Before we start: Classical electromagnetism

Literature: Classical electromagnetism is discussed in L2 and B11, with the mathematical background covered by B4.

0.1 Electric field, electric charge, electric dipole

Objects having a property known as the *electric charge* (Q) experience forces ($\vec{F}$) described as the *electric field*. Since the force depends on both charge and field, a quantity $\vec{E} = \vec{F}/Q$ known as the *electric intensity* has been introduced:

$$\vec{F} = Q\vec{E}. \tag{1}$$

Field lines are often used to visualize the fields: direction of the line shows the direction of $\vec{E}$, density of the lines describes the size of $\vec{E}$ ($|E|$). A homogeneous static electric field is described by straight parallel field lines.

Two point electric charges of the same size and opposite sign ($+Q$ and $-Q$) separated by a distance $2r$ constitute an *electric dipole*. Electric dipoles in a homogeneous static electric field experience a *moment of force*, or *torque* $\vec{\tau}$:

$$\vec{\tau} = 2\vec{r} \times \vec{F} = 2\vec{r} \times Q\vec{E} = 2Q\vec{r} \times \vec{E} = \vec{\mu}_{\mathrm{e}} \times \vec{E}, \tag{2}$$

where $\vec{\mu}_{\mathrm{e}}$ is the *electric dipole moment*.

$$\vec{\tau} = \vec{\mu}_{\mathrm{e}} \times \vec{E} \tag{3}$$

is another possible definition of $\vec{E}$. As derived in Section 0.6.1, potential energy of an electric dipole is

$$\mathcal{E} = -\vec{\mu}_{\mathrm{e}} \cdot \vec{E}. \tag{4}$$

0.2 Magnetic field and magnetic dipole

There is no "magnetic charge", but magnetic moments exist:

$$\vec{\tau} = \vec{\mu}_{\mathrm{m}} \times \vec{B}, \tag{5}$$

where $\vec{\mu}_{\mathrm{m}}$ is the *magnetic dipole moment* (because this course is about magnetic resonance, we will write simply $\vec{\mu}$). This is *the definition* of the *magnetic induction* $\vec{B}$ as a quantity describing magnetic field. As a consequence, potential energy of a magnetic dipole can be derived as described by Eq. 27 for the electric dipole.

Potential energy of a magnetic moment $\vec{\mu}$ is

$$\mathcal{E} = -\vec{\mu} \cdot \vec{B}. \tag{6}$$

The magnetic induction $\vec{B}$ is related to the force acting on a charged object, but in a different way than the electric intensity $\vec{E}$ (cf. Eq. 1). The magnetic force depends not only on the electric charge Q but also on the speed of the charge $\vec{v}$ (i.e., on the *electric current*)

$$\vec{F} = Q(\vec{v} \times \vec{B}). \tag{7}$$

Therefore, the torque $\vec{\tau}$ cannot be described by an equation similar to Eq. 2. Instead,

$$\vec{\tau} = \vec{r} \times \vec{F} = Q\vec{r} \times (\vec{v} \times \vec{B}). \tag{8}$$

Due to the fundamental difference between Eqs. 2 and 8, it is more difficult to describe relation between the magnetic force, magnetic moment and energy. We experience it in Sections 0.6.2 and 4.9.1.

0.3 Source of the electric field

The source of the electric field is the *electric charge*. The charge (i) feels (a surrounding) field and (ii) makes (its own) field. Charge at rest is a source of a static electric field. Parallel plates with homogeneous distribution of charges (a capacitor) are a source of a homogeneous static electric field.

Force between charges is described by the *Coulomb's law*. The force between two charges is given by

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{Q_1 Q_2}{r^2}\frac{\vec{r}}{|r|}, \tag{9}$$

where $\epsilon_0 = 8.854187817 \times 10^{-12}\,\mathrm{F\,m^{-1}}$ is the vacuum electric permittivity.

Consequently, the electric intensity generated by a point charge is

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\frac{\vec{r}}{|r|}. \tag{10}$$

The electric intensity generated by a charge density ρ is

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\int\limits_V \mathrm{d}V \frac{\rho}{r^2}\frac{\vec{r}}{|r|} \tag{11}$$

Coulomb's law implies that electric fields lines of a resting charge

1. are going out of the charge (diverge), i.e., the static electric field has a source (the charge)
2. are not curved (do not have curl or rotation), i.e., the static electric field does not circulate

This can been written mathematically in the form of *Maxwell equations*:[1]

$$\mathrm{div}\ \vec{E} = \frac{\rho}{\epsilon_0}, \tag{12}$$

$$\mathrm{rot}\ \vec{E} = 0. \tag{13}$$

where div $\vec{E}$ is a scalar equal to $\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$ and rot $\vec{E}$ is a vector with the x, y, z components equal to $\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}$, $\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}$, $\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}$, respectively. These expressions can be written in a much more compact form, if we introduce a vector operator $\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$. Using this formalism, the Maxwell equations have the form

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}, \tag{14}$$

$$\vec{\nabla} \times \vec{E} = 0. \tag{15}$$

0.4 Origin of the magnetic field

Electric charge at rest does not generate a magnetic field, but a *moving* charge does. The magnetic force is a *relativistic effect* (consequence of the contraction of distances in the direction of the motion, described by Lorentz transformation).[2] Magnetic field of a moving point charge is moving with the charge. Constant *electric current* generates a stationary magnetic field. Constant electric current in an *ideal solenoid* generates a *homogeneous* stationary magnetic field inside the solenoid.

Magnetic induction generated by a current density $\vec{j}$ (*Biot-Savart law*):

$$\vec{B} = \frac{1}{4\pi\epsilon_0 c^2} \int\limits_V \mathrm{d}V \frac{\vec{j}}{r^2} \times \frac{\vec{r}}{|r|} = \frac{\mu_0}{4\pi} \int\limits_V \mathrm{d}V \frac{\vec{j}}{r^2} \times \frac{\vec{r}}{|r|} \tag{16}$$

Biot-Savart law implies that magnetic field lines of a constant current in a straight wire

1. do not diverge, i.e., the static magnetic field does not have a source

[1] The first equation is often written using *electric induction* $\vec{D}$ as div $\vec{D} = \rho$. If electric properties are described in terms of individual charges in vacuum, $\vec{D} = \epsilon_0 \vec{E}$. If behavior of charges bound in molecules is described in terms of polarization $\vec{P}$ of the material, $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$.

[2] A charge close to a very long straight wire which is uniformly charged experiences an electrical force $F_\perp$ in the direction perpendicular to the wire. If the charges in the wire move with a velocity v_0 and the charge close to the wire moves along the wire with a velocity v_1, the perpendicular force changes to $F_\perp(1 - \frac{v_0 v_1}{c^2})$, were c is the speed of light in vacuum. The modifying factor is clearly relativistic (B11.5).

2. make closed loops around the wire (have curl or rotation), i.e., the magnetic field circulates around the wire

This can been written mathematically in the form of *Maxwell equations*:[3]

$$\vec{\nabla} \cdot \vec{B} = 0, \tag{17}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j}. \tag{18}$$

A simple example of a moving charge is a circular loop with an electric current. As derived in Section 0.6.2, magnetic moment of a current loop is proportional the angular momentum of the circulating charge.

Magnetic dipolar moment $\vec{\mu}$ is proportional to the angular momentum $\vec{L}$

$$\vec{\mu} = \gamma \vec{L}, \tag{19}$$

where γ is known as the *magnetogyric ratio*.

The classical theory does not explain why particles like electrons or nuclei have their own magnetic moments, even when they do not move in circles (because the classical theory does not explain why such particles have their own angular momenta). However, if we take the nuclear magnetic moment as a fact (or if we obtain it using a better theory), the classical results are useful. It can be shown that the magnetic moment is *always* proportional to the angular momentum,[4] but the proportionality constant is not always $Q/2m$; it is difficult to obtain for nuclei.

Analysis of the current loop in a static homogeneous external magnetic field, presented in Section 0.6.2, shows that if the direction of the magnetic moment $\vec{\mu}$ of the loop differs from the direction of $\vec{B}$, a torque trying to align $\vec{\mu}$ with $\vec{B}$. However, the magnetic dipole does not adopt the energetically most favored orientation (with the same direction of $\vec{\mu}$ as $\vec{B}$), but rotates around $\vec{B}$ without changing the angle between $\vec{\mu}$ and $\vec{B}$. This motion on a cone is known as *precession*.

This is not a result of quantum mechanics, but a classical consequence of the relation between the magnetic moment and angular momentum of the current loop. The spinning top also precess in the Earth's gravitational field and riding a bicycle is based on the same effect.[5] The precession frequency can be derived easily for the classical current loop in a magnetic field (see Section 0.6.3):

Angular frequency of the precession of a magnetic dipolar moment $\vec{\mu}$ in a magnetic field $\vec{B}$ is

$$\vec{\omega} = -\gamma \vec{B}. \tag{20}$$

[3]The second equation is often written using *magnetic intensity* $\vec{H}$ as $\vec{\nabla} \times \vec{H} = \vec{j}$. If magnetism is described as behavior of individual charges and magnetic moments in vacuum, $\vec{H} = \vec{B}/\mu_0$. If properties of a magnetic materials are described in terms of its *magnetization* $\vec{M}$, then $\vec{H} = \vec{B}/\mu_0 - \vec{M}$.

[4]A consequence of the rotational symmetry of space described mathematically by the Wigner-Eckart theorem.

[5]If you sit on a bike which does not move forward, gravity soon pulls you down to the ground. But if the bike has a certain speed and you lean to one side, you do not fall down, you just turn a corner. A qualitative discussion of precession using the spinning top and riding a bicycle is presented in L2.4–L2.5.

0.5 Electrodynamics and magnetodynamics

Similarly to the electric charge, the magnetic dipole (i) feels the surrounding magnetic field and (ii) generates its own magnetic field. The magnetic field generated by a precessing magnetic dipole is not stationary, it varies. To describe variable fields, the Maxwell equations describing rotation must be modified:[6]

$$\vec{\nabla} \times \vec{E} = -\frac{\mathrm{d}\vec{B}}{\mathrm{d}t}, \tag{21}$$

$$\vec{\nabla} \times \vec{B} = \frac{1}{c^2}\frac{\mathrm{d}\vec{E}}{\mathrm{d}t} + \mu_0\vec{j}. \tag{22}$$

Note that electric and magnetic fields are coupled in the dynamic equations. Not only electric currents current, but also temporal variation of $\vec{E}$ induces circulation of $\vec{B}$, and circulation of $\vec{E}$ is possible if $\vec{B}$ varies. This has many important consequences: it explains electromagnetic waves in vacuum and has numerous fundamental applications in electrical engineering, including those used in NMR spectroscopy.

Eq. 21 shows us how the frequency of the precession motion can be measured. A magnetic dipole in a magnetic field $\vec{B}_0$ generates a magnetic field $\vec{B}'$ with the component $\parallel \vec{B}_0$ constant and the component $\perp \vec{B}_0$ rotating around $\vec{B}_0$. If we place a loop of wire next to the precessing dipole, with the axis of the loop perpendicular to the axis of precession, the rotating component of $\vec{B}'$ induces circulation of $\vec{E}$ which creates a measurable oscillating electromotive force (voltage) in the loop (see Section 0.6.4).

$$U = \frac{\mu_0}{4\pi}\frac{2|\mu|S}{r^3}\omega\sin(\omega t). \tag{23}$$

As a consequence, an oscillating electric current flows in the loop (L2.8).

HOMEWORK

First check that you understand Section 0.6.1. Then, derive how is the magnetic moment of a current loop related to the angular momentum (Section 0.6.2) and what defines the precession frequency of a magnetic moment of a current loop in a homogeneous magnetic field (Section 0.6.3).

[6]The second equation can be written as $\vec{\nabla} \times \vec{H} = \frac{\mathrm{d}\vec{D}}{\mathrm{d}t} + \vec{j}$.

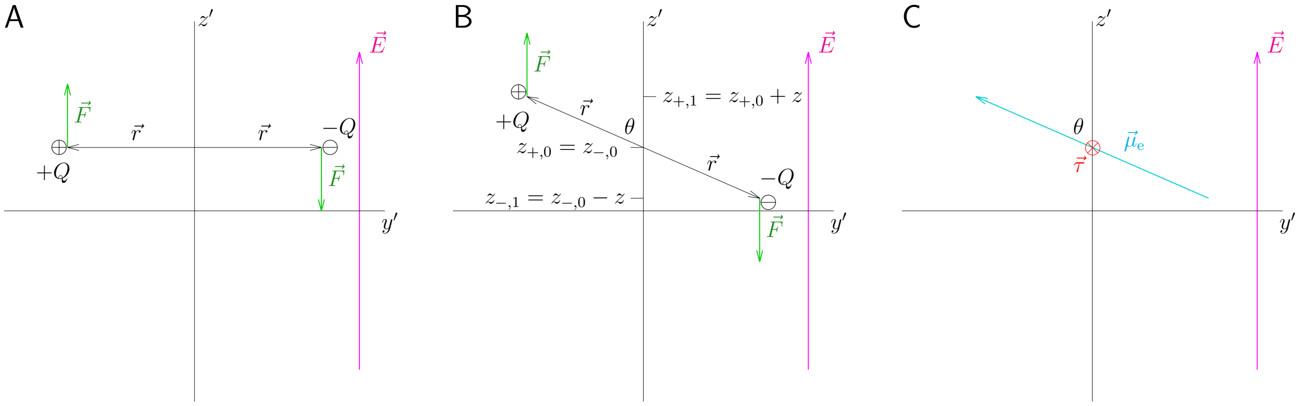

Figure 1: Potential energy of an electric dipole in a homogeneous electric field described by the intensity $\vec{E}$. The reference position of the dipole (0) is shown in Panel A, the actual position of the dipole (1) is shown in Panel B. Individual charges and forces are shown in panels A and B, the dipolar moment $\vec{\mu}_\mathrm{e}$ and the torque $\vec{\tau}$ (its direction $-x$ is depicted using the symbol $\otimes$) are shown in Panel C. Note that the direction of $\vec{\mu}_\mathrm{e}$ follows the convention used in physics, the convention used in chemistry is opposite.

0.6 SUPPORTING INFORMATION

0.6.1 Potential energy of an electric dipole

Potential energy[7] of the electric dipole can be calculated easily as a sum of potential energies of the individual charges. Potential energy is defined as the work done by the field moving the charge from a position (1) to a reference position (0). If we choose a coordinate system as defined in Figure 1, then the force acts only in the z'-direction ($F_{z'} = |\vec{F}| = Q|\vec{E}|$ for the positive charge and $F_{z'} = -|\vec{F}| = -Q|\vec{E}|$ for the negative charge). Therefore, it is sufficient to follow only how the z'-coordinates of the charges change because changes of other coordinates do not change the energy. The natural choice of the reference position is that the z' coordinates are the same for both charges, $z_{+,0} = z_{-,0}$. Changing the z' coordinate of the positive charge from $z_{+,0}$ to $z_{+,1} = z_{+,0} + z$ results in a work

$$Q|\vec{E}|(z_{+,0} - z_{+,1}) = -Q|\vec{E}|z. \tag{24}$$

Changing the z' coordinate of the negative charge from $z_{-,0}$ to $z_{-,1} = z_{-,0} - z$ results in a work

$$-Q|\vec{E}|(z_{-,0} - z_{-,1}) = -Q|\vec{E}|z. \tag{25}$$

Adding the works

$$\mathcal{E} = -2Q|\vec{E}|z = -2Q|\vec{E}|r\cos\theta = -\vec{\mu}_\mathrm{e}\cdot\vec{E}, \tag{26}$$

where θ is the angle between $\vec{E}$ and $\vec{\mu}_\mathrm{e}$.

Equivalently, the potential energy can be defined as the work done by the torque $\vec{\tau}$ on $\vec{\mu}_\mathrm{e}$ (Figure 1C) when rotating it from the reference orientation to the orientation described by the angle θ (between $\vec{E}$ and $\vec{\mu}_\mathrm{e}$). The reference angle for $z_{+,0} = z_{-,0}$ is $\pi/2$, therefore,

$$\mathcal{E} = \int\limits_{\frac{\pi}{2}}^{\theta} |\vec{\tau}|\mathrm{d}\theta' = \int\limits_{\frac{\pi}{2}}^{\theta} |\vec{\mu}_\mathrm{e}||\vec{E}|\sin\theta'\mathrm{d}\theta' = -|\vec{\mu}_\mathrm{e}||\vec{E}|\cos\theta = -\vec{\mu}_\mathrm{e}\cdot\vec{E}. \tag{27}$$

0.6.2 Current loop as a magnetic dipole

Now we derive what is the magnetic dipole of a circular loop with an electric current. The magnetic moment is defined by the torque $\vec{\tau}$ it experiences in a magnetic field $\vec{B}$ (Eq. 5):

$$\vec{\tau} = \vec{\mu}\times\vec{B}, \tag{28}$$

[7] Do not get confused: $\mathcal{E}$ (scalar) is the energy and $\vec{E}$ (vector) is electric intensity.

Therefore, we can calculate the magnetic moment of a current loop if we place it in a magnetic field $\vec{B}$. Let us first define the geometry of our setup. Let the axis z is the normal of the loop and let $\vec{B}$ is in the xz plane ($\Rightarrow B_y = 0$). The vector product in Eq. 5 then simplifies to

$$\tau_x = \mu_y B_z, \tag{29}$$
$$\tau_y = \mu_z B_x - \mu_x B_z, \tag{30}$$
$$\tau_z = -\mu_y B_x. \tag{31}$$

Note that we assume that the electric current in the loop and the magnetic field are independent. The current is not induced by $\vec{B}$ but has another (unspecified) origin, and $\vec{B}$ is not a result of the current, but is introduced from outside.

As the second step, we describe the electric current in the loop. The electric current is a motion of the electric charge. We describe the current as a charge Q homogeneously distributed in a ring (loop) of a mass m which rotates with a circumferential speed v. Then, each element of the loop of a infinitesimally small length $\mathrm{d}l = r\mathrm{d}\varphi$ contains the same fraction of the mass $\mathrm{d}m$ and of the charge $\mathrm{d}Q$, moving with the velocity $\vec{v}$. The direction of the vector $\vec{v}$ is tangent to the loop and the amount of the charge per the length element is $Q/2\pi r$. The motion of the charge element $\mathrm{d}Q$ can be described, as any circular motion, by the *angular momentum*

$$\mathrm{d}\vec{L} = \vec{r} \times \mathrm{d}\vec{p} = \mathrm{d}m(\vec{r} \times \vec{v}), \tag{32}$$

where $\vec{r}$ is the vector defining the position of the charge element $\mathrm{d}Q$ (Figure 2A). In our geometry, $\vec{r}$ is radial and therefore always perpendicular to $\vec{v}$. Since both $\vec{r}$ and $\vec{v}$ are in the xy plane, $\mathrm{d}\vec{L}$ must have the same direction as the normal of the plane. Therefore, the x and y components of $\mathrm{d}\vec{L}$ are equal to zero and the z component is constant and identical for all elements (note that $\vec{r}$ and $\vec{v}$ of different elements differ, but $\vec{r} \times \vec{v}$ is constant, oriented along the normal of the z axis and with the size equal to rv for all elements). It is therefore easy to integrate $\mathrm{d}\vec{L}$ and calculate $\vec{L}$ of the loop

$$L_x = 0, \tag{33}$$
$$L_y = 0, \tag{34}$$
$$L_z = rv \int_{\mathrm{loop}} \mathrm{d}m = mrv. \tag{35}$$

As the third step, we examine forces acting on $\mathrm{d}Q$. The force acting on a moving charge in a magnetic field (the Lorentz force) is equal to

$$\vec{F} = Q(\vec{E} + \vec{v} \times \vec{B}), \tag{36}$$

but we are now only interested in the magnetic component $\vec{F} = Q(\vec{v} \times \vec{B})$. The force acting on a single charge element $\mathrm{d}Q$ is

$$\mathrm{d}\vec{F} = \mathrm{d}Q(\vec{v} \times \vec{B}) = \frac{Q}{2\pi r}\mathrm{d}l(\vec{v} \times \vec{B}) = \frac{Q}{2\pi}(\vec{v} \times \vec{B})\mathrm{d}\varphi. \tag{37}$$

The key step in our derivation is the definition of the torque

$$\vec{\tau} = \vec{r} \times \vec{F} = Q\vec{r} \times (\vec{v} \times \vec{B}), \tag{38}$$

which connects our analysis of the circular motion with the definition of $\vec{\mu}$ (Eq. 5). The torque acting on a charge element is (Figure 2B)

$$\mathrm{d}\vec{\tau} = \vec{r} \times \mathrm{d}\vec{F} = \frac{Q}{2\pi}\vec{r} \times (\vec{v} \times \vec{B})\mathrm{d}\varphi = \frac{Q}{2\pi}\left(\vec{v}(\vec{r} \cdot \vec{B}) - \vec{B}\underbrace{(\vec{r} \cdot \vec{v})}_{=0}\right)\mathrm{d}\varphi = \frac{Q}{2\pi}(\vec{r} \cdot \vec{B})\vec{v}\mathrm{d}\varphi. \tag{39}$$

where a useful vector identity $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$ helped us to simplify the equation because $\vec{r} \perp \vec{v}$. Eq. 39 tells us that the torque has the same direction as the velocity $\vec{v}$ ($\vec{v}$ is the only vector on the right-hand side because $\vec{r} \cdot \vec{B}$ is a scalar). In our coordinate frame, $v_x = -v \sin\varphi$, $v_y = v \cos\varphi$, $v_z = 0$, and $\vec{r} \cdot \vec{B} = r_x B_x + r_y B_y + r_z B_z = r_x B_x = B_x r \cos\varphi$ ($\vec{r} \cdot \vec{B}$ is reduced to $r_x B_x$ in our coordinate frame because $B_y = 0$ and $r_z = 0$). Therefore, we can calculate the components of the overall torque $\vec{\tau}$ as (Figure 2C)

$$\tau_x = -\frac{Qrv}{2\pi}B_x \int_0^{2\pi} \sin\varphi \cos\varphi \mathrm{d}\varphi = -\frac{Qrv}{4\pi}B_x \int_0^{2\pi} \sin(2\varphi)\mathrm{d}\varphi = 0, \tag{40}$$
$$\tau_y = \frac{Qrv}{2\pi}B_x \int_0^{2\pi} \cos^2\varphi \mathrm{d}\varphi = \frac{Qrv}{4\pi}B_x \int_0^{2\pi} (1 + \cos(2\varphi))\mathrm{d}\varphi = \frac{Qrv}{2}B_x, \tag{41}$$
$$\tau_z = 0. \tag{42}$$

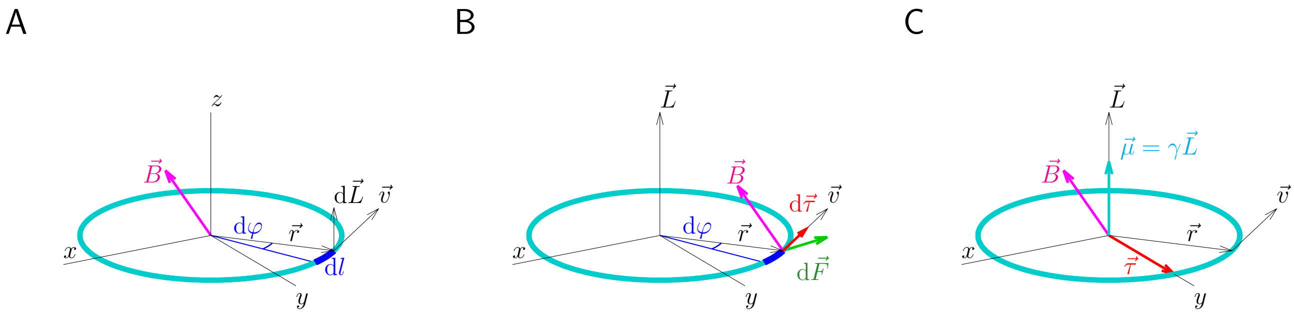

Figure 2: Current loop as a magnetic dipole. The loop of radius r and length $2\pi r$, charge Q and mass m is shown in cyan. A magnetic induction $\vec{B}$ of an external field is shown in magenta. The coordinates are chosen such that the loop is placed in the xy plane and $\vec{B}$ in the xz. An element of charge $\mathrm{d}Q$ (moving with the velocity $\vec{v}$), mass $\mathrm{d}m$ and length $\mathrm{d}l = r\mathrm{d}\varphi$ is shown in blue. The angular momentum of the blue element is $\mathrm{d}\vec{L} = \vec{r} \times \vec{v}\mathrm{d}m$ (Panel A). The total angular momentum is $\vec{L} = \vec{r} \times \vec{v}m$ (Panel B). The force $\mathrm{d}\vec{F} = \vec{v} \times \vec{B}\mathrm{d}Q$ and the torque $\mathrm{d}\tau = \vec{r} \times \mathrm{d}\vec{F}$ acting on the blue element are depicted as the green and red arrows in Panel B. The torque acting on the whole loop and the magnetic moment experiencing the torque in the field $\vec{B}$ are shown as the red and cyan arrows in Panel C.

Comparison with Eqs. 29–31 immediately shows that

$$\mu_x = 0, \tag{43}$$
$$\mu_y = 0, \tag{44}$$
$$\mu_z = \frac{Qrv}{2} \tag{45}$$

and comparison with Eqs. 33–35 reveals that the magnetic dipole moment of the current loop is closely related to the angular momentum $\vec{L} = \vec{r} \times m\vec{v}$:

$$\vec{\mu} = \frac{Q}{2m}\vec{L}. \tag{46}$$

0.6.3 Precession

Angular momentum of a particle moving in a circle is defined as $\vec{L} = m\vec{r} \times \vec{v}$ (Eq. 32), where $\vec{r}$ defines position of the particle and m and $\vec{v}$ are the mass and the velocity of the particle, respectively (Figure 3A). The change of $\vec{L}$ is described by the time derivative of $\vec{L}$.

$$\frac{\mathrm{d}\vec{L}}{\mathrm{d}t} = m\frac{\mathrm{d}(\vec{r} \times \vec{v})}{\mathrm{d}t} = m\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} \times \vec{v} + m\vec{r} \times \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = m\underbrace{(\vec{v} \times \vec{v})}_{0} + \vec{r} \times m\vec{a}. \tag{47}$$

According the second Newton's law, $m\vec{a}$ is equal to the force acting on the particle (changing $\vec{L}$)

$$\frac{\mathrm{d}\vec{L}}{\mathrm{d}t} = \vec{r} \times m\vec{a} = \vec{r} \times \vec{F} = \vec{\tau}, \tag{48}$$

where $\vec{F}$ is the force and $\vec{\tau}$ is the corresponding torque. The change of the angular momentum of a current loop due to an external force can be calculated in the same manner (Figure 3). For an infinitesimal element of the loop,

$$\frac{\mathrm{d}(\mathrm{d}\vec{L})}{\mathrm{d}t} = \vec{r} \times \vec{a}\,\mathrm{d}m = \vec{r} \times \mathrm{d}\vec{F} = \mathrm{d}\vec{\tau}. \tag{49}$$

In a homogeneous magnetic field, the force acting on all elements is the same and integration of the individual elements is as easy as in Eq. 35, resulting in Eq. 48, where the force $\vec{F}$ and the torque $\vec{\tau}$ now act on the angular momentum of the whole loop. Because $\vec{\mu} = \gamma\vec{L}$ (Eq. 19) and $\vec{\tau} = \vec{\mu} \times \vec{B}$ (Eq. 5, the the magnetic moment of a current loop in a homogeneous magnetic field changes as

$$\frac{\mathrm{d}\vec{\mu}}{\mathrm{d}t} = \gamma\vec{r}\times\vec{F} = \gamma\vec{\tau} = \gamma\vec{\mu}\times\vec{B} = -\gamma\vec{B}\times\vec{\mu}. \tag{50}$$

Rotation of any vector, including $\vec{\mu}$ can be described using the angular frequency $\vec{\omega}$ (its magnitude is the speed of the rotation in radians per second and its direction is the axis of the rotation):

$$\frac{\mathrm{d}\vec{\mu}}{\mathrm{d}t} = \vec{\omega}\times\vec{\mu}. \tag{51}$$

Comparison with Eq. 50 immediately shows that $\vec{\omega} = -\gamma\vec{B}$.

0.6.4 Electromotive force (voltage)

We can use a simple example to analyze the induced voltage quantitatively. This voltage (the electromotive force) is an integral of the electric intensity along the detector loop. Stokes' theorem (see B9) allows us to calculate such integral from Eq. 21.

$$\oint_L \vec{E}\mathrm{d}\vec{l} = -\int_S \frac{\partial\vec{B}}{\partial t}\mathrm{d}\vec{S} = S\frac{\partial\vec{B}}{\partial t}\cdot\vec{n}, \tag{52}$$

where S is the area of the loop and $\vec{n}$ is the normal vector to the loop. If the distance r of the magnetic moment from the detector is much larger than the size of the loop, the magnetic induction of a field which is generated by a magnetic moment $\vec{\mu}$ rotating in a plane perpendicular to the detector loop and which crosses the loop (let us call it B_x) is[8]

$$B_x = \frac{\mu_0}{4\pi}\frac{2\mu_x}{r^3}. \tag{53}$$

As $\vec{\mu}$ rotates with the angular frequency ω, $\mu_x = |\mu|\cos(\omega t)$, and

$$\frac{\partial B_x}{\partial t} = -\frac{\mu_0}{4\pi}\frac{2}{r^3}|\mu|\omega\sin(\omega t). \tag{54}$$

Therefore, the oscillating induced voltage is

$$\oint_L \vec{E}\mathrm{d}\vec{l} = \frac{\mu_0}{4\pi}\frac{2|\mu|S}{r^3}\omega\sin(\omega t). \tag{55}$$

[8] We describe the field generated by a magnetic moment in more detail later in Section 8.1 when we analyze mutual interactions of magnetic moments of nuclei.

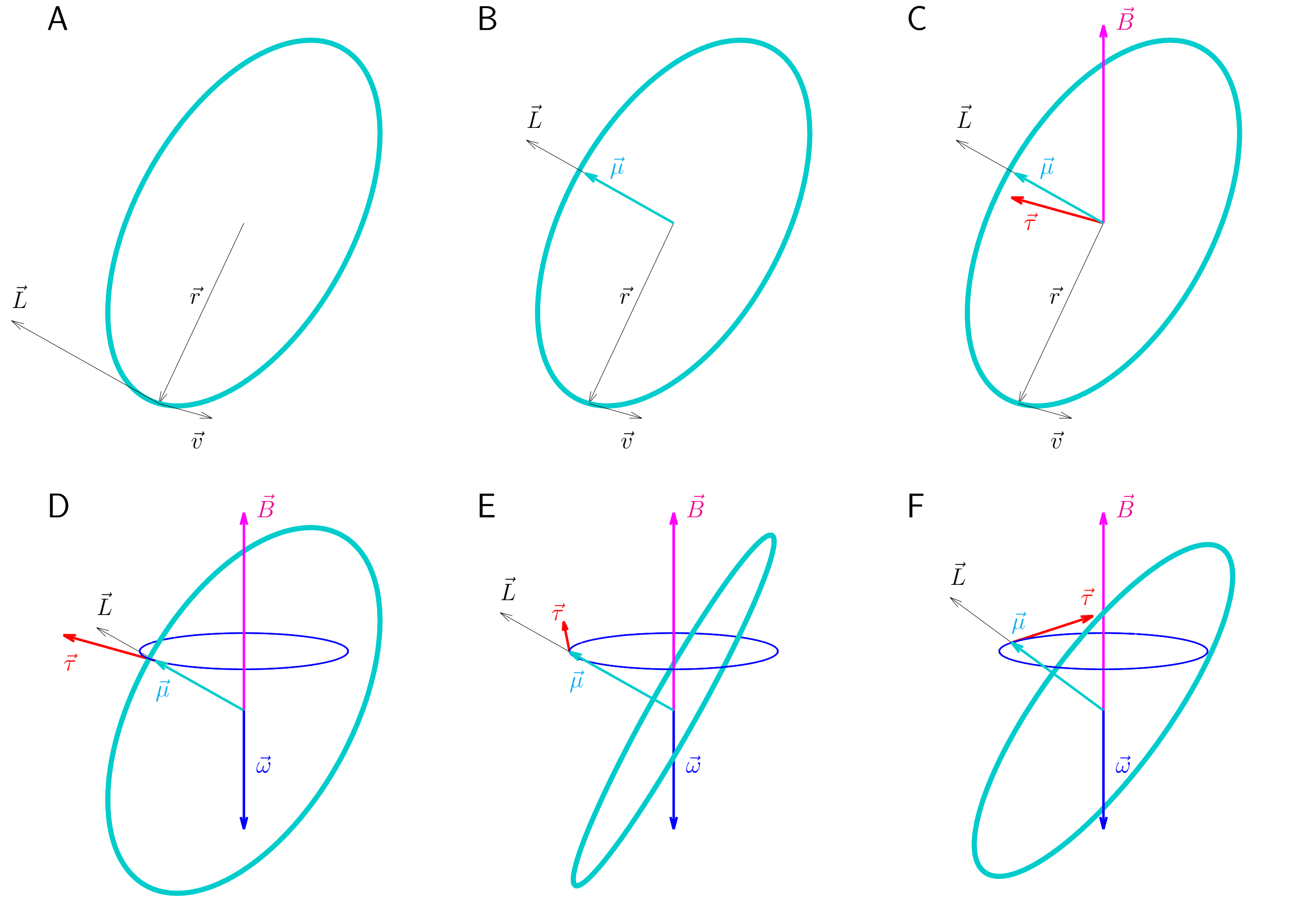

Figure 3: Classical description of precession of a current loop in a homogeneous magnetic field. Angular momentum $\vec{L}$ of a charged particle of the mass m moving in a circular loop (shown in cyan in Panel A) randomly oriented in space is given by the vector product of the actual position vector of the particle $\vec{r}$ and the actual particle's velocity $\vec{v}$ ($\vec{L} = m\vec{r} \times \vec{v}$). Note that size and direction of $\vec{L}$ is the same for all positions of the particle along the circle (for all possible vectors $\vec{r}$). The angular momentum $\vec{L}$ of a current loop of the same mass and the magnetic moment $\vec{\mu}$ (cyan arrow), proportional to $\vec{L}$ are shown in Panel B. The proportionality constant is γ (Eq. 19). In a presence of a vertical static magnetic field $\vec{B}$ (magenta arrow in Panel C), the loop experiences a torque $\vec{\tau} = \vec{\mu} \times \vec{B}$ (Eq. 5), shown as the red arrow in Panel C. This torque (red arrow moved to the tip of the cyan arrow in Panel D) acts on $\vec{\mu}$, which precesses about $\vec{B}$. Two snapshots of the precessing $\vec{\mu}$ (with the loop) are shown in Panels E and F. The tip of the cyan arrow representing $\vec{\mu}$ rotates about $\vec{B}$ (the blue circle) with the angular frequency $\vec{\omega} = -\gamma B$.