that all these vectors are part of straight worldlines of observers. Since the expression contains only squares it only involves the proper times measured by these observers. With four spaceships traveling along these straight worldlines it is then possible to arrange an experiment (see Figure 2.2) to test whether the proper times they measure satisfy the above identity, i.e. whether $\tau_1^2 + \tau_2^2 = 2\tau_3^2 + 2\tau_4^2$. One finds that it is indeed satisfied.

2.2 Three classes of spacetime vectors

Let $\bar{u}$, $\bar{v}$ be two straight worldline vectors. Then

$$\bar{u}^2 = -\tau_u^2 \,, \qquad \bar{v}^2 = -\tau_v^2 \,. \tag{2.7}$$

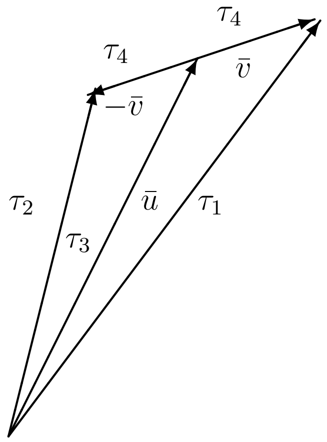


Figure 2.2: Experiment involving four spaceships to test equation (2.6).

We now define a new vector which is a linear combination of $\bar{u}$ and $\bar{v}$,

$$\bar{y} = a\bar{u} + b\bar{v}\,, \tag{2.8}$$

for some a, b. Its inner product with $\bar{u}$ is

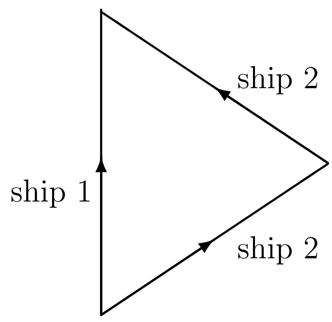
$$\bar{u} \cdot \bar{y} = a\bar{u}^2 + b\bar{u} \cdot \bar{v}\,. \tag{2.9}$$

Taking $a = -b\frac{\bar{u}\cdot\bar{v}}{\bar{u}^2}$ we find

$$\bar{u} \cdot \bar{y} = 0\,. \tag{2.10}$$

We say that $\bar{y}$ is *orthogonal* to $\bar{u}$.

Now consider the following situation where two spaceships part and then meet again:



Spaceship 1 is unaccelerated throughout the duration of its journey, while spaceship 2 travels unaccelerated for a while, then accelerates hard for a short time to reverse its direction of motion and then again floats freely until it meets spaceship 1 again.

We will take the spacetime vectors corresponding to this situation as in Figure 2.3 with

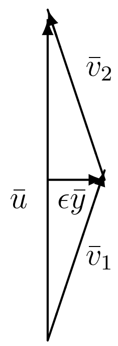


Figure 2.3: Spacetime vectors corresponding to two spaceships parting and meeting again.

$$\bar{v}_1 = \frac{1}{2}\bar{u} + \epsilon\bar{y}\,, \qquad \bar{v}_2 = \frac{1}{2}\bar{u} - \epsilon\bar{y}\,, \tag{2.11}$$

where $\bar{y}$ is the vector introduced above which is orthogonal to $\bar{u}$ and ϵ is a small number. Note that $\bar{v}_1 + \bar{v}_2 = \bar{u}$ so that the spaceships indeed meet at the end. The proper time for the journey of ship 1 is

$$\tau_1 = \sqrt{-\bar{u}^2}\,. \tag{2.12}$$

The proper time for the journey of ship 2 is the sum of the proper time for the two segments of the journey

$$\tau_2 = \sqrt{-\bar{v}_1^2} + \sqrt{-\bar{v}_2^2}\,. \tag{2.13}$$

Since $\bar{u} \cdot \bar{y} = 0$ we have $\bar{v}_1^2 = \frac{1}{4}\bar{u}^2 + \epsilon^2\bar{y}^2 = \bar{v}_2^2$ so that

$$\tau_2 = 2\sqrt{-\tfrac{1}{4}\bar{u}^2 - \epsilon^2\bar{y}^2} = \sqrt{-\bar{u}^2}\sqrt{1 + \frac{4\epsilon^2\bar{y}^2}{\bar{u}^2}} = \tau_1\sqrt{1 - \frac{4\epsilon^2\bar{y}^2}{\tau_1^2}}\,. \tag{2.14}$$

The principle of maximum proper time says that the unaccelerated observer measures the longest proper time, i.e. $\tau_1 > \tau_2$ (we assume $\epsilon\bar{y} \neq 0$). This in turn implies that

$$\sqrt{1 - \frac{4\epsilon^2\bar{y}^2}{\tau_1^2}} < 1 \qquad \Rightarrow \qquad \bar{y}^2 > 0\,, \tag{2.15}$$

the vector $\bar{y}$ has positive square. This result was obtained assuming $\bar{u}^2 = -\tau_1^2 < 0$. If we had decided instead to take the opposite convention, i.e. $\bar{u}^2 = +\tau_1^2 > 0$, the same calculation would give $\bar{y}^2 < 0$. We see that, in contrast to what we are used to from Euclidean space, it is not possible for all spacetime vectors to have positive square. This fact follows from the principle of maximum proper time. Clearly no observer can travel along $\bar{y}$ because then his clock would need to show an imaginary time, which is absurd.

Consider now the vector

$$\bar{w} = c\bar{u} + d\bar{y}\,, \tag{2.16}$$

with $\bar{u}$ and $\bar{y}$ orthogonal as before. Squaring this we find

$$\bar{w}^2 = c^2\bar{u}^2 + d^2\bar{y}^2\,. \tag{2.17}$$

If we take $c^2 = -\frac{d^2\bar{y}^2}{\bar{u}^2}$ (note that the RHS is positive which is consistent with c, d being real numbers) we get $\bar{w}^2 = 0$! We conclude that there also exist spacetime vectors $\bar{w} \neq \bar{0}$ such that $\bar{w}^2 = 0$.

To summarize we have learned that there are 3 classes of spacetime vectors:

- $\bar{v}^2 < 0$: **Timelike**
- $\bar{v}^2 > 0$: **Spacelike**
- $\bar{v}^2 = 0$: **Null**

Vectors that are part of a straight worldline of an observer are timelike. We have seen above that the principle of maximum proper time implies that if $\bar{u}$ is timelike and $\bar{u} \cdot \bar{y} = 0$ then $\bar{y}$ is spacelike (or $\bar{y} = \bar{0}$). This is a very useful result to remember when working with spacetime vectors:

$\bar{u}$ timelike and $\bar{u} \cdot \bar{v} = 0 \Rightarrow \bar{v}$ spacelike (or $\bar{v} = \bar{0}$).

Chapter 3

Simultaneity and spatial distance

An observer traveling along in a spaceship only has direct access to the interior of the spaceship. Nevertheless they must be able to make statements and inferences about what happens in the outside world. To be able to do this they need in particular to be able to say when an event, which is not on their worldline, occurred. Another way to say it is that they need to have a way to determine whether an event far away is *simultaneous* with an event on their worldline, e.g. a supernova explosion far away happens when their clock shows 10:23.

The natural way to do this is via the construction in figure 3.1. The observer sends out a probe, which travels on a straight worldline to the event and on a straight worldline back. She arranges it so that the probe reaches the event precisely when half the proper time of its journey has elapsed. Then she will say that the event on her worldline halfway between sending out and receiving the probe is simultaneous with the distant event.

From the figure we have

$$\tau^2 = -(\bar{v} + \bar{r})^2 = -(\bar{v} - \bar{r})^2 \qquad \Rightarrow \qquad \bar{v} \cdot \bar{r} = 0\,, \tag{3.1}$$

so that $\bar{r}$, being orthogonal to a timelike vector, must be spacelike.

We also need a way to measure spatial distances. To see how to do this let us consider a family of straight parallel worldlines L_0, L_1, ... defined by the equation

$$\bar{R}_n = \lambda_n \bar{u} + n\bar{\rho}\,, \qquad n = 0, 1, 2, \ldots \tag{3.2}$$

where $\bar{u}$, $\bar{\rho}$ are timelike vectors and $\lambda_n \in \mathbb{R}$ parametrizes a point on the n'th worldline, L_n. This is illustrated in figure 3.2. This could be a fleet of identical spaceships traveling unaccelerated and arranged head to tail. Consider now an observer traveling from the front of the fleet to the back, counting how many ships he passes. This number is a measure of how far it is from the head of the fleet to the tail. The distance is expressed in units of "standard spaceship".

There is an alternative way to measure this distance. We first note that there is only

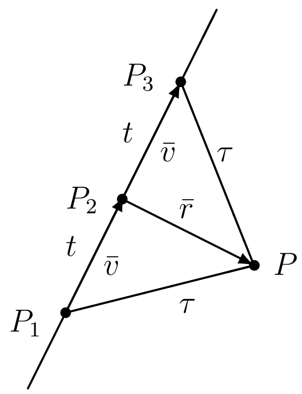


Figure 3.1: Via this construction the observer decides that P_2, halfway between P_1 and P_3, is simultaneous with P.

one vector going from L_0 to L_n with the property that it is orthogonal to $\bar{u}$.[1] It is given by

$$\bar{r}_n = n\bar{\rho} - \left(\frac{n\bar{\rho}\cdot\bar{u}}{\bar{u}^2}\right)\bar{u} = n\left(\bar{\rho} - \frac{\bar{\rho}\cdot\bar{u}}{\bar{u}^2}\bar{u}\right) . \tag{3.3}$$

Note that $\bar{r}_n$, and therefore also its magnitude, is proportional to n, the number of spaceships. We can therefore use the magnitude $\sqrt{\bar{r}_n^2}$ as a measure of the distance. All we need to do is work out the conversion factor to go between $\sqrt{\bar{r}_n^2}$ and the number of spaceships.

Looking at figure 3.1 we read off

$$\tau^2 = -(\bar{v}+\bar{r})^2 = t^2 - \bar{r}^2\,, \qquad \text{or} \qquad \bar{r}^2 = t^2 - \tau^2\,. \tag{3.4}$$

The advantage of this method is that we don't need the fleet of spaceships (other than to fix the unit of distance). Later we will find an even more practical way to measure distance.

How we pick the unit of distance is up to us. Nothing prevents us from choosing units such that $\sqrt{\bar{r}^2}$ itself is the distance. This is in fact the most natural choice to make and we will stick to it in this course. From (3.4) we see that now space and time acquire the same dimensions. In the theory of relativity this is as natural as say height and width having the same dimensions and being measured in the same units.

[1]Proof: It is clear that at least one such vector exists. Assume there are two such vectors $\bar{r}_1$, $\bar{r}_2$. We may assume their foot-point is the same point on L_0. The fact the $\bar{u}\cdot\bar{r}_1 = \bar{u}\cdot\bar{r}_2 = 0$ implies $\bar{u}\cdot(\bar{r}_1-\bar{r}_2) = 0$. But $\bar{r}_1 - \bar{r}_2 = \lambda\bar{u}$ for some λ and the previous equation implies $\lambda = 0$ so that $\bar{r}_1 = \bar{r}_2$. $\square$

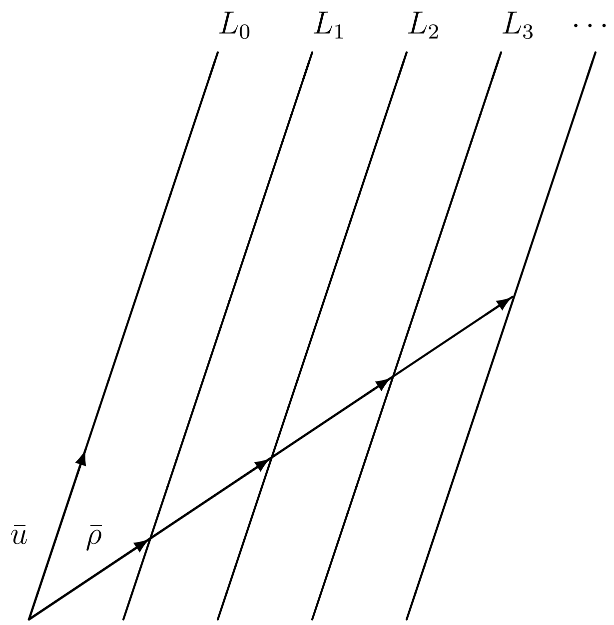


Figure 3.2: A family of parallel worldlines.

3.1 Orthogonal space

To every unaccelerated observer there corresponds a straight worldline. Such a worldline is characterized by a timelike vector which we can normalize to a *unit vector* $\hat{u}$. We will always us a 'hat' to denote unit vectors. A timelike unit vector satisfies $\hat{u}^2 = -1$ and a spacelike unit vector $\hat{r}^2 = 1$. Given an observer with worldline unit vector $\hat{u}$ there exist orthogonal vectors $\bar{r}$,

$$\hat{u} \cdot \bar{r} = 0 \,. \tag{3.5}$$

They form the *orthogonal space* to the observer's worldline. This is a vector space since linear combinations of such vectors clearly belong to the space. In fact, since all such $\bar{r}$ are spacelike (or zero), it is a Euclidean vector space. Since we are imposing one condition on the four components of $\bar{r}$ the orthogonal space is three-dimensional. Recall that $\sqrt{\bar{r}^2}$ is the (spatial) distance from the observer. The orthogonal space is the space used in Newtonian physics. The difference is that in the theory of relativity each observer has their own orthogonal space.

Given the worldline of an observer with direction $\hat{u}$, we can split any spacetime vector $\bar{R}$ into a component along $\hat{u}$ and a component orthogonal to it as

$$\bar{R} = t\hat{u} + \bar{r} \qquad \text{with} \qquad \hat{u} \cdot \bar{r} = 0\,. \tag{3.6}$$

This is illustrated in figure 3.3. According to the figure, an observer following the worldline

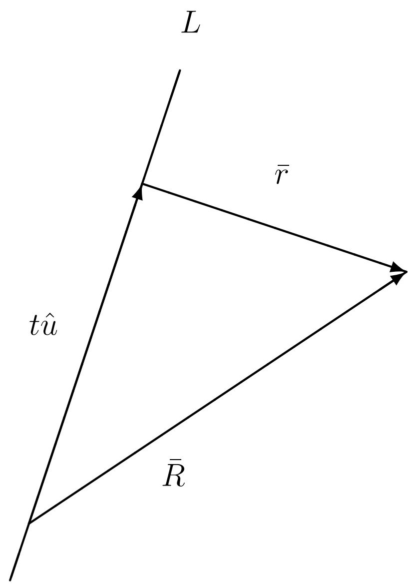


Figure 3.3: Split of a spacetime vector $\bar{R}$ with respect to a timelike direction $\hat{u}$.

L measures the event corresponding to $\bar{R}$ to happen at a time t and spatial position $\bar{r}$. The distance to the event is $\ell = \sqrt{\bar{r}^2}$. From the equation we find $t = -\hat{u} \cdot \bar{R}$ and $\bar{r} = \bar{R} + (\hat{u} \cdot \bar{R})\hat{u}$. We see that t and $\bar{r}$ are uniquely fixed in terms of $\hat{u}$ and $\bar{R}$.

3.2 Linearly independent vectors

Consider four spacetime vectors

$$\bar{A},\ \bar{B},\ \bar{C},\ \bar{D}\,. \tag{3.7}$$

They are linearly independent if none of them can be expressed as a linear combination of the others, or equivalently if the equation

$$a\bar{A} + b\bar{B} + c\bar{C} + d\bar{D} = 0 \tag{3.8}$$

has only the trivial solution $a = b = c = d = 0$. Note that since spacetime is four-dimensional we cannot have more than four linearly independent vectors.

In Euclidean space we are used to two orthogonal vectors being linearly independent. This is **not** true in spacetime, e.g. a null vector is orthogonal to itself but clearly not linearly independent of itself. What is true is that if $\bar{v} \cdot \bar{u} = 0$ for *all* $\bar{u}$ then $\bar{v} = 0$. To see this take $\bar{u}$ timelike. We conclude that $\bar{v}$ must be spacelike, but since it is orthogonal to all other spacelike vectors it must vanish since these form a Euclidean vector space.

To test if $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{D}$ are linearly independent we form the determinant of the matrix of inner products

$$\begin{vmatrix} \bar{A}\cdot\bar{A} & \bar{A}\cdot\bar{B} & \bar{A}\cdot\bar{C} & \bar{A}\cdot\bar{D} \\ \bar{B}\cdot\bar{A} & \bar{B}\cdot\bar{B} & \bar{B}\cdot\bar{C} & \bar{B}\cdot\bar{D} \\ \bar{C}\cdot\bar{A} & \bar{C}\cdot\bar{B} & \bar{C}\cdot\bar{C} & \bar{C}\cdot\bar{D} \\ \bar{D}\cdot\bar{A} & \bar{D}\cdot\bar{B} & \bar{D}\cdot\bar{C} & \bar{D}\cdot\bar{D} \end{vmatrix} \tag{3.9}$$

The vectors are linearly dependent if and only if this determinant vanishes. To see this assume they are linearly dependent. Then (3.8) holds for some non-zero coefficients. Taking this linear combination of rows in the matrix we obtain a row of zeros so that the determinant vanishes. Conversely, if the determinant vanishes there exists a linear combination of the rows that gives zero. This means that there exists a vector $a\bar{A} + b\bar{B} + c\bar{C} + d\bar{D}$, with a, b, c, d not all zero, which is orthogonal to $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{D}$. Assuming they are linearly independent leads to a contradiction since this vector would then be orthogonal to all vectors and would therefore have to vanish, therefore they must be linearly dependent.

Note that this test works only for four vectors. It does not work for lower-dimensional subspaces of spacetime.