

# Advanced Topics in Cosmology

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## Abstract

These are lecture notes of Advanced Topics of Cosmology

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# 1 Cosmology:Preview

## 1.1 Units

We mostly use the natural system of units where the Planck constant, speed of light and the Boltzman constant are equal to one

$$\hbar = c = k_B = 1 . \quad (1)$$

Then the mass  $M$ , energy  $E$  and temperature  $T$  have the same dimensions since

$$[E] = [Mc^2] = [M] \quad (2)$$

and also we have

$$[E] = [k_B T] = [T] = [M] . \quad (3)$$

Time  $t$  and length  $l$  have in natural system dimension  $[M]^{-1}$  as follows from the fact that

$$[E] = [\hbar\omega] = [\omega] = [t^{-1}] \quad (4)$$

so that  $[t] = [M]^{-1}$ . In the same way we have

$$[l] = [ct] = [t] = [M]^{-1} . \quad (5)$$

It is useful to know coefficients that relate various units

Quantity	SI dimensions	Natuatural dimensions	Conversions
mass	$kg$	$M$	$1GeV = 1.8 \times 10^{-27}kg$
length	$m$	$M^{-1}$	$1GeV^{-1} = 0.197 \times 10^{-15}m$
time	$s$	$M^{-1}$	$1GeV^{-1} = 6.58 \times 10^{-25}s$
energy	$kg \cdot m^2 \cdot s^{-2}$	$M$	$1GeV = 5.39 \times 10^{-19}kg \cdot m \cdot s^{-1}$
momentum	$kg \cdot m \cdot s^{-1}$	$M$	$1GeV = 5.39 \times 10^{-19}kg \cdot m \cdot s^{-1}$
velocity	$m \cdot s^{-1}$		$1 = 2.998 \times 10^8 m \cdot s^{-1}$
cross section	$m^2$	$M^{-2}$	$1GeV^{-2} = 0.389 \times 10^{-31}m^2$
force	$kg \cdot m \cdot s^{-2}$	$M^2$	$1GeV^2 = 8.19 \times 10^5 \text{Newton}$

The traditional unit of length in cosmology is Megaparsec

$$1 \text{ Mpc} = 3.1 \times 10^{22}m . \quad (6)$$

It is interesting to mention the several units of length that are used in astronomy. Besides the metric system in use are the astronomical unit ( $a.u.$ ) which is the average distance from the Earth to the Sun

$$1 \text{ a.u.} = 1.5 \times 10^{11}m \quad (7)$$

Further, there is a light year, the distance that a photon travels in one year

$$1 \text{ year} = 3.16 \times 10^7s , 1 \text{ light year} = 0.95 \times 10^{16}m \quad (8)$$

parsec ( $pc$ )-distance from which an object of size  $1a.u.$  is seen at angle  $1arc$  second

$$1 \text{ pc} = 2.1 \cdot 10^5 a.u. = 3.3 \text{ light year} = 3.1 \times 10^{16}m \quad (9)$$

It is instructive to give distances of various objects expressed in above units.  $10a.u.$  is the average disance to Saturn,  $30a.u.$  is the same for Pluto,  $100a.u.$  is the estimate of the maximum distance which can be reached by solar wind (particles emitted by the Sun). The nearest stars-Proxima and Alpha Centauri are at  $1.3pc$  from the Sun. The distance to Arcturus, which is one of the brightest stars on the night sky, is about 36 light years. The distance to Capella, which is one of the most brightest stars on night sky, is more than 42 light years. The distances to Canopus and Betelgeuse, which are again one of the most brightest stars on the night sky, are about  $100pc$  ( $300 \text{ l.y.}$ ) and  $200pc$  ( $642 \text{ l.y.}$ ) respectively. Crab Nebula-the remnant of supernova  $SN1054$  is  $2kpc$  ( $6523 \text{ l.y.}$ ) away from us.

The next point on the scale of distance is  $8kpc$ . This is the distance from the Sun to the center of our Galaxy. Our Galaxy is of spiral type, the diameter of its disc is about  $30kpc$  and the thickness of the disc is about  $250pc$ . The distance to the nearest dwarf galaxies that are satelites of our Galaxy is about  $30kpc$ . Fifteen of these satellites are known, the largest of them are Large and Small Magellanic Clouds which are about  $50kpc$  away. It is also interesting to note that only eight Milky Way satellites were known by 1994.

The mass density of the usual matter in usual (not dwarf) galaxies is about  $10^5$  higher than the average over Universe.

The nearest usual galaxy-the spiral galaxy  $M31$  in Andromeda constellation- is  $800kpc$  away from the Milky Way. Another nearby galaxy is in Triangulum constellation. Our Galaxy together with Andromeda and Triangulum galaxies , their satellites and other 35 smaller galaxies constitute the Local Group which is the gravitationally bound object consisting of about 50 galaxies.

The next scale is the size of clusters of galaxies which is  $1 - 3Mpc$ . Rich clusters contain thousands of galaxies. The mass density in clusters exceeds the average density over the Universe by a factor of a hundred and even sometimes a thousand. The distance to the center of the nearest cluster, which is the Virgo constellation (souhvězdí Panny), is about  $15Mpc$ . Clusters of galaxies are the largest gravitationally bound systems in the Universe.

## 1.2 The Universe Today

In this subsection we give a brief overview of properties of the present Universe.

### 1.2.1 Homogeneity and isotropy

When we look at the night sky we see that Universe looks the same from all directions which is sign of isotropy. Further, it is natural to presume that our place in the Universe is not exceptional. Mathematically we say that Universe is homogeneous and isotropic on large scales. We should stress that this claim holds on the large spatial scales only. We can obtain a rough estimate how large this scale is we know that the sizes of the largest structures in the Universe which are superclusters of galaxies are tens of Megaparsec. Then we say that on larger scales all parts of the Universe look the same which is a physical explanation of homogeneity.

### 1.2.2 Expansion

The Universe expands which means that the space stretches out. This expansion is characterized by scale factor  $a(t)$  which is a growing function of time. Now the distance between two objects is proportional to  $a(t)$  and the number density of particles is proportional to  $a^{-3}(t)$ . The rate of the cosmological expansion is characterized by the Hubble parameter

$$H(t) = \frac{\dot{a}(t)}{a(t)} \quad (10)$$

It is clear that this parameter depends on time  $t$  where its present value is denoted by  $H_0$ . The expansion of the Universe implies that the wavelength of the photon emitted in the past is growing. This fact is known as redshift and denoted as  $z$  and it corresponds to the ratio of photon wavelengths at absorption and emission

$$\frac{\lambda_{ab}}{\lambda_{em}} = 1 + z . \quad (11)$$

Clearly this expression depends on time of emission which means that it depends on the distance to source. Further, which is very important, it is directly measurable

quantity when the wavelength of emission is determined by the physics of emission process while  $\lambda_{ab}$  is measured wavelength.

For  $z \ll 1$  the distance to the source  $r$  and the redshift are related by Hubble law

$$z = H_0 r, \quad z \ll 1. \quad (12)$$

For large  $z$  the dependence is more complicated and we will analyse it in more details in this lecture. The present value of the Hubble constant is

$$H_0 = (70, 5 \pm 1.3) \frac{km}{s \cdot Mpc}. \quad (13)$$

It is also convenient to use following parameterization of the constant  $H_0$  as

$$H_0 = h \cdot 100 \frac{km}{s \cdot Mpc}. \quad (14)$$

Hubble constant is directly related to the age of the Universe and to the size of its observable part. In fact, Hubble constant has physical dimension  $[t^{-1}]$  and hence the present Universe is characterized by time scale

$$H_0^{-1} = \frac{1}{h \cdot 100} \frac{s \cdot Mpc}{km} = h^{-1} \cdot 3 \cdot 10^{17} s \approx 1.4 \cdot 10^{10} \text{ years}. \quad (15)$$

Further,  $H_0^{-1}$  is roughly the size of the observable part of the Universe. These notions suggest that it is natural to trace the evolution of the Universe back in time and we reach the time event known as Big Bang which is the moment when the classical evolution begins. Then the size of observable part of the Universe is the distance traveled by signals emitted at the Big Bang that move with the speed of light. At this place it is important to stress that the actual size of the Universe is larger than the horizon size. In fact, according to General Relativity the spatial size of the Universe could be infinite. This claim is directly related to the notion of **Spatial flatness**. As we will see later, homogeneity and isotropy does not imply that the 3-dimensional spatial subspace of the Universe is flat (Euclidean) or equivalently, it has zero spatial curvature. We can also find solutions that are homogeneous and isotropic even for 3-sphere with positive spatial curvature and 3-hyperboloid with negative spatial curvature. In order to decide which of these possibilities is realized in nature we have to perform appropriate observation. Fundamental observation result of the recent years is the fact that the spatial curvature of our Universe is very small or even equal to zero. Further, data from Cosmic microwave background (CMB) measurement are consistent with the fact that the Universe has trivial spatial topology. Note that the flat space-time could have non-trivial topology as for example torus  $S^1 \times S^1 \times S^1$ . However if it were true we should observe some regular pattern in the CMB. However such pattern is absent.

### 1.2.3 Hubble Tension

As we know, Universe is expanding and in fact, its expansion is accelerating. Generally the rate of expansion is given by function  $H(z)$  where the Hubble factor  $H_0$

gives the rate at which the scale factor  $a(z)$  in a Friedmann equation is changing today, corresponding to  $z = 0$ . There are several methods how we can find value of  $H_0$  but we are not going into details. The determination of  $H_0$  from the cosmic microwave background corresponds to the early time  $z = 1000$  yields  $H_0 = 67.36 \pm 0.54 \text{ kms}^{-1}\text{Mpc}^{-1}$ . On the other hand measurements based on late-time observables  $z \leq 10$ , for example, on type Ia supernovova, give the value  $H_0 = 73,3 \pm 0.8 \text{ kms}^{-1}\text{Mpc}^{-1}$ . This disagreement between high-redshift and low-redshift measurements is known as Hubble tension which was discovered rather recently. Indeed, this tension has existed since the first release of results from Planck in 2013 and has grown in significance with the improvement of the data. The resolution of this tension is not known and it is remarkable that there are about  $10^2$  theories that want to explain Hubble tension that however require untested modifications of the standard model of particle physics or general relativity. On the other hand there is more conservative possibility that is based on the idea that the treatment of systematic errors is inadequate.

#### 1.2.4 Warm Universe

It is well known that the present Universe is filled by the gas of non-interacting photons which is known as Cosmic Microwave Background (CMB) which was discovered in 1964. The number density of CMB photons is about 400 in cubic centimeter. It is also very remarkable that the energy distribution of photons is thermal and has Planckian spectrum with present temperature

$$T_0 = 2.726 \pm 0.001K . \quad (16)$$

Further, the temperature of photons as we measure on celestial sphere is the same at the level of accuracy  $10^{-4}$  which means that the photons come from different directions have the same properties which is again evidence of the homogeneity and isotropy of the Universe. On the other hand the temperature still depends on the direction in the sky where it was shown that the angular anisotropy is of order

$$\frac{\delta T}{T_0} \sim 10^{-4} - 10^{-5} . \quad (17)$$

It is a very remarkable fact that this anisotropy gives us important information about the early Universe and also can serve as the check of the validity of all cosmological models. Another important fact about the cosmic microwave background is that its presence defines a special reference frame in the Universe which is the frame where the gas of photons is at rest. Further, the Solar system moves with respect to this frame towards the Hydra constellation. The velocity of this motion determines the dipole component of the measured CMB anisotropy

$$\delta T_{dipol} = 3.346 \text{ mK} . \quad (18)$$

CMB temperature  $T$  depends on the direction  $\mathbf{n}$  on the celestial sphere. Then it is natural to perform decomposition over spherical harmonics  $Y_{lm}(\mathbf{n})$ . These functions

define a basis of function on a sphere. Let us now define the temperature fluctuation  $\delta T$  in the direction  $\mathbf{n}$  in the form

$$\delta T(\mathbf{n}) = T(\mathbf{n}) - T_0 - \delta T_{dipole} . \quad (19)$$

According to the previous discussion it is natural to perform following decomposition

$$\delta T(\mathbf{n}) = \sum_{l,m} a_{l,m} Y(\mathbf{n}) , \quad (20)$$

where the coefficients  $a_{l,m}$  obey the condition  $a_{l,m}^* = (-1)^m a_{l,-m}$  that ensure that  $\delta T$  is real. The observation data show that temperature fluctuations  $\delta T(\mathbf{n})$  are Gaussian random field that means that the coefficients  $a_{l,m}$  are statistically independent for different  $l$  and  $m$

$$\langle a_{l,m} a_{l',m'}^* \rangle = C_{lm} \delta_{ll'} \delta_{mm'} , \quad (21)$$

where brackets mean averaging over an ensemble of Universes. The coefficients  $C_{lm}$  do not depend on  $m$  in isotropic Universe so that  $C_{lm} = C_l$ . Then we obtain

$$\langle \delta T(\mathbf{n}_1) \delta T(\mathbf{n}_2) \rangle = \sum_l \frac{2l+1}{4\pi} C_l P_l(\cos \theta) . \quad (22)$$

where  $P_l$  are the Legendre polynomials which are functions of the angle  $\theta$  between two vectors  $\mathbf{n}_1$  and  $\mathbf{n}_2$ . For example, in case of the temperature fluctuation when  $\mathbf{n}_1 = \mathbf{n}_2$  and hence  $\theta = 0$  we obtain

$$\langle \delta T^2 \rangle = \sum_l \frac{2l+1}{4\pi} C_l \quad (23)$$

When we measure CMB anisotropy we get large set of data which are the values of  $C_l$  for different  $l$ . It is important that this set is determined by numerous parameters of the present and early Universe and hence we gain many informations about early Universe.

### 1.2.5 The Energy Balance in the Present Universe

Today the present energy density in spatial flat Universe is given by the formula

$$\rho_c = \frac{3}{8\pi} H_0^2 M_{pl}^2 , \quad (24)$$

which is the energy density in our Universe today

$$\rho_c = 1.04 \cdot h^2 \cdot 10^{-5} \frac{GeV}{cm^3} \approx 0.52 \cdot 10^{-5} \frac{GeV}{cm^3} . \quad (25)$$

It is very remarkable that the cosmological observations imply that the contribution of baryons (protons and nuclei) into the present energy density is

$$\Omega_B = \frac{\rho_B}{\rho_c} = 0.046 . \quad (26)$$

It is also very interesting that only 10 % of baryons are in the stars. Remaining 90% are in hot gas. Further, the contribution of relic neutrinos of all types is even smaller and it is restricted as

$$\Omega_\nu \equiv \frac{\sum \rho_{\nu_i}}{\rho_c} < 0.004 - 0.02 , \quad (27)$$

where the sum runs over three species of neutrinos  $\nu_e, \nu_\mu, \nu_\tau$  and anti-neutrinos  $\bar{\nu}_e, \bar{\nu}_\mu, \bar{\nu}_\tau$ . It is also important to stress that cosmological observations do not bring contribution to the debate about the mass of neutrinos so that the contribution given above could be even smaller. Finally other known stable particles give very small fractions to the total energy content of the Universe. In other words, the dominant material in the present Universe is unknown. The unknown form of the matter can be divided into two fractions where one part can cluster while the second one not. The matter that clusters is known as **dark matter**.

It is remarkable that the results of Big Bang Nucleosynthesis, CMB anisotropy and structure formation suggest that the dark matter cannot consist of known particles. The most popular scenario is that it is made from new stable particles that were non-relativistic in the past and remain non-relativistic today. We call such a form of matter as *cold* dark matter. Unfortunately such a particle has not been detected yet and hence the origin of dark matter is unsolved problem of particle physics.

Even more serious problem is the problem of the remaining 75% of energy in the present Universe that is homogeneously spread over space. This form of matter does not contain known or unknown particles but rather some unknown form of energy of vacuum type. At present we do not know the origin of this mysterious matter and there are many conjectures, for example, we can imagine that this vacuum energy corresponds to cosmological constant which is constant in time and space. Another possibility is that the dark energy corresponds to the energy density of some specific scalar field which is called as quintessence. In this case the vacuum density is not constant while it weakly depends on time.

As we mentioned above one of the candidates to dark energy is vacuum. In particle theory we often ignore vacuum energy since it is always possible to consider this vacuum energy as the reference point for energy while we are interested in the masses and energies of particles which are excitations above vacuum. The situation changes when we take general relativity into account since every form of energy is source of gravity. The problem with the vacuum energy is that the energy density has to have dimension  $M^4$  where  $M$  is some energy or mass scale. It is natural that this mass scale corresponds to the mass scale that characterizes given interaction. These scales are  $1 GeV$  for strong interactions,  $100 GeV$  for electroweak interactions and  $M_{pl} \sim 10^{19} GeV$  for gravitational interactions. Hence we have following estimate

$$\begin{aligned} \rho_{vac} &\sim 1 GeV^4 , && \text{strong interactions} , \\ &\sim 10^8 GeV^4 , && \text{electroweak interactions} , \\ &\sim 10^{76} GeV^4 , && \text{gravitational interactions} . \end{aligned} \quad (28)$$



It is fundamental problem of contemporary physics that these estimates exceeds by many orders of magnitude of the experimental value of dark energy density

$$\rho_{\Lambda} \sim \rho_c \sim 10^{-5} , \frac{GeV}{cm^3} \sim 10^{-46} GeV^4 . \quad (29)$$

This problem is known as *cosmological constant problem*. It is not known why this experimental value is so different from theoretical estimates and it is also great mystery why it is different from zero. Further, if we imagine that the vacuum energy density is larger than the previous one that this scale would be incompatible with our existence since large and positive vacuum energy would lead to very fast cosmological expansion. Such a formation would not allow formation of galaxies and stars. On the other hand the Universe with large and negative vacuum energy would recollapse very quickly so that no structure could form. It is also interesting problem that the three different energy components-dark energy, dark matter and baryons are of the same order of magnitude in the present Universe. In fact, these components have different origins so that we should expect that they would give contributions of different orders of magnitude.

Another possibility is that the dark matter could be formed from black holes even it is not clear whether total mass of all black holes could be sufficient for all the dark matter. In fact, new analysis of supernovae implies that this is unprobably. Let us outline mechanism how we could detect such a black hole. A black hole of sufficient mass that passes in front of supernova acts as magnifying lens due to its gravitational field. As a result the the star should appear brighter and hence this black hole could be spotted as a dark dot in bright region. However the search for such an effect in over a thousand of supernovae brought empty result. This analysis was performed for black holes with masses greater than 0.01 times that our Sun and show that such objects can account for at most 40% of dark matter in Universe.

We should stress that the idea that the dark matter is consist from black holes is not new. In fact, S. Hawking was the first who proposed it already in 1974 and it attracted an attention of other physicists since it does not require some new particles in order to explain dark matter. As we know black hole are formed when massive stars implode however there are not enough stars that could be responsible for the generation of black holes in such an amount to consist dark matter. On the other hand current theories predict that black holes could be formed in early Universe (primordial black holes) from the gravitational colapse of dense regions of matter. Moreover, it turns out that the total mass of these black holes could be enough to account for dark matter.

There is a crucial difference between black holes that originate from the colapse of star and primordial black holes. In the first case the black holes have visible halo of gas remnants of collapsed star that swirl around them. In case of the primordial black holes the situation is different since they were formed before the first atoms were created and hence there are no disc around them and they are completely dark. Then in order to detect such black hole we should search for their gravitational effect on light. Previous studies of primordial black holes with the help of gravitational effects ruled out the large number of primordial black holes with the

masses in the ranges  $10 - 10^{-8}$  times that of our Sun. Then in 2015 LIGO-Virgo collaboration detected the first gravitational signal from two merging black holes and their analysis showed that the black holes they detected each had a mass more than 25 time greater than that of our Sun. This mass is much heavier than the mass expected for a black hole that is a result of dying star. Then it was natural to presume that LIGO collaboration has detected primordial black hole.

In order to detect black holes using the effect of gravitational lensing we have to analyze an effect when the path of light that is emitted by an object is bent by gravity or some massive object, as for example black hole, that is localized between the star and observers living on Earth. The black hole acts as lens since it focuses light rays closer and hence star appears brighter. Clearly more massive black hole makes the star to appear brighter. The similar analysis can be performed in case of supernovae that can serve as standard candle. However recent analysis of signals from over 1300 supernovae that are spread across most of the sky in the Northern hemisphere did not find supernovae that are brighter than expected. The conclusion from this calculation is that the cumulative mass of these black holes can account for only 40% of dark matter. In other words, we should abandon the idea that dark matter could be formed from the black holes only even if there is still a possibility that black holes are responsible for some small fraction of dark matter while the rest is coming from some other potential candidates as are for example weakly interacting massive particles (WIMP), sterile neutrinos.

### 1.2.6 Future of the Universe

As we will see in more details in following section, the future of the Universe is determined by its geometry and properties of dark energy. In more details, Einstein equations predict that the contribution of the spatial curvature into effective density is proportional to  $a^{-2}$  so that even if the spatial curvature is non-zero so that it will dominate over energy density of the non-relativistic matter that is proportional to  $a^{-3}$ . In other words at late times the main contribution to the energy is given by spatial curvature and dark energy which could be time dependent. Let us imagine situation that this dark energy depends on time and goes to zero sufficiently rapidly in future and the Universe has positive curvature (closed model). Then the expansion will slow down and eventually the Universe will end in singularity. On the other hand when the Universe has spatial negative curvature it will expand forever even if its expansion will slow down. The remarkable property of this model is that all distances will be larger and larger. For example, all systems that are not gravitationally bound disappear.

The situation is different in case when dark energy does not depend on time. Then the dark energy determines future of the Universe. In case of the positive energy the Universe will expand exponentially. On the other hand in case, even rather exotic, that the dark energy will become negative in future, the dark energy will slow the expansion down and the Universe will collapse to singularity.

It is important to stress that it is *in principle* impossible to predict the future

of Universe on the basis of cosmological observations that are performed at present. The reason for this claim is that these observations can determine the dependence or independence of dark energy on time in the past but cannot predict its dependence in future. In order to know future of the Universe we should know the nature of dark energy. However from present observations we can extrapolate the future evolution of the Universe in 10-20 mld years. During this period the Universe will expand at rate comparable to the present Hubble rate. To complete this discussion it is good to mention another possibility that the dark energy will *grow* in future. If this growth will be sufficiently fast the Universe will end in **Big Rip**. In this case the scale factor becomes infinite in finite time. As a result the interaction (for example, electromagnetic) will not be sufficient to keep them in the bound states that means that all bound states, as for example nuclei and atoms will disintegrate.

### 1.3 Universe in the Past

The fact that the Universe expands implies that it was denser and warmer in the past. Then the time evolution of the Universe can be described using GR and standard thermodynamics with the main outcome that matter had higher and higher temperature and density when we approach the beginning of the Universe. It is also important that at most stages of its evolution matter was in thermal equilibrium. **The Hot Big Bang Theory** is theory of this time evolution of the Universe. However it turns out that there are some exceptional moments in the evolution of the Universe.

#### 1.3.1 The Recombination

At very early times when the temperature was high enough, the binding energy was not able to keep electrons in atoms and the electron was in the state of baryon-electron- photon plasma. The temperature of recombination-where recombination means transition from plasma to gas-is determined by binding energy in hydrogen atom which is  $13.6 eV$ . However this is only rough estimate and it can be shown taht the recombination occurs at lower temperature  $T \sim 0.3eV$ . Before the epoch of recombination the photons actively interact with electrons and they are scatered by these electrons. On the other hand after recombination the neutral gas is transparent for photons. This is an important consequence since CMB that we observe today is a consequence of the recombination epoch and hence it gives us information about the properties of the Universe at the epoch when its temperature as about  $0.26 eV$  that corresponds to temperature  $T = 3000K$  and the age of the Universe was about 370 000 years. We also saw that high degree of isotropy of CMB shows that the Universe was homogenous at recombination. Further, the density perturbations  $\frac{\delta\rho}{\rho}$  were comparable to the temperature fluctuations and were of order  $10^{-5}$ . However these small pertubations were crucial for the origin of structures that were the first stars, then galaxies and clusters of galaxies.

### 1.3.2 Big Bang Nucleosynthesis(BBN)

Another important epoch occurs at higher temperatures whose temperature is roughly determined by binding energies in nuclei which is  $1 - 10 \text{ MeV}$ . At higher temperatures protons and neutrons were free in cosmic plasma. However in the process of expansion of the Universe the temperature decreases and hence neutrons were bounded into nuclei. Then there are following light nuclei in primordial plasma: hydrogen, helium-4 (formed from two protons and two neutrons) and small amount of deuterium (one proton and one neutron in atomic nucleus), helium-3 (two protons and one neutron in atomic nucleus) and lithium-7 (three protons and four neutrons in atomic nucleus). In other words, heavier elements were not formed in the early Universe. In fact, heavy elements were produced during stellar evolution. For example, carbon, which is an important element in the nucleosynthesis, arises in the fusion of three  $^4\text{He}$ -nuclei. It is important that this process occurs at high densities that are reached in stellar interiors after hydrogen was burned out. Then all other elements are produced from carbon. For example, iron is produced in thermonuclear reaction in star. Iron is relatively light element so that in case of heavier elements can be produced during supernova explosions. It is important to stress that the BBN is the epoch that can be studied directly since it can be calculated with the help of GR and known microscopic physics which is physics of nuclei and weak interactions. As a consequence the good agreement between BBN theory and observation is one of the key arguments for the support of our theory of early Universe. Finally, BBN epoch lasted from  $1 - 100$  seconds after the Big Bang that corresponds to energies from  $1 \text{ MeV}$  to  $50 \text{ keV}$ .

### 1.3.3 Neutrino Decoupling

Neutrinos are very interesting particles with many remarkable properties. One of the most interesting facts is that they decouple from cosmic plasma at temperature  $2 - 3 \text{ MeV}$  as opposite to the case of photons when it happens at temperature  $0.26 \text{ eV}$ . Before this temperature the neutrinos are in thermal equilibrium with the rest of matter and after temperature of decoupling they freely propagate through the Universe. However we should stress that it is very difficult to detect these relic neutrinos. Further, even if the role of neutrinos in present Universe is not important, they played very significant role in the BBN theory. More precisely, neutrino components affect the expansion rate and hence cooling rate of plasma at the time of BBN. In summary, success of BBN theory gives decisive evidence for the existence of relic neutrinos.

### 1.3.4 Cosmological phase transitions

If we go back in time we come to the epochs whose results were not probed by observations so far. Then in order to describe these epochs we have to use our models of high energy physics beyond Standard model. As we will see in more details below, General Relativity predicts that at early times the Universe was much

hotter. In fact, in very short time intervals after Big Bang the temperature was of order of hundreds GeV. In fact, if we extrapolate time evolution as it is described by standard GR to the beginning of the Universe we find that the temperature diverges. However there we are beyond domain of our current physical theories and certainly physics beyond standard model is necessary. It turns out that very early times in the evolution of Universe can be called as epochs of phase transitions. Before we proceed to the description of these phenomena let us give simple example of phase transitions in solid state physics which is change of properties some material around Curie temperature  $T_c$ . For  $T < T_c$  there are ordered magnetic moments in material and hence it has ferromagnetic properties while for  $T > T_c$  these moments have chaotic positions and magnetic properties are lost.

- *Transition from quark-gluon plasma to hadronic matter*

This phase transition is determined by the energy scale of strong interaction and it is about  $200 \text{ MeV}$ . At much higher temperatures quark and gluons behave as individual particles even if they are strongly interacting when the temperature decreases. For temperatures lower than  $200 \text{ MeV}$  they are confined in colorless bound states known as **hadrons**. Hadrons are divided into two groups. The first one, **baryons** which are formed from three quarks (antibaryons from three antiquarks) and they have half-integer spins and hence are fermions. The second one are **mesons** which are formed from quark and anti-quark and hence they have integer spin so that they are bosons.

- *Electroweak transition*

At temperatures above  $100 \text{ GeV}$  which is an energy scale of electroweak interaction the Higgs condensation is zero and hence  $W-$  and  $Z-$  bosons have zero masses. The present phase corresponds to situation when the temperature is lower than  $100 \text{ GeV}$ , Higgs field has non-zero vacuum expectation value (in other words, Higgs field condensates) and  $W-$  and  $Z-$  bosons are massive.

- *Grand Unified Transition*

Physics strongly believe (and there are strong arguments for that) that at energies and temperatures above  $10^{16} \text{ GeV}$  the strong, weak and electromagnetic interactions unify into single force. In other words, there is a phase transition at temperature  $T_{GUT} \sim 10^{16} \text{ GeV}$ . However there is still a possibility that the maximum temperature at Universe is below  $T_{GUT}$  so that the phase of Grand Unification may not exist in the early Universe. Such a possibility occurs in many models of inflation where the reheat temperature is below  $T_{GUT}$ .

### 1.3.5 Baryon Asymmetry

The present Universe contains baryons which are protons (two  $u$ -quarks and one  $d$ -quark), neutrons (formed from two  $d$ -quarks and one  $u$ -quark) and practically no

antibaryons <sup>2</sup>. Anti-baryons are particles that have the same properties as baryons with exceptional property that they have opposite charges. In order to describe this abundance of baryons over anti-baryons we introduce a ration of baryon over photon densities. It is given as

$$\eta_B \equiv \frac{n_B}{n_\gamma} = 6.2 \cdot 10^{-10} . \quad (30)$$

The baryon number is conserved at sufficiently low energies and temperatures and that in the early Universe  $n_B/n_\gamma$  was of the same order o magnitude.

The situation changes at temperatures that are higher than hunderds MeV. In this case there were many quarks and anti-quarks that were continuously pair created and annihilated. In other words there were almost as many particles with negative baryon number (antiquarks) as those with positive baryon number (quarks). It can be shown that the number of quark-antiquark part at high energies was the same order as the number of photons so the baryon asymmetry is determined by the following ratio

$$\eta_B \sim \frac{n_q - n_{\bar{q}}}{n_q + n_{\bar{q}}} \sim 10^{-10} . \quad (31)$$

where  $n_q$  an  $n_{\bar{q}}$  are number densities of quarks and antiquarks. The previous ration can be manipulated to  $n_q \sim (1 + 10^{-10})n_{\bar{q}}$  that means that there was very tiny excess of baryons over anti-baryons. It is remarkable that this fact has origin of baryonic asymmetry since as the Universe expands and cools down antiquarks annihilate with quarks while uncompensated quarks remain and they form protons and neutrons.

Key problem of cosmology is to explain the very existence of the baryon asymmetry and also to understand its value which is given in the formula above. It turns out that this asymmetry existed in the Universe from the beginning which means that this is one of the initial date of the cosmological evolution. In fact, physicists feel that it is more natural to presume that the Universe was baryon symmetric at the beginning of its existence. The assymetry given in (30) was generated in the course of the cosmological evolution due to processes with baryon number non-conservation. There are many models that explain generation of the baryon assymetry. Unfortunately there is not unique answer to the problem of its origin. Most probably the baryon asymetry was generated at very high temperature at least 100 *GeV*. Further, the problem of baryon asymetry cannot be solved in with the help of Standard model of particle model.

### 1.3.6 Generation of Dark Matter

Today we do not know the origin of non-baryonic clustered dark matter. One possibility that the content of the dark matter is formed by stable particles that exist in models beyond Standard Model. In other words an existence of Dark Matter is strong argument for incompletness of the Standard Model. However the fact that there is no experimental evidence of the properties of these particles that form

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<sup>2</sup>Note that electic charge of *u*-quark is 2/3 while electric charge of *d*-quark is  $-1/3$ .

the dark matter makes very difficult to answer the question how dark matter was generated in the early Universe.

### 1.3.7 Structure Formation in the Universe

As we said above there are many important stages of the cosmological evolution of the Universe. The characteristic properties of these stages, nucleosynthesis or recombination is that they have finite time of duration. However it is important to stress that there is process in the Universe that began at a very early epoch but continues at present. This is the process of the *formation of structures* which were first stars, galaxies, clusters of galaxies and superclusters, where the order in which they are written means that smaller objects were formed earlier.

The theory that describes structure formation is based on the *Jeans instability* which is gravitational instability of matter density perturbations. However the crucial point is that these perturbations should exist at the very early stage of the cosmological evolutions even if they were very small in amplitude. Further, Hot Big Bang theory cannot explain an existence of these perturbations and predict their properties. In fact, inflation conjecture predicts the mechanism of generation of perturbations. What it is even more remarkable inflation is consistent with all cosmological data. The density of perturbations grows at the stage of the expansion of the Universe when the dominant component of energy is contained in the non-relativistic matter. The transition to this stage occurs 80000 years after Big Bang. Before this transition the Universe was so hot that the dominant contribution to the energy was hidden in the relativistic matter known as radiation. The transition from radiation to matter occurs at the epoch which is known as radiation matter equality. At this time the ratio of density perturbation to matter density is  $\frac{\delta\rho}{\rho} \sim 10^{-3} - 10^{-5}$ . Due to the fluctuation of matter there are regions of the higher density and they are sources of gravitational potential that attract surrounding matter. As a consequence matter flow to this region and hence the density becomes even higher which is physical description of gravitational instability. In case when the density is sufficiently high these regions become gravitationally bound and effectively they are not affected by cosmological expansion. For example, the size of this region does not grow despite of the expansion of the Universe. In fact, the gravitational interaction in this region implies its collapse to an object of even smaller size. This is the origin of the formation of *protostars* and *protogalaxies*. First stars were formed at  $z \sim 10$  and somewhat earlier and first galaxies somewhat later.

As it is clear from the analysis performed above the mass of object as galaxy or clusters of galaxies is determined by the size of the primordial overdense region. As a result number densities of galaxies and clusters and their mass distribution are consequences of the spectrum of primordial perturbations. Experimental data that are determined on structures in the Universe are consistent with the simplest flat primordial spectrum known as *Harrison-Zeldovich spectrum*. When we say flat we mean that the spectrum is scale invariant which means that in some sense perturbations of different sizes have one and the same amplitude.

There are another kind of perturbations which are perturbations in cosmic medium at recombination. These perturbations give famous CMB temperature anisotropy and polarization. In other words primordial spectrum can be also determined from CMB observations. It is very nice consistency check that the spectra found from CMB and structures are in good agreement with each other.

It is also very remarkable that structure formation gives an argument for the presence of the dark matter. In the absence of dark matter density perturbations would start to grow after recombinations only and hence they would not have developed to the structures. Further, the theory of structure formation implies that the major part of dark matter must be cold which means that they have to consist particles that become non-relativistic at very early epoch. Say differently, if the dark matter were hot which means that they consist particles that remain relativistic until late times, then the formation of object of relatively small sizes would be suppressed which is in contradiction with observation.

### 1.3.8 Inflationary Epoch

As we will show in more details further, standard Hot Big Bang Theory has own problems. The most important issue is related to initial conditions that should be fine tuned appropriately in order to describe present Universe.

Let us be more precise. Since we work with hot Universe it means that it has to be warm. Then it is natural to introduce its entropy, or more precisely, entropy density. It can be shown that entropy density is of order of photon number density which, at present Universe, is

$$s \sim 10^3 cm^{-3} . \quad (32)$$

Then the entropy in the observable part of the Universe with the size  $R_0 \sim 10^4 Mpc \sim 10^{28} cm$  is

$$S \sim sR_0^3 \sim 10^{88} . \quad (33)$$

This is certainly very huge number and one can ask the question why our Universe has such a large entropy. The problem with Hot Big Bang Theory is that entropy is almost conserved during the hot stage. Then it is clear that such a huge entropy has to be introduced into theory as one of the initial data. This is certainly not very satisfactory and it is known as entropy problem. It turns out that there are more problems with standard cosmological theory and which are related to the fact that theory cannot explain why Universe is so large, spatially flat, homogeneous and isotropic.

It turns out that solution of these problems can be found in inflationary theory. The main idea of inflation is simple. Before the standard cosmological epoch as is known in Hot Big Bang theory there is an epoch of exponential expansion known as *inflation*. The remarkable property of inflation is that during exponential expansion initial small region of the Universe with the spatial size comparable to the Planck length expands to very large size of many order of magnitude larger than the size of the part of the Universe we see today. This naturally explain flatness homogeneity



and isotropy of observable part of the Universe. Further, the duration of inflationary epoch can be very short. It can be shown that problems of the Hot Big Bang Theory can be solved when the duration is greater than  $(50 - 70)H_{infl}^{-1}$ , where  $H_{infl}$  is the Hubble parameter at inflation, for example  $H_{infl} \sim 10^{-6}M_{pl}$ . As a result minimum duration of inflation is of order  $10^{-35}s$ .

Inflation is possible when the energy density is almost constant during its epoch. However this can be ensured when we introduce new field known as inflation, since conventional matter does not have this property. This field has to change slowly during inflation. Also its potential energy changes slowly making expansion exponential character.

Since we know that standard cosmological description given by hot big bang model gives excellent description it is clear that inflation has to terminate at some point. Then inflation is followed by post-inflationary reheating when inflaton energy is transferred to the energy of conventional matter. At the same time the Universe heats up to very high temperature and Hot Big Bang epoch begins.

It is remarkable that inflation can answer all problems of standard cosmological theory.

## 1.4 Quantum Mechanics and Cosmology

The standard text book interpretation of Quantum Mechanics is known as Copenhagen interpretation. It is very important for our understanding many physical phenomena including atoms, molecules, chemistry and also solid state physics. Today's general picture is that the Copenhagen interpretation of quantum mechanics, where there is sharp separation between measurement apparatus and measured object. However this fact implies that we can consider Copenhagen interpretation as an approximation suitable for laboratory applications. More precisely, we give four reasons for this strong claim.

- **There are no classical realm, no observables and no measurements in the Early Universe**

Standard quantum mechanics is based on presumption that there are instruments that are used as measurement apparatus. In other words they transform microscopic phenomena to macroscopic which are determined by classical laws of physics. In other words orthodox Copenhagen formulation of quantum mechanics ask for boundary between classical and quantum worlds. However there is clearly no such a division in cosmology.

- **Observers are Physical Systems that exist in the Universe**

As is well known Copenhagen quantum mechanics predicts probabilities of the results of measurements that are performed on subsystems of the Universe by another subsystem which we call as observer or as apparatus. This subsystem is outside the system which is measured.

Certainly this description could be useful for experiments performed in laboratory. However the question is can be useful for early Universe where there is no measurements and where are no observers. Does it mean that quantum mechanics cannot be applied to the physical processes that occurred before first observers emerge? Certainly this is not true and hence we need more general formulation of quantum mechanics than Copenhagen interpretation is and where observers is inside the Universe. However one can argue that such a generalization is not necessary since present observers measure present records that are results of the processes that performed in the past and whose time evolutions is governed known quantum mechanics deterministic law. However the question is what record exactly means. As we argued above record is physical quantity that is correlated with physical processes in the past with high probability. However such a probability depends on two times, one corresponding to the formation of the event in the past and the second one corresponding to the time when a consequence of this event is recorded. Such a probabilities do not follow from standard Copenhagen interpretation and hence we should search for its generalization.

- **Fixed Classical Space-time**

It is well known that quantum mechanics needs fixed space-time structure that defines possible direction of time evolution. States are defined on space-like surfaces (surfaces at constant  $t$ ) at fixed space-times and evolve according to Schrödinger equations through family such a space-like surfaces.

However at the early Universe we are at very small distances where quantum gravity becomes important. It is not exactly known what means that the geometry fluctuates at the Planck scale however it is believed that our description of this realm should change in some way.

## 2 The Expansion of the Universe

If we look at the night sky we see that it seems the same from all directions. More precisely, we know that the visible universe seems to be the same in all directions at least if we analyze distances larger than 300 million of light years. The more precise support for this claim follows from the analysis of the cosmic microwave background that we will discuss below. The main outcome of this analysis is that the radiation has been traveling to us for 14 billion years and this radiation is the same in all directions which support the claim that the universe at large distances seems to be the same at all directions. Certainly there is no reason why we should presume that something special is with our position at the Universe, as for example we could imagine that we are in the centre of the sphere so that the universe will look isotropic from all directions naturally. Then we should accept the fact that the universe should appear isotropic to observers in the whole universe. Of course, this is not precisely true as we know that for observers that are moving with high

velocity through our galaxy the universe will not be the same in all directions due to the famous Doppler effect. Explicitly, for such observers the starlight coming toward them from the direction where they are traveling is much more intensive than the starlight that they see from behind. In other words when we say that the universe is isotropic we should specify which observers we mean. We say that the universe is isotropic for the family of freely falling observers which we mean observers that move with the average velocity of typical galaxies in their neighbourhoods. We say that the conditions have to be the same at the same time at any points that can be mapped each other by rotation about typical galaxies. However then it is clear that any point can be mapped to any other by sequence of rotations about various typical galaxies and consequently the universe has to be homogeneous. In other words the properties of the universe at all points at the same time look to be the same. This crucial presumption allows us to choose a simple form of the space-time metric which was firstly discovered by Friedmann as a solution of the Einstein equations and then derived by Robertson and Walker from the requirement of homogeneity and isotropy. As far as we know current modern cosmology is based on this (FRW) metric. Let us discuss properties of this metric in more details.

## 2.1 Spacetime geometry

Let us consider three dimensional homogeneous and isotropic space and its geometry which is characterized by a metric  $g_{ij}(\mathbf{x})$  where  $i, j = 1, 2, 3$  and  $\mathbf{x} \equiv (x^1, x^2, x^3)$  where  $x^i$  label spatial coordinates of points in three dimensional space time. The line element is defined as

$$ds^2 = g_{ij} dx^i dx^j \quad (34)$$

where  $ds$  is *proper distance* between  $\mathbf{x}$  and  $\mathbf{x} + d\mathbf{x}$  which means that it is a distance measured in a coordinate system that is cartesian in a small neighborhood of the point  $\mathbf{x}$ . The simplest possibility of the homogeneous and isotropic three dimensional space is flat space where

$$ds^2 = d\mathbf{x}^2 . \quad (35)$$

The transformations that leave this line element invariant are three dimensional rotations

$$x'^i = \Lambda^i_j x^j , \Lambda^i_k \Lambda^j_m \delta_{ij} = \delta_{km} \quad (36)$$

and translations

$$x'^i = x^i + b^i . \quad (37)$$

Another more interesting possibility of the homogeneous and isotropic space is spherical surface embedded in four-dimensional Euclidean space with radius  $a$ . The four dimensional Euclidean space has line element

$$ds^2 = d\mathbf{x}^2 + dz^2 \quad (38)$$

while the spherical surface is described by the equation

$$z^2 + \mathbf{x}^2 = a^2 \quad (39)$$

This space is invariant under four dimensional rotation or it is invariant under three dimensional rotation that leaves the point  $z$  invariant.

The last possibility is hyperspherical suface in four dimensional pseudo-Euclidean space with line element

$$ds^2 = d\mathbf{x}^2 - dz^2, \quad z^2 - \mathbf{x}^2 = a^2, \quad (40)$$

where  $a^2$  is an arbitrary positive constant. This space is left invariant by four-dimensional pseudo-rotation that has the form of Lorentz transformations where the time  $t$  is replaced by  $z$ .

Let us now perform rescaling

$$z = az', \quad \mathbf{x} = a\mathbf{x}' \quad (41)$$

and we obtain that the spherical and hyperspherical line elements and surfaces have the form

$$ds^2 = a^2(dz'^2 \pm d\mathbf{x}'^2), \quad z'^2 \pm \mathbf{x}'^2 = 1 \quad (42)$$

In what follows we drop primes on  $z'$  and  $\mathbf{x}'$  and use ordinary symbols  $z$  and  $\mathbf{x}$ .

Let us perform diferential of the equation  $z^2 \pm \mathbf{x}^2 = 1$  and we obtain

$$zdz = \mp \mathbf{x} \cdot d\mathbf{x} \quad (43)$$

so that we can write the line element as

$$ds^2 = a^2 \left( d\mathbf{x}^2 \pm \frac{(\mathbf{x} \cdot d\mathbf{x})^2}{z^2} \right) = a \left( d\mathbf{x}^2 \pm \frac{(\mathbf{x} \cdot d\mathbf{x})^2}{1 \mp \mathbf{x}^2} \right) \quad (44)$$

We can use common notation for all three spaces when we introduce parameter  $K$  equal to

$$K = \begin{cases} +1 & \text{spherical} \\ -1 & \text{hyperspherical} \\ 0 & \text{Euclidean} \end{cases} \quad (45)$$

and hence we can write the spatial line element as

$$ds^2 = a^2(t) \left[ d\mathbf{x}^2 + K \frac{(\mathbf{x} \cdot d\mathbf{x})^2}{1 - K\mathbf{x}^2} \right] \quad (46)$$

As the final step we extend this geometry to the full spacetime when we include the spatial line element with  $a$  now arbitrary function of time and hence we obtain

$$d\tau^2 \equiv -g_{\mu\nu}dx^\mu dx^\nu = dt^2 - a^2 \left[ d\mathbf{x}^2 + K \frac{(\mathbf{x} \cdot d\mathbf{x})^2}{1 - K\mathbf{x}^2} \right] \quad (47)$$

where  $\tau$  is the proper time that obeys well known relation

$$ds_{f.d.}^2 = -d\tau^2 \quad (48)$$

where  $ds_{f.d.}^2$  is line element in four dimensional space-time defined as

$$ds_{f.d.}^2 = g_{\mu\nu} dx^\mu dx^\nu . \quad (49)$$

where  $x^\mu = (x^0, \mathbf{x})$  label points in four dimensional space-time. It is important to stress that (47) contains the scale factor  $a$  which is arbitrary function of time (known as Robertson-Walker scale factor).

We are not going to prove it but it can be shown that the metric (47) is the unique metric if the universe is spherically symmetric and isotropic to a set of freely falling observers. Note that the components of metric have the form

$$g_{00} = -1 , \quad g_{0i} = 0 , \quad g_{ij} = a^2(\delta_{ij} + K \frac{x^i x^j}{1 - K \mathbf{x}^2}) \quad (50)$$

where  $i, j, k = 1, 2, 3$  and  $x^0 = t$

## 2.2 Alternative derivation of the metric

As argued before there is observation evidence for isotropy and the Copernican principle says that we are not the center of the Universe and therefore observers elsewhere should also observe an isotropy all cosmological models are based on the existence of homogeneity and isotropy of manifold. However it is important to stress that this claim is not certainly true. The Universe is apparently not static, but changing in time. Therefore the cosmological models are based on the idea that the Universe is homogeneous and isotropic in space but not in time. This means that the Universe can be foliated into space-like surfaces such that each slice is homogeneous and isotropic. Then it is natural to consider our space-time to be  $R \times \Sigma$  where  $R$  represents the time direction and  $\Sigma$  is a homogeneous and isotropic three-manifold. Since we may think of isotropy as invariance under rotation and homogeneity as invariance under translation we get that  $\Sigma$  must be a maximally symmetric space. More precisely, the homogeneity and isotropy imply that the space has its maximum possible number of Killing vectors. Therefore we can write the metric in the form

$$ds^2 = -dt^2 + a^2(t) \gamma_{ij}(\mathbf{x}) dx^i dx^j . \quad (51)$$

Here  $t$  is time-like coordinate and  $(x^1, x^2, x^3)$  are the coordinates on  $\Sigma$  where  $\gamma_{ij}$  is the maximally symmetric metric on  $\Sigma$ . The function  $a(t)$  is known as **scale factor** that tells us how big the space-like slice  $\Sigma$  is at the moment  $t$ . The coordinates used here in which the metric is free of cross terms  $dt dx^i$  and the space-like components are proportional to a single function of  $t$  are known as **comoving coordinates** and an observer who stays at constant  $x^i$  is also called as “comoving”. Only comoving observer will think that the Universe looks isotropic.

It is important to stress that these observers, that are at rest to this frame are *geodesic* which means that they are free. Note that for these particles (observers) we have  $ds^2 = -dt^2$  as follows from the fact that  $dx^i = 0$  which implies that  $t$  has the meaning of the proper time for particles at rest.

We show that the world-line  $x^i = \text{const}$  obeys the geodesic equation in the metric (277). Note that the geodesic equation takes the form

$$\frac{du^\mu}{d\lambda} + \Gamma_{\nu\lambda}^\mu u^\nu u^\lambda = 0 , \quad (52)$$

where  $u^\mu$  is 4-velocity

$$\frac{dx^\mu}{d\lambda} \quad (53)$$

and where  $\lambda$  is the parameter along the world-line of the particle. To begin with we calculate the Christoffel symbols

$$\Gamma_{\nu\lambda}^\mu = \frac{1}{2} g^{\mu\sigma} (\partial_\nu g_{\lambda\sigma} + \partial_\lambda g_{\nu\sigma} - \partial_\sigma g_{\nu\lambda}) . \quad (54)$$

For the metric (277) we have following non-zero components

$$g_{00} = -1 , \quad g_{ij} = a^2(t) \gamma_{ij} \quad (55)$$

with the inverse components

$$g^{00} = -1 , \quad g^{ij} = \frac{1}{a^2(t)} \gamma^{ij} , \quad (56)$$

where

$$\gamma^{ij} \gamma_{jk} = \delta^i_k . \quad (57)$$

Explicitly, for the metric in the form (50) the metric  $\gamma_{ij}$  is equal to

$$\gamma_{ij} = \delta_{ij} + K \frac{x^i x^j}{1 - K \mathbf{x}^2} . \quad (58)$$

It can be shown that the only non-zero components of  $\Gamma_{\nu\lambda}^\mu$  are

$$\Gamma_{0j}^i = \frac{1}{2} g^{ik} \partial_0 g_{jk} = \frac{\dot{a}}{a} \delta^i_j , \quad \Gamma_{ij}^0 = a \dot{a} \gamma_{ij} , \quad \Gamma_{jk}^i = {}^{(3)}\Gamma_{jk}^i , \quad (59)$$

where  ${}^{(3)}\Gamma_{jk}^i$  are the Christoffel symbols for metric  $\gamma_{ij}$ .

Let us now again consider the equation (278). The only non-zero component of the 4-velocity  $u^\mu = \frac{dx^\mu}{d\lambda}$  of the particle at rest is

$$u^0 = \frac{dx^0}{d\lambda} \quad (60)$$

Now the on-shell condition implies

$$u^\mu u^\nu g_{\mu\nu} = -1 \Rightarrow \frac{dx^0}{d\lambda} = 1 . \quad (61)$$

Then clearly (278) is obviously satisfied since  $\frac{du^0}{d\lambda} = 0$  and  $\Gamma_{00}^\mu = 0$  for all  $\mu$ . In other words the world-lines of particles which are at rest in our reference frame are indeed geodesic.

As we have shown in introduction the maximally symmetric Euclidean three-metric  $\gamma_{ij}$  obey

$$R_{ijkl}^{(3)} = k(\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk}) , \quad (62)$$

where  $k$  is some constant and the superscript on the Riemann tensor reminds to us that it is associated with the three metric  $\gamma_{ij}$  not to the metric of entire space-time. Then the Ricci tensor is

$$R_{jl}^{(3)} = \gamma^{ik} R_{ijkl}^{(3)} = 2k\gamma_{jl} . \quad (63)$$

Since the space is maximally symmetric then it will certainly be spherically symmetric as well. For such a space-time the metric can be put in the form

$$ds^2 = \gamma_{ij}dx^i dx^j = e^{2\beta}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) . \quad (64)$$

The Ricci tensor for the metric given above has components

$$\begin{aligned} R_{11}^{(3)} &= \frac{2}{r}\partial_r\beta , \\ R_{22}^{(3)} &= e^{-2\beta}(r\partial_r\beta - 1) + 1 \\ R_{33}^{(3)} &= [e^{-2\beta}(r\partial_r\beta - 1) + 1] \sin^2\theta . \end{aligned}$$

If we compare these expressions to (288) we can solve for  $\beta(r)$ :

$$\begin{aligned} \frac{2}{r}\partial_r\beta &= 2ke^{2\beta} \Rightarrow 2d\beta e^{-2\beta} = 2kr \Rightarrow \beta = -\frac{1}{2}\ln(C - kr^2) , \\ e^{-2\beta}(r\partial_r\beta - 1) + 1 &= 2kr^2 \Rightarrow e^{-2\beta}(r^2ke^{2\beta} - 1) + 1 = 2kr^2 \Rightarrow \\ \Rightarrow -e^{-2\beta} + 1 &= kr^2 \Rightarrow C = 1 \end{aligned} \quad (65)$$

and the third equation is identically solved. Then we obtain following metric on space-time:

$$ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right] . \quad (66)$$

This form of metric is known as **Friedman-Robertson-Walker metric** (FRW). Then the Einstein equations will determine the behavior of the scale factor  $a(t)$ . We can also easily see that the metric is invariant under the scaling transformations:

$$\begin{aligned} k &\rightarrow \frac{k}{|k|} , \\ r &\rightarrow \sqrt{|k|}r , \\ a &\rightarrow \frac{a}{\sqrt{|k|}} . \end{aligned} \quad (67)$$

Therefore it is clear that the only relevant parameter is  $k/|k|$  and there are three cases of interest:  $k = -1$  ,  $k = 0$  and  $k = 1$ . The case  $k = -1$  corresponds to

constant negative curvature on  $\Sigma$  and is called **open**, the case  $k = 0$  corresponds no curvature on  $\Sigma$  and is called **flat** ; the case  $k = 1$  corresponds to positive curvature on  $\Sigma$  and is called **closed**. Now we will examine these possibilities in more details:

- For  $k = 0$  the metric on  $\Sigma$  is

$$d\sigma^2 = dx_i dx^i, i = 1, 2, 3 \quad (68)$$

that is simply the Euclidean space. Globally, it could describe  $R^3$  or more complicated manifold, as for example three torus  $S^1 \times S^1 \times S^1$ .

- For  $k = 1$  we define

$$r = \sin \xi, dr = \cos \xi d\xi \quad (69)$$

and hence the metric on  $\Sigma$  can be written as

$$d\sigma^2 = d\xi^2 + \sin^2 \xi d\Omega^2 \quad (70)$$

which is the metric of three sphere. In this case the only possible global structure is actually three sphere.

- The case  $k = -1$  we can write

$$r = \sinh \psi \quad (71)$$

and the metric on  $\Sigma$  is

$$d\sigma^2 = d\psi^2 + \sinh^2 \psi d\Omega^2 \quad (72)$$

which is the metric of three dimensional space of constant negative curvature. Globally such a space can extend forever but it can also describe a non-simply connected compact space.

Now we have to explain the meaning of the scale factor  $a(t)$ . To do this we calculate the proper distance at time  $t$  from the origin to a comoving object at radial coordinate  $r$

$$d(r, t) = a(t) \int_0^r \frac{dr'}{\sqrt{1 - Kr'^2}} = a(t) \times \begin{cases} \sin^{-1} r & K = 1 \\ \sinh^{-1} r & K = -1 \\ r & K = 0 \end{cases} \quad (73)$$

We see that that the proper distance from us to a comoving object increases with  $a(t)$ . Generally we say that the proper distance between any two comoving observes in the universe is proportional  $a(t)$ . The rate of change is equal to

$$\frac{dr}{dt} = \frac{\dot{a}}{a} d \quad (74)$$



### 3 Probe Dynamics in Expanding Background

In this section we will analyze dynamics of probe particle, massive or massless, in general background. The massive particle in general gravitational field is described by the action

$$S = -m \int d\lambda \sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}, \quad \dot{x}^\mu = \frac{dx^\mu}{d\lambda}, \quad (75)$$

where  $\lambda$  is parameter that labels the world-line. We see that this action cannot describe massless particle ( $m = 0$ ) since in this case the action naively vanishes. In order to describe massless particle we use a standard trick and introduce an einbain  $e(\tau)$  so that the action (75) takes the form

$$S = \frac{1}{2} \int d\lambda \left( \frac{1}{e} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu - m^2 e \right). \quad (76)$$

To see an equivalence between these two formulations we perform the variation of the action (76) with respect to  $e$  that gives

$$-\frac{1}{e^2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu - m^2 = 0 \quad (77)$$

which can be solved for  $e$  as

$$e = \frac{1}{m} \sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}. \quad (78)$$

Inserting (78) into (76) we obtain the action (75).

that inserting back to the action we obtain the original action. Finally note that the action (76) can describe massless particle as well since we can easily take the limit  $m \rightarrow 0$  in (76). We return to this problem below.

Let us now derive equations of motion from (76). Since  $g_{\mu\nu}$  generally depend on  $x^\mu$  we should perform its variation as well and we obtain

$$-2 \frac{d}{d\lambda} \left( \frac{1}{e} g_{\mu\nu} \dot{x}^\nu \right) + \frac{1}{e} \partial_\mu g_{\rho\sigma} \dot{x}^\rho \dot{x}^\sigma = 0, \quad (79)$$

while the equation of motion for  $e$  gives the condition (77). Note that the equation of motion (79) can be written as

$$2 \frac{\dot{e}}{e^2} g_{\mu\nu} \dot{x}^\nu - 2 \frac{1}{e} g_{\mu\nu} \ddot{x}^\nu - 2 \frac{1}{e} \partial_\rho g_{\mu\sigma} \dot{x}^\rho \dot{x}^\sigma + \frac{1}{e} \partial_\mu g_{\rho\sigma} \dot{x}^\rho \dot{x}^\sigma = 0 \quad (80)$$

To proceed further we use the fact that

$$\begin{aligned} & - 2 \partial_\rho g_{\mu\sigma} \dot{x}^\rho \dot{x}^\sigma + \partial_\mu g_{\rho\sigma} \dot{x}^\rho \dot{x}^\sigma = \\ & = -(\partial_\rho g_{\mu\sigma} \dot{x}^\rho \dot{x}^\sigma + \partial_\sigma g_{\mu\rho} \dot{x}^\sigma \dot{x}^\rho - \partial_\mu g_{\rho\sigma} \dot{x}^\rho \dot{x}^\sigma) = -2g_{\mu\nu} \Gamma_{\rho\sigma}^\nu \end{aligned} \quad (81)$$

Then (80) can be rewritten into the form

$$\ddot{x}^\mu + \Gamma_{\rho\sigma}^\mu \dot{x}^\rho \dot{x}^\sigma - \frac{\dot{e}}{e} \dot{x}^\mu = 0 . \quad (82)$$

Let us now study consequence of the fact that the action (76) is invariant under reparametrization

$$\lambda' = f(\lambda) , \quad x'^\mu(\lambda') = x^\mu(\lambda) , \quad e'(\lambda') = \frac{e(\lambda)}{\dot{f}} , \quad \frac{dx'^\mu(\lambda')}{d\lambda'} = \frac{dx^\mu(\lambda)}{d\lambda} \frac{1}{\dot{f}} . \quad (83)$$

Using this invariance of the action we have a freedom in identification of  $\lambda$  with another physical quantity. A particular useful choice is to impose condition  $\epsilon = \frac{1}{m}$  so that we obtain on-shell condition

$$g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = -1 \quad (84)$$

which can be written as

$$g_{\mu\nu} dx^\mu dx^\nu = -d\lambda_{\epsilon=m-1}^2 \quad (85)$$

Comparing this relation with the definition of the proper time we see that we should identify  $\lambda_{\epsilon=m-1}$  with proper time  $\tau$ . In this case it is convenient to introduce vector of four velocity defined as

$$u^\mu = \frac{dx^\mu}{d\tau} , \quad u^\mu u^\nu g_{\mu\nu} = -1 . \quad (86)$$

It is also important to stress that we could identify  $\lambda$  with proper time *after* extremization of the action. Then of course the equation of motion for  $x^\mu$  take the familiar form

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma_{\rho\sigma}^\mu \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau} = 0 . \quad (87)$$

Note that this form of the equation of motion is invariant under transformations

$$\tau' = a\tau + b , \quad (88)$$

where  $a$  and  $b$  are constants known as *affine* parameters. This statement follows from the fact that

$$\frac{dx'^\mu}{d\tau'} = \frac{dx^\mu}{d\tau} \frac{1}{a} , \quad \frac{d^2 x'^\mu}{d\tau'^2} = \frac{1}{a^2} \frac{d^2 x^\mu}{d\tau^2} \quad (89)$$

Note however that this is true only for constant  $a$  and  $b$ . To see this we will prove the covariance of the general equations of motion. To see this let us start with the equations of motion in the form

$$\frac{d^2 x'^\mu(\lambda')}{d\lambda'^2} + \Gamma_{\rho\sigma}^\mu \frac{dx'^\rho}{d\lambda'} \frac{dx'^\sigma}{d\lambda'} - \frac{de'(\lambda')}{d\lambda'} \frac{dx'^\mu}{d\lambda'} = 0 \quad (90)$$

and use the transformation rules

$$\lambda' = f(\lambda), \quad x'^{\mu}(\lambda') = x^{\mu}(\lambda), \quad e'(\lambda') = \frac{e(\lambda)}{\dot{f}}, \quad \frac{dx'^{\mu}(\lambda')}{d\lambda'} = \frac{dx^{\mu}(\lambda)}{d\lambda} \frac{1}{\dot{f}}. \quad (91)$$

so that we can write

$$\begin{aligned} \frac{dx'^{\mu}}{d\lambda'} &= \frac{dx^{\mu}}{d\lambda} \frac{d\lambda}{d\lambda'} = \frac{dx^{\mu}}{d\lambda} \frac{1}{\dot{f}}, \\ \frac{d^2x'^{\mu}}{d\lambda'^2} &= \frac{d^2x^{\mu}}{d\lambda^2} \left(\frac{1}{\dot{f}}\right)^2 - \frac{dx^{\mu}}{d\lambda} \frac{\ddot{f}}{\dot{f}^3}, \\ \frac{\frac{de'(\lambda')}{d\lambda'} dx'^{\mu}}{e'(\lambda')} &= \frac{\frac{d}{d\lambda} \left(\frac{e}{\dot{f}}\right) \frac{1}{\dot{f}}}{\frac{e}{\dot{f}}} \frac{dx^{\mu}}{d\lambda} \frac{1}{\dot{f}} = \frac{dx^{\mu}}{d\lambda} \frac{\dot{e}\dot{f} - e\ddot{f}}{\dot{f}^3 e} \end{aligned} \quad (92)$$

using the fact that

$$\lambda = f^{-1}(\lambda'), \quad \frac{d\lambda}{d\lambda'} = \frac{df^{-1}}{d\lambda'} = \frac{1}{\dot{f}(f^{-1}(\lambda'))} \quad (93)$$

Note that in equation (92) we did not write explicitly the dependence of  $\dot{f}$  on  $f^{-1}(\lambda')$ . Then (90) has the form

$$\begin{aligned} &\frac{d^2x'^{\mu}(\lambda')}{d\lambda'^2} + \Gamma_{\rho\sigma}^{\mu} \frac{dx'^{\rho}}{d\lambda'} \frac{dx'^{\sigma}}{d\lambda'} - \frac{\frac{de'(\lambda')}{d\lambda'} dx'^{\mu}}{e'(\lambda')} \frac{1}{d\lambda'} = \\ &= \frac{d^2x^{\mu}}{d\lambda^2} \frac{1}{\dot{f}^2} - \frac{dx^{\mu}}{d\lambda} \frac{\ddot{f}}{\dot{f}^3} + \Gamma_{\rho\sigma}^{\mu} \frac{dx^{\rho}}{d\lambda} \frac{dx^{\sigma}}{d\lambda} \frac{1}{\dot{f}^2} - \frac{dx^{\mu}}{d\lambda} \frac{\dot{e}}{e} \frac{1}{\dot{f}^2} + \frac{dx^{\mu}}{d\lambda} \frac{\ddot{f}}{\dot{f}^3} = \\ &= \frac{d^2x^{\mu}}{d\lambda^2} \frac{1}{\dot{f}^2} + \Gamma_{\rho\sigma}^{\mu} \frac{dx^{\rho}}{d\lambda} \frac{dx^{\sigma}}{d\lambda} \frac{1}{\dot{f}^2} - \frac{dx^{\mu}}{d\lambda} \frac{\dot{e}}{e} \frac{1}{\dot{f}^2} = 0 \end{aligned} \quad (94)$$

and this result proves the covariance of the equations of motion.

Let us now consider massless particle ( $m = 0$ ). Performing the same calculations as above we find that the equations of motion for  $x^{\mu}$  have the same form as in (82) since  $m$  only affect the equation of motion for  $\epsilon$  in that now have the form

$$\frac{1}{e^2} g_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu} = 0. \quad (95)$$

Of course, in this case we cannot identify  $\lambda$  with proper time since proper time is not well defined due to the fact that  $d\tau = 0$  for massless particle. On the other hand we will argue that it is always possible to choose such parameter (*affine*) so that the equation of motion for massless particle has the form (87). To see this let us proceed as follows. Let  $t^{\mu} \equiv \frac{dx^{\mu}}{d\lambda}$  obeys the geodesic equations

$$\frac{dt^{\mu}}{d\lambda} + \Gamma_{\rho\sigma}^{\mu} t^{\rho} t^{\sigma} = \kappa t^{\mu}, \quad \kappa = \frac{\dot{e}}{e}. \quad (96)$$

Let us define vector  $u^\mu = \frac{dx^\mu}{d\lambda^*} = t^\mu \frac{d\lambda}{d\lambda^*}$  and calculate

$$\begin{aligned}
\frac{du^\mu}{d\lambda^*} + \Gamma_{\rho\sigma}^\mu u^\rho u^\sigma &= \frac{d}{d\lambda} \left( t^\mu \frac{d\lambda}{d\lambda^*} \right) \frac{d\lambda}{d\lambda^*} + \Gamma_{\rho\sigma}^\mu t^\rho t^\sigma \left( \frac{d\lambda}{d\lambda^*} \right)^2 = \\
&= \left( \frac{dt^\mu}{d\lambda} + \Gamma_{\rho\sigma}^\mu t^\rho t^\sigma \right) \left( \frac{d\lambda}{d\lambda^*} \right)^2 + t^\mu \frac{d}{d\lambda} \left( \frac{d\lambda}{d\lambda^*} \right) \frac{d\lambda}{d\lambda^*} = \\
&= \kappa t^\mu \left( \frac{d\lambda}{d\lambda^*} \right)^2 + t^\mu \frac{d}{d\lambda} \left( \frac{d\lambda}{d\lambda^*} \right) \frac{d\lambda}{d\lambda^*}
\end{aligned} \tag{97}$$

which is equal to zero when

$$-\frac{d\lambda}{d\lambda^*} \kappa = \frac{d}{d\lambda} \left( \frac{d\lambda}{d\lambda^*} \right) \tag{98}$$

or equivalently

$$-d\lambda \kappa(\lambda) = \frac{d\left(\frac{d\lambda}{d\lambda^*}\right)}{\frac{d\lambda}{d\lambda^*}} \tag{99}$$

This equation has the solution (up to integration constant)

$$\frac{d\lambda}{d\lambda^*} = \exp\left(-\int^\lambda d\lambda' \kappa(\lambda')\right), \quad \frac{d\lambda^*}{d\lambda} = \exp\left(\int^\lambda d\lambda' \kappa(\lambda')\right). \tag{100}$$

We see that it is possible to define *affinely parametrized* null geodesic that obeys the equation

$$\frac{du^\mu}{d\lambda^*} + \Gamma_{\rho\sigma}^\mu u^\rho u^\sigma = 0. \tag{101}$$

Let us now return to the case of massive particle with the mass  $m_0$  and consider the gauge fixed form where the parameter  $\lambda = \tau$ . In this case the equation of motion for  $e$  allows us to write

$$-1 = \left( g_{00} + g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} + 2g_{0i} \frac{dx^i}{dt} \right) \left( \frac{dt}{d\lambda} \right)^2 \tag{102}$$

so that we obtain an important relation

$$\frac{dt}{d\lambda} = \frac{1}{\sqrt{-g_{00} - 2g_{0i}v^i - g_{ij}v^i v^j}}, \quad v^i \equiv \frac{dx^i}{dt}. \tag{103}$$

It is also interesting to insert the solution of the equation of motion  $\epsilon$  into the action so that it takes the form ( $e = \frac{1}{m_0} \sqrt{-g_{\mu\nu} u^\mu u^\nu}$ )

$$\begin{aligned}
S &= \frac{1}{2} \int d\lambda \left[ \frac{1}{\epsilon} g_{\mu\nu} u^\mu u^\nu - m_0^2 \epsilon \right] = \\
&= -m_0 \int dt \sqrt{-g_{00} - 2g_{0i}v^i - g_{ij}v^i v^j}.
\end{aligned} \tag{104}$$

It is instructive to find corresponding Hamiltonian. From the action above we obtain conjugate momenta

$$p_i = \frac{\partial L}{\partial v^i} = m_0 \frac{g_{0i} + g_{ij}v^j}{\sqrt{-g_{00} - 2g_{0i}v^i - g_{ij}v^i v^j}} \quad (105)$$

and hence

$$H = p_i \dot{x}^i - L = m_0 \frac{-g_{00} - g_{0i}v^i}{\sqrt{-g_{00} - 2g_{0i}v^i - g_{ij}v^i v^j}} \quad (106)$$

that, of course, we have to express using canonical variables. Using definition of momenta we obtain

$$p_i g^{ij} p_j = m_0^2 \frac{g_{0i} g^{ij} g_{j0} + 2g_{0i} v^i + g_{ij} v^i v^j}{-g_{00} - 2g_{0i} v^i - g_{ij} v^i v^j} \quad (107)$$

where we introduced  $g^{ij}$  which is metric inverse to  $g_{ij}$  so that  $g_{ij} g^{jk} = \delta_i^k$ . From the previous equation we obtain

$$2g_{0i} v^i + g_{ij} v^j = - \frac{\frac{1}{m_0^2} p_i g^{ij} p_j g_{00} + g_{0i} g^{ij} g_{j0}}{1 + \frac{1}{m_0^2} p_i g^{ij} p_j} \quad (108)$$

so that

$$p_i = \frac{g_{0i} + g_{ij} v^j}{\sqrt{-g_{00} + g_{0i} g^{ij} g_{j0}}} \sqrt{m_0^2 + p_i g^{ij} p_j} \quad (109)$$

and finally

$$v^i = g^{ij} \left( \frac{p_i \sqrt{-\det g_{\mu\nu}}}{\sqrt{\det g_{ij}} \sqrt{m_0^2 + p_i g^{ij} p_j}} - g_{j0} \right) \quad (110)$$

using also the fact that

$$-\det g_{\mu\nu} = -(g_{00} - g_{0i} g^{ij} g_{j0}) \det g_{ij} \quad (111)$$

Now we are finally ready to find corresponding Hamiltonian

$$\begin{aligned} H &= \frac{-g_{00} + g_{0i} g^{ij} g_{j0} - \frac{g_{0i} g^{ij} p_j \sqrt{-\det g_{\mu\nu}}}{\sqrt{\det g_{ij}} \sqrt{m_0^2 + p_i g^{ij} p_j}}}{\sqrt{-g_{00} - 2g_{0i} v^i - g_{ij} v^i v^j}} = \\ &= \frac{1}{m_0} \frac{\sqrt{-\det g_{\mu\nu}}}{\sqrt{\det g_{ij}}} \sqrt{m_0^2 + p_i g^{ij} p_j} - \frac{1}{m_0} g_{0i} g^{ij} p_j \end{aligned} \quad (112)$$

Now using this Hamiltonian we can derive canonical equations of motion

$$\begin{aligned} \dot{x}^i &= \{x^i, H\} , \\ \dot{p}_i &= \{p_i, H\} , \end{aligned} \quad (113)$$

where Poisson bracket between two functions on the phase space is defined as

$$\{F, G\} = \frac{\partial F}{\partial x^i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial x^i} . \quad (114)$$

Generally the equations of motion for  $p_i$  are rather complicated due to the fact that metric components generally depend on  $x^i$ . The situation simplifies considerably in flat FRW background

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j . \quad (115)$$

when  $\det g_{\mu\nu} = -a^6$ ,  $\det g_{ij} = a^6$ ,  $g_{0i} = 0$  and hence the Hamiltonian is equal to

$$H = \frac{1}{m_0} \sqrt{m_0^2 + \frac{1}{a^2} p_i \delta^{ij} p_j} . \quad (116)$$

Now the equation of motion takes the form

$$\begin{aligned} \dot{x}^i &= \{x^i, H\} = \frac{1}{m_0 a^2} \frac{\delta^{ij} p_j}{\sqrt{a^{-2} p_i \delta^{ij} p_j + m_0^2}} , \\ \dot{p}_i &= \{p_i, H\} = 0 \Rightarrow p_i = k_i . \end{aligned} \quad (117)$$

We see that the momentum  $p_i$  is constant. On the other hand the norm of state slows since the norm is given as  $p_i g^{ij} p_j = \frac{1}{a^2} k_i \delta^{ij} k_j$ .

On the other hand let us proceed in slightly different way and introduce following variable

$$X^i = a x^i , \dot{x}^i = \frac{1}{a} (\dot{X}^i - H X^i) . \quad (118)$$

Note that this procedure is suitable for the flat FRW background. Using these variables we find the action in the form

$$S = -m \int dt \sqrt{1 - (\dot{X}^i - H X^i) \delta_{ij} (\dot{X}^j - H X^j)} . \quad (119)$$

The meaning of the variables  $X^i$  can be found when we take the non-relativistic limit where we replace  $\sqrt{1 - A} = 1 - \frac{1}{2} A^2$  so that the action

$$\begin{aligned} S_{nonrel} &= -m \int dt + \int dt \frac{m}{2} (\dot{X}^i - H X^i) \delta_{ij} (\dot{X}^j - H X^j) = \\ &= \int dt \frac{m}{2} \dot{X}^i \dot{X}_i + \dots , \end{aligned} \quad (120)$$

where we neglected the remaining terms. Comparing this expression with the standard form of the non-relativistic Lagrangian we can interpret  $X^i = a(t)x^i$  as the physical variable even if we mean that both variables are physical.

Now from (368) we determine the momenta conjugate to  $X^i$

$$P_i = \frac{\delta L}{\delta \dot{X}^i} = m \frac{\delta_{ij}(\dot{X}^j - HX^j)}{\sqrt{(\dots)}} \quad (121)$$

and hence the Hamiltonian takes the form

$$\mathcal{H} = \dot{X}^i P_i - L = \frac{m}{\sqrt{(\dots)}} + P_i X^i H = \sqrt{m^2 + P_i P^i} + P_i X^i H \quad (122)$$

Using this Hamiltonian we derive the equation of motion

$$\begin{aligned} \dot{X}^i &= \{X^i, H\} = \frac{P^i}{\sqrt{m^2 + P_i P^i}} + X^i H, \\ \dot{P}_i &= \{P_i, H\} = -P_i H. \end{aligned} \quad (123)$$

The last equation can be integrated as

$$dP_i = -P_i \frac{da}{a} \Rightarrow \ln P_i = -\ln a + \ln K_i \Rightarrow P_i = \frac{K_i}{a}. \quad (124)$$

We see that the "physical" momentum  $P_i$  is red shifted as the universe expands. Note that we can also find the time dependence of  $X^i$  by integrating the first equation since it takes generally of the form

$$\dot{X}^i = F^i(t) + G(t)X^i, \quad G(t) = \frac{\dot{a}}{a}, \quad F^i = \frac{P^i}{\sqrt{m^2 + P_i P^i}}. \quad (125)$$

We solve this equation in an standard way. First of all we solve the homogeneous equation

$$\dot{X}^i = G(t)X^i \Rightarrow X^i = C^i \exp\left(\int dt' G(t')\right) \quad (126)$$

Note that we have

$$\int dt' G(t') = \int \frac{da}{a} \frac{1}{a} dt = \int \frac{da}{a} = \ln a \Rightarrow e^{\int dt' G(t')} = e^{\ln a} = a \quad (127)$$

and hence

$$X^i = aC^i. \quad (128)$$

This is solution of homogeneous equation. In order to find solution of non homogeneous equation we promote  $C^i$  to depend on time and hence

$$\dot{X}^i = \dot{C}^i \exp\left(\int dt' G(t')\right) + CG \exp\left(\int dt' G(t')\right) = \dot{C}^i \exp\left(\int dt' G(t')\right) + GX^i \quad (129)$$

Comparing this equation with the right side of equation (125) we obtain that  $C^i$  has to obey the equation

$$\frac{dC^i}{dt} = e^{-\int dt' G(t')} F(t) \Rightarrow \frac{dC^i}{dt} = \frac{K^i}{a\sqrt{m^2 a^2 + K_i K^i}} \quad (130)$$

that can be in principle integrated if we know the time dependence of  $a$ . There is a particular simple solution corresponding to the particle with zero physical momentum when  $K_i = 0$ . From upper equation we immediately find that  $C^i = \mathcal{C}^i = \text{const}$  and hence

$$X^i = \mathcal{C}^i a \quad (131)$$

that is an expected result. The physical interpretation of this result is that particle slows down with respect to comoving coordinates as the Universe expands (since  $a \rightarrow \infty$ ). In fact this is an actual slowing down, in the sense that a gas of particles with initially high relative velocities will cool down as the Universe expands.

### 3.1 Fields and Currents in the Universe

As we will show below the matter content of the Universe is described with the help of various vector and tensor fields instead of an action for one particle or collection of particles. Examples of these fields is the current of galaxies and the energy-momentum tensor whose mean values have to satisfy the requirements of isotropy and homogeneity. Isotropy requires that the mean value of any three vector has to vanish while the requirement of homogeneity means that any three scalar (which is quantity invariant under spatial coordinate transformations) can depend on time only. More explicitly, let us consider a system of particles that are labeled by index  $n$  where each particle has space-time coordinates  $x_n^\mu(\lambda)$  where  $\lambda$  parameterizes the particle trajectory. Let us each particle has electric charge  $e_n$ . Then we can define electric current four-vector as

$$\begin{aligned} J^\mu(x) &\equiv \frac{1}{\sqrt{-g(x)}} \int d\lambda \sum_n e_n \delta^4(x_n(\lambda) - x) \frac{dx_n^\mu}{d\lambda} = \\ &= \frac{1}{\sqrt{-g(x)}} \int d\lambda \sum_n e_n \delta^3(\mathbf{x}_n(u) - \mathbf{x}) \delta(t_n(u) - t) \frac{dx_n^\mu}{dt_n} \frac{dt_n}{d\lambda} = \\ &= \frac{1}{\sqrt{-g(x)}} \sum_n e_n \int d\lambda \delta^3(\mathbf{x}_n(\lambda) - \mathbf{x}) \frac{\delta(\lambda - \lambda_n)}{\frac{d\lambda}{d\lambda}} \frac{dx_n^\mu}{dt_n} \frac{dt_n}{d\lambda} = \\ &= \frac{1}{\sqrt{-g(x)}} \sum_n e_n \delta^3(\mathbf{x}_n(t) - \mathbf{x}) \frac{dx_n^\mu}{dt} , \end{aligned} \quad (132)$$

where  $\delta^4$  is four-dimensional delta function that is normalized as

$$\int d^4x f(y) \delta(y - x) = f(x) . \quad (133)$$



Note that  $\delta^3(\mathbf{z})$  is delta function in three spatial dimensions. We also used the fact that

$$\delta(g(x)) = \sum_i \frac{\delta(x - x_i)}{|g'(x_i)|} \quad (134)$$

where  $x_i$  are roots of the equation  $g(x) = 0$ . Then  $\lambda_n$  is root of the equation  $t_n(\lambda_n) = t$ . For example, let us consider a integral of  $\sqrt{-\det g} J^0$  over finite three dimensional volume

$$\int d^3 \mathbf{x} \sqrt{-\det g} J^0 = \sum_n e_n \int d^3 \mathbf{x} \delta(\mathbf{x}_n(t) - \mathbf{x}) = \sum_n e_n \quad (135)$$

which is clearly total electric charge in that volume. Of course, we can consider another type of current, as for example baryon current where the baryon number of  $n$ -th particle of  $n$ -th galaxy appears instead of  $e_n$ . It is also important to explain why we introduced the factor  $\frac{1}{\sqrt{-g}}$  in the definition of current. The reason is that the delta function is not a scalar. To see this note that under change of coordinates  $x'^\mu = x'^\mu(x^\nu)$  we have

$$d^4 x' = \left| \frac{\partial x'}{\partial x} \right| d^4 x \quad (136)$$

where  $\left| \frac{\partial x'}{\partial x} \right|$  is Jacobian of the coordinate transformations. On the other hand we know that the metric transform us

$$g'_{\mu\nu}(x') = g_{\rho\sigma} \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} . \quad (137)$$

Taking the determinant of this expression we obtain

$$\det g'_{\mu\nu} \equiv g' = \left| \frac{\partial x}{\partial x'} \right|^2 g = \left| \frac{\partial x'}{\partial x} \right|^{-2} g . \quad (138)$$

and we see that  $d^4 x \sqrt{-g}$  transforms as a scalar. Let us consider definition of the delta function

$$f(y) = \int d^4 x f(x) \delta^4(x - y) = \int (d^4 x \sqrt{-g}) f(x) \left( \frac{\delta^4(x - y)}{\sqrt{-g}} \right) . \quad (139)$$

Now since  $f(y)$ ,  $d^4 x \sqrt{-g}$ ,  $f(x)$  are scalars we find that  $\frac{\delta^4(x-y)}{\sqrt{-g}}$  is scalar rather than the delta function itself. Returning to our current we find that it obeys the conservation law

$$\begin{aligned} \partial_\mu (\sqrt{-g} J^\mu) &= \int d\lambda \sum_n e_n \frac{\partial}{\partial x^\mu} \left( \frac{dx_n^\mu}{d\lambda} \delta(x_n^\mu - x^\mu) \right) = \\ &= \int d\lambda \sum_n e_n \frac{\partial}{\partial x_n^\mu} \delta^4(x_n^\mu - x) \frac{dx_n^\mu}{d\lambda} = - \int d\lambda \sum_n e_n \frac{d}{d\lambda} \delta^4(x_n - x) = 0 \end{aligned} \quad (140)$$

using

$$\frac{\partial}{\partial x^\mu} \delta^4(x_n - x) = - \frac{\partial}{\partial x_n^\mu} \delta^4(x_n - x) \quad (141)$$

Note that this conservation law can be written into the form

$$\begin{aligned}\partial_\mu(\sqrt{-g}J^\mu) &= \partial_\mu J^\mu \sqrt{-g} + \frac{1}{2}\partial_\mu g_{\rho\sigma}g^{\sigma\rho}\sqrt{-g}J^\mu = \\ \sqrt{-g}(\partial_\mu J^\mu + \Gamma_{\mu\nu}^\mu J^\nu) &= \sqrt{-g}\nabla_\mu J^\mu = 0\end{aligned}\tag{142}$$

using the fact that

$$\begin{aligned}\Gamma_{\rho\sigma}^\mu &= \frac{1}{2}g^{\mu\omega}(\partial_\rho g_{\omega\sigma} + \partial_\sigma g_{\omega\rho} - \partial_\omega g_{\rho\sigma}) , \\ \Gamma_{\mu\nu}^\mu &= \frac{1}{2}g^{\mu\omega}(\partial_\mu g_{\omega\nu} + \partial_\nu g_{\omega\mu} - \partial_\omega g_{\mu\nu}) = \frac{1}{2}g^{\mu\omega}\partial_\nu g_{\omega\mu} .\end{aligned}\tag{143}$$

In other words the covariant conservation law for the current  $J^\mu$  in general curved space-time has the form

$$\nabla_\mu J^\mu = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}J^\mu) = 0 .\tag{144}$$

Another important object that characterises the matter content of the Universe is the stress energy tensor  $T^{\mu\nu}$  where the conservation law has the form

$$\nabla_\nu T^{\mu\nu} = \partial_\nu T^{\mu\nu} + \Gamma_{\kappa\nu}^\mu T^{\kappa\nu} + \Gamma_{\kappa\nu}^\nu T^{\mu\kappa} = 0 .\tag{145}$$

Let us now write the form of this stress energy tensor for an ideal gas of particles that move freely the stress energy tensor has the form

$$\begin{aligned}T^{\mu\nu}(x) &= \frac{1}{\sqrt{-g}} \int du \sum_n \delta^4(x_n(u) - x) \frac{dx^m u_n(u)}{du} p_n^\nu(u) = \\ &= \frac{1}{\sqrt{-g}(x)} \sum_n \delta^3(\mathbf{x}_n(t) - \mathbf{x}) p_n^\mu(t) \frac{p_n^\nu(t)}{E_n(t)} ,\end{aligned}\tag{146}$$

where

$$p_n^\mu = m \frac{dx_n^\mu}{du} = m \frac{dx_n^\mu}{dt_n} \frac{dt_n}{du} = m \frac{dx_n^\mu}{dt_n} E_n , \quad p_n^0 = E_n = m \frac{dt_n}{du} .\tag{147}$$

A perfect fluid is defined as a medium for which at every point we can introduce a locally inertial Cartesian frame that moves with the fluid. In this frame of reference the fluid appears the same at all directions. In this reference frame the components of the energy momentum tensor must take the form

$$T^{ij} = p\delta^{ij} , \quad T^{0i} = T_{0i} = 0 , \quad T^{00} = \rho .\tag{148}$$

We have to demand that  $T^{i0} = 0$  since in the opposite case the non-zero value of  $T^{0i}$  would determine special directions in space as the direction of  $T^{0i}$  which would be

in clash with the presumption of the spatial isotropy. For the same reason we have to demand that  $T^{ij}$  is proportional to the unit matrix  $\delta^{ij}$ . The functions  $p$  and  $\rho$  are known as the pressure and energy density evaluated at the rest frame. Then in locally inertial Cartesian frame with an arbitrary velocity the energy momentum tensor takes the form

$$T^{\alpha\beta} = p\eta^{\alpha\beta} + (p + \rho)u^\alpha u^\beta , \quad (149)$$

where  $\rho$  and  $p$  are defined to be the same as in the co-moving inertial frame. Further,  $u^\alpha$  is defined by the condition that it transforms as a four-vector under Lorentz transformations. Since locally Cartesian co-moving inertial frame is the frame that moves with the fluid it is clear that it is determined by the condition  $u^0 = 1$  and  $u^i = 0$ . Note that for this form of the four vector (149) reduces to (148) since

$$T^{00} = -p + (p + \rho) = \rho , \quad T^{ij} = p\delta^{ij} , \quad T^{0i} = 0 . \quad (150)$$

The four-vector  $u^\alpha$  is normalized so that in any inertial frame has the form  $\eta_{\alpha\beta}u^\alpha u^\beta = -1$ . Now since we have determined the form of the stress energy tensor in locally inertial frame we have to proceed to the next step and find its form in the general frame defined by the metric  $g_{\mu\nu}$ . The straightforward generalization leads to the following form of the stress energy tensor for the perfect fluid in general background

$$T^{\mu\nu} = pg^{\mu\nu} + (p + \rho)u^\mu u^\nu , \quad u^\mu u^\nu g_{\mu\nu} = -1 . \quad (151)$$

It is important that  $p$  and  $\rho$  correspond to the pressure and the energy density in a locally co-moving inertial coordinate frame. Since apparently they do not change when we move from one frame to another they are scalars from the space-time transformations as opposite to the four vector  $u^\mu$  that transforms as

$$u'^\mu(x') \frac{\partial}{\partial x'^\mu} = u^\nu(x) \frac{\partial}{\partial x^\nu} \Rightarrow u'^\mu(x') = u^\nu(x) \frac{\partial x'^\mu}{\partial x^\nu} . \quad (152)$$

The formula (151) is apparently correct since it is manifestly covariant and since it reduces to (148) in local rest frame. Further, using the form of the stress energy tensor (151) we can derive the equations of motion of relativistic hydrodynamics for ideal fluid simply from the requirement of the conservation of the stress energy tensor

$$\nabla_\mu T^{\mu\nu} = 0 . \quad (153)$$

It is also important to stress that there is another conservation law which expresses the conservation of some quantity, as for example the number of baryons. In local inertial frame such a conservation law has to have to form of the divergence of four vector that for the density  $n$  has the form

$$\frac{\partial}{\partial x^\alpha} (nu^\alpha) = 0 . \quad (154)$$

The generalization to the general gravitation fields is straightforward when we replace ordinary partial derivative with covariant one

$$\nabla_\mu (nu^\mu) = 0 . \quad (155)$$

Now we can ask the question how the situation changes when the fluid is not perfect. Generally this is very difficult problem as the full theory of dissipative relativistic hydrodynamics is not known however we can gain some interesting physical results as follows. Let us presume that the effect of dissipation is small so that we can describe it by small corrections  $\Delta T^{\alpha\beta}$  and  $\Delta N$  to the stress energy tensor and current in the locally inertial frame. Explicitly, we have

$$T_{dis}^{\alpha\beta} = p\eta^{\alpha\beta} + (\rho + p)u^\alpha u^\beta + \Delta T^{\alpha\beta}, \quad \frac{\partial}{\partial x^\alpha} T_{dis}^{\alpha\beta} = 0. \quad (156)$$

In case of the current we have

$$\frac{\partial}{\partial x^\alpha} J_{dis}^\alpha = \frac{\partial}{\partial x^\alpha} (nu^\alpha + \Delta N^\alpha) = 0. \quad (157)$$

We define  $\rho$  as the scalar that corresponds to the energy density that is observed in the co-moving frame where  $u^i = 0$ . In this frame we have

$$T^{00} = \rho + \Delta T^{00} \quad (158)$$

and since  $T^{00}$  corresponds to the energy density we immediately see that in this frame  $\Delta T^{00} = 0$ . Clearly this relation can be written as

$$u_0 u_0 \Delta T^{00} = 0 \quad (159)$$

since in the local rest frame  $u_i = 0$ ,  $u_0 = -1$ . Hence the covariant form of this relation in any locally inertial Cartan frame is

$$u_\alpha u_\beta \Delta T^{\alpha\beta} = 0. \quad (160)$$

In case of the scalar  $n$ , we define it as the value of some conserved density which is observed in co-moving frame. In this frame we again find  $\Delta N^0 = 0$  that using the same arguments as above implies that in all locally inertial Cartesian frames we have

$$u_\alpha \Delta N^\alpha = 0. \quad (161)$$

On the other hand there is a freedom in the definition of the four-velocity  $u^\alpha$ . The first possibility is to say that  $u^i$  is the velocity of the particle transport, so that in the comoving frame where  $u^i = 0$  we would have  $\Delta N^i = 0$  together with  $\Delta N^0 = 0$ . However this result implies that in general local inertial Cartesian frame we have  $\Delta N^\alpha = 0$ . This option is known as Eckart gauge. The second possibility is to define  $u^i$  as the velocity of the energy transport so that in the comoving frame we have  $T^{i0} = \Delta T^{i0} = 0$  that implies

$$u_\beta \Delta T^{\beta\alpha} = \Delta T^{\alpha 0} = 0 \quad (162)$$

which implies that in all local Cartesian inertial frames we have the condition

$$u_\beta \Delta T^{\beta\alpha} = 0 \quad (163)$$

but generally  $\Delta N^\alpha \neq 0$ . Note that this definition of the velocity is convenient in the cosmology since it imposes the maximum possible constraint on the energy-momentum while it puts less constraints on the current of conserved quantities since in cosmology we have situations where there are either no non-zero conserved quantities at all. Such an example is the early Universe before cosmological leptogenesis or baryogenesis. Or there is the second possibility where there are conserved quantities with small values so that they cannot affect the relation between pressure and density, as in radiation-dominated era at temperatures above about  $10^4 K$ . Now with this definition of the velocity we can determine the correction terms  $\Delta T^{\alpha\beta}$  and  $\Delta N^\alpha$  using the second law of thermodynamics. We are not going into details of this procedure.

### 3.1.1 Action Principle and Stress Energy Tensor

In previous section we derived stress energy tensor for the ideal fluid using reasonable physical arguments. On the other hand we should be able to define stress energy tensor for any physical system. The fundamental physical principle says that we should be able to introduce an action for all physical systems and that the equations of motion that govern the evolutions of these systems are determined by extremization of this action. Clearly this situation holds in case of general relativity as well where the stress energy tensor appears on the right side of Einstein equations and that determines the matter content in the Universe. In other words the stress energy tensor is related to the infinitesimal change  $\delta g^{\mu\nu}$ <sup>3</sup> in the metric in the matter part of the action  $S_m$ . Explicitly we have

$$\delta S_m = - \int d^4x \sqrt{-g} T_{\mu\nu} \delta g^{\mu\nu} , \quad (164)$$

Since  $\delta g^{\mu\nu}$  is symmetric we immediately see from the expression above that  $T^{\mu\nu}$  is symmetric as well and we identify this tensor with the stress energy tensor.

## 3.2 Gravitational Field Equations

As we know in General Relativity (GR) the metric tensor is dynamical field and the equations of GR arise as extremum conditions for the action functional. The principle of equivalence means that all equations have to have the same form in all reference frames. In other words we require that the action function has to be the same in all reference frames which means that the action is a scalar. Since the action is given as the integral over time of the Lagrangian we find also that the Lagrangian has to be given as the integral over space section of the spacetime. In summary we postulate that the gravity action has the form

$$S_{gr} = \int d^4x \sqrt{-g} \mathcal{L}_{gr} , \quad (165)$$

---

<sup>3</sup>It is slightly easier to use the variation  $\delta g^{\mu\nu}$  instead of  $\delta g_{\mu\nu}$  even if it is clear that these two variations are related due to the fact that  $g_{\mu\nu} g^{\nu\rho} = \delta_\mu^\rho$  and hence  $\delta g_{\mu\nu} g^{\nu\rho} + g_{\mu\nu} \delta g^{\nu\rho} = 0$  which implies  $\delta g^{\mu\nu} = -g^{\mu\rho} \delta g_{\rho\sigma} g^{\sigma\nu}$ .

where the Lagrangian density  $\mathcal{L}_{gr}(x)$  transforms as under coordinate transformations  $x'^{\mu} = x^{\mu}(x)$

$$\mathcal{L}'(x') = \mathcal{L}(x) \quad (166)$$

and due to the fact that  $d^4x' \sqrt{-g'(x')} = d^4x \sqrt{-g(x)}$  we really see that  $S_{gr}$  does not change under diffeomorphism transformations.

The simplest possibility is to take the Lagrangian density to be equal to constant  $\mathcal{L} = -\Lambda$  so that

$$S_{\Lambda} = -\Lambda \int d^4x \sqrt{-g} . \quad (167)$$

However this action does not contain the time derivatives of the metric and hence the dynamics that would follow from this action is trivial. For that reason we should search more complicated form of the Lagrangian density.

The Lagrange density is a tensor density, which can be written as  $\sqrt{-g}$  times a scalar that is function of the metric and its derivatives. The question is the form of given scalar. Since we know that the metric can be set equal to its canonical form and its first derivatives set to zero at any one point, any nontrivial scalar must involve at least second derivatives of the metric. The Riemann tensor is of course made from second derivatives of the metric, and we argued earlier that the only independent scalar we could construct from the Riemann tensor was the Ricci scalar  $R$ . What we did not show, but is nevertheless true, is that any nontrivial tensor made from the metric and its first and second derivatives can be expressed in terms of the metric and the Riemann tensor. Therefore, the only independent scalar constructed from the metric, which is no higher than second order in its derivatives, is the Ricci scalar. Hilbert figured that this was therefore the simplest possible choice for a Lagrangian, and proposed

$$\mathcal{L}_H = \sqrt{-g} R . \quad (168)$$

The equations of motion should come from varying the action with respect to the metric. In fact let us consider variations with respect to the inverse metric  $g^{\mu\nu}$ , which are slightly easier but give an equivalent set of equations. Using  $R = g^{\mu\nu} R_{\mu\nu}$ , in general we will have

$$\begin{aligned} \delta S &= \int d^4x \left[ \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} + \sqrt{-g} R_{\mu\nu} \delta g^{\mu\nu} + R \delta \sqrt{-g} \right] = \\ &= (\delta S)_1 + (\delta S)_2 + (\delta S)_3 . \end{aligned} \quad (169)$$

The second term  $(\delta S)_2$  is already in the form of some expression times  $\delta g^{\mu\nu}$ ; let's examine the others more closely.

Recall that the Ricci tensor is the contraction of the Riemann tensor, which is given by

$$R^{\rho}{}_{\mu\lambda\nu} = p_{(\lambda)} \Gamma_{\nu\mu}^{\lambda} + \Gamma_{\lambda\sigma}^{\rho} \Gamma_{\nu\mu}^{\sigma} - (\lambda \leftrightarrow \nu) . \quad (170)$$

We perform the variation of the Riemann tensor in such a way that we firstly perform variation of the connection coefficients and then we substitute into this expression.

In fact, after some calculations we find the variation of the Riemann tensor in the form

$$\delta R^\rho{}_{\mu\lambda\nu} = \nabla_\lambda(\delta\Gamma^\rho_{\nu\mu}) - \nabla_\nu(\delta\Gamma^\rho_{\lambda\mu}) . \quad (171)$$

Therefore, the contribution of the first term in (169) to  $\delta S$  can be written

$$\begin{aligned} (\delta S)_1 &= \int d^4x \sqrt{-g} g^{\mu\nu} [\nabla_\lambda(\delta\Gamma^\lambda_{\nu\mu}) - \nabla_\nu(\delta\Gamma^\lambda_{\lambda\mu})] \\ &= \int d^4x \sqrt{-g} \nabla_\sigma [g^{\mu\sigma}(\delta\Gamma^\lambda_{\lambda\mu}) - g^{\mu\nu}(\delta\Gamma^\sigma_{\mu\nu})] , \end{aligned} \quad (172)$$

where we have used metric compatibility. However the integral above is an integral with respect to the natural volume element of the covariant divergence of a vector; by Stokes's theorem, this is equal to a boundary contribution at infinity which we can set to zero by making the variation vanish at infinity. Therefore this term does not contribute to the total variation.

In order to calculate the  $(\delta S)_3$  term we have to use the variation

$$\delta(g^{-1}) = \frac{1}{g} g_{\mu\nu} \delta g^{\mu\nu} . \quad (173)$$

and consequently

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu} . \quad (174)$$

If we now return back to (169), and remembering that  $(\delta S)_1$  does not contribute, we find

$$\delta S = \int d^4x \sqrt{-g} \left[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right] \delta g^{\mu\nu} . \quad (175)$$

However this should vanish for arbitrary variations and consequently we derive Einstein's equations in vacuum:

$$\frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0 . \quad (176)$$

However we would like to get the non-vacuum field equations as well. In other words we consider an action of the form

$$S = \frac{1}{8\pi G} S_H + S_m , \quad (177)$$

where  $S_m$  is the action for matter, and we have presently normalized the gravitational action (although the proper normalization is somewhat convention-dependent). Following through the same procedure as above leads to

$$\frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} = \frac{1}{8\pi G} \left( R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) + \frac{1}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\mu\nu}} = 0 , \quad (178)$$

and we recover Einstein's equations if we set

$$T_{\mu\nu} = -\frac{1}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\mu\nu}} . \quad (179)$$

In fact (179) turns out to be the best way to define a symmetric energy-momentum tensor.

Einstein's equations may be thought of as second-order differential equations for the metric tensor field  $g_{\mu\nu}$ . There are ten independent equations (since both sides are symmetric two-index tensors), which seems to be exactly right for the ten unknown functions of the metric components. However, the Bianchi identity  $\nabla^\mu G_{\mu\nu} = 0$  which we prove below represents four constraints on the functions  $R_{\mu\nu}$ , so there are only six truly independent equations. In fact this is appropriate, since if a metric is a solution to Einstein's equation in one coordinate system  $x^\mu$  it should also be a solution in any other coordinate system  $x^{\mu'}$ . This means that there are four unphysical degrees of freedom in  $g_{\mu\nu}$  (represented by the four functions  $x^{\mu'}(x^\mu)$ ), and we should expect that Einstein's equations only constrain the six coordinate-independent degrees of freedom.

It is important to stress that as differential equations, these are extremely complicated; the Ricci scalar and tensor are contractions of the Riemann tensor, which involves derivatives and products of the Christoffel symbols, which in turn involve the inverse metric and derivatives of the metric. Furthermore, the energy-momentum tensor  $T_{\mu\nu}$  will generally involve the metric as well. The equations are also nonlinear, that implies that two known solutions cannot be superposed to find a third. It is therefore very difficult to solve Einstein's equations in any sort of generality. Then in order to solve them we have to perform some simplifying assumptions. The most popular sort of simplifying assumption is that the metric has a significant degree of symmetry, and we will talk later on about how symmetries of the metric make life easier.

An important property of the energy momentum tensor is that it is conserved. In the flat background the conservation equation takes the form

$$\partial_\mu T^{\mu\nu} = 0 , \quad (180)$$

where the first equation  $\partial_\mu T^{\mu i} = 0$  expresses the conservation of the energy density while the remaining three equations  $\partial_\mu T^{\mu i} = 0$  defines the conservation of the momentum density. In general relativity the conservation equation takes the form

$$\nabla_\mu T^{\mu\nu} = 0 . \quad (181)$$

This equation can be proved using the equation of motion for the metric when we apply the covariant derivative on both sides of this equation

$$\nabla^\mu \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = 8\pi G \nabla^\mu T_{\mu\nu} . \quad (182)$$

We show that the left side of this equation is *identically zero*. Note that generally the matter fields do not have to be on shell since this equation follows from the variation of the action with respect to the metric. To see this we recall the Bianchi identity for the Riemann tensor

$$\nabla_\rho R^\lambda_{\sigma\mu\nu} + \nabla_\nu R^\lambda_{\sigma\rho\mu} + \nabla_\mu R^\lambda_{\sigma\nu\rho} = 0 . \quad (183)$$



Now we contract  $\lambda$  and  $\mu$  indices and by definition  $R^\mu_{\sigma\mu\nu} = R_{\sigma\nu}$  we obtain the identity

$$\nabla_\rho R_{\sigma\nu} - \nabla_\nu R_{\rho\sigma} + \nabla_\lambda R^\lambda_{\sigma\nu\rho} = 0 . \quad (184)$$

Then we contract this equation with  $g^{\rho\sigma}$  and we obtain

$$0 = \nabla_\rho R^\rho_\nu - \nabla_\nu R + \nabla^\lambda R_{\lambda\nu} = 2\nabla^\mu (R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R) = 0 . \quad (185)$$

which implies that the covariant conservation law of the stress energy-tensor is a necessary condition for the consistency of the Einstein equation.

On the other hand the stress energy tensor is determined by the matter action. Clearly when we search the extremum of the action we perform the variation of the action with respect to the matter fields so that the energy momentum tensor should be conserved as the consequence of the matter equations of motions as well. Alternatively, we can presume the evolution of the matter fields on the fixed background and in this case the energy-momentum tensor should be conserved as well.

To proceed note that the matter action is diffeomorphism invariant so that the conservation of the energy momentum tensor should follow from the invariance of the action under general diffeomorphism transformation. In fact, under transformation

$$x'^\mu = x^\mu + \xi^\mu . \quad (186)$$

Then

$$\begin{aligned} g'^{\mu\nu}(x') &= g^{\rho\sigma} \frac{\partial x'^\mu}{\partial x^\rho} \frac{\partial x'^\nu}{\partial x^\sigma} \Rightarrow \\ g'^{\mu\nu}(x') &= g^{\mu\nu}(x) + g^{\nu\lambda}(x) \partial_\lambda \xi^\mu + \partial_\lambda x^\mu g^{\lambda\nu}(x) \end{aligned} \quad (187)$$

If we expand

$$g'^{\mu\nu}(x') = g'^{\mu\nu}(x + \xi) = g^{\mu\nu}(x) + \partial_\lambda g'^{\mu\nu} \xi^\lambda = g^{\mu\nu}(x) + \partial_\lambda g^{\mu\nu} \xi^\lambda \quad (188)$$

we find the variation  $g^{\mu\nu}$  as

$$\delta g^{\mu\nu}(x) = g'^{\mu\nu}(x) - g^{\mu\nu}(x) = -\partial_\lambda g^{\mu\nu}(x) \xi^\lambda + g^{\mu\lambda} \partial_\lambda \xi^\nu + \partial_\lambda \xi^\mu g^{\lambda\nu} . \quad (189)$$

Now we proceed to the transformation property of the matter fields. Their form depends on the character of these fields, whether they are scalars, vectors,..... For example, in case of the scalar field we find

$$\phi'(x') = \phi(x) \Rightarrow \phi'(x) - \phi(x) = -\partial_\lambda \phi \xi^\lambda \quad (190)$$

Since the action is invariant under the diffeomorphism invariance we obtain

$$\delta_\xi S_m = \frac{1}{2} \int d^4x \sqrt{-g} T_{\mu\nu} (\nabla^\mu \xi^\nu + \nabla^\nu \xi^\mu) + \int d^4x \sqrt{-g} \frac{\delta \mathcal{L}_m}{\delta \psi} \delta \psi_\xi = 0 , \quad (191)$$

where we also used the fact that the variation of the metric can be written as

$$g'^{\mu\nu} - g^{\mu\nu} = \nabla^\mu \xi^\nu + \nabla^\nu \xi^\mu \quad (192)$$

Note that the equation (191) has to be zero of shell. Let us now presume that the matter field equations are satisfied which implies that the second term in (191) vanishes. Then using integration by parts we can rewrite (191) into the form

$$\delta_\xi S_m(\text{on shell}) = - \int d^4x \sqrt{-g} \xi^\mu \nabla^\mu T_{\mu\nu} = 0 \quad (193)$$

that using the fact that  $\xi^\mu$  is arbitrary implies the conservation of the stress energy tensor.

We continue with the study of the Einstein equations where we now discuss the possibility of the introduction of a cosmological constant. In order to introduce it we add it to the conventional Hilbert action. We therefore consider an action given by

$$S = \int d^4x \sqrt{-g} (R - 2\Lambda) , \quad (194)$$

where  $\Lambda$  is some constant. The resulting field equations are

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0 , \quad (195)$$

and of course there would be an energy-momentum tensor on the right hand side if we had included an action for matter.  $\Lambda$  is the cosmological constant. In order to find its meaning it is convenient to move the additional term in (195) to the right hand side, and think of it as a kind of energy-momentum tensor, with  $T_{\mu\nu} = -\Lambda g_{\mu\nu}$  (it is automatically conserved by metric compatibility). Then  $\Lambda$  can be interpreted as the “energy density of the vacuum,” a source of energy and momentum that is present even in the absence of matter fields. This interpretation is important because quantum field theory predicts that the vacuum should have some sort of energy and momentum. In ordinary quantum mechanics, an harmonic oscillator with frequency  $\omega$  and minimum classical energy  $E_0 = 0$  upon quantization has a ground state with energy  $E_0 = \frac{1}{2} \hbar \omega$ . A quantized field can be thought of as a collection of an infinite number of harmonic oscillators, and each mode contributes to the ground state energy. The result is of course infinite, and must be appropriately regularized, for example by introducing a cutoff at high frequencies. The final vacuum energy, which is the regularized sum of the energies of the ground state oscillations of all the fields of the theory, has no good reason to be zero and in fact would be expected to have a natural scale

$$\Lambda \sim m_P^4 , \quad (196)$$

where the Planck mass  $m_P$  is approximately  $10^{19}$  GeV, or  $10^{-5}$  grams. Observations of the universe on large scales allow us to constrain the actual value of  $\Lambda$ , which turns out to be smaller than (196) by at least a factor of  $10^{120}$ . This is the largest known discrepancy between theoretical estimate and observational constraint in physics,

and convinces many people that the ‘‘cosmological constant problem’’ is one of the most important unsolved problems today. On the other hand the observations do not tell us that  $\Lambda$  is strictly zero, and in fact allow values that can have important consequences for the evolution of the universe.

Now it is time to give a concrete example of the stress energy tensor. Very natural situation occurs when we have a gas of particles with mass  $m_n$ , charges  $e_n$  and trajectories  $x_n^\mu(\lambda)$  that interact with electromagnetic field  $A_\mu$ . This system also propagates on the manifold with the metric  $g_{\mu\nu}$ . Then principles of equivalence implies that the action of the matter part of the action has the form

$$\begin{aligned}
I_m = & -\frac{1}{4} \int d^4x \sqrt{-\det g} F_{\mu\nu} F_{\rho\sigma} g^{\mu\rho} g^{\nu\sigma} - \\
& - \sum_n m_n \int d\lambda_n \sqrt{-g_{\mu\nu}(x_n(\lambda))} \frac{dx_n^\mu(\lambda)}{d\lambda} \frac{dx_n^\nu(\lambda_n)}{d\lambda_n} + \\
& + \sum_n e_n \int d\lambda \frac{dx_n^\mu(\lambda)}{d\lambda} A_\mu(x_n(\lambda)) .
\end{aligned} \tag{197}$$

where  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ . Note that the  $n$ -th particle couples to the electric field evaluated at her position  $x_n^\mu$ . Finally note that  $\lambda_n$  is the parameter that labels the world-line of  $n$ -th particle. Now the variation of the action with respect to  $x_n^\mu(\lambda_n)$  gives

$$\begin{aligned}
& m_n \frac{1}{2\sqrt{-g_{\mu\nu}(x_n(\lambda_n))} \frac{dx_n^\mu(\lambda_n)}{d\lambda_n} \frac{dx_n^\nu(\lambda_n)}{d\lambda_n}} \partial_\mu g_{\rho\sigma}(x_n) \frac{dx_n^\rho}{d\lambda_n} \frac{dx_n^\sigma}{d\lambda_n} - \\
& - m_n \frac{d}{d\lambda_n} \left[ \frac{1}{\sqrt{-g_{\mu\nu}(x_n(\lambda_n))} \frac{dx_n^\mu(\lambda_n)}{d\lambda_n} \frac{dx_n^\nu(\lambda_n)}{d\lambda_n}} g_{\mu\nu}(x_n) \frac{dx_n^\nu}{d\lambda_n} \right] + e_n F_{\mu\nu}(x_n(\lambda_n)) \frac{dx_n^\nu}{d\lambda_n} = 0
\end{aligned} \tag{198}$$

As the next step we explicitly perform the derivative with respect to  $\lambda_n$  at the first expression on the second line and after some steps we obtain

$$\begin{aligned}
& -\frac{m_n}{\sqrt{-g_{\mu\nu}(x_n(\lambda_n))} \frac{dx_n^\mu(\lambda_n)}{d\lambda_n} \frac{dx_n^\nu(\lambda_n)}{d\lambda_n}} \frac{1}{2} [\partial_\rho g_{\mu\nu} + \partial_\nu g_{\mu\rho} - \partial_\mu g_{\rho\nu}] \frac{dx_n^\rho}{d\lambda} \frac{dx_n^\nu}{d\lambda} - \\
& - m_n \frac{1}{\sqrt{-g_{\mu\nu}(x_n(\lambda_n))} \frac{dx_n^\mu(\lambda)}{d\lambda_n} \frac{dx_n^\nu(\lambda)}{d\lambda_n}} g_{\mu\nu}(x_n) \frac{dx_n^\nu}{d\lambda_n} - \\
& - m_n \frac{d}{d\lambda} \left[ \frac{1}{\sqrt{-g_{\mu\nu}(x_n(\lambda))} \frac{dx_n^\mu(\lambda)}{d\lambda} \frac{dx_n^\nu(\lambda)}{d\lambda_n}} \right] + e_n F_{\mu\nu}(x_n) \frac{dx_n^\nu}{d\lambda_n} = 0
\end{aligned} \tag{199}$$

Now we multiply this equation with  $g^{\omega\mu}$  and perform relabeling indices. Further, there is still square root factor. On the other hand we know that proper time of  $n$ -th particle is defined as

$$d\tau_n^2 = -g_{\mu\nu}(x_n)dx_n^\mu dx_n^\nu \quad (200)$$

that implies

$$\frac{d\tau_n}{d\lambda_n} = \sqrt{-g_{\mu\nu}(x_n(\lambda_n)) \frac{dx_n^\mu}{d\lambda_n} \frac{dx_n^\nu}{d\lambda_n}} \quad (201)$$

Due to the world-line reparameterization invariance we can choose the parameter  $\lambda_n$  in such a way that  $\frac{d\tau_n}{d\lambda_n} = 1$ . Then we have

$$\frac{dx_n^\mu}{d\lambda_n} = \frac{dx_n^\mu}{d\tau_n} \frac{d\tau_n}{d\lambda_n} \quad (202)$$

and the equation of motion above takes final form

$$m_n \left[ \frac{dx_n^\mu}{d\tau_n} + \Gamma_{\nu\rho}^\mu(x_n) \frac{dx_n^\nu}{d\tau_n} \frac{dx_n^\rho}{d\tau_n} \right] = e_n g^{\mu\nu} F_{\nu\sigma} \frac{dx_n^\sigma}{d\tau_n} . \quad (203)$$

With the help of the action we can determine stress energy tensor for the electromagnetic field and for the collections of particles. Since  $T_{\mu\nu} = -\frac{1}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\mu\nu}}$  we obtain (using also the fact that  $\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}$ ) for the action in the form  $S = -\int d^4x \sqrt{-g} \mathcal{L}$

$$T_{\mu\nu} = -\frac{1}{2}g_{\mu\nu}\mathcal{L} + \text{frac}\delta\mathcal{L}\delta g^{\mu\nu} . \quad (204)$$

In case of the electromagnetic field we find

$$T_{\mu\nu}^{e.f.} = -\frac{1}{8}g_{\mu\nu}F_{\rho\sigma}F_{\gamma\delta}g^{\rho\gamma}g^{\sigma\delta} + \frac{1}{2}F_{\mu\sigma}F_{\nu\rho}g^{\sigma\rho} . \quad (205)$$

Observe that this stress energy tensor obeys the condition

$$g^{\nu\mu}T_{\mu\nu}^{e.f.} = -\frac{1}{8}g^{\mu\nu}g_{\nu\mu}F_{\rho\sigma}F_{\gamma\delta}g^{\rho\gamma}g^{\sigma\delta} + \frac{1}{2}F_{\mu\nu}g^{\mu\rho}g^{\nu\sigma}F_{\rho\sigma} = 0 . \quad (206)$$

In other words the stress energy tensor of the electromagnetic field is traceless. In case of the collection of particles we write the relevant part of the action as

$$\begin{aligned} & -\sum_n m_n \int d\lambda_n \sqrt{-g_{\mu\nu}(x_n(\lambda)) \frac{dx_n^\mu(\lambda)}{d\lambda} \frac{dx_n^\nu(\lambda)}{d\lambda}} = \\ & = -\sum_n m_n \int d^4x \int d\lambda_n \sqrt{-g_{\mu\nu}(x)} \frac{dx_n^\mu(\lambda)}{d\lambda} \frac{dx_n^\nu(\lambda)}{d\lambda} \delta^4(x - x_n) \end{aligned} \quad (207)$$

so that we obtain

$$\begin{aligned}
T^{\mu\nu} &= -\frac{1}{\sqrt{-\det g}} \frac{\delta I_m}{\delta g_{\mu\nu}} = \\
&= \frac{1}{\sqrt{-\det g}} \sum_n m_n \int d\lambda_n \frac{1}{\sqrt{-g_{\mu\nu}(x)} \frac{dx_n^\mu(\lambda)}{d\lambda_n} \frac{dx_n^\nu(\lambda)}{d\lambda_n}} \frac{dx_n^\mu}{d\lambda_n} \frac{dx_n^\nu}{d\lambda_n} \delta^4(x - x_n) .
\end{aligned} \tag{208}$$

### 3.3 Currents in Cosmology

Due to the principles of homogeneity and isotropy we require that the mean value of any three vector has to vanish while homogeneity requires that the mean value of any three scalar is function of time only. In other words we require that the current of galaxies, or baryons has components

$$J^i = 0 , J^0 = n(t) . \tag{209}$$

Since this current is conserved it obeys the equation

$$\nabla_\mu J^\mu = \partial_\mu J^\mu + \Gamma_{\mu\nu}^\mu J^\nu = \frac{dn}{dt} + \Gamma_{i0}^i n = \frac{dn}{dt} + 3 \frac{da}{dt} \frac{n}{a} = 0 . \tag{210}$$

since the non-zero components of the Chritoffel symbols are

$$\Gamma_{j0}^i = \frac{\dot{a}}{a} \delta_j^i , \Gamma_{ij}^0 = a\dot{a}\gamma_{ij} , \Gamma_{jk}^i = {}^{(3)}\Gamma_{jk}^i . \tag{211}$$

so that  $\Gamma_{\mu 0}^\mu = \Gamma_{i0}^i = 3\frac{\dot{a}}{a}$ . From (210) we obtain following time dependence of the scalar density

$$\frac{dn}{n} = -3 \frac{da}{a} \tag{212}$$

that can be easily integrated with the result

$$n = \frac{n_0 a_0^3}{a^3} , \tag{213}$$

where  $n_0$  is density at the value of the scale factor  $a_0$ .

Let us now consider the case of the stress energy tensor. The principle of spatial isotropy means that the mean value of any three vector  $t^{ij}$  at  $\mathbf{x} = 0$  to be proportional to  $\delta_{ij}$  which however means that it is proportional to  $g_{ij}$  that is equal to  $\frac{1}{a^2}\delta_{ij}$  at  $\mathbf{x} = 0$ . Further, homogeneity requires that this tensor cannot depend on spatial coordinates and the only dependence is on time  $t$ . Further, since this proportionality is the relation between two spatial tensors clearly it is preserved under any transformations of space coordinates that include also transformations that transform the origin into any other point. In other words the principles of homogeneity and isotropy requires that the stress energy tensor has the form

$$T^{00} = \rho(t) , \quad T^{0i} = 0 , \quad T^{ij} = \frac{1}{a^2(t)} \gamma^{ij}(\mathbf{x}) p(t) . \tag{214}$$

Let us now analyze the conservation equation  $\nabla_\nu T^{\nu\mu} = 0$  for this stress energy tensor. First of all we have

$$\nabla_\nu T^{\rho\mu} = \partial_\nu T^{\rho\mu} + \Gamma_{\nu\lambda}^\rho T^{\lambda\mu} + \Gamma_{\nu\lambda}^\mu T^{\rho\lambda} = 0 \quad (215)$$

that implies

$$\nabla_\nu T^{\nu\mu} = \partial_\nu T^{\nu\mu} + \Gamma_{\nu\lambda}^\nu T^{\lambda\mu} + \Gamma_{\nu\lambda}^\mu T^{\nu\lambda} = 0 \quad (216)$$

Let us start with the case  $\mu = i$ . Then the conservation equation has the form

$$\begin{aligned} \partial_0 T^{0i} + \partial_j T^{ji} + \Gamma_{\mu j}^\mu T^{ji} + \Gamma_{jk}^i T^{jk} &= \partial_j T^{ji} + {}^{(3)}\Gamma_{kj}^k T^{ji} + {}^{(3)}\Gamma_{jk}^i T^{jk} = \\ &= \frac{p(t)}{a^2(t)} (\partial_j \gamma^{ji} + {}^{(3)}\Gamma_{kj}^k \gamma^{ji} + {}^{(3)}\Gamma_{jk}^i \gamma^{jk}) = 0 \end{aligned} \quad (217)$$

using the fact that non-zero components of affine connection in FRW metric have the form

$$\Gamma_{j0}^i = \frac{\dot{a}}{a} \delta_{ij}, \quad \Gamma_{ij}^0 = a\dot{a}\gamma_{ij}, \quad \Gamma_{jk}^i = {}^{(3)}\Gamma_{jk}^i. \quad (218)$$

and using the fact that the expression in the bracket is the covariant derivative compatible with the metric  $\gamma_{ij}$  so that it is equal to zero. Let us now consider more interesting case when  $\mu = 0$  so that we have

$$\begin{aligned} \nabla_\nu T^{\nu 0} &= \partial_0 T^{00} + \Gamma_{0\lambda}^\nu T^{\lambda 0} + \Gamma_{\nu\lambda}^0 T^{\nu\lambda} = \\ &= \partial_0 T^{00} + \Gamma_{i0}^i T^{00} + \Gamma_{ij}^0 T^{ij} = \partial_0 \rho + 3\frac{\dot{a}}{a}\rho + \frac{\dot{a}}{a}\gamma_{ij}\gamma^{ji}p = \\ &= \partial_0 \rho + 3\frac{\dot{a}}{a}(\rho + p) = 0 \end{aligned} \quad (219)$$

In other words we obtain famous equation

$$\boxed{\partial_0 \rho + 3\frac{\dot{a}}{a}(\rho + p) = 0} \quad (220)$$

This equation can be easily integrated when we know an state equation that expresses pressure as function of energy density  $p = p(\rho)$  so that we obtain differential equation

$$\frac{d\rho}{\rho + p(\rho)} + 3\frac{da}{a} = 0 \quad (221)$$

that can be integrated, at least in principle. The most simplest examples however correspond the situation where we have linear dependence of pressure on energy density

$$p = w\rho, \quad (222)$$

where  $w$  is time-independent. In this case the equation (221) implies

$$\rho = \rho_0 \frac{a_0^{3(1+w)}}{a^3(1+w)}, \quad (223)$$

where  $\rho_0$  is energy density at the time  $t_0$  when the scale factor is equal to  $t_0$ .

In fact, this symple relation can be applied for the most important cases in modern cosmology:

- **Cold Matter**

Cold matter corresponds non-relativistic matter when we can neglect its pressure  $p = 0$ . In this case we have  $w = 0$  and hence

$$\rho = \rho_0 \frac{a_0^3}{a^3} .$$

- **Hot Matter** This is the case of relativistic matter when we can neglect the rest mass. In this case we have

$$p = \frac{\rho}{3} , w = \frac{1}{3}$$

and hence

$$\rho = \rho_0 \frac{a_0^4}{a^4} .$$

- **Vacuum Energy** This is the sort of matter where the stress energy tensor is proportional to  $g^{\mu\nu}$  and hence  $p = -\rho$ . In this case we have  $w = -1$  and hence the energy density is constant known as *cosmological constant* or *vacuum energy*.

It is important to stress one subtle point. The conservation of the stress energy tensor holds for the whole system including all components, as for example radiation, matter and cosmological constant. On the other hand we derived the time evolution of corresponding components with the help of the presumption that each component is conserved separately. In other words we implicitly introduced presumption that there is no interchange of energy between the different components.

There is another important point that we have to discuss in more details. It is clear from previous analysis that all equations and their solutions describe local properties of space-time. On the other hand it is certainly very instructive to know more informations about global properties of space-time. For  $K = +1$  space is finite since it is spherical surface which does not have a boundary.

The situation is different in case  $K = 0$  or  $K = -1$ . Common practise is to consider these spaces as infinite but there is a possibility to have finite spaces with the same local geometry. We can find very nice example in two dimensional geometry when we consider torus which is topologically the space  $S^1 \times S^1$ . This is the space with flat metric but which is finite with non-trivial topology. The similar situation can occur in case of the space  $K = 0$  as well when we could identify the points  $\mathbf{x}$  and  $\mathbf{x} + n_1 \mathbf{L}_1 + n_2 \mathbf{L}_2 + n_3 \mathbf{L}_3$ , where  $n_1, n_2, n_3$  are integers and  $\mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3$  are fixed three vectors. Then the space is finite with volume  $a^3 \mathbf{L}_1 \cdot (\mathbf{L}_2 \times \mathbf{L}_3)$ . However such a finite space proposal should have a consequence on the pattern of distribution of matter and radiation where we should see some periodicity. On the other hand the fact that there is no sign of periodicity in our observation implies that any periodicity lengths as  $|\mathbf{L}_i|$  have to be larger than  $10^{10}$  light years.

### 3.4 The cosmological redshift

The analysis presented above is certainly valid for any derivative  $\dot{a}$  which, of course, it is crucial to determine which is the physical one. In other words, we have to determine whether Universe is expanding, contracting or whether it does not change at all ( $\dot{a} = 0$ ). The way how we extract this information follows from the spectral lines that come to us from distant galaxies when we compare shift in frequencies with frequencies that we measure at terrestrial laboratories.

In order to calculate shift in frequencies let us presume (without loss of generality) that we are in the center of FRW coordinate system and consider a light ray that comes to us along radial direction. Since the light ray obeys the equation  $d\tau^2 = 0$  we obtain from the invariance of the line element following equation

$$0 = -dt^2 + a^2 \frac{1}{1 - Kr^2} dr^2 \quad (224)$$

that implies differential equation

$$\frac{dt}{a(t)} = \pm \frac{dr}{\sqrt{1 - Kr^2}} \quad (225)$$

This equation determines the propagation of the photon in expanding Universe. On the other hand we are mainly interested in the shift of the frequencies of the light. To see the origin of this phenomena let us consider an action of free electromagnetic field

$$S = -\frac{1}{4} \int d^4x \sqrt{-g} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma}, F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (226)$$

Consider now the propagation of a photon in the homogeneous isotropic Universe. Since the photon wavelength is small compared to the spatial curvature radius even if the Universe is open or closed. Then we can consider the metric that is spatially flat with the metric

$$ds^2 = -dt^2 + a^2(t) \delta_{ij} dx^i dx^j. \quad (227)$$

Let us introduce conformal time  $\eta$  instead of  $t$  that is defined as

$$dt = a d\eta \quad (228)$$

or equivalently

$$\eta = \int \frac{dt}{a(t)}. \quad (229)$$

This result can be generally integrated so that we have  $\eta = \eta(t)$  and we presume that this relation can be inverted so that  $t = t(\eta)$  and consequently  $a = a(\eta)$ . Now the metric has the form

$$ds^2 = a^2(\eta) [-d\eta^2 + \delta_{ij} dx^i dx^j] \quad (230)$$



and we see that the metric element in FRW spacetime is conformally flat in the sense that

$$g_{\mu\nu} = a^2(\eta)\eta_{\mu\nu} , \quad (231)$$

where the Minkowski metric is spanned by coordinates  $(\eta, x^i)$ . Then we clearly have

$$g^{\mu\nu} = a^{-2}\eta^{\mu\nu} , \sqrt{g} = a^4 \quad (232)$$

and we find that in  $\eta, x^i$  coordinates the action of the electromagnetic field has the form

$$S = -\frac{1}{4} \int d^4x \eta^{\mu\rho} \eta^{\nu\sigma} F_{\mu\rho} F_{\nu\sigma} . \quad (233)$$

Now it is clear that the solution of the equation of motion for the free electromagnetic field in the Universe is given as the superposition of the plane waves

$$A_\mu^{(\alpha)} = e_\mu^{(\alpha)} e^{ik\eta - i\mathbf{k}\mathbf{x}} , \quad (234)$$

where  $\mathbf{k}$  is constant vector,  $|\mathbf{k}| = k$  and  $e_\mu^{(\alpha)}$  is the standard polarization vector of photons with  $\alpha = 1, 2$ . Note that  $k$  is not the physical frequency as follows from following arguments. The quantity  $\Delta x = \frac{2\pi}{k}$  is the coordinate wavelength of a photon while the physical wavelength at time  $t$  is

$$\lambda(t) = a(t)\Delta x = 2\pi \frac{a(t)}{k} . \quad (235)$$

In the same way we define period  $\Delta\eta = \frac{2\pi}{k}$  of electromagnetic wave in conformal time while the period of the physical time is

$$T = a(t)\Delta\eta = 2\pi \frac{a(t)}{k} . \quad (236)$$

Then we see that the frequency is equal to

$$\omega(t) = \frac{2\pi}{T} = \frac{k}{a(t)} \quad (237)$$

and since we know that the frequency is equal to the magnitude of the physical momentum of photon we obtain

$$\mathbf{p} = \frac{\mathbf{k}}{a(t)} . \quad (238)$$

This relation implies that the physical momentum either decreases or increases with dependence on the fact whether Universe is expanding or contracting. We see that in the expanding universe the scale factor  $a(t)$  is growing and hence the physical wavelength grows. On the other hand the physical momentum is decreasing function of time. The phenomena when the wavelength is growing during the expansion of the Universe is named as the redshift. Explicitly, if the photon was emitted at time  $t_i$  with physical wave length  $\lambda_i$  in the physical process as for example when the

electron in the excited state in the atom drops to the ground state which is certainly physical process. Now we know that the state propagates freely as in (387) and then it is again detected in time  $t_0$  where  $t_0$  we means the present time in the reversed physical process when its physical wave length now is

$$\lambda(t_0) = a(t_0) \frac{2\pi}{k} \quad (239)$$

Now expressing  $\frac{2\pi}{k}$  using the physical wave length at time of emission we find the famous relation

$$\lambda(t_0) = \frac{a(t_0)}{a(t_i)} \lambda_i \equiv \lambda_i (1 + z(t_i)) . \quad (240)$$

The quantity

$$z(t_i) = \frac{a(t_0)}{a(t_i)} - 1 \quad (241)$$

is called *redshift*. The earlier the object emits the photon then this photon has to travel longer and consequently  $a(t_i)$  is smaller and hence object at larger distances have the larger redshifts.

Note that these formulas are valid in general for all  $z$ . Let us now consider objects that are not in large distance. Then the difference  $t - t_0$  is not very large and we can expand

$$a(t_i) = a(t_0) - \dot{a}(t_0)(t_0 - t_i) \quad (242)$$

Using the present value of the Hubble parameter  $H_0 = \frac{\dot{a}(t_0)}{a(t_0)} \equiv \frac{\dot{a}_0}{a_0}$  we can write

$$a(t_i) = a_0 [1 - H_0(t_0 - t_i)] \quad (243)$$

so that to the linear order we find following expression for the redshift

$$z(t_i) = \frac{1}{1 - H_0(t_0 - t_i)} - 1 \simeq H_0(t_0 - t_i) . \quad (244)$$

Finally the travel time is equal to

$$\begin{aligned} 0 &= -dt^2 + a(t)^2 dr^2 = -dt^2 + (a_0 - \dot{a}_0(t_0 - t))^2 dr^2 \approx \\ &= -dt^2 + a_0^2 dr^2 \Rightarrow (t_0 - t) = a_0(r_i - r_0) \equiv R \end{aligned} \quad (245)$$

where  $R$  is the physical distance of the object from the our observer. Inserting this expression into (397) we derive famous Hubble law

$$z = H_0 R , z \ll 1 . \quad (246)$$

The redshift of light from other galaxies was firstly observed in the 1910's by Vesto Melvin Slipher at the Lowell Observatory in Flagstaff, Arizona. From 1918 to 1925 C. Wirtz and K. Lundmark discovered a number of spiral nebulae with redshifht that seem to increase with distance. On the other hand these earlier date

do not have a clear linear relation between redshift and distance. The origin of this fact is the problem that has not been solved until present which is the fact that real galaxies generally do not move only with general expansion of universe but they typically have additional velocities of hundreds of kilometers per second that is caused by gravitational fields of near galaxies. Then in order to see linear relation between redshift and distance we have to analyze galaxies with  $|z| \gg 10^{-3}$ . For these galaxies their cosmological velocities are thousands of kilometers per second and hence their additional velocities can be neglected.

In 1929 Hubble published his result that he had found almost linear relation between redshift and distance. The problem was that at that time redshifts and distances had been measured only for galaxies in the constellation Virgo. In this case the redshift was about  $1000 \text{ km/sec}$  which is not much larger than typical peculiar velocities. For that reason these data did not really support a linear relation. However by the early 1930's he measured redshifts and distances out to the Coma cluster with redshift  $z \sim 0.02$  that corresponds to a velocity of about  $7000 \text{ km/sec}$  where now linear relation between redshift and distance was clear with the main conclusion that the universe really is expanding.

The research of the redshift of galaxies has continued until today and we observe galaxies with even larger redshift where however we have to take into account relativistic effects and hence the linear dependence is not fully valid. Currently, the object with the highest known redshifts are galaxies that produce gamma ray burst. The highest confirmed spectroscopic redshift of a galaxy is that of *GN-z11* galaxy in constellation Ursa Major (Great Bear) with a redshift  $z = 11.1$  corresponding to 400 millions years after the Big Bang. The cosmic microwave background has a redshift of  $z = 1089$  corresponding to 379000 years after the Big Bang. We are also still waiting for the observation of the light from the oldest Population III stars, which are stars formed not long after first atoms were formed, may have a redshift  $20 < z < 100$ .

### 3.5 Distances at small redshift

We met the notion of distance in previous section. Even if this notion can be intuitively clear from everyday life we should be more careful when we work in cosmology. To see this let us start with simpler case when we consider objects that are relatively close with  $z < 0.1$ . In this case we can neglect the effects of the space-time curvature and we can also neglect an effect of cosmic expansion on the determinations of distances. These measurements are also very important since they allow us to determine the value of the Hubble constant  $H_0$ . Further, distance measurements at larger redshifts that determine the shape of the curve  $a(t)$ , depend on the observations of standard candles, which are objects of known intrinsic luminosity that has to be identified and calibrated at these small redshifts.

Today it is convenient to separate the objects that we use for the determination of distances in cosmology into two categories known as primary and secondary distance indicators. The absolute luminosities of the primary distance indicators in our local

group of galaxies are mainly measured directly with the help of kinematic methods that do not depend on a priori knowledge of absolute luminosities. The problem with these primary distance indicators is that they are not bright enough so that it cannot be studied at distances where  $z \geq 0.01$ . At these distances cosmological velocities  $cz$  are larger than typical random velocities of galaxies. For that reason primary indicators cannot be used directly for the analysis of function  $a(t)$ . For that reason we have to use *secondary distance indicators* that are bright enough to be studied at these large distances and whose absolute velocities are known when we associate the closer ones with primary distance indicators.

**Primary distance indicators** We are not going into details of the description of the primary distance indicators and we restrict ourselves to their list. These indicators are

- Trigonometric parallax
- Proper motions
- Apparent luminosity The measurement of the apparent luminosity is the most common method of determining distances in cosmology. It is based on the measurement of apparent luminosity of objects where we know or suppose that we know they absolute luminosity.

## 3.6 Brief Review of General Relativity

### 1. The Equivalence Principle

#### 3.7 Local Flatness

For a given point  $P$  in spacetime it is always possible to find a coordinate system  $x'^\alpha$  such that

$$g_{\alpha'\beta'}(P) = \eta_{\alpha'\beta'} , \Gamma_{\beta'\gamma'}^{\alpha'}(P) = 0 . \quad (247)$$

where  $\eta_{\alpha'\beta'} = \text{diag}(-1, 1, 1, 1)$ . Such a coordinate system are called *local Lorentz frame* at  $P$ . It is important that it is not possible to have the derivatives of the connection to zero when the space-time is curved. The physical meaning of the local flatness is that free falling observers see no effect of gravity in his immediate vicinity.

**Proof of the theorem** Let  $x^\alpha$  is an arbitrary coordinate system and we presume that  $P$  is at the origin of both coordinate systems. Then the coordinates of the point near  $P$  are related by

$$x^{\alpha'} = A^{\alpha'}_{\beta} x^{\beta} + O(x^2) , \quad x^{\alpha} = A^{\alpha}_{\beta'} x^{\beta'} + O(x'^2) , \quad (248)$$

where  $A^{\alpha'}_{\alpha}, A^{\alpha}_{\beta'}$  are constant matrices. Using these two expression we find

$$x^{\alpha'} = x^{\beta'} \delta_{\beta'}^{\alpha'} = A_{\gamma}^{\alpha'} A^{\gamma}_{\beta'} x^{\beta'} \Rightarrow \delta_{\beta'}^{\alpha'} = A_{\gamma}^{\alpha'} A^{\gamma}_{\beta'} \quad (249)$$

so that these two matrices are inverse. Note that the metric components in the coordinate system  $x^{\alpha'}$  are  $g_{\alpha'\beta'}$  so that

$$g_{\alpha'\beta'} dx^{\alpha'} dx^{\beta'} = g_{\alpha\beta} dx^{\alpha} dx^{\beta} \Rightarrow g_{\alpha'\beta'} = g_{\alpha\beta} \frac{dx^{\alpha}}{dx^{\alpha'}} \frac{dx^{\beta}}{dx^{\beta'}} = g_{\alpha\beta} A^{\alpha}_{\alpha'} A^{\beta}_{\beta'} \quad (250)$$

Now we demand that the left-hand side is equal to  $\eta_{\alpha'\beta'}$ . **The question: what happens when we demand that the left side is equal to Rindler Space-time.** This gives 10 equations for 16 unknown components of the matrix  $A^{\alpha}_{\beta'}$ . Solutions can be found with 6 undetermined components. This corresponds to the freedom of performing a Lorentz transformation (3 rotation parameters and 3 boost parameters) that leave the Minkowski metric invariant.

Let us presume that we choose the particular form of  $A^{\alpha}_{\alpha'}$ . Then we also know  $A^{\alpha'}_{\alpha}$  by inverting the matrix and the coordinate transformation is known to the first order. Then we can proceed to the second order as

$$x^{\alpha'} = A^{\alpha'}_{\beta} x^{\beta} + \frac{1}{2} B^{\alpha'}_{\beta\gamma} x^{\beta} x^{\gamma} \quad (251)$$

Now we can study the transformation properties of the connections

$$\Gamma^{\alpha'}_{\beta'\gamma'} = A^{\alpha'}_{\alpha} A^{\beta}_{\beta'} A^{\gamma}_{\gamma'} \Gamma^{\alpha}_{\beta\gamma}(P) - B^{\alpha'}_{\beta\gamma} A^{\beta}_{\beta'} A^{\gamma}_{\gamma'} \quad (252)$$

If we demand that left side is equal to zero we find the equation

$$B^{\alpha'}_{\beta\gamma} = A^{\alpha'}_{\alpha} \Gamma^{\alpha}_{\beta\gamma} \quad (253)$$

These equations determine  $B$  uniquely.

### 3.7.1 Geodesic deviation

Consider two geodesics  $\gamma_0, \gamma_1$  each described by relations  $x^{\alpha}(t)$  where  $t$  is an affine parameter. Let us develop the notion of deviation vector. To do this we introduce family of geodesics  $x^{\alpha}(s, t)$ , where  $s \in [0, 1]$  where  $s = 0$  corresponds to  $\gamma_0$  and  $s = 1$  corresponds to  $\gamma_1$ . Note that the affine parameter along the specific geodesic is  $t$ . The vector field is  $u^{\alpha} = \frac{\partial x^{\alpha}}{\partial t}$  that is tangent to the geodesics and it satisfies the geodesic equation

$$\nabla_{\beta} u^{\alpha} u^{\beta} = 0. \quad (254)$$

Keeping  $t$  fixed and vary  $s$  instead we obtain family of curves that are generally not geodesics. We denote these curves as

$$\zeta^{\alpha} = \frac{\partial x^{\alpha}}{\partial s} \quad (255)$$

and its restriction to this vector  $\gamma_0$  is  $\zeta^{\alpha}(s = 0)$  and it gives a clear notion of deviation from  $\gamma_0$  to  $\gamma_1$ . We would like to derive an expression for its acceleration along geodesics

$$\frac{D^2 \zeta^{\alpha}}{dt^2} = \frac{d}{dt} (\nabla_{\beta} \zeta^{\alpha} u^{\beta}) = \nabla_{\gamma} (\nabla_{\beta} \zeta^{\alpha} u^{\beta}) u^{\gamma} \quad (256)$$

where these quantities are evaluated at  $\gamma_0$ . In flat space-time geodesics are straight. Now we show an important relations

$$\mathcal{L}_u \zeta^\alpha = \mathcal{L}_\zeta u^\alpha = 0 \Rightarrow \nabla_\beta \zeta^\alpha u^\beta = \nabla_\beta u^\alpha \zeta^\beta \quad (257)$$

as follows from the fact that  $u^\alpha = \frac{\partial x^\alpha}{\partial t}$ ,  $\zeta^\alpha = \frac{\partial x^\alpha}{\partial s}$  and  $\partial_s u^\alpha = \frac{\partial^2 x^\alpha}{\partial s \partial t} = \partial_t \zeta^\alpha$ . Then

$$\begin{aligned} \frac{d}{dt}(\zeta^\alpha u_\alpha) &= \nabla_\beta(\zeta^\alpha u_\alpha)u^\beta = \\ &= \nabla_\beta \zeta^\alpha u^\beta u_\alpha + \zeta^\alpha \nabla_\beta u_\alpha u^\beta = \\ &= \nabla_\beta u^\alpha \zeta^\beta u_\alpha = \frac{1}{2} \nabla_\beta(u^\alpha u_\alpha) \zeta^\beta = 0 \end{aligned} \quad (258)$$

due to the fact that  $u^\alpha u_\alpha$  is constant. Then we can choose the parameterization of the geodesic where  $\zeta^\alpha$  is everywhere orthogonal to  $u^\alpha$

$$\zeta^\alpha u_\alpha = 0 . \quad (259)$$

This equation implies that curves  $t = \text{const}$  cross  $\gamma_0$  orthogonally. Let us now calculate the relative acceleration of  $\gamma_1$  with respect to  $\gamma_0$

$$\begin{aligned} \frac{D^2 \zeta^\alpha}{dt^2} &= \nabla_\gamma(\nabla_\beta \zeta^\alpha u^\beta)u^\gamma = \\ &= \nabla_\gamma(\nabla_\beta u^\alpha \zeta^\beta)u^\gamma = \\ &= \nabla_\gamma \nabla_\beta u^\alpha \zeta^\beta u^\gamma + \nabla_\beta u^\alpha \nabla_\gamma \zeta^\beta u^\gamma = \\ &= (\nabla_\beta \nabla_\gamma u^\alpha - R^\alpha_{\mu\beta\gamma} u^\mu) \zeta^\beta u^\gamma + \nabla_\beta u^\alpha \nabla_\gamma u^\beta \zeta^\gamma = \\ &= \nabla_\beta(\nabla_\gamma u^\alpha u^\gamma) \zeta^\beta - \nabla_\gamma u^\alpha \nabla_\beta u^\gamma \zeta^\beta - R^\alpha_{\mu\beta\gamma} u^\mu \zeta^\beta u^\gamma + \nabla_\beta u^\alpha \nabla_\gamma u^\beta \zeta^\gamma = \\ &= -R^\alpha_{\beta\gamma\delta} u^\beta \zeta^\gamma u^\delta . \end{aligned} \quad (260)$$

which is geodesic deviation equations. It shows that curvature produces a relative acceleration between two neighbouring geodesic.

### 3.7.2 Fermi normal coordinates

This is related to the problem of the local flatness when we extend given theorem from single point  $P$  to an entire geodesic  $\gamma$ . Let us presume that the geodesic is time-like.

We show that we can introduce coordinates  $x^\alpha = (t, x^a)$  such that near  $\gamma$  the metric can be expressed as

$$\begin{aligned} g_{tt} &= -1 - R_{tatb}(t)x^a x^b + O(x^3) , \\ g_{ta} &= -\frac{2}{3} R_{tbac}(t)x^b x^c + O(x^3) , \\ g_{ab} &= \delta_{ab} - \frac{1}{3} R_{acbd}(t)x^c x^d + O(x^3) . \end{aligned} \quad (261)$$

These coordinates are known as *Fermi normal coordinates* and  $t$  is proper time along the geodesic  $\gamma$  on which all spatial coordinates  $x^a$  are all zero. In previous expression the components of the Riemann tensor are evaluated on  $\gamma$  and they depend on  $t$  only. We see from the previous expression that  $g_{\alpha\beta}|_\gamma = g_{\alpha\beta}(x=0) = \eta_{\alpha\beta}$  along geodesic. We also see that the first derivative of  $g$  evaluated at  $\gamma$  are zero so that we also obtain  $\Gamma_{\alpha\beta}^\mu = 0$  on  $\gamma$ . In other words we have local flatness theorem everywhere on the geodesic.

Let us give the proof. Let  $x^\alpha = (t, x^a)$  are the Fermi normal coordinates and  $x^{\alpha'}$  are arbitrary coordinate system. Let us imagine that we have a space-time with a metric  $g_{\alpha'\beta'}$  in these coordinates. Let us consider time-like geodesic  $\gamma$  in this space-time. Its tangent vector is  $u^{\alpha'}$  and let  $t$  is the proper time along  $\gamma$ . Let us choose a point  $O$  at this geodesic where we set  $t = 0$ . We also introduce an orthonormal basis  $\hat{e}_\mu^{\alpha'}$  at this point where index  $\mu$  serves to label the four basis vectors. We identify  $\hat{e}_t^{\alpha'}$  with the tangent vector  $u^{\alpha'}$  at  $O$ . From this we construct a basis everywhere on  $\gamma$  by parallel transporting  $\hat{e}_\mu^{\alpha'}$  away from  $O$ . The basis vector satisfy

$$\nabla_{\beta'} \hat{e}_\mu^{\alpha'} u^{\beta'} = 0, \hat{e}_t^{\alpha'} = u^{\alpha'} \quad (262)$$

and also

$$g_{\alpha'\beta'} \hat{e}_\mu^{\alpha'} \hat{e}_\nu^{\beta'} = \eta_{\mu\nu}, \quad (263)$$

everywhere on  $\gamma$ .

Consider now a space-like geodesic  $\beta$  that has origin at a point  $P$  on  $\gamma$  at which  $t = t_P$ . This geodesic has a tangent vector  $v^{\alpha'}$  and let  $s$  is proper distance along  $\beta$  and we set  $s = 0$  at  $P$ . We assume that at  $P$   $v^{\alpha'}$  is orthogonal to  $u^{\alpha'}$  so that it admits the decomposition

$$v_\gamma^{\alpha'} = \Omega^a \hat{e}_a^{\alpha'} \quad (264)$$

in order to have  $v^{\alpha'} v_{\alpha'} = 1$  we has to have

$$v^{\alpha'} v_{\alpha'} = \Omega^a \hat{e}_a^{\alpha'} g_{\alpha'\beta'} \hat{e}_b^{\beta'} \Omega^b = \Omega^a \delta_{ab} \Omega^b = 1 \quad (265)$$

The Fermi normal coordinates of a point  $Q$  located away from the geodesic  $\gamma$  are constructed as follows. First we find the unique geodesic that passes through  $Q$  and intersects  $\gamma$  orthogonally. We label the intersection point  $P$  and we call this geodesic  $\beta(t_P, \Omega_Q^a)$  with  $t_P$  denoting proper time at the intersection point and  $\Omega_Q^a$  the expansion coefficients of  $v^{\alpha'}$  at that point. We then assign to  $Q$  the new coordinates

$$x^0 = t_P, x^a = \Omega_Q^a S_Q, \quad (266)$$

where  $S_Q$  is proper distance from  $P$  to  $Q$ . These are the Fermi normal coordinates of the point  $Q$ . We therefore have  $x^\alpha = (t, \Omega^a s)$  and we have to find how they are related to  $x^{\alpha'}$  the original system.

**Coordinate Transformation** Our goal is to describe the family of geodesics  $\beta(t, \Omega^a)$  by relations of the form  $x^{\alpha'}(t, \Omega^a, s)$ . In these parameters  $t$  and  $\Omega^a$  specify

which geodesic and  $s$  is proper distance along this geodesic. The tangent to the geodesics  $\beta(t, \Omega^a)$  is

$$v^{\alpha'} = \left( \frac{\partial x^{\alpha'}}{\partial s} \right)_{t, \Omega^a} , \quad (267)$$

where the notation indicates explicitly that the derivative with respect to  $s$  is taken while keeping  $t$  and  $\Omega^a$  fixed. This vector is solution to the geodesic equation subject to the initial condition

$$v^{\alpha'}|_{s=0} = \Omega^a \hat{e}_a^{\alpha'} . \quad (268)$$

### 3.8 The Rindler horizon in flat space-time

Consider flat space-time with Cartesian coordinates in the  $X - T$  plane given by

$$ds^2 = -dT^2 + dX^2 + dL_{\perp}^2 , \quad (269)$$

where  $dL_{\perp}^2$  means line element in the transverse space. The lines  $X = \pm T$  divide  $X - T$  plane into four quadrants that we denote as right ( $\mathcal{R}$ ), left ( $\mathcal{L}$ ) wedges as well as the past ( $\mathcal{P}$ ) and future ( $\mathcal{F}$ ) of the origin. Let us now introduce two coordinates  $(t, l)$  instead of  $(T, X)$  in all the four quadrants through the transformations

$$\kappa T = \sqrt{2\kappa l} \sinh(\kappa t) , \quad \kappa X = \pm \sqrt{2\kappa l} \cosh(\kappa t) \quad (270)$$

for  $|X| > |T|$  with positive sign in  $\mathcal{R}$  and negative sign in  $\mathcal{L}$  and

$$\kappa T = \pm \sqrt{-2\kappa l} \cosh(\kappa t) , \quad \kappa X = \sqrt{-2\kappa l} \sinh(\kappa t) \quad (271)$$

for  $|X| < |T|$  with the positive sign in  $\mathcal{F}$  and negative sign in  $\mathcal{P}$ , where clearly  $l < 0$  in  $\mathcal{F}$  and in  $\mathcal{P}$ .

In fact, for  $|X| > |T|$  we have

$$|X| > |T| \Rightarrow X^2 > T^2 \Rightarrow \frac{2l}{\kappa} (\cosh^2(\kappa t) - \sinh^2(\kappa t)) = \frac{2l}{\kappa} > 0 \quad \text{for } l > 0 \quad (272)$$

In the same way we can show the situation in  $|X| < |T|$ . With the help of these transformations we obtain

$$\begin{aligned} \kappa dT &= \frac{\sqrt{2\kappa}}{2\sqrt{l}} dl \sinh(\kappa t) + \sqrt{2\kappa l} \kappa \cosh(\kappa t) dt , \\ \kappa dX &= \frac{\sqrt{2\kappa}}{2\sqrt{l}} dl \cosh(\kappa t) + \sqrt{2\kappa l} \kappa \sinh(\kappa t) dt \\ ds^2 &= -dT_d^2 X^2 + dL_{\perp}^2 = -\frac{1}{\kappa^2} \left( \frac{\sqrt{2\kappa}}{2\sqrt{l}} \sinh(\kappa t) dl + \sqrt{2\kappa l} \kappa \cosh(\kappa t) dt \right)^2 + \\ &\quad + \frac{1}{\kappa^2} \left( \frac{\sqrt{2\kappa}}{2\sqrt{l}} dl \cosh(\kappa t) + \sqrt{2\kappa l} \kappa \sinh(\kappa t) dt \right)^2 + dL_{\perp}^2 = \\ &= -2\kappa l dt^2 + \frac{1}{2\kappa l} dl^2 + dL_{\perp}^2 \end{aligned} \quad (273)$$



For  $l > 0$  that corresponds to  $\mathcal{R}$  and  $\mathcal{L}$  we have that the coordinate  $t$  is timelike and  $l$  is space-like, while their role reverses in  $\mathcal{F}$  and  $\mathcal{P}$  where  $l < 0$ . A given value of  $(t, l)$  corresponds to a pair of point in  $\mathcal{R}$  and  $\mathcal{L}$  for  $l > 0$  and to pair of point in  $\mathcal{F}$  and  $\mathcal{P}$  for  $l < 0$ . The surface  $l = 0$  acts as a horizon for observers in  $\mathcal{R}$ . Let us consider stationary observers in the new coordinates with

$$l = \text{const}, x_{\perp} = \text{const} \quad (274)$$

will have the trajectory

$$X^2 - T^2 = 2l/\kappa \quad (275)$$

that are trajectories of observers moving with constant proper acceleration in the inertial frame. In other words, Rindler observer is defined as an observer that is at rest in Rindler coordinates. We see that the closer to the horizon is the greater proper acceleration he has.

**Coordinates for the Accelerated Frame** We mean frame with proper acceleration. Let us consider fixed inertial frame  $S$  with coordinates  $x, t$  and the observer in his frame  $S'$ . We want to find world-line of this observer in the inertial frame  $S$  so that we want to find the  $S$  coordinates  $t, x$  of our motion parameterized as function  $\tau$  that is shown on our clock. In our local frame we want to feel a constant force so in this frame we have

$$\frac{dx'^2}{dt'^2} = g \quad (276)$$

Now we have to find the relations between these two frames.

Since there is observation evidence for isotropy and the Copernican principle says that we are not the center of the Universe and therefore observers elsewhere should also observe an isotropy all cosmological models are based on the existence of homogeneity and isotropy of manifold. However it is important to stress that this claim is not certainly true. The Universe is apparently not static, but changing in time. Therefore the cosmological models are based on the idea that the Universe is homogeneous and isotropic in space but not in time. This means that the Universe can be foliated into space-like surfaces such that each slice is homogeneous and isotropic. Then it is natural to consider our space-time to be  $R \times \Sigma$  where  $R$  represents the time direction and  $\Sigma$  is a homogeneous and isotropic three-manifold. Since we may think of isotropy as invariance under rotation and homogeneity as invariance under translation we get that  $\Sigma$  must be a maximally symmetric space. More precisely, the homogeneity and isotropy imply that the space has its maximum possible number of Killing vectors. Therefore we can write the metric in the form

$$ds^2 = -dt^2 + a^2(t)\gamma_{ij}(x)dx^i dx^j. \quad (277)$$

Here  $t$  is time-like coordinate and  $(x^1, x^2, x^3)$  are the coordinates on  $\Sigma$  where  $\gamma_{ij}$  is the maximally symmetric metric on  $\Sigma$ . The function  $a(t)$  is known as **scale factor** that tells us how big the space-like slice  $\Sigma$  is at the moment  $t$ . The coordinates used here in which the metric is free of cross terms  $dt dx^i$  and the space-like components are proportional to a single function of  $t$  are known as **comoving coordinates** and

an observer who stays at constant  $x^i$  is also called as “comoving”. Only comoving observer will think that the Universe looks isotropic.

It is important to stress that these observers, that are at rest to this frame are *geodesic* which means that they are free. Note that for these particles (observers) we have  $ds^2 = -dt^2$  as follows from the fact that  $dx^i = 0$  which implies that  $t$  has the meaning of the proper time for particles at rest.

We show that the world-line  $x^i = \text{const}$  obeys the geodesic equation in the metric (277). Note that the geodesic equation takes the form

$$\frac{du^\mu}{d\lambda} + \Gamma_{\nu\lambda}^\mu u^\nu u^\lambda = 0, \quad (278)$$

where  $u^\mu$  is 4-velocity

$$\frac{dx^\mu}{d\lambda} \quad (279)$$

and where  $\lambda$  is the parameter along the world-line of the particle. To begin with we calculate the Christoffel symbols

$$\Gamma_{\nu\lambda}^\mu = \frac{1}{2}g^{\mu\sigma}(\partial_\nu g_{\lambda\sigma} + \partial_\lambda g_{\nu\sigma} - \partial_\sigma g_{\nu\lambda}). \quad (280)$$

For the metric (277) we have following non-zero components

$$g_{00} = -1, \quad g_{ij} = a^2(t)\gamma_{ij} \quad (281)$$

with the inverse components

$$g^{00} = -1, \quad g^{ij} = \frac{1}{a^2(t)}\gamma^{ij}, \quad (282)$$

where

$$\gamma^{ij}\gamma_{jk} = \delta^i_k. \quad (283)$$

It can be shown that the only non-zero components of  $\Gamma_{\nu\lambda}^\mu$  are

$$\Gamma_{0j}^i = \frac{1}{2}g^{ik}\partial_0 g_{jk} = \frac{\dot{a}}{a}\delta^i_j, \Gamma_{ij}^0 = -a\dot{a}\gamma_{ij}, \Gamma_{jk}^i = {}^{(3)}\Gamma_{jk}^i, \quad (284)$$

where  ${}^{(3)}\Gamma_{jk}^i$  are the Christoffel symbols for metric  $\gamma_{ij}$ .

Let us now again consider the equation (278). The only non-zero component of the 4-velocity  $u^\mu = \frac{dx^\mu}{d\lambda}$  of the particle at rest is

$$u^0 = \frac{dx^0}{d\lambda} \quad (285)$$

Now the on-shell condition implies

$$u^\mu u^\nu g_{\mu\nu} = -1 \Rightarrow \frac{dx^0}{d\lambda} = 1. \quad (286)$$

Then clearly (278) is obviously satisfied since  $\frac{du^0}{d\lambda} = 0$  and  $\Gamma_{00}^\mu$  for all  $\mu$ . In other words the world-lines of particles which are at rest in our reference frame are indeed geodesic.

As we have shown in introduction the maximally symmetric Euclidean three-metric  $\gamma_{ij}$  obey

$$R_{ijkl}^{(3)} = k(\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk}) , \quad (287)$$

where  $k$  is some constant and the superscript on the Riemann tensor reminds to us that it is associated with the three metric  $\gamma_{ij}$  not to the metric of entire space-time. Then the Ricci tensor is

$$R_{jl}^{(3)} = \gamma^{ik} R_{ijkl}^{(3)} = 2k\gamma_{jl} . \quad (288)$$

Since the space is maximally symmetric then it will certainly be spherically symmetric as well. For such a space-time the metric can be put in the form

$$d\sigma^2 = \gamma_{ij} dx^i dx^j = e^{2\beta} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) . \quad (289)$$

The Ricci tensor for the metric given above has components

$$\begin{aligned} R_{11}^{(3)} &= \frac{2}{r} \partial_r \beta , \\ R_{22}^{(3)} &= e^{-2\beta} (r \partial_r \beta - 1) + 1 \\ R_{33}^{(3)} &= [e^{-2\beta} (r \partial_r \beta - 1) + 1] \sin^2 \theta . \end{aligned}$$

If we compare these expressions to (288) we can solve for  $\beta(r)$ :

$$\begin{aligned} \frac{2}{r} \partial_r \beta &= 2ke^{2\beta} \Rightarrow 2d\beta e^{-2\beta} = 2kr \Rightarrow \beta = -\frac{1}{2} \ln(C - kr^2) , \\ e^{-2\beta} (r \partial_1 \beta - 1) + 1 &= 2kr^2 \Rightarrow e^{-2\beta} (r^2 k e^{2\beta} - 1) + 1 = 2kr^2 \Rightarrow \\ \Rightarrow -e^{-2\beta} + 1 &= kr^2 \Rightarrow C = 1 \end{aligned} \quad (290)$$

and the third equation is identically solved. Then we obtain following metric on space-time:

$$ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right] . \quad (291)$$

This form of metric is known as **Friedman-Robertson-Walker metric** (FRW). Then the Einstein equations will determine the behavior of the scale factor  $a(t)$ . We can also easily see that the metric is invariant under the scaling transformations:

$$\begin{aligned} k &\rightarrow \frac{k}{|k|} , \\ r &\rightarrow \sqrt{|k|} r , \\ a &\rightarrow \frac{a}{\sqrt{|k|}} . \end{aligned} \quad (292)$$

Therefore it is clear that the only relevant parameter is  $k/|k|$  and there are three cases of interest:  $k = -1$ ,  $k = 0$  and  $k = 1$ . The case  $k = -1$  corresponds to constant negative curvature on  $\Sigma$  and is called **open**, the case  $k = 0$  corresponds to no curvature on  $\Sigma$  and is called **flat**; the case  $k = 1$  corresponds to positive curvature on  $\Sigma$  and is called **closed**. Now we will examine these possibilities in more details:

- For  $k = 0$  the metric on  $\Sigma$  is

$$d\sigma^2 = dx_i dx^i, i = 1, 2, 3 \quad (293)$$

that is simply the Euclidean space. Globally, it could describe  $R^3$  or more complicated manifold, as for example three torus  $S^1 \times S^1 \times S^1$ .

- For  $k = 1$  we define

$$r = \sin \xi, dr = \cos \xi d\xi \quad (294)$$

and hence the metric on  $\Sigma$  can be written as

$$d\sigma^2 = d\xi^2 + \sin^2 \xi d\Omega^2 \quad (295)$$

which is the metric of three sphere. In this case the only possible global structure is actually three sphere.

- The case  $k = -1$  we can write

$$r = \sinh \psi \quad (296)$$

and the metric on  $\Sigma$  is

$$d\sigma^2 = d\psi^2 + \sinh^2 \psi d\Omega^2 \quad (297)$$

which is the metric of three dimensional space of constant negative curvature. Globally such a space can extend forever but it can also describe a non-simply connected compact space.

In order to solve the Einstein's equations of motion we have to calculate the Christoffel's symbols for the metric ansatz (291). If we denote  $\dot{a} \equiv \frac{da}{dt}$  then these symbols are given by

$$\begin{aligned} \Gamma_{11}^0 &= \frac{a\dot{a}}{1 - kr^2}, \quad \Gamma_{22}^0 = a\dot{a}r^2, \quad \Gamma_{33}^0 = a\dot{a}r^2 \sin^2 \theta, \\ \Gamma_{01}^1 &= \Gamma_{02}^2 = \Gamma_{20}^2 = \Gamma_{03}^3 = \Gamma_{30}^3 = \frac{\dot{a}}{a}, \\ \Gamma_{22}^1 &= -r(1 - kr^2), \quad \Gamma_{33}^1 = -r(1 - kr^2) \sin^2 \theta, \\ \Gamma_{12}^2 &= \Gamma_{21}^2 = \Gamma_{13}^3 = \Gamma_{31}^3 = \frac{1}{r}, \\ \Gamma_{33}^2 &= -\sin \theta \cos \theta, \quad \Gamma_{23}^3 = \Gamma_{32}^3 = \sin \theta. \end{aligned} \quad (298)$$

After simple calculations we can find following nonzero components of the Ricci tensor

$$\begin{aligned}
R_{00} &= -3\frac{\ddot{a}}{a} , \\
R_{11} &= \frac{a\ddot{a} + 2\dot{a}^2 + 2k}{1 - kr^2} , \\
R_{22} &= r^2(a\ddot{a} + 2\dot{a}^2 + 2k) , \\
R_{33} &= r^2(a\ddot{a} + 2\dot{a}^2 + 2k) \sin^2 \theta .
\end{aligned} \tag{299}$$

Then the Ricci scalar is equal to

$$R = g^{\mu\nu} R_{\nu\mu} = \frac{6}{a^2}(a\ddot{a} + \dot{a}^2 + k) . \tag{300}$$

Since Universe is not empty we are not interested in the vacuum Einstein equations. Rather we must study the solutions of the Einstein's equations that contain the nontrivial right hand side. The standard model with we begin is the Universe filled by a perfect fluid that is defined as fluids that are isotropic in their rest frame. The energy momentum tensor for a perfect fluid can be written

$$T_{\mu\nu} = (p + \rho)U_\mu U_\nu + pg_{\mu\nu} , \tag{301}$$

where  $p$  and  $\rho$  are energy density and pressure as measured in the rest frame and  $U_\mu$  is the four-velocity of the fluid. It is clear that if a fluid which is isotropic in some frame leads to a metric which is isotropic in some frame, the two frames will coincide, that is the fluid will be in rest frame in comoving coordinates. The four-velocity is then

$$U^\mu = (1, 0, 0, 0) , \tag{302}$$

and the energy tensor is

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & & & \\ 0 & g_{ij}p & & \\ 0 & & & \end{pmatrix} . \tag{303}$$

If we raise its index we obtain

$$T^\mu{}_\nu = g^{\mu\kappa} T_{\kappa\nu} = \text{diag}(-\rho, p, p, p) \tag{304}$$

and note that the trace is equal to

$$T \equiv T^\mu{}_\mu = -\rho + 3p . \tag{305}$$

For letter purposes it is also instructive to consider the zero component of the conservation of the stress energy tensor

$$\begin{aligned}
0 &= \nabla_\mu T^\mu{}_0 = \partial_\mu T^\mu{}_0 + \Gamma^\mu_{\mu 0} T^\mu{}_0 - \Gamma^\lambda_{\mu 0} T^\mu{}_\lambda = \\
&= -\partial_0 \rho - 3\frac{\dot{a}}{a}(\rho + p) .
\end{aligned} \tag{306}$$

To proceed it is necessary to choose the equation of state, the relation between  $\rho$  and  $p$ . It appears that all perfect fluids relevant to cosmology obey the simple equation of state

$$p = w\rho , \quad (307)$$

where  $w$  is constant independent on time. Then the conservation of energy becomes

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a} \quad (308)$$

that can be integrated and we obtain

$$\rho = a^{-3(1+w)} . \quad (309)$$

The most interesting examples of cosmological are **dust** and **radiation**. Dust is characterized with  $w = 0$ . Examples include ordinary stars and galaxies where the pressure is negligible in comparison with the energy density. Dust is also known as matter and Universes whose energy is mostly due to dust are known as **matter-dominated**. The energy density in matter falls as

$$\rho \sim a^{-3} \quad (310)$$

that can be interpreted as the decrease in the number density of particles as the Universe expands. (For dust the energy density is dominated by the rest energy that is proportional to the number density.)

The second form of the fluid, **Radiation** may be used to describe either actual electromagnetic radiation, or massive particles moving at relative velocities sufficiently close to the speed of light so that they become indistinguishable from photons. The stress energy tensor of the radiation can be expressed in terms of the field strength as

$$T^{\mu\nu} = \frac{1}{4\pi} \left( F^{\mu\lambda} F_{\lambda}^{\nu} - \frac{1}{4} g^{\mu\nu} F^{\lambda\sigma} F_{\lambda\sigma} \right) . \quad (311)$$

Then the trace of this stress energy tensor is

$$T = T^{\mu\nu} g_{\nu\mu} = \frac{1}{4\pi} \left[ F^{\mu\lambda} F_{\mu\lambda} - \frac{(4)}{4} F^{\lambda\sigma} F_{\lambda\sigma} \right] = 0 \quad (312)$$

Since this should be also equal to (305) we get that

$$p = \frac{1}{3}\rho . \quad (313)$$

An Universe in which most of the energy density is in the form of radiation is known as **radiation-dominated**. The energy density in radiation then falls off as

$$\rho \sim a^{-4} . \quad (314)$$

This result implies that the energy density of radiation falls of faster than that in matter. It is believed that today the energy density of the Universe is dominated by

matter with  $\rho_{mat}/\rho_{rad} \sim 10^6$ . However in the past the Universe was much smaller and the energy density in radiation would have dominated at very early times.

There is also one important form of energy density that is sometimes considered, namely that of the vacuum itself. Introducing energy into the vacuum is equivalent to introducing a cosmological constant so that Einstein's equations with cosmological constant are

$$E_{\mu\nu} = 8\pi GT_{\mu\nu} - \Lambda g_{\mu\nu} \quad (315)$$

that is clearly the same form as the equations with no cosmological constant but an energy-momentum tensor for the vacuum

$$T_{\mu\nu}^{vac} = -\frac{\Lambda}{8\pi G} g_{\mu\nu} . \quad (316)$$

This has form of the perfect fluid with

$$\rho = -p = \frac{\Lambda}{8\pi G} \quad (317)$$

that implies that  $w = -1$  and from (309) we see that the energy density is independent on  $a$ . Since the energy density of matter and the radiation decreases as the Universe expands, if there is nonzero vacuum energy it tends to wind over the long term. If this happens we say that the Universe became **vacuum-dominated**.

Now we turn to the Einstein's equations. Recall that they can be written in the form

$$R_{\mu\nu} = 8\pi G \left( T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) . \quad (318)$$

The  $\mu\nu = 00$  components is

$$-3\frac{\ddot{a}}{a} = 4\pi G(\rho + 3p) , \quad (319)$$

and the  $\mu\nu = ij$  equations give

$$\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{k}{a^2} = 4\pi G(\rho - p) . \quad (320)$$

Using (319) we simplify (320) as

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} . \quad (321)$$

(321) together with (319) are known as **Friedmann equations**.

Now we introduce some terminology considering cosmological parameters. The rate of expansion is characterized by the **Hubble parameter**

$$H = \frac{\dot{a}}{a} . \quad (322)$$

The value of the Hubble parameter at present epoch is the Hubble constant,  $H_0$ . There is also the **deceleration parameter**

$$q = -\frac{a\ddot{a}}{\dot{a}^2} \quad (323)$$

that measures the rate of change of the rate of expanding. Another useful parameter is the **density parameter**

$$\Omega = \frac{8\pi G}{3H^2}\rho = \frac{\rho}{\rho_{crit}} \quad , \quad (324)$$

where the critical density is defined by

$$\rho_{crit} = \frac{3H^2}{8\pi G} \quad . \quad (325)$$

This quantity, that is generally time dependent, is called critical density because the Friedmann equation (321) can be written as

$$\Omega - 1 = \frac{k}{H^2 a^2} \quad , \quad (326)$$

where generally  $H$  is time dependent. The sign of  $k$  is therefore determined by whether  $\Omega$  is greater than, equal to, or less than one. In other words, we have

$$\begin{aligned} \rho < \rho_{crit} &\Rightarrow \Omega < 1 \Rightarrow k = -1 \rightarrow \text{open} \quad , \\ \rho = \rho_{crit} &\Rightarrow \Omega = 1 \Rightarrow k = 0 \rightarrow \text{flat} \quad , \\ \rho > \rho_{crit} &\Rightarrow \Omega > 1 \Rightarrow k = 1 \rightarrow \text{closed} \quad . \end{aligned} \quad (327)$$

It is useful to know the qualitative behavior of various possibilities of the solutions of the Friedman equations. Let us for the moment set  $\Lambda = 0$  and consider the behavior of Universe filled with fluids of positive energy  $\rho > 0$  and nonnegative pressure  $p > 0$ . Then (319) implies that  $\ddot{a} < 0$ . Since we know from observation that the Universe is expanding ( $\dot{a} > 0$ ) this means that the Universe is decelerating which could be intuitively expected since the gravitation attraction of the matter in the Universe works against the expanding. The fact that the Universe is decelerating means that it must have been expanding even faster in the past; if we trace the evolution backward in time, we reach the singularity at  $a = 0$ . Notice that if  $\ddot{a}$  were exactly zero,  $a(t)$  would be straight line  $a(t) = Ct$  (we have chosen the integration constant that at  $t = 0, a(0) = 0$  and hence  $H(t) = \frac{\dot{a}}{a} = \frac{1}{t}$  so that  $H_0^{-1}$  would determine the age of the Universe.

The singularity at  $a = 0$  is known as **Big Bang**. It represents the creation of Universe from a singular space, not explosion of matter into a pre-existing space-time. Since for  $a \rightarrow 0$  the energy density becomes arbitrary high we do not expect classical general relativity to give a correct description of nature in this regime.



The future evolution is different for different  $k$ . For the open and flat cases  $k = -1, 0$  the (321) implies

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 + |k|. \quad (328)$$

Since the right hand side is strictly positive so  $\dot{a}$  never passes through zero. Since  $\dot{a} > 0$  today it follows that  $\dot{a} > 0$  for all time. Thus open and flat Universes expand forever-they are temporally and spatially open. It is however important to keep in mind that this works on the presumption of nonzero positive energy density. Negative energy density Universes do not have to expand forever, even if they are open.

The question is how fast these Universes keep expanding? Let us now consider the quantity  $\rho a^3$  (recall that this is constant in matter dominated Universe). Using the conservation of energy (306) we get

$$\frac{d}{dt}(a^3\rho) = a^3(3\frac{\dot{a}}{a}\rho + \dot{\rho}) = -3pa^2\dot{a} \quad (329)$$

that implies that

$$\frac{d}{dt}(a^3\rho) < 0. \quad (330)$$

This result implies that  $a^2\rho$  must go to zero in an ever-expanding Universe where  $a \rightarrow \infty$ <sup>4</sup> Then (328) implies that

$$\dot{a}^2 \rightarrow |k|. \quad (331)$$

(We must stress that it holds for  $k = -1, 0$ . Thus for  $k = -1$  an expanding approaches the limiting value  $\dot{a} \rightarrow 1$  while for  $k = 0$  the Universe keeps expanding but more and more slowly.

For the closed Universe ( $k = 1$ ) (321) implies

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 - 1. \quad (332)$$

It is clear that the argument that  $\rho a^2 \rightarrow 0$  as  $a \rightarrow \infty$  still holds. In this case the right hand side of the upper equation becomes negative which clearly cannot happen. Therefore the Universe does not expand indefinitely,  $a$  posses an upper bound  $a_{max}$ . As  $a$  approaches  $a_{max}$  the equation (319) implies

$$\ddot{a} = -\frac{4\pi G}{3}(\rho + 3p)a_{max} < 0 \quad (333)$$

and hence  $\ddot{a}$  is finite and negative at this point, so  $a$  reaches  $a_{max}$  and starts decreasing. Since  $\ddot{a} < 0$  it will inevitably continue to contract to zero- the Big Crunch.

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<sup>4</sup>For example, when  $a(t) \sim t$  we should have  $\rho \sim t^{-4}$  at least and hence  $a^2\rho \sim t^{-2} \rightarrow 0$  for  $t \rightarrow \infty$ .

Thus, the closed Universe (on presumption of positive  $\rho$  and non negative  $p$ ) is closed in time as well as space.

We will now list some of the exact solutions corresponding to only one type of energy density. For dust-only Universe ( $p = 0$ ) it is convenient to define a **development angle**  $\phi(t)$ , rather than using  $t$  as a parameter directly. The solutions are then, for open Universes;

$$a = \frac{C}{2}(\cosh \phi - 1) , \quad t = \frac{C}{2}(\sinh \phi - \phi) , \quad k = -1 , \quad (334)$$

for flat Universes

$$a = \left(\frac{9C}{4}\right)^{1/3} t^{2/3} , \quad k = 0 , \quad (335)$$

and for closed Universes

$$a = \frac{C}{2}(1 - \cos \phi) , t = \frac{C}{2}(\phi - \sin \phi) , \quad k = +1 , \quad (336)$$

where we have defined

$$C = \frac{8\pi G}{3}\rho a^3 = \text{constant} . \quad (337)$$

For Universes filled with nothing but radiation,  $p = \frac{1}{3}\rho$ , we have once again open Universes,

$$a = \sqrt{C'} \left[ \left(1 + \frac{t}{\sqrt{C'}}\right)^2 - 1 \right]^{1/2} , \quad k = -1 \quad (338)$$

flat Universes,

$$a = (4C')^{1/4} t^{1/2} , \quad k = 0 \quad (339)$$

and closed Universes,

$$a = \sqrt{C'} \left[ 1 - \left(1 - \frac{t}{\sqrt{C'}}\right)^2 \right]^{1/2} , \quad k = +1 \quad (340)$$

where we have defined

$$C' = \frac{8\pi G}{3}\rho a^4 = \text{constant} . \quad (341)$$

Let us now consider the case of nonzero cosmological constant. We start with  $\Lambda < 0$ . In this case  $\Omega$  is negative and we get that  $k = -1$ . The solution in this case is

$$a = \sqrt{\frac{-3}{\Lambda}} \sin \left( \sqrt{\frac{-\Lambda}{3}} t \right) . \quad (342)$$

There is also an open ( $k = -1$ ) solution for  $\Lambda > 0$  given by

$$a = \sqrt{\frac{3}{\Lambda}} \sinh \left( \sqrt{\frac{\Lambda}{3}} t \right) . \quad (343)$$

A flat vacuum-dominated Universe must have  $\Lambda > 0$  and the solution is

$$a \sim \exp\left(\pm\sqrt{\frac{\Lambda}{3}}t\right) \quad (344)$$

while the closed Universe must also have  $\Lambda > 0$  and satisfies

$$a = \sqrt{\frac{3}{\Lambda}} \cosh\left(\sqrt{\frac{\Lambda}{3}}t\right) . \quad (345)$$

These solutions are a little misleading. In fact the three solutions for  $\Lambda > 0$  - (343),(344),(345)-all represent the same space-time, just in different coordinates. This space-time, known as **de Sitter space** is maximally symmetric as a space-time. The  $\Lambda < 0$  solution is also maximally symmetric and is known as **anti-de Sitter space**

Before we conclude this section we spend some time with the discussion of the situation when the matter sector in Universe constitutes more general form of matter. For example, we can presume that all components of the matter are present. Then the total density parameter takes the form

$$\Omega = \sum_i \Omega_i \quad (346)$$

and the Friedman equation can be written as

$$\Omega - 1 = \frac{k}{H^2 a^2} . \quad (347)$$

As in the particular previous example we obtain that the sign of  $k$  is determined whether  $\Omega$  is greater than, equal to, or less than one. Explicitly, we have

$$\begin{aligned} \rho < \rho_{crit} &\Rightarrow \Omega < 1 \rightarrow k = -1 , \text{ open} , \\ \rho = \rho_{crit} &\Rightarrow \Omega = 1 \rightarrow k = 0 , \text{ flat} , \\ \rho > \rho_{crit} &\Rightarrow \Omega > 1 \rightarrow k = 1 , \text{ closed} . \end{aligned} \quad (348)$$

Since  $\rho_i \sim a^{-n_i}$  we have

$$\frac{\rho_i}{\rho_j} = \frac{\Omega_i}{\Omega_j} = a^{-(n_i - n_j)} \quad (349)$$

so that relative amount of energy in different components changes as the Universe evolves.

### 3.9 Motion of the probe in the FRW Universe

In order to understand properties of given background it is common strategy to study the dynamics of the probe in given background. Let us then consider the motion of particle in the FRW Universe.

Let us consider the action for the massive particle

$$S = - \int d\lambda \sqrt{-g_{\mu\nu} u^\mu u^\nu} , u^\mu = \frac{dx^\mu}{d\lambda} . \quad (350)$$

where  $\lambda$  is parameter that labels the world-line. We introduce einbain  $e(\tau)$  so that the action takes the form

$$S = \frac{1}{2} \int d\lambda \left[ \frac{1}{\epsilon} g_{\mu\nu} u^\mu u^\nu - m^2 \epsilon \right] , \quad (351)$$

To see the equivalence between these two formulations we perform the variation with respect to  $\epsilon$  that gives

$$-\frac{1}{\epsilon^2} g_{\mu\nu} u^\mu u^\nu - m^2 = 0 \Rightarrow \epsilon = \frac{1}{m} \sqrt{-g_{\mu\nu} u^\mu u^\nu} \quad (352)$$

that inserting back to the action we obtain the original action. Further, the equation of motion with respect to  $x^\mu$  gives

$$-2 \frac{d}{d\lambda} \left( \frac{1}{\epsilon} g_{\mu\nu} u^\nu \right) + \frac{1}{\epsilon} \partial_\mu g_{\rho\sigma} u^\rho u^\sigma = 0 \quad (353)$$

It is important to stress that the action is invariant under  $\tau' = f(\tau)$  so that  $d\tau' = \frac{df}{d\tau} d\tau$ . We can fix the gauge by imposing  $\epsilon = \frac{1}{m}$  so that we obtain on-shell condition

$$g_{\mu\nu} u^\mu u^\nu = -1 \quad (354)$$

Note that this relation allows us to write (when  $g_{0u} = 0$ )

$$-1 = (-g_{00} + g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt}) \left( \frac{dt}{d\lambda} \right)^2 \Rightarrow \frac{dt}{d\lambda} = \frac{1}{\sqrt{g_{00} - g_{ij} v^i v^j}} , v^i \equiv \frac{dx^i}{dt} \quad (355)$$

Then the equation of motion for  $x^\mu$  takes the form

$$\begin{aligned} & \frac{du^\mu}{d\lambda} + g^{\mu\nu} \partial_\rho g_{\nu\sigma} u^\sigma u^\rho - \frac{1}{2} g^{\mu\nu} \partial_\nu g_{\rho\sigma} u^\rho u^\sigma \Rightarrow \\ & \frac{du^\mu}{d\lambda} + \frac{1}{2} g^{\mu\nu} (\partial_\rho g_{\nu\sigma} + \partial_\sigma g_{\nu\rho} - \frac{1}{2} \partial_\nu g_{\rho\sigma}) u^\rho u^\sigma = 0 \Rightarrow \\ & \frac{d^2 x^\mu}{d^2 \lambda} + \Gamma_{\rho\sigma}^\mu \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda} = 0 . \end{aligned} \quad (356)$$

It is also interesting to insert the solution of the equation of motion  $\epsilon$  into the action so that it takes the form

$$\begin{aligned} S &= \frac{1}{2} \int d\lambda \left[ \frac{1}{\epsilon} g_{\mu\nu} u^\mu u^\nu - m^2 \epsilon \right] = \\ &= \frac{m}{2} \int dt \sqrt{g_{00} - g_{ij} v^i v^j} [-(g_{00} - g_{ij} v^i v^j) \left( \frac{dt}{d\lambda} \right)^2 - 1] = \\ &= -m \int dt \sqrt{g_{00} - g_{ij} v^i v^j} . \end{aligned} \quad (357)$$

It is also interesting to analyze the equation of motion that follows from the original action

$$S = -m \int d\lambda \sqrt{-g_{MN} \dot{X}^M \dot{X}^N} \quad (358)$$

The equations of motion have the form

$$-\frac{\partial_K g_{MN} \dot{X}^M \dot{X}^N}{2\sqrt{-g_{MN} \dot{X}^M \dot{X}^N}} + \partial_\tau \left( \frac{g_{KN} \dot{X}^N}{\sqrt{-g_{MN} \dot{X}^M \dot{X}^N}} \right) = 0 . \quad (359)$$

This is equation of motion for  $X$ . Let us denote the variation of the action with respect to  $X$  as  $\frac{\delta S}{\delta X}$ . If we multiply given expression with  $\dot{X}^K$  we obtain

$$\begin{aligned} \frac{\delta S}{\delta X^K} \dot{X}^K &= -\frac{\partial_\tau g_{MN} \dot{X}^M \dot{X}^N}{2\sqrt{-g_{MN} \dot{X}^M \dot{X}^N}} + \partial_\tau \left( \frac{1}{\sqrt{-g_{MN} \dot{X}^M \dot{X}^N}} \right) \dot{X}^K g_{KM} \dot{X}^N + \\ &+ \frac{1}{\sqrt{-g_{MN} \dot{X}^M \dot{X}^N}} \dot{X}^K \partial_\tau g_{KM} \dot{X}^N + \frac{1}{\sqrt{-g_{MN} \dot{X}^M \dot{X}^N}} \dot{X}^K g_{KM} \ddot{X}^N = 0 \end{aligned} \quad (360)$$

Note that it holds as identity and not as a consequence of the equations of motion.

Let us now consider the flat FRW background

$$ds^2 = -dt^2 + a^2(t) \delta_{ij} dx^i dx^j . \quad (361)$$

so that the action takes the form

$$S = -m \int dt \sqrt{1 - a^2 \delta_{ij} \dot{x}^i \dot{x}^j} . \quad (362)$$

It is interesting to determine the Hamiltonian formulation of this system

$$p_i = \frac{\delta L}{\delta \dot{x}^i} = a^2 m \frac{\delta_{ij} \dot{x}^j}{\sqrt{1 - a^2 \delta_{ij} \dot{x}^i \dot{x}^j}} . \quad (363)$$

Then we find

$$\mathcal{H} = p_i \dot{x}^i - L = \frac{ma^2}{\sqrt{1 - a^2 \delta_{ij} \dot{x}^i \dot{x}^j}} = a^2 \sqrt{\frac{1}{a^2} p_i \delta^{ij} p_j + m^2} \quad (364)$$

using

$$a^2 \dot{x}^i \delta_{ij} \dot{x}^j = \frac{p_i \delta^{ij} p_j}{m^2 a^2 + p_i \delta^{ij} p_j} \quad (365)$$

Now the equation of motion takes the form

$$\begin{aligned}\dot{x}^i &= \{x^i, H\} = \frac{\delta^{ij} p_j}{\sqrt{a^{-2} p_i \delta^{ij} p_j + m^2}} , \\ \dot{p}_i &= \{p_i, H\} = 0 \Rightarrow p_i = k_i .\end{aligned}\tag{366}$$

We see that the momentum  $p_i$  is constant. On the other hand the norm of state slows since the norm is given as  $p_i g^{ij} p_j = \frac{1}{a^2} k_i \delta^{ij} k_j$ .

On the other hand let us introduce following variable

$$X^i = ax^i , \dot{x}^i = \frac{1}{a} (\dot{X}^i - H X^i)\tag{367}$$

Using these variables we find the action in the form

$$S = -m \int dt \sqrt{1 - (\dot{X}^i - H X^i) \delta_{ij} (\dot{X}^j - H X^j)} .\tag{368}$$

The meaning of the variables  $X^i$  can be found when we take the non-relativistic limit where we replace  $\sqrt{1 - A} = 1 - \frac{1}{2} A^2$  so that the action

$$\begin{aligned}S_{nonrel} &= -m \int dt + \int dt \frac{m}{2} (\dot{X}^i - H X^i) \delta_{ij} (\dot{X}^j - H X^j) = \\ &= \int dt \frac{m}{2} \dot{X}^i \dot{X}_i + \dots ,\end{aligned}\tag{369}$$

where we neglected the remaining terms. Comparing this expression with the standard form of the non-relativistic Lagrangian we can interpret  $X^i = a(t)x^i$  as the physical variable even if we mean that both variables are physical.

Now from (368) we determine the momenta conjugate to  $X^i$

$$P_i = \frac{\delta L}{\delta \dot{X}^i} = m \frac{\delta_{ij} (\dot{X}^j - H X^j)}{\sqrt{(\dots)}}\tag{370}$$

and hence the Hamiltonian takes the form

$$\mathcal{H} = \dot{X}^i P_i - L = \frac{m}{\sqrt{(\dots)}} + P_i X^i H = \sqrt{m^2 + P_i P^i} + P_i X^i H\tag{371}$$

Using this Hamiltonian we derive the equation of motion

$$\begin{aligned}\dot{X}^i &= \{X^i, H\} = \frac{P^i}{\sqrt{m^2 + P_i P^i}} + X^i H , \\ \dot{P}_i &= \{P_i, H\} = -P_i H\end{aligned}\tag{372}$$

The last equation can be integrated as

$$dP_i = -P_i \frac{da}{a} \Rightarrow \ln P_i = -\ln a + \ln K_i \Rightarrow P_i = \frac{K_i}{a} . \quad (373)$$

We see that the "physical" momentum  $P_i$  is red shifted as the universe expands. Note that we can also find the time dependence of  $X^i$  by integrating the first equation since it takes generally the form

$$\dot{X}^i = F^i(t) + G(t)X^i \quad (374)$$

so that we search the solution of the homogeneous equation

$$\dot{X}^i = G(t)X^i \Rightarrow X^i = C^i \exp\left(\int dt G(t)\right) \quad (375)$$

Note that we have

$$\int dt G(t) = \int \frac{da}{a} \frac{1}{a} dt = \int \frac{da}{a} = \ln a \Rightarrow e^{\int dt G(t)} = e^{\ln a} = a . \quad (376)$$

Then we say that  $C^i$  depends on time so we obtain that it has to obey the equation

$$\frac{dC^i}{dt} = e^{-\int dt' G(t')} F(t) \Rightarrow \frac{dC^i}{dt} = \frac{K^i}{a\sqrt{m^2 a^2 + K_i K^i}} \quad (377)$$

that can be in principle integrated if we know the time dependence of  $a$ . There is a particular simple solution corresponding to the particle with zero physical momentum when  $K_i = 0$ . From upper equation we immediately find that  $C^i = \mathcal{C}^i = \text{const}$  and hence

$$X^i = \mathcal{C}^i a \quad (378)$$

that is an expected result. The physical interpretation of this result is that particle slows down with respect to comoving coordinates as the Universe expands (since  $a \rightarrow \infty$ ). In fact this is an actual slowing down, in the sense that a gas of particles with initially high relative velocities will cool down as the Universe expands.

Very interesting is the case of the particle with null mass which is photon. In principle we could use the the action for the massive particle written without the square root and then take the limit  $m \rightarrow 0$  however we will be more conservative and consider the standard treatment of the electromagnetic wave in curved background.

We consider the action of free electromagnetic field

$$S = -\frac{1}{4} \int d^4x \sqrt{-g} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} , F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (379)$$

Consider now the propagation of a photon in the homogeneous isotropic Universe. Since the photon wavelength is small compared to the spatial curvature radius even if the Universe is open or closed. Then we can consider the metric that is spatially flat with the metric

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j . \quad (380)$$

Let us introduce conformal time  $\eta$  instead of  $t$  that is defined as

$$dt = a d\eta \quad (381)$$

or equivalently

$$\eta = \int \frac{dt}{a(t)} . \quad (382)$$

This result can be generally integrated so that we have  $\eta = \eta(t)$  and we presume that this relation can be inverted so that  $t = t(\eta)$  and consequently  $a = a(\eta)$ . Now the metric has the form

$$ds^2 = a^2(\eta)[-d\eta^2 + \delta_{ij}dx^i dx^j] \quad (383)$$

and we see that the metric element in FRW spacetime is conformally flat in the sense that

$$g_{\mu\nu} = a^2(\eta)\eta_{\mu\nu} . \quad (384)$$

where the Minkowski metric is spanned by coordinates  $(\eta, x^i)$ . Then we clearly have

$$g^{\mu\nu} = a^{-2}\eta^{\mu\nu} , \sqrt{g} = a^4 \quad (385)$$

and we find that in  $\eta, x^i$  coordinates the action of the electromagnetic field has the form

$$S = -\frac{1}{4} \int d^4x \eta^{\mu\rho} \eta^{\nu\sigma} F_{\mu\rho} F_{\nu\sigma} . \quad (386)$$

Now it is clear that the solution of the equation of motion for the free electromagnetic field in the Universe is given as the superposition of the plane waves

$$A_\mu^{(\alpha)} = e_\mu^{(\alpha)} e^{ik\eta - i\mathbf{k}\mathbf{x}} \quad (387)$$

where  $\mathbf{k}$  is constant vector,  $|\mathbf{k}| = k$  and  $e_\mu^{(\alpha)}$  is the standard polarization vector of photons with  $\alpha = 1, 2$ . Note that  $k$  is not the physical frequency as follows from following arguments. The quantity  $\Delta x = \frac{2\pi}{k}$  is the coordinate wavelength of a photon while the physical wavelength at time  $t$  is

$$\lambda(t) = a(t)\Delta x = 2\pi \frac{a(t)}{k} . \quad (388)$$

In the same way we define period  $\Delta\eta = \frac{2\pi}{k}$  of electromagnetic wave in conformal time while the period of the physical time is

$$T = a(t)\Delta\eta = 2\pi \frac{a(t)}{k} . \quad (389)$$

Then we see that the frequency is equal to

$$\omega(t) = \frac{2\pi}{T} = \frac{k}{a(t)} \quad (390)$$



and since we know that the frequency is equal to the magnitude of the physical momentum of photon we obtain that the physical momentum depends on time as in case of the massive particle namely

$$\mathbf{p} = \frac{\mathbf{k}}{a(t)} \quad (391)$$

We see that in the expanding universe the scale factor  $a(t)$  is growing and hence the physical wavelength grows. On the other hand the physical momentum is decreasing function of time. The phenomena when the wavelength is growing during the expansion of the Universe is named as the redshift. Explicitly, if the photon was emitted at time  $t_i$  with physical wave length  $\lambda_i$  in the physical process as for example when the electron in the excited state in the atom drops to the ground state which is certainly physical process. Now we know that the state propagates freely as in (387) and then it is again detected in time  $t_0$  where  $t_0$  we means the present time in the reversed physical process when its physical wave length now is

$$\lambda(t_0) = a(t_0) \frac{2\pi}{k} \quad (392)$$

Now expressing  $\frac{2\pi}{k}$  using the physical wave length at time of emission we find the famous relation

$$\lambda(t_0) = \frac{a(t_0)}{a(t_i)} \lambda_i \equiv \lambda_i (1 + z(t_i)) . \quad (393)$$

The quantity

$$z(t_i) = \frac{a(t_0)}{a(t_i)} - 1 \quad (394)$$

is called *redshift*. The earlier the object emits the photon then this photon has to travel longer and consequently  $a(t_i)$  is smaller and hence object at larger distances have the larger redshifts.

Note that these formulas are valid in general for all  $z$ . Let us now consider objects that are not in large distance. Then the difference  $t - t_0$  is not very large and we can expand

$$a(t_i) = a(t_0) - \dot{a}(t_0)(t_0 - t_i) \quad (395)$$

Using the present value of the Hubble parameter  $H_0 = \frac{\dot{a}(t_0)}{a(t_0)} \equiv \frac{\dot{a}_0}{a_0}$  we can write

$$a(t_i) = a_0 [1 - H_0(t_0 - t_i)] \quad (396)$$

so that to the linear order we find following expression for the redshift

$$z(t_i) = \frac{1}{1 - H_0(t_0 - t_i)} - 1 \simeq H_0(t_0 - t_i) . \quad (397)$$

Finally the travel time is equal to

$$\begin{aligned} 0 &= -dt^2 + a(t)^2 dr^2 = -dt^2 + (a_0 - \dot{a}_0(t_0 - t))^2 dr^2 \approx \\ &= -dt^2 + a_0^2 dr^2 \Rightarrow (t_0 - t) = a_0(r_i - r_0) \equiv R \end{aligned} \quad (398)$$

where  $R$  is the physical distance of the object from the our observer. Inserting this expression into (397) we derive famous Hubble law

$$z = H_0 R, z \ll 1. \quad (399)$$

The redshift is something that can be measured, we know the rest-frame wavelengths of various spectral lines in the radiation of distant galaxies, so that we can determine how much their wavelengths have changed along the path from time  $t_i$  when they were emitted to time  $t_0$  when they were observed. We therefore know the ratio of the scale factors at these two times however we do not know the times themselves.

### 3.10 Horizons

One of the most crucial concepts of the FRW Universe is the existence of *horizons*.

Suppose a emitter,  $\mathbf{e}$  sends a light signal to an observer  $\mathbf{o}$ , who is at  $r = 0$ . Restricting to the radial geodesic (that means that  $d\phi = d\theta = 0$  we obtain from the vanishing of the metric elements the equation for null geodesics in the form

$$ds^2 = 0 = a^2(\eta)(-d\eta^2 + dr^2) \Rightarrow \eta = \pm r + r_0, \quad (400)$$

where  $\eta$  is conformal time. Let us presume that the light hits the observer at time  $\eta_0$  that is larger than  $\eta_e$  where  $\eta_e$  is time when this signal was emitted. Since for  $\eta = \eta_0$  we have  $r = 0$  we get  $\eta_0 = r_0$  and consequently  $\eta - \eta_0 = \pm r$ . Since also for  $\eta_e$  this equation implies

$$\eta_0 - \eta_e = \mp r_e$$

and we obtain that we should choose the positive sign in front of  $r$  since  $\eta_0 - \eta_e > 0$  and  $r$  is positive. Finally we get the relation

$$\eta_0 - \eta_e = r_e. \quad (401)$$

Let us now presume that  $\eta_e$  is bounded from below by  $\tilde{\eta}_e$ ; for example  $\tilde{\eta}_e$  might represent the Big Bang singularity. Then there exist a maximum distance to which the observer can see, known as a *particle horizon distance* given by

$$r_{ph}(\eta_0) = \eta_0 - \tilde{\eta}_e \quad (402)$$

Similarly, suppose that  $\eta_0$  is bounded from above by  $\tilde{\eta}_0$ . Then there exists a limit to space-time events which can be influenced by the emitter. This limit is known as the *event horizon distance* given by

$$r_{eh}(\eta_0) = \tilde{\eta}_0 - \eta_e \quad (403)$$

These horizon distance may be converted to *proper horizon distances* at cosmic time  $t$ . For example, we have an emitter at time  $\tilde{\eta}_e$  at  $r_e = 0$ . Then at time  $\eta$ . Then from the equation for geodesics we obtain

$$\eta - \tilde{\eta}_e = r(\tau) \quad (404)$$

since  $d\eta = \frac{dt}{a(t)}$  we obtain

$$\eta - \tilde{\eta} = \int_{t_e}^t \frac{dt'}{a(t')} \quad (405)$$

using also the fact that the proper distance at time  $t$  is given by multiplication with  $a(t)$  we get the *proper horizon distance* as

$$dh = a(t) \int_{t_e}^t \frac{dt'}{a(t')} . \quad (406)$$

## 4 Our Universe Today

In this section we will discuss the remarkable properties that have been discovered in past few years. Most remarkable among them is the fact that the universe is dominated by a uniformly- distributed and slowly varying source of "dark energy" which may be a vacuum energy (cosmological constant), a dynamical field or something completely different.

### 4.1 Matter

The inventory of constituencies comprising actual Universe is complicated by the fact that they are not at all equally visible. In the years before we knew the dark energy was an important constituent of the Universe and before observations of galaxy and distributions and CMB anisotropies observational cosmology measured two numbers: The Hubble constant  $H_0$  and the matter density parameter  $\Omega_M$ . Measuring the extragalactic distances is very difficult, but most current measurement of the Hubble constant performed Planck experiment in 2013 gives the value of the cosmological constant to be equal to

$$H_0 = 67.80 \pm 0.77 \text{ km/sec/Mpc} , \quad (407)$$

where

$$1 \text{ Mpc} = 10^6 \text{ parsec} = 3 \times 10^{24} \text{ cm} . \quad (408)$$

We see that the Hubble parameter in fact has the dimension  $[t^{-1}]$  so that it has the value

$$H_0^{-1} = h^{-1} \cdot 3 \cdot 10^7 \text{ s} = h^{-1} \cdot 10^{10} \text{ yrs} \approx 1.4 \cdot 10^{10} \text{ yrs} , \quad (409)$$

where  $h$  is a dimensionless parameter

$$h = 0.678 . \quad (410)$$

In particle physics units ( $\hbar = c = 1$ ) this is equal to

$$H_0 \sim 10^{-33} \text{ eV} . \quad (411)$$

It is convenient to express the Hubble constant as

$$H_0 = 100 h \text{ km/sec/Mpc} . \quad (412)$$

It turns out that the scale  $H_0^{-1}$  gives order of magnitude of the age of the Universe and the distance scale  $H_0^{-1}$  is roughly the size of the observable part equal to

$$H_0^{-1} \approx h^{-1} \cdot 3000 Mpc \approx 4.3 \cdot 10^3 Mpc . \quad (413)$$

At this place it is natural to speak about speed of expansion of Universe. In fact, this notion does not make sense in its own meaning since Hubble constant has dimension velocity per distance. The Universe is expanding, but the expansion doesn't have a speed; it has a speed-per-unit-distance. Explicitly, for each megaparsec [3.3 million light years from Earth, the universe is expanding an extra 73.325 kilometers per second. It is also important to stress that no object can travel with velocity greater than velocity of light.

Note that since  $\rho_i = 3H_0^2\Omega_i/8\pi G$  measurement of  $\rho_i$  is often expressed as measurement of  $\Omega_i h^2$ . The Hubble constant provides the rough measure of the scale of the Universe since in the matter or radiation dominated Universe is  $t_0 \sim H_0^{-1}$ .

For years, determinations of  $\Omega_M$  based on dynamics of galaxies and clusters have led to values of  $\Omega_M$  between 0.1 and 0.4. Alternatively, the determination of  $\Omega_M$  is the same as the determination of the **baryons**. Recent measurements suggest that baryons contribute to  $\Omega$  as

$$\Omega_B = 0.05 . \quad (414)$$

In other words baryons constitute rather small fraction of the present energy density in the Universe. It is also important to stress that the most of the baryons in our Universe are dark: direct measurements of the mass density of stars give an estimate

$$\Omega_{stars} \sim 0.005 \quad (415)$$

that is about an order of magnitude smaller than  $\Omega_B$ . The fact that most of the baryons are dark follows from the dynamics of individual galaxies implies that there is even matter there. The implied existence of this celebrated **dark matter** is confirmed by applying the virial theorem to clusters of galaxies, by looking at the temperature profiles of clusters, by "weighing" clusters by gravitational lensing and by large-scale motions of clusters between galaxies. On the other hand there is nothing dramatic about this observation: baryons may hide in dust and neutral gas clouds, brown dwarfs etc.

The next form of matter are **Photons**. They however contribute even smaller fraction

$$\Omega_\gamma \approx 6 \cdot 10^{-4} . \quad (416)$$

From electric neutrality the number density of **electrons** is about the same <sup>5</sup> as that of baryons, but then due to their very small mass their contribution to the total mass fraction is negligible.

The remaining known stable particles are **neutrinos**. As we will sketch below their number density is calculable in Hot Big Bang theory and these calculations

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<sup>5</sup>There are also neutrons whose number is somewhat smaller than the number of protons.

are confirmed by Big Bang Nucleosynthesis. The number density of each type of neutrinos is

$$n_{\nu_a} = 115 \frac{1}{cm^3} , \quad (417)$$

where  $\nu_a = \nu_e, \nu_\mu, \nu_\tau$ . Direct limit on the mass of electron neutrino  $m_{\nu_e} < 2.6 eV$  together with the observations of neutrino oscillations suggests that every type of neutrino has mass smaller than  $2.6 eV$ . Then the estimation of the energy density of neutrinos is

$$\rho_{\nu, total} = \sum_{\alpha} m_{\nu_\alpha} n_{\nu_\alpha} < 8 \cdot 10^{-7} \frac{GeV}{cm^3} \quad (418)$$

that implies

$$\Omega_{\nu, total} < 0.16 . \quad (419)$$

However this estimate does not make use any cosmological date. In fact cosmological observations give stronger bound

$$\Omega_{\nu, total} < 0.01 . \quad (420)$$

In terms of the neutrino masses this bound reads

$$\sum m_{\nu_a} < 0.42 eV \quad (421)$$

so that every neutrino has to be lighter than  $0.14 eV$ . On the other hand atmospheric neutrino data and further experiments tell that the mass of at least one neutrino must be larger than  $0.02 eV$ . These results suggest that there is window for measuring neutrino masses by cosmological observations.

We see that most of the energy density in the present Universe is not in the form of known particles, most energy in the present Universe has to be in something “unknown”. In fact essentially every known particle in the Standard Model of particle physics has been ruled out as a candidate for this “unknown” matter. Moreover, there is a strong evidence that this “something unknown” has two components: **clustered dark energy** and **unclustered dark energy**.

It is believed that **Clustered dark matter** consists of new stable massive particles. These make clumps of energy density that encounter for much of the mass of galaxies and most of the mass of galactic clusters. There are number of ways of estimating the contribution of non-baryonic dark matter into the total density of the Universe:

- Composition of the Universe affects the angular anisotropy of cosmic microwave background (CMB). The present measurements of the CMB anisotropy enable to estimate the total mass density of dark matter.
- The density of non-baryonic dark matter is crucial for structure formation of the Universe. If we compare the results of numerical simulations of structure formation with observational data gives reliable estimate of the mass density of non-baryonic clustered dark matter.

One of the few things we know about the dark matter is that it must be “cold”-not only is it non-relativistic today, but it must have been that way for a very long time. The other thing we know about cold dark matter (CDM) is that it should interact very weakly with ordinary matter, so as to have escaped detection thus far. In summary the non-baryonic cold dark matter has

$$\Omega_{CDM} \approx 0.25 . \quad (422)$$

There is a direct evidence that dark matter exists in the largest gravitationally bound objects-clusters of galaxies. There are various methods to determine the gravitating mass of a cluster and even mass distribution in a cluster, which give consistent results, for example:

- We measure velocities of galaxies in galactic clusters and make use of the gravitational virial theorem

$$\text{Kinetic energy of a gravity} = \frac{1}{2} \text{ Potential energy} .$$

In this way we obtain the gravitational potential and thus the distribution of the total mass in a cluster.

- The second example of the measurement of masses of clusters use the notion of intra-cluster gas. Its temperature that is determined from  $X$ -ray measurements is also related to the gravitational potential through the virial theorem.
- The third example of measurement is based on observation of gravitational lensing of background galaxies by clusters.

Finally, dark matter exists also in galaxies. Its distribution is measured by the observations of rotation velocities of distant stars and gas clouds around a galaxy.

At present there are many hypotheses considering candidates for this form of dark matter. One such an idea is that the natural candidates are particles which participate in weak interactions that of course needs more detailed justification.

### **Unclustered dark energy**

Non-baryonic clustered dark matter is not the whole story. If we use the above estimates we obtain an estimate for the energy density of all particles

$$\Omega_{\gamma} + \Omega_B + \Omega_{\mu_{total}} + \Omega_{CDM} \approx 0.3 . \quad (423)$$

Since the observation that  $\Omega_T \approx 1$  implies that 70 percent of the energy density is unclustered.

In fact this result nicely fits recent observations. Indeed, it can be shown that neither relativistic nor non-relativistic matter can lead to the accelerated expansion of the Universe <sup>6</sup>. In other words the accelerated expansion requires energy stored in something dramatically different from conventional particles and it has to have

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<sup>6</sup>We will discuss this problem in the next subsection.

negative pressure. In fact the analysis of the entire set of cosmological data in terms of dark energy with phenomenological equation of state

$$p = w\rho, w = \text{const} \quad (424)$$

gives

$$\Omega_\Lambda = 0.72 \pm 0.02 \quad (425)$$

(here subscript  $\Lambda$  refers to dark energy) and

$$-1.2 < w < -0.8. \quad (426)$$

It is worth noting that the vacuum value,  $w = -1$  is right in the middle of the allowed region that corresponds to a vacuum energy density

$$\rho_\Lambda \sim (10^{-3} \text{eV})^4. \quad (427)$$

Given the significance of these results it is natural to ask what level of confidence we should have in them. There are potential sources of systematic error and these were discussed in the original papers [?, ?]. On the other hand the recent measurements of the cosmic microwave background confirmed the picture outlined above with the matter density and nonzero cosmological constant.

In summary, the composition of the present Universe is fairly complex. It is challenging for future physics that most of the energy density comes from species which particle physicists are unfamiliar with: vacuum or vacuum-like dark energy and non-baryonic clumped dark matter. This poses serious problems for both fundamental physics and cosmology:

- **What are the particles of non-baryonic dark matter?**

Currently popular option is the lightest supersymmetric particle that is stable in many supersymmetric extensions of the Standard model. Of course there are many other options, such as axions, gravitinos and so on. In any case experimental discovery of the dark matter particle would be great achievement of both particle physics and cosmology.

- **Why there are baryons and no anti-baryons in our Universe?**

Alliteratively, what is the origin of matter-antimatter asymmetry of the Universe? We will discuss this issue later and here we notice only that the solution of this problem is based on extension of the Standard Model.

- **Why the mass density of the non-baryonic dark matter is so similar to the mass density of baryons?**

Both these densities scale as  $a^{-3}(t)$  so their ratio stays constant during most of the evolution of the Universe. Then it is possible that mechanism which create baryons and dark matter particles in the early Universe are related to each other so that the approximate equality of the mass densities is not a mere coincidence. On the other hand it is difficult to construct corresponding particle model.

- **What is the origin of dark energy? If this is vacuum, why vacuum has non-zero energy density, which, however, is very small by particle physics standard?**

This is one of the most fundamental problems of the microscopic physics. In natural units the vacuum density is about

$$\rho_c \sim 10^{-46} GeV^4 . \quad (428)$$

On the other hand we would expect on the basis of the dimensional grounds that the vacuum energy takes value  $1 GeV^4$  (QCD-scale) or  $10^8 GeV^4$  (electroweak scale). It is great challenge to explain this enormous discrepancy but despite numerous attempts it remains an open problem.

- **Why now?**

The energy density of non-relativistic dark matter and dark energy scales differently: The non-relativistic dark matter scales as  $a^{-3}(t)$  while the latter stays approximately constant. Hence at early times (small  $a(t)$ ) the energy density of non-relativistic matter exceeded by far the dark energy density. Conversely, future expansion of the Universe will be dominated by dark energy. On the other hand these energy densities are of the same order of magnitude today. The question is why is this the case? What is special about the present epoch of the evolution of the Universe?

## 4.2 Supernovae and the Accelerating Universe

The first hint that the matter does not dominate the Universe came from the studies of the Type Ia supernovae that are commonly recognized as "standard candles". The special property of Supernovae Type Ia is that it has nearly uniform intrinsic luminosity (absolute magnitude  $M \sim -19.5$ ). It turns out that they can be detected at high redshifts ( $z \sim 1$ ) that allows in principle a good handle on cosmological effects.

The importance of the supernovae measurements began to be clear from the works of two independent groups that observed distant supernovae in order to measure cosmological parameters: the High-Z Supernova Team and the Supernova Cosmology Project. These groups obtained the dependence of the redshift on apparent magnitude. These data are much better fit by a universe dominated by a cosmological constant than by a flat matter-dominated model. In fact, the supernova results alone allow huge range of possible values of  $\Omega_M$  and  $\Omega_\Lambda$ . On the other hand if we presume that we know something about one of these parameters the second one will be tightly constrained and in particular they imply (425).

Since these observations are very fundamental one has to ask the question about the level of confidence of them. In fact there are number of potential sources of systematic error that have been considered by these two research teams. In summary these results are commonly accepted with their significant predictions considering the vacuum energy of the Universe.



### 4.3 Dark Energy

It appears that the most difficult problem to solve is the origin of the dark energy. The most disappointing possibility would be that the carrier of dark energy is **vacuum**: The difficulties with this option will be discussed below.

Another option, more promising from the observational viewpoint is that dark energy is due to some light field. In fact, there are good reasons to consider the this dynamical dark matter as an alternative to cosmological constant. Firstly, the dynamical energy density can evolve slowly to zero so that we can solve the cosmological constant problem .

The simplest possibility how to describe dark matter is the same kind of source that is involved in models of inflation in the very early Universe; a scalar field  $\phi$  rolling slowly in a potential, something known as *quintessence*.

As an example, consider a homogeneous scalar field  $\phi(t)$  in an expanding Universe. The action of the scalar field is

$$S = - \int d^4x \sqrt{-g} \left( \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + V(\phi) \right) , \quad (429)$$

where  $V(\phi)$  is potential. The equations of motions that follow from the action above have the form

$$\partial_\mu [\sqrt{-g} g^{\mu\nu} \partial_\nu \phi] - \sqrt{-g} \frac{\delta V}{\delta \phi} = 0 \quad (430)$$

that for homogeneous field in an expanding Universe takes the form

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0 . \quad (431)$$

In order to take the back-reaction of this scalar field on the Einstein equations into account we have to determine the components of the stress energy tensor. In field theory the stress energy tensor is defined as

$$T_{\mu\nu} = - \frac{2}{\sqrt{-g}} \frac{\delta S_{matter}}{\delta g^{\mu\nu}} \quad (432)$$

that for the action of the form  $S = - \int d^4x \sqrt{-g} \mathcal{L}$  takes the form

$$T_{\mu\nu} = -g_{\mu\nu} \mathcal{L} + 2 \frac{\delta \mathcal{L}}{\delta g^{\mu\nu}} , \quad (433)$$

where we have used

$$\frac{\delta \sqrt{-g}}{\delta g^{\mu\nu}} = -\frac{1}{2} \sqrt{-g} g_{\mu\nu} . \quad (434)$$

More precisely, for the action (429) the stress energy tensor takes the form

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \left[ \frac{1}{2} g^{\alpha\beta} (\nabla_\alpha \phi) (\nabla_\beta \phi) + V(\phi) \right] . \quad (435)$$

Let us now restrict to the homogeneous case in which all quantities depend only on cosmological time  $t$  and we also set  $k = 0$ . A homogeneous real scalar field behaves as a perfect fluid with

$$\rho = T_{00} = \frac{\dot{\phi}^2}{2} + V(\phi) . \quad (436)$$

The other components of the stress energy tensor take the form

$$T_{ij} = -g_{ij} \left( \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + V \right) + \partial_i \phi \partial_j \phi . \quad (437)$$

If we define pressure as

$$p = \frac{1}{3} \sum_{i=1}^3 T_{ii} \quad (438)$$

we get

$$p = \frac{\dot{\phi}^2}{2} - V(\phi) . \quad (439)$$

Thus any state which is dominated by the potential energy of a scalar field will have negative pressure.

If the slope of the potential  $V$  is quite flat we will have solutions for which  $\phi$  is nearly constant and only evolving very gradually with time, the energy density in such a configuration is

$$\rho_\phi \approx V(\phi) \approx \text{const.} \quad (440)$$

Thus we see that slowly-rolling scalar field is an appropriate candidate for dark energy with the vacuum equation of state

$$p_\phi = -\rho_\phi \quad (441)$$

but the energy density  $\rho_\phi$  slowly decreases in time. But this proposal raises several questions: why the genuine vacuum energy density is zero (constant part of the potential  $V_0$ ) so that it does not contribute to dark energy density? What is the physics behind the field  $\phi$ ? Where does the small energy scale,  $V(\phi) \sim 10^{-46} \text{GeV}$  today, come from? All these questions remain unanswered <sup>7</sup>.

In fact, it is important to stress that introducing dynamics opens up the possibility to bring new problems that depend on form and specific kind of model being considered. Most quintessence models feature scalar fields  $\phi$  with masses of order the current Hubble scale

$$m_\phi \sim H_0 \sim 10^{-33} \text{eV} . \quad (442)$$

In quantum field theory the light scalar fields are unnatural, renormalization effects tend to drive scalar masses up to the scale of new physics. It is then very difficult to understand the origin of masses of such a small value when we know that the scale of new physics is approximately  $10^{11} \text{eV}$ . Moreover, light scalar fields give rise to long-range forces and time-dependent coupling constant that should be

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<sup>7</sup>For certain scalar potentials the fourth question can be explained.

observable. Therefore we have to invoke additional fine-tunings to explain why the quintessence field has not already been experimentally detected.

Another possibility, how to explain today acceleration of Universe, is that there is nothing special about the present era; rather acceleration is just something that happens from time to time. This can be enforced by oscillating dark energy. In these models the potential takes the form of a decaying exponential with small perturbations

$$V(\phi) = e^{-\phi}[1 + \alpha \cos \phi] . \quad (443)$$

Another models of quintessence are *k-essence models* that are based on presumption that the scalar field  $\phi$  has the form

$$K = f(\phi)g(\dot{\phi}^2) , \quad (444)$$

where  $f, g$  are functions specified by the model. Unfortunately, in neither the *k-essence* models nor the oscillating models do we have a compelling particle-physics motivation for the chosen dynamics and in both cases the behavior still depends sensitively on the precise form of parameters and interactions chosen.

Given the challenge of the problem it is worthwhile considering the possibility that cosmic acceleration is not due to some kind of stuff but rather arise from new gravitational physics.

As a first attempt, consider the simplest correction to the Einstein-Hilbert action,

$$S = \frac{M_p^2}{2} \int d^4x \sqrt{-g} \left( R - \frac{\mu^4}{R} \right) + \int d^4x \sqrt{-g} \mathcal{L}_M , \quad (445)$$

where  $\mu$  is a new parameter with units of  $[mass]$  and  $\mathcal{L}_M$  is the Lagrangian density for matter. The equations arising from this action are complicated and it is difficult to solve them. It is convenient to transform from the action used in (445) which we call the *matter frame* to the *Einstein frame* where the gravitational Lagrangian takes the Einstein-Hilbert form and the additional degrees of freedom ( $\ddot{H}$  and  $\dot{H}$ ) are represented by a fictitious scalar field  $\phi$ . In terms of the new metric  $g_{\mu\nu}$  the theory is that of a scalar field  $\phi(x)$  minimally coupled to Einstein gravity and non-minimally coupled to matter with the potential

$$V(\phi) = \mu^2 M_p^2 \exp \left( -2\sqrt{\frac{2}{3}} \frac{\phi}{M_p} \right) \sqrt{\exp \left( \sqrt{\frac{2}{3}} \frac{\phi}{M_p} \right) - 1} . \quad (446)$$

Yet another option for the explaining the accelerated expansion of our Universe is that gravity deviates from General Relativity at cosmological distances and time scales so that the Friedmann equation is not valid at present epoch. Finally, any modification of the Einstein-Hilbert action must, of course, be consistent with the classic solar system tests of gravity theory as well as numerous other astrophysical dynamical tests. In known Lorentz-Invariant examples of such a theory there either exist ghosts (fields with negative energy unbounded from below) or gravity becomes strongly coupled at quantum level. A consistent theory of this sort would probably

require “gravitational Higgs mechanism” and violation of Lorentz-invariance but even this-rather exotic idea- has not yet lead to a consistent model that would be able to explain the accelerated expansion of the Universe.

In summary, there are many models whose aim is to explain current acceleration area. All of these models have many problems however it is certainly very important to study them.

## 4.4 Observational Evidence for Dark Energy

In this section we briefly review facts considering observational evidence for dark energy. The first one is based on so named *Luminosity distance*

### 4.4.1 Luminosity distance

In 1998 the accelerated expansion of the Universe was reported on the observations of Type Ia Supernova (SN Ia). This observations are based on the existence of redshift in the expanding Universe that is related to the fact that the light emitted by a stellar object becomes red-shifted due the expanding of the Universe. The wavelength  $\lambda$  increases proportionality to the scale factor  $a$  according to the formula

$$1 + z = \frac{\lambda_0}{\lambda} = \frac{a_0}{a} , \quad (447)$$

where  $z$  is named as redshift and where the subscript zero denotes the quantities given at present epoch.

Another important concept that is related to the observational tools in an expanding background is the definition of the distance. In fact there are many ways how to define distance in expanding Universe. For example, we can consider comoving distance as a distance measured in comoving variables. It turns out that this distance does not change during the evolution of the Universe. On the other hand we can define physical distance that scales proportionally to the scale factor. An alternative way of defining of distance is through the luminosity distance that plays a very important role in astronomy, including supernova observations.

Let us consider for a moment Minkowski space-time and define an absolute luminosity  $L_s$  of source that is related to the energy flux  $\mathcal{F}$  at the distance  $d$  from the source by the formula

$$\mathcal{F} = \frac{L_s}{4\pi d^2} . \quad (448)$$

We can generalize this relation to the expanding Universe and define the luminosity distance  $d_L$  as

$$d_L^2 \equiv \frac{L_s}{4\pi\mathcal{F}} . \quad (449)$$

Let us consider an object with an absolute luminosity  $L_s$  located at coordinate

distance  $\chi$ <sup>8</sup> from an observer located at  $\chi = 0$ . The energy of object that is emitted in time interval  $\Delta t_1$  let is denoted as  $\Delta E_1$  while the energy that reaches the sphere at radius  $\chi$  is written as  $\Delta E_0$ . From the basic principles it is clear that  $\Delta E_1$  and  $\Delta E_0$  are proportional to the frequencies of light at  $\chi = \chi_s$  and  $\chi = 0$  respectively. In other words,  $\Delta E_1 \sim \nu_1$ ,  $\Delta E_0 \sim \nu_0$ . We also define the luminosity  $L_s$  and  $L_0$  through the relations

$$L_s = \frac{\Delta E_1}{\Delta t_1}, \quad L_0 = \frac{\Delta E_0}{\Delta t_0}. \quad (452)$$

The speed of light is given by  $c = \nu_1 \lambda_1 = \nu_0 \lambda_0$  where  $\lambda_1, \lambda_0$  are wavelengths at  $\chi = \chi_s$  and  $\chi = 0$ . Then (447) implies

$$\frac{\lambda_0}{\lambda_1} = \frac{\nu_1}{\nu_0} = \frac{\Delta E_1}{\Delta E_0} = \frac{\Delta t_0}{\Delta t_1} = 1 + z, \quad (453)$$

using also the fact that  $\nu_0 \Delta t_0 = \nu_1 \Delta t_1$ . If we now combine (453) and (452) we obtain

$$\frac{L_s}{L_0} = \frac{\Delta E_1}{\Delta E_0} \frac{\Delta t_0}{\Delta t_1} = (1 + z)^2. \quad (454)$$

The light travelling along  $\chi$  direction satisfies the geodetic motion  $ds^2 = -dt^2 + a^2(t)d\chi^2 = 0$  that implies

$$\chi_s = \int_0^{\chi_s} d\xi = \int_{t_1}^{t_0} \frac{dt}{a(t)} = \frac{1}{a_0 H_0} \int_0^z \frac{dz'}{h(z')}, \quad h(z) = \frac{H(z)}{H_0}, \quad (455)$$

where we have take  $t_0$  as the time at present epoch and consequently  $\chi_0 = 0$ . We have also used the fact that

$$1 + z = \frac{a_0}{a} \Rightarrow \frac{dz}{dt} = -\frac{a_0}{\dot{a}} \Rightarrow dt = -\frac{dz \dot{a}}{a_0}. \quad (456)$$

Now the form of the metric (450) implies that the area of two sphere at  $t = t_0$  is given by  $S = 4\pi(a_0 f_K(\chi_s))^2$ , where  $\chi_s$  corresponds to the fact that we observe signal from the distance  $\chi_s$ . Hence the observed energy flux is

$$\mathcal{F} = \frac{L_0}{4\pi(a_0 f_K(\chi_s))^2}. \quad (457)$$

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<sup>8</sup>Recall that the metric has following form:

$$ds^2 = -dt^2 + a^2(t)[d\chi^2 + f_K^2(\chi)(d\theta^2 + \sin^2 \theta d\phi^2)], \quad (450)$$

where

$$\begin{aligned} f_K &= \sin \chi, \quad k = 1, \\ f_K &= \chi, \quad k = 0, \\ f_K &= \sinh \chi, \quad k = -1. \end{aligned} \quad (451)$$

Using these results we obtain

$$d_{L_s}^2 = \frac{L_s}{4\pi\mathcal{F}} = \frac{L_s 4\pi (a_0 f_K(\chi_s))^2}{4\pi L_0} = a_0^2 f_K(\chi_s)^2 (1+z)^2 . \quad (458)$$

If we combine (455) with (458) and use the fact that in FRW background  $f_K(\chi) = \chi$  we obtain

$$d_L = \frac{1+z}{H_0} \int_0^z \frac{dz'}{h(z')} . \quad (459)$$

We can invert this result and express  $H(z)$  as function of  $d_L(z)$  and  $z$

$$H(z) = \left( \frac{d}{dz} \left[ \frac{d_L(z)}{1+z} \right] \right)^{-1} . \quad (460)$$

If we measure the luminosity distance observationally we can determine the expanding rate of the Universe.

As we know the energy density on the right hand side of the Friedmann equations includes all components that are presented in Universe, namely non-relativistic particles, relativistic particles, cosmological constant:

$$\rho = \sum_i \rho_i^{(0)} (a/a_0)^{-3(1+w_i)} = \sum_i (1+z)^{3(1+w_i)} , \quad (461)$$

where we have used (447). Here  $w_i$  and  $\rho_i^{(0)}$  correspond to the equation of state and the present energy density of each component.

Then the Friedmann equation takes standard form

$$H^2 = H_0^2 \sum_i \Omega_i^{(0)} (1+z)^{3(1+w_i)} , \quad \Omega_i^{(0)} = \frac{8\pi G \rho_i^{(0)}}{3H_0^2} = \frac{\rho_i^{(0)}}{\rho_c^{(0)}} . \quad (462)$$

Hence the luminosity distance in a flat geometry is given by

$$d_L = \frac{(1+z)}{H_0} \int_0^z \frac{dz'}{\sqrt{\sum_i \Omega_i^{(0)} (1+z')^{3(1+w_i)}}} . \quad (463)$$

The formula above is the basic theoretical ingredient for the direct evidence of the current acceleration of the Universe that is related to the observation of luminosity distances of high redshift supernovae.

The Type Ia supernova (SN Ia) can be observed when the white dwarf starts exceed the mass of the Chandrasekhar limit and explode. The common belief is that SN Ia are formed in the same way irrespective of where they are in the Universe that means that they have a common absolute magnitude  $M$  independent of the redshift  $z$ . This implies that they can be treated as an ideal standard candle. We do not go to these details but it is important that using these methods the luminosity distance of the SN Ia supernovae that was observed is

$$H_0 d_L \simeq 1.16 , \quad \text{for } z = 0.83 . \quad (464)$$

On the other hand the theoretical estimate that follows from (463) is

$$\begin{aligned} H_0 d_L &\simeq 0,95, & \Omega_m^{(0)} &\simeq 1, \\ H_0 d_L &\simeq 1.23, & \Omega_m^{(0)} &\simeq 0.3, & \Omega_\Lambda^{(0)} &\simeq 0.7. \end{aligned} \tag{465}$$

for two-component form of matter. There are of course lot of literature considering the fitting the estimate date and the form of the matter that is present in Universe. The conclusion is that the present experimental date suggests the form of the matter given above.

## 4.5 The age of the Universe and the cosmological constant

Another important evidence for the existence of the cosmological constant emerges when we compare the age of the Universe  $t_0$  to the age of the oldest stellar populations  $t_s$ . It is clear that the consistency demands that  $t_0 > t_s$ . On the other hand it is difficult to satisfy this condition for a flat cosmological model with normal form of matter. On the other hand the presence of cosmological constant can resolve this problem.

To begin with we review the estimates of the oldest stellar objects. It was estimated that the age of the oldest objects lay in the interval 11–13 Gyr. Consequently the age of the Universe needs to satisfy the lower bound  $t_0 > 11 - 12$  Gyr. Let us calculate the age of the Universe from the Friedmann equations where we consider three contributions to the matter: radiation ( $w_r = 1/3$ ), pressure-less dust ( $w_m = 0$ ) and cosmological constant  $w_\Lambda = -1$ .

$$\begin{aligned} H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} &= H_0^2 \left[ \Omega_r^0 \left( \frac{a}{a_0} \right)^{-4} + \Omega_m^{(0)} \left( \frac{a}{a_0} \right)^{-3} + \right. \\ &\quad \left. + \Omega_\Lambda^{(0)} - k_0 \left( \frac{a}{a_0} \right)^{-2} \right], \quad k_0 = \frac{k}{a_0^2 H_0^2}. \end{aligned} \tag{466}$$

Then using the fact that  $1 + z = \frac{a_0}{a}$  we can determine the age of the Universe as

$$\begin{aligned} t_0 &= \int_0^{t_0} dt' = \int_0^{a_0} \frac{da}{Ha} = \left( -dz = \frac{a_0 da}{a^2} \right) = \\ &= \int_0^\infty \frac{dz}{H(1+z)} = \int_0^\infty \frac{dz}{H_0 x [\Omega_r^0 x^4 + \Omega_m^{(0)} x^3 + \Omega_\Lambda^{(0)} - k_0 x^2]^{1/2}}, \end{aligned} \tag{467}$$

where  $x = 1 + z$ . Since the radiation dominated period is much shorter than the total age of the Universe it is a natural to neglect its contribution to the formula above. In other words the integral coming from the region  $z \geq 1000$  does not affect too strongly the integral (467). Hence we set  $\Omega_r^{(0)} = 0$  when we evaluate  $t_0$ .

Let us start with the case when the cosmological constant is absent ( $\Omega_\Lambda^{(0)} = 0$ ). Since  $k_0 = \Omega_m^{(0)} - 1$  the integral (467) is equal to

$$t_0 = \int_0^\infty \frac{dz}{H_0 x \sqrt{\Omega_m^{(0)} x^3 - k_0 x^2}} = \int_0^\infty \frac{dz}{H_0 (1+z)^2 \sqrt{1 + \Omega_m^{(0)} z}} . \quad (468)$$

For a flat Universe that is characterized with  $k_0 = 0$  and  $\Omega_m^0 = 1$  we obtain

$$t_0 = \frac{2}{3H_0} . \quad (469)$$

As we know the present Hubble parameter is constrained to be

$$H_0^{-1} = 9.776 h^{-1} \text{ Gyr} , \quad 0.64 < h < 0.8 . \quad (470)$$

Then (469) gives

$$t_0 = 8 - 10 \text{ Gyr} . \quad (471)$$

However this does not satisfy the stellar age bound

$$t_0 < 11 - 12 \text{ Gyr} .$$

In other words the flat Universe without a cosmological constant suffers from a serious age problem.

For arbitrary  $\Omega_m^{(0)}$  the equation (467) can be integrated and we obtain

$$H_0 t_0 = \frac{1}{1 - \Omega_m^{(0)}} - \frac{\Omega_m^{(0)}}{2(1 - \Omega_m^{(0)})^{3/2}} \ln \left( \frac{1 - \sqrt{1 - \Omega_m^{(0)}}}{1 + \sqrt{1 - \Omega_m^{(0)}}} \right) \quad (472)$$

that is of course valid for  $\Omega_m^{(0)} < 1$  only. Let us consider various limits of the equation above. For  $\Omega_m^{(0)} \rightarrow 0$  we obtain  $H_0 t_0 \rightarrow 1$  while for  $\Omega_m^{(0)} \rightarrow 1$  we obtain  $t_0 H_0 \rightarrow 2/3$ . As we know the observation of the CMB constraints the curvature of the Universe to be close to be flat  $|k_0| = |\Omega_m^{(0)} - 1| \ll 1$ . However since then  $\Omega_m^{(0)} \approx 1$  in this case we again obtain

$$t_0 = \frac{2}{3H_0} \simeq 8 - 10 \text{ Gyr} \quad (473)$$

that is again consistent with the time of the stellar age bound.

On the other hand the age problem can be easily solved in a flat Universe ( $k_0 = 0$ ) with a cosmological constant ( $\Omega_\Lambda \neq 0$ ). In this case the equation (467) gives

$$\begin{aligned} H_0 t_0 &= \int_0^\infty \frac{dz}{(1+z) \sqrt{\Omega_m^{(0)} (1+z)^3 + \Omega_\Lambda^{(0)}}} = \\ &= \frac{2}{3\sqrt{\Omega_\Lambda^{(0)}}} \ln \left( \frac{1 + \sqrt{\Omega_\Lambda^{(0)}}}{\sqrt{\Omega_m^{(0)}}} \right) , \end{aligned} \quad (474)$$



where  $\Omega_m^{(0)} + \Omega_\Lambda^{(0)} = 1$ . We see that  $H_0 t_0 \rightarrow \infty$  for  $\Omega_m^{(0)} \rightarrow 0$  and  $H_0 t_0 \rightarrow 2/3$  for  $\Omega_m^{(0)} \rightarrow 1$ . When  $\Omega_m^{(0)} = 0.3$  and  $\Omega_\Lambda^{(0)} = 0.7$  one has

$$t_0 = 0.964 H_0^{-1} = 13.1 \text{ Gyr} , \text{ for } h = 0.72 . \quad (475)$$

Hence this easily satisfies the constraint  $t_0 > 11 - 12$  Gyr that arises from the observation the oldest stellar populations. Thus the presence of  $\Lambda$  solves the age-crisis problem.

## 4.6 The Cosmological Constant Problem

In classical general relativity the cosmological constant  $\Lambda$  is a completely free parameter. Let us determine corresponding dimension of given constant. Note that it appears in the action in the form

$$\frac{1}{8\pi G} \int d^4x \sqrt{-g} \Lambda . \quad (476)$$

Since the dimension of  $G$  is  $[G] = M^{-2}$  where  $M$  is mass scale and since  $[d^4x] = M^{-4}$  we find from the requirement that the action is dimensionless that the dimension of  $\Lambda$  is given by the equation

$$[\Lambda] = M^2 \quad (477)$$

while

$$\rho_\Lambda = \frac{1}{8\pi G} \Lambda \quad (478)$$

has dimension  $[\rho_\Lambda] = M^4$  as it is expected for the energy density. In fact,  $\Lambda$  is completely free and its value should be determined by experiment.

The introduction of quantum mechanics changes the situation in some way. Firstly, the Planck's constant allows us to define the reduced Planck mass  $M_P \sim 10^{18} \text{ GeV}$ , as well as reduced Planck length

$$L_P = (8\pi G)^{1/2} \sim 10^{-32} \text{ cm} . \quad (479)$$

Hence the natural guess for the value of the cosmological constant is

$$\Lambda_P^{\text{guess}} \sim L_P^{-2} , \quad (480)$$

or as an energy density

$$\rho_{\text{vac}}^{\text{guess}} \sim M_P^4 = (10^{18} \text{ GeV})^4 . \quad (481)$$

We can find support for this guess by thinking about the quantum fluctuation of vacuum. As we know any quantum field can be considered as collection of infinite number of harmonic oscillators. From quantum mechanics we know that harmonic oscillator with frequency  $\omega$  has the vacuum energy  $\frac{1}{2} \hbar \omega$ . Since each mode of the quantum field contributes to the vacuum energy and the net result should be an integral over all of these modes. Usually we perform an integration over infinite

interval and hence this integral diverges so that the vacuum energy appears to be infinite. However, the infinity arises from contribution of modes with very small wavelengths, it is possible to be mistake to include such a modes since we do not know what happens at these scales. In other words we do not have any justification whether the quantum field theory approach can be applied in these small scales as well. To account for our ignorance we should include the cut-off energy above which we ignore any potential contributions and hope that some more complete theory could justify this approach. If the cut-off is at the Planck scale we get the value given above.

However, we claim to have measured the vacuum energy. The observed value is different from the theoretical estimate:

$$\rho_{vac}^{obs} \sim 10^{-120} \rho_{vac}^{guess} . \quad (482)$$

In other words, we can express the vacuum energy in terms of the mass scale

$$\rho_{vac} = M_{vac}^4 \quad (483)$$

so that the observed result is

$$M_{vac}^{obs} \sim 10^{-3} eV. \quad (484)$$

The discrepancy is thus

$$M_{vac}^{obs} \sim 10^{-30} M_{vac}^{guess} . \quad (485)$$

In addition to the fact that it is very small to its natural value the vacuum energy at present poses an additional puzzle. The coincidence between observed vacuum energy and current matter density. It can be shown that the ratio of vacuum energy to matter density depends on time as follows from

$$\frac{\Omega_{\Lambda}}{\Omega_M} = \frac{\rho_{\Lambda}}{\rho_M} \sim a^3 . \quad (486)$$

As a consequence, at early times the vacuum energy was negligible with respect in comparison to matter and radiation while at late times matter and radiation are negligible.

To date the value of the cosmological constant is one of the most mysterious problems in current physics, perhaps it could be compared with the mysterious radiation of the black body at the end of 19' century. On the other hand it is instructive to consider an example of supersymmetry which relates to the cosmological constant problem in interesting way. The main idea of supersymmetry is that for each fermionic degree of freedom there is corresponding bosonic degree of freedom and vice-versa. For example, for spin 1/2 electron there should be spin 0 electron of the same mass and charge. The good news is that while bosons contribute positively to the vacuum energy the fermion contributions is negative. Hence, if the degrees of freedom exactly match the vacuum energy is zero.

We do not, however, live in supersymmetric state. If supersymmetry exists, then it must be broken at some scale  $M_{susy}$ . In other words, for physical processes where

the characteristic energy is much smaller than  $M_{susy}$  we do not see any supersymmetry and this is the case how our world looks like. On the other hand when we probe physics with energy scale higher with  $M_{susy}$  we can expect that supersymmetry is restored. More precisely, we can explain this situation as follows. We expect that SUSY is broken in nature, for example spontaneously broken which means that there is one ground state. The fluctuation above states gain masses and one expect that super-partners of known particles, get masses of order  $M_{susy}$ . Then for energies much smaller than  $M_{susy}$  these particles are not visible, on the other hand for energies larger than  $M_{susy}$  we can neglect their masses and these particles look like massless again. Then we say that supersymmetry is restored at higher energies. This has an consequence for the vacuum energy. Recall that the vacuum energy was defined as sum over infinite number of oscillators. For modes with energy much larger than  $M_{susy}$  these modes find their super-partners and hence their contribution to the vacuum energy vanishes. This is of course does not happen for modes with energy smaller than  $M_{susy}$ . In other words we can expect that the vacuum energy will be equal to

$$\rho_{vac} \sim M_{susy}^4 . \quad (487)$$

The question is how high  $M_{susy}$  should be. Nice property of SUSY is that it helps us to understand *hierarchy problem*- why scale of electroweak symmetry breaking is much smaller than the scales of quantum gravity or grand unification. For SUSY to be relevant to the hierarchy problem we need the SUSY breaking scale to be just above the electroweak breaking scale

$$M_{susy} \sim 10^3 \text{ GeV} . \quad (488)$$

Since this is very close to the experimental bound it is now common belief that SUSY should be discovered soon at Fermilab or CERN, if it is connected to electroweak physics. However considering relation between SUSY and cosmological constant we again see that we are in discrepancy with observation:

$$M_{vac}^{(obs)} \sim 10^{-15} M_{susy} \text{ (Experiment)}. \quad (489)$$

Of course there exists a possibility that our estimate  $M_{vac} \sim M_{susy}$  is incorrect. For example let us guess following formula

$$M_{vac} \sim \left( \frac{M_{susy}}{M_P} \right) M_{susy} . \quad (490)$$

Interestingly, since  $M_P$  is fifteen orders of magnitude larger than  $M_{susy}$  and  $M_{susy}$  is fifteen orders of magnitude larger than  $M_{vac}$  this guess gives up the correct answer. Unfortunately this is simple numerology, we do not know how this formula should come from.

Another possibility how to explain the value of the cosmological constant is the presumption that it is simply feature of our local environment. This is the idea commonly known as **anthropic principle**.

In order to give this idea concrete meaning let us presume that there are many different regions of the Universe in which the vacuum energy takes different values. Then we can expect that we find ourselves in a region which was suitable for our own existence. Larger value of cosmological constant than we presently observe would either have led to a rapid re collapse of the universe (if  $\rho_{vac}$  were negative) or an inability to form galaxies (if  $\rho_{vac}$  were positive).

The idea environmental selection is based on certain special conditions and we do not understand whether these conditions hold in our Universe. In particular we have to show that there can be a huge number of different domains with slightly different values of the vacuum energy and that these domains are big enough that our entire observable Universe is in a single domain. Further we also have to show that the possible variation of other physical quantities from domain to domain is consistent with observations.

Recent work in string theory whose pure essence is the currently very popular idea of **String Landscape** supports the idea that there are huge number of possible vacuum states rather than a unique one. Unfortunately the detailed discussion of this idea is beyond the scope of this introduction review.

To conclude, at present, unfortunately, there is not any theory that could explain the mysterious facts considering cosmological constant. To find such a theory is one of the most prominent goals of physical community.

However there is another, maybe deeper problem related to the cosmological constant. This is the problem of radiative stability of the cosmological constant. For example, let us consider unimodular gravity where cosmological constant can be simply interpreted as integration constant with value  $\lambda_0$ . Then we should take into account quantum nature of the matter keeping in mind that we can still treat gravity as classical theory. In other words we do not need to carry about quantum nature of gravity and consider matter in the context of effective field theory.

As we argued above the vacuum in quantum field theory has its own energy that we call as  $\rho_{vac}$ . In principle we can add arbitrary constant value to the action  $\Lambda$  so that the combination  $\frac{\Lambda}{8\pi G} + \rho_{vac}$  truly gravities. In principle we could fine tuned  $\Lambda$  in such a way that  $\frac{\Lambda}{8\pi G} + \rho_{vac}$  is equal to observed value. However the true problem of cosmological constant is its radial instability. In fact, in quantum field theory we cancel divergences in physical parameters by fixing finite value by empirical observation. The real problem arises when these calculations are unstable again additional loop corrections.

## 4.7 The Cosmic Microwave Background

Most of the radiation we observe in Universe today is in the form of the almost isotropic black body spectrum with temperature approximately  $2.7K$  known as *Cosmic Microwave Background (CMB)*. The small angular fluctuations in temperature of the CMB reveal a great deal about the constituents of the Universe.

We have seen previously that the radiation gas evolves and sources the evolution of the expanding Universe. Since the radiation and dusts have different evolution

laws that as we approach earlier and earlier times in the Universe with smaller and smaller scale factors the ratio of the energy density in radiation to that in matter grows proportionally to  $1/a(t)$ . Furthermore, even particles which are now massive and contribute to matter used to be hotter, at sufficiently early times were relativistic and thus contributed to radiation. In summary, we say that the early Universe was dominated by radiation. More precisely, at early times the CMB photons were easily energetic enough to ionize hydrogen atoms and therefore the Universe was filled with a charged plasma. This phase lasted until the photons red shifted enough to allow protons and electrons to combine during the era of *recombination*. Shortly after this time the photons decoupled from the now neutral plasma and free streamed through the Universe.

More precisely, the concept of an expanding Universe provides us with a clear explanation of the origin of the CMB. Black body radiation is emitted by bodies in thermal equilibrium. The present Universe is certainly not in this state, and so without an evolving space-time we should have no explanation for the origin of this radiation. However, at early times, the density and energy densities in the Universe were high enough that matter was in approximate thermal equilibrium at each point in space, yielding a blackbody spectrum at early times. Then there is crucial thermodynamic fact about the CMB. A blackbody distribution, such as that generated at early Universe, is such that at temperature  $T$ , the energy flux in the frequency range  $[\nu, \nu + d\nu]$  is given by Planck distribution

$$P(\nu, T)d\nu = 8\pi h \left(\frac{\nu}{c}\right)^3 \frac{1}{e^{h\nu/kT} - 1} d\nu , \quad (491)$$

where  $h$  is Planck's constant and  $k$  is the Boltzmann constant. Note that this is energy flux of photons which are in equilibrium with matter at temperature  $T$ . Under recalling  $\nu \rightarrow \lambda\nu$  , with  $\lambda = \text{constant}$  the shape of the spectrum is unaltered if  $T \rightarrow T/\lambda$ . We know that the wave length are stretched with the cosmic expansion and therefore the frequencies will scale inversely due to the same effect. We then see that the effect of cosmic expanding on an initial blackbody spectrum is to retain its blackbody nature, but just at lower and lower temperatures

$$T \sim \frac{1}{a} . \quad (492)$$

This is what we mean when we say that the Universe is cooling as it expands.

More precisely, let us presume that radiation begins free expansion at time  $t_L$  where radiation suddenly went from being in thermal equilibrium with matter and then expands freely. The subscript  $L$  means *last scattering*. Then let us presume that photon has frequency  $\nu$  at late time  $t$  after decoupling when the photon travels freely. Due to the arguments given previously we have that it had a frequency

$$\nu_L = \nu \frac{a(t)}{a(t_L)} \quad (493)$$

To proceed further we observe that energy flux  $P(\nu, T)$  is related to the number density  $n(\nu, T)$  as  $P(\nu, T) = h\nu n(\nu, T)$ . Further, conservation of number of photons

gives

$$n(t, \nu) a^3(t) d\nu = n_{T(t_L)}(\nu_L) a^3(t_L) d\nu_L \quad (494)$$

we get, using results given above we obtain number density at the time  $t$  to be equal to

$$n(t, \nu) = \left( \frac{a(t_L)}{a(t)} \right)^3 \frac{a(t)}{a(t_L)} n_{T(t_L)} \quad (495)$$

Now  $n_{T(t_L)}$  is equal to

$$n_{T(t_L)} = \frac{8\pi \nu^2(t_L)}{c^3} \frac{1}{e^{\frac{h\nu(t_L)}{kT(t_L)}} - 1} d\nu = \frac{8\pi \nu^2 a^2(t)}{c^3 a^2(t_L)} \frac{1}{e^{\frac{h\nu}{kT(t_L)} \frac{a(t)}{a(t_L)}} - 1} \quad (496)$$

so that we find that the number density of photons with frequency  $\nu$  at time  $t$  is equal to

$$n(t, \nu) = \frac{8\pi}{c^3} \frac{\nu^2 d\nu}{\exp \frac{h\nu}{kT(t)} - 1} \quad (497)$$

where

$$T(t) = T(t_L) \frac{a(t_L)}{a(t)}. \quad (498)$$

We should stress that this conclusion does not change even in case when the decoupling of photons occurs at finite time interval on condition that the interaction between photons and matter occurs during a finite time interval when these interactions are limited to elastic scattering processes where photon frequencies do not change.

The energy density in this radiation is given

$$\int_0^\infty h\nu n(\nu) d\nu = a_B T^4, \quad (499)$$

where  $a_B$  is the radiation energy constant

$$a_B = \frac{8\pi^4 k_B^4}{15h^3 c^3} \quad (500)$$

that using the present temperature of CMB given by  $T_{\gamma_0} = 2,725K$  gives an equivalent mass density (in units where  $c = 1$ )

$$\rho_{\gamma_0} = a_B T_{\gamma_0}^4 = 4.64 \times 10^{-34} \text{ g cm}^{-3}. \quad (501)$$

Then the ratio with the critical density gives

$$\Omega_\gamma = \frac{\rho_{\gamma_0}}{\rho_{0crit}} = 2.47 \times 10^{-5} h^{-2}. \quad (502)$$

It is also interesting that there are another light particles that have similar properties as photons in CMB which are neutrinos and antineutrinos of three different types that gives a total energy density in the radiation. It is important to stress that

radiation is ment all massless or nearly massless particles. Then the total energy density of radiation is

$$\rho_{R0} = \left[ 1 + 3 \left( \frac{7}{8} \right) \left( \frac{4}{11} \right) \right] \rho_{\gamma 0} = 7.80 \times 10^{-34} gcm^{-3} . \quad (503)$$

or equivalently

$$\Omega_R \equiv \frac{\rho_{R0}}{\rho_{0crit}} = 4.15 \times 10^{-5} h^{-2} . \quad (504)$$

Further, we can calculate number of densitz of photons at present to be equal to

$$n_{\gamma 0} = \int_0^\infty \frac{8\pi\nu^2 d\nu}{\exp(h\nu/kT) - 1} = \frac{30\zeta(3)}{\pi^4} \frac{A_B T^3}{k} = 20.28 [T(degK)]^3 cm^{-3} . \quad (505)$$

where  $\zeta(3) = 1.202077\dots$ . For  $T = 2.275K$  this gives a present number density equal to

$$n_{\gamma 0} = 410 \text{ photons}/cm^3 . \quad (506)$$

Let us compare this number with the present number density of nucleons  $n_{B0}$  that is equal to

$$n_{B0} = \frac{3\Omega_B H_0^2}{8\pi G m_N} = 1.123 \times 10^{-5} \Omega_B h^2 \text{ nucleons}/cm^3 , \quad (507)$$

where now  $h$  is the Hubble constant in units of  $100 \text{ kms}^{-1} Mpc^{-1}$ . Since both  $n_\gamma$  and  $n_B$  depend on time through the factor  $a^{-3}(t)$  we see that the rasion  $n_\gamma/n_B$  is the same at least during the period when the photons travel freely.

## 4.8 The equilibrium era

In this section we study era when the radiation and matter were in thermal equilibrium. This happens for sufficiently high temperatures and densitites.

Let us consider equilibrium where entropy and the baryon number which, for temperatures  $\ll 10^{13}K$  is given as the number of protons and neutrons, in any comoving frame are constants and hence their rasion of entropy per baryon was also constant. Let us denote  $k_B\sigma$  as the rasion of entropy per baryon. We start with the second law of thermodynamics which is known as

$$TdS = dU + pdV \quad (508)$$

In our case we should apply it for densities. More precisely, we relate these quantities to the number of baryons so that the second law of thermodynamics has the form

$$d(k\sigma) = \frac{d(\epsilon/n_B) + pd(1/n_B)}{T} , \quad (509)$$

where  $n_B$  is baryon number density so that  $1/n_B$  is the volume per baryon,  $\epsilon$  is the thermal density and  $p$ – is the pressure. Let us presume that photons and non-relativistic particles which are protons, helium nuclei and electrons correspond to ideal gas. Then we can write

$$\epsilon = a_B T^4 + \frac{3}{2} n_B \mathcal{N} k T , \quad p = \frac{1}{3} a_B T^4 + n_B \mathcal{N} k T , \quad (510)$$

where  $\mathcal{N}$  is fixed number of the non-relativistic particles per baryon. Then the second thermodynamics law takes the form

$$d(\sigma k) = 4a_B \frac{T^2}{n_B} dT - \frac{1}{3} a_B T^3 \frac{dn_B}{n_B^2} + \frac{3}{2} dn_B \mathcal{N} k + \frac{3}{2} n_B \mathcal{N} k \frac{dT}{T} - \frac{1}{n_B} \mathcal{N} k_B T dn_B \quad (511)$$

that can be integrated as

$$\sigma = \frac{4a_B T^3}{3n_B k} + \mathcal{N} \ln \left( \frac{T^{3/2}}{n_B C} \right), \quad (512)$$

where  $C$  is integration constant. Now in thermal equilibrium  $\sigma$  is constant. We know that at present the number of photons to number of nucleons is of order  $10^8$ . In other words the first term in the expression for  $\sigma$  is larger than the second one. It is natural to presume that this quantity is also much larger than unity when photons were in equilibrium with matters. Then since  $\sigma$  is constant at this era we should also have that the ration  $T^3/n_B$  was constant at this time. We also saw that this ratio was constant when the photons were travelling freely so that we can say that this ratio is constant from the beginning of the equilibrium era to present. Then

$$\sigma = \frac{4a_B T^3}{3n_B k} = \frac{3.6n_{\gamma 0}}{n_{B0}} = 1.31 \times 10^8 h^{-2} \Omega_B^{-1}, \quad (513)$$

where we used the fact that  $\sigma$  is constant and hence it can be determined by values of  $n_{\gamma 0}$  and  $n_{B0}$  at present. Since baryon number is conserved we have that  $n_B a^3$  is constant and hence previous relation implies that  $T \sim \frac{1}{a}$ .

It is important to stress that even if collisions cannot change distribution of photon energies when the photon number is much graeter than the number of charged particles at sufficient high temperatures collisions change the energy of individual photon. Let us now perform estimate when photons stop exchange energies with electrons at this era.

It is well known that the rate of the interaction of individual photons with electron is given by number

$$\Lambda_\gamma = \sigma_T n_e c, \quad (514)$$

where  $n_e$  is number density of electrons,  $\sigma_T = 0.666 \times 10^{-24} \text{cm}^2$  is the cross section of Thomson scattering that is elastic scattering of photons by non-relativistic electrons.

The matter content at this era was following. 76% of matter is ionized hydrogen and the rest was helium that were ionized at temperatures above 20000K. Since hydrogen and helium were ionized we have one electron for nucleon for hydroen and half an electron per nucleon for helium. Then the net number of electrons per nucleon is

$$0.76 + \frac{1}{2} 0.24 = 0.88 \quad (515)$$

Then the number density of electrons at temperature  $T$  is

$$n_e = 0.88 n_B = 0.88 n_{B0} \frac{T^3}{T_{\gamma 0}^3} \quad (516)$$



Then

$$\Lambda_\gamma = 0.88 n_{B0} \frac{T^3}{T_{\gamma 0}^3} \sigma_{TC} = 1.97 \times 10^{-19} s^{-1} \times \Omega_B^2 \frac{T^3}{T_{\gamma 0}^3} . \quad (517)$$

where we used the fact that

$$n_{B0} = \frac{3\Omega_B H_0^2}{8\pi G m_N} = 1.123 \times 10^{-5} \Omega_B h^2 \text{ nucleons}/\text{cm}^3 . \quad (518)$$

In other words  $\Lambda_\gamma$  determines number of collision of one photons per electrons which, of course, does not determine energy transfer between matter and radiation. In fact, let us consider a photon with energy much less than  $m_e c^2$ . This photon will transfer a momentum to the non-relativistic electron of order of its own momentum which is typically about  $kT$  that implies that electron gains energy of order  $(kT)^2/m_e$ . Then the rate for energy transfer of order  $kT$  between photon and electrons is equal to the rate of collisions given above multiplied by the fraction of the energy  $kT$  transferred per collision

$$\Gamma_\gamma = \left( \frac{kT}{m_e} \right) \Lambda_\gamma \approx 9.0 \times 10^{-29} s^{-1} \Omega_B h^2 \left( \frac{T}{T_{\gamma 0}} \right)^4 \quad (519)$$

As the next step we have to compare this with the cosmic expansion rate that is determined by  $H$ . If we presume that at this time the universe was dominated by photons and neutrinos so that the cosmic expansion rate is equal to

$$H = \frac{\dot{a}}{a} = H_0 \sqrt{\Omega_R \frac{T^4}{T_{\gamma 0}^4}} = 2.1 \times 10^{-20} s^{-1} \left( \frac{T}{T_{\gamma 0}} \right)^2 \quad (520)$$

Then  $\Gamma_\gamma$  is greater than  $H$  when

$$\begin{aligned} 9.0 \times 10^{-29} s^{-1} \Omega_B h^2 \frac{T^4}{T_{\gamma 0}^4} &> 2.1 \times 10^{-20} s^{-1} \frac{T^2}{T_{\gamma 0}^2} \Rightarrow \\ \frac{T^2}{T_{\gamma 0}^2} &> 0.23 \times 10^9 \frac{1}{\Omega_B h} \end{aligned} \quad (521)$$

so that  $\Gamma_\gamma$  is greater than  $H$  until temperature dropped below temperature  $T_{freeze}$  given by expression above equal to

$$T_{freeze} = 1.5 \times 10^4 K (\Omega_B h^2)^{-1/2} \quad (522)$$

that, for  $\Omega_B h^2 \sim 0.02$  is equal to  $10^5 K$ .

Then, when the temperature is lower than  $10^5$  photons do not exchange energy with electrons but  $\Lambda_\gamma$  is still  $10^5$  greater than  $H$ . In other words  $\Lambda_\gamma$  is larger than  $H$  until temperature drops to much lower temperatures. To see this note that for  $3K \ll T \ll 10^4 K$  the universe was matter dominated where  $H$  depends on matter as

$$H = H_0 \sqrt{\Omega_M T^3 / T_{\gamma 0}^3} = 3.3 \times 10^{-18} s^{-1} \sqrt{\Omega_M h^2} \left( \frac{T}{T_{\gamma 0}} \right)^{3/2} . \quad (523)$$

This is equal to  $\Lambda_\gamma$  for temperatures

$$T \approx 18K(\Omega_M h^2)^{1/3}/(\Omega_B h^2)^{2/3} \quad (524)$$

This is equal to  $T = 130K$  for  $\Omega_M h^2 = 0.15$  and  $\Omega_B h^2 = 0.02$ . Then when temperature will be low enough it happens that electrons and nuclei will be bounded together to form neutral atoms. As a result the scattering rate  $\Lambda_\gamma$  drops rapidly which happens at temperature about  $3000K$ . This happens at time when era of rapid scattering of photons and electrons ends. This is known also as the time of last scattering.

## 4.9 Recombination and last scattering

We argued that for temperatures below  $T < 10^4 K$  photons do not exchange energy with electrons. Then they still continue scattering with free electrons but without gain or loss energy. This situation stops when electrons will be bound into hydrogen and helium. This is called as *recombination*.

Let us consider situation at time when protons, electrons and hydrogen and helium atoms are in thermal equilibrium at the temperature of radiation. In such a situation the number density of non-relativistic particles of type  $-i$  at temperature  $T$  is given by Maxwell-Boltzman formula

$$n_i = g_i(2\pi\hbar)^{-3} e^{\mu_i/kT} \int d^3q \exp \left[ - \left( m_i + \frac{q^2}{2m_i} \right) /kT \right], \quad (525)$$

where  $m_i$  is the particle mass  $g_i$  is the number of spin states and  $\mu_i$  is the chemical potential of particle of type  $i$ . Main property of chemical potential is that it is conserved in any reaction that occurs in the gas. We consider situation when  $i$  correspond protons- $p$ , electrons- $e$  and hydrogen atoms in any bound states that we denote as  $1s, 2s, 2p, \dots$

We should also say few words about helium that was about 24% of the mass of the early universe. It is important to stress that helium atoms are more bounded than hydrogen atoms so that for  $T < 4.400K$  they are neutral atoms. As a result they are not important for what we discuss here.

Further, electron and protons have spin 1/2 so that they are two states for them and hence  $g_p = g_e = 2$ . On the other hand the  $1s$  ground state of hydrogen atoms have two states with spins 0 and 1 so that they have  $g_{1s} = 1 + 3 = 4$ .

Let us firstly consider recombination and ionization reaction



so that chemical potential obeys following law

$$\mu_p + \mu_e = \mu_{1s} . \quad (527)$$

On the other hand photons can be freely created and destroyed in reactions



so that their chemical potential vanishes. Then using the fact that

$$(2\pi\hbar)^{-3} = \int d^3p \exp\left(-\frac{p^2}{2mkT}\right) = \left(\frac{mkT}{2\pi\hbar}\right)^{3/2} \quad (529)$$

so that the number of particles  $n_i$  is equal to

$$\begin{aligned} n_i &= g_i e^{\mu_i/kT - m_i/kT} (2\pi\hbar)^{-3} \int dq^2 \exp\left(-\frac{q^2}{2m_i kT}\right) = \\ &= g_i e^{\mu_i/kT - m_i/kT} \left(\frac{m_i kT}{2\pi\hbar}\right)^{3/2} \end{aligned} \quad (530)$$

so that

$$\begin{aligned} \frac{n_{1s}}{n_p n_e} &= \frac{4e^{\mu_{1s}/kT - m_{1s}/kT} \left(\frac{m_{1s} kT}{2\pi\hbar}\right)^{3/2}}{2e^{\mu_p/kT - m_p/kT} \left(\frac{m_p kT}{2\pi\hbar}\right)^{3/2} 2e^{\mu_e/kT - m_e/kT} \left(\frac{m_e kT}{2\pi\hbar}\right)^{3/2}} = \\ &= \left(\frac{m_e kT}{2\pi\hbar}\right)^{-3/2} e^{2(\mu_{1s} - \mu_p - \mu_e)/kT} \exp(-(m_{1s} - m_p - m_e)/kT) = \\ &= \left(\frac{m_e kT}{2\pi\hbar}\right)^{-3/2} \exp(B_1/kT), \end{aligned} \quad (531)$$

where we ignored difference between mass of hydrogen atom and mass of proton except of exponential so that  $m_{1s} = m_p$ . We further used the conservation of the chemical potential and introduced the binding energy  $B_1$  of the  $1s$  ground state of hydrogen defined as

$$B_1 = m_p + m_e - m_H = 13.6 \text{ eV} \quad (532)$$

Further, since the cosmic matter is charge neutral we have

$$n_e = n_p. \quad (533)$$

Let us now say few words about excited states. If we have equilibrium then the number of excited states differs from the number density by a factor  $\exp(-\Delta E/kT)$  where  $\Delta E$  is the excitation energy and for temperatures below  $4200K$  this exponential factor is less than  $6 \times 10^{-13}$  so that we can neglect the presence of excited hydrogen atoms in case the thermal equilibrium is maintained.

Further, at the time of recombination the amount of neutral or ionized hydrogen, which are  $1s$  and  $p$  states respectively, correspond to 76% of matter. In other words,

$$n_p + n_{1s} = 0.76n_B \quad (534)$$

where  $n_B$  is the number of density of baryons which are, at these temperatures neutrons and protons. Let us introduce quantity  $X$  defined as

$$X = \frac{n_p}{n_p + n_{1s}}. \quad (535)$$

This is known as fractional hydrogen ionization. This quantity satisfies *Saha equation*

$$X(1 + SX) = 1 . \quad (536)$$

where

$$S = \frac{(n_p + n_{1s})n_{1s}}{n_p^2} . \quad (537)$$

In fact, by definition we have

$$X(1 + XS) = \frac{n_p}{n_p + n_{1s}} \left( 1 + \frac{(n_p + n_{1s})n_{1s}}{n_p^2} \frac{n_p}{n_p + n_{1s}} \right) = 1 \quad (538)$$

On the other hand from using (531) we find that  $S$  is equal to

$$S = \frac{(n_p + n_{1s})n_{1s}}{n_p^2} = 0.76n_B \frac{n_{1s}}{n_p^2} = 0.76n_B \left( \frac{m_e kT}{2\pi\hbar} \right)^{-3/2} \exp(B_1/kT) , \quad (539)$$

where we used the neutrality of universe  $n_p = n_e$  in (531). For  $n_B = n_{B0} \left( \frac{T}{T_{\gamma 0}} \right)^3$  and for  $n_{B0} = \frac{3\Omega_B H_0^2}{8\pi G m_N} = 1.123 \times 10^{-5} \Omega_B h^2$  nucleons/cm<sup>3</sup> we obtain

$$S = 1/1747 \times 10^{-22} e^{157894/T} T^{3/2} \Omega_B h^2 . \quad (540)$$

This is very steep function where for example the equilibrium value of the ionization dropped from 97% for  $T = 4.200K$  to less than 1% for  $T = 3000K$ .

This description gives correct temperature of the fractional ionization however detailed treatment of this process is very complicated and we recommend [1] for more details.

## 4.10 Anisotropy in CMB

It is also well known that CMB is not a perfectly isotropic radiation bath. Deviations from isotropy at the level of one part in 10<sup>5</sup> have developed over the last decade into one of our most precise observation tool in cosmology. The small temperature anisotropies on the sky are usually analyzed by decomposing the signal into spherical harmonics via

$$\frac{\Delta T}{T} = \sum_{l,m} a_{lm} Y_{lm}(\phi, \theta) , \quad (541)$$

where  $a_{lm}$  are expansion coefficients and  $\theta$  and  $\phi$  are spherical polar angles on the sky. Next we define the power spectrum as

$$C_l = \langle |a_{lm}|^2 \rangle . \quad (542)$$

The fluctuations in the CMB spectrum are useful for the study of cosmology from many reasons. To understand why, we should show at the first place why they arise. Matter today in the Universe exists in the form of clusters of stars, galaxies, and clusters and super-clusters of galaxies. Our understanding how large scale structures developed is that initially small density perturbations in the otherwise homogeneous Universe grew through the gravitational instability to the objects we observe today. Such picture requires that from place to place there were small variations in the density of matter at the time when CMB firstly decoupled from the photon-baryon plasma. Then CMB photons propagated freely through the Universe nearly unaffected by anything except the cosmic expanding itself. However at the time of their decoupling different photons were released from regions of space with slightly different gravitational potentials. Since the gravitational potential affects the photon redshift, photons from some regions redshift slightly more than those from other regions, giving rise to a small temperature anisotropy in the CMB observed today. In this sense CMB reflects the initial conditions that ultimately gave rise to structure in the Universe.

It is important that CMB fluctuations give us the value of  $\Omega_{total}$ . In fact, careful analysis of all of the features of the CMB power spectrum provide constraints on essentially all of the cosmological parameters. For example, let us consider recent result from *WMAP*. For total density of the Universe they find

$$0.98 \leq \Omega_{total} \leq 1.08 \quad (543)$$

at 0.95 confidence which is a strong evidence for a flat Universe. Nevertheless, much tighter constraints on the remaining values can be derived by assuming either an exactly a flat Universe or a reasonable value of Hubble constant. When for example we presume a flat Universe, we can derive values for the Hubble constant, matter density (which then implies the vacuum density from  $\Omega_{total} = 1$ ) and baryon density:

$$\begin{aligned} h &= 0.72 \pm 0.05, \\ \Omega_M &= 1 - \Omega_\Lambda = 0.29 \pm 0.07, \\ \Omega_B &= 0.047 \pm 0.006. \end{aligned} \quad (544)$$

If we instead assume that the Hubble constant is given by the value determined by HST project

$$H_0 = 100 h \text{ km sec}^{-1} \text{ Mpc}^{-1}, \quad h = 0.71 \pm 0.06 \quad (545)$$

we can derive separate tight constraints on  $\Omega_M$  and  $\Omega_\Lambda$ .

In summary, taking all of the data together we obtain a remarkably consistent picture of the current constituents of our Universe:

$$\begin{aligned} \Omega_B &= 0.04, \\ \Omega_{DM} &= 0.26, \\ \Omega_\Lambda &= 0.7. \end{aligned} \quad (546)$$

There are many mysterious things considering these values. Firstly, the baryon density is mysterious due to the asymmetry between baryons and antibaryons. Secondly, the problem with dark matter is that we have never detected it directly and only have promising ideas as to what it might be. However the biggest mystery is the vacuum energy, we now try to explain why it is mysterious and what kinds of mechanism might be responsible for its value.

## 5 Early Times in the Standard Cosmology

Early times at the in the Standard Cosmology are characterized by very high temperatures and densities with many particle species kept in (approximate) thermal equilibrium by rapid interactions. Our goal is then to develop some tools of the thermodynamics in expanding Universe. In fact, up the mid-1960 it was not clear whether the early Universe had been hot or cold. This situation changed with the Penzias and Wilson's 1964-1965 discovery of 2.7 K microwave background radiation arriving from the farthest reaches of the Universe since the existence of the microwave background has been predicted by the hot Universe theory.

### 5.1 Review of the building blocks of the standard cosmology and matter

For reader's convenience we review some basics facts considering the standard models of cosmology.

- **The Classical general relativity:**

The classical general relativity provides good description of the geometry of space-time for scales  $l \gg l_P = M_P^{-1} = 10^{-33} \text{cm}$  or equivalently for energy scales below the Planck scale  $M_P$ .

- Physical scales are stretched by the scale factor  $a(t)$  with respect to the comoving scales

$$l_{phys}(t) = a(t)l_{com} . \quad (547)$$

A physical wavelength redshifts proportional to the scale factor where its time derivative obeys the Hubble law

$$\frac{dl_{phys}(t)}{dt} = \frac{\dot{a}}{a}al_{com} = H(t)l_{phys}(t) = \frac{l_{phys}}{d_H(t)} . \quad (548)$$

- The equilibrium temperature decreases as the Universe expands as

$$T(t) = \frac{T_0}{a(t)} . \quad (549)$$

- **The Standard Model of Particle Physics:**

The current standard model of particle physics that is experimentally tested with remarkable precision describes the theory of strong (QCD), weak and electroweak interactions (EW) as a gauge theory based on the gauge group

$$SU(3)_c \otimes SU(2) \otimes U(1)_Y . \quad (550)$$

The particle content is: three generations of quarks and leptons:

$$\begin{pmatrix} u \\ d \end{pmatrix} \begin{pmatrix} c \\ s \end{pmatrix} \begin{pmatrix} t \\ b \end{pmatrix} ; \begin{pmatrix} \nu_e \\ e \end{pmatrix} \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix} \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix} \quad (551)$$

vector Bosons: 8 gluons (massless) that mediate the strong interactions in QCD,  $Z^0, W^\pm$  that are massive with masses  $M_Z = 91.18 \pm 0.02 \text{ GeV}$  and  $M_W = 80.4 \pm 0.06 \text{ GeV}$  that mediate the electroweak interactions, the photon (massless)-the mediator of electromagnetic interaction and the scalar Higgs that was discovered in 2011 at LHC with the mass  $M_H = 125.09 \text{ GeV}$ .

- It is known that the couplings associated with strong, weak and electrodynamics interactions depend on the mass scale that characterize given process. The current theoretical ideas propose that these couplings are unified in a grand unified theory (GUT) at the scale

$$M_{GUT} \sim 10^{16} \text{ GeV} .$$

Further, the UV scale where the Gravity is eventually unified with the rest of particle physics is the Planck scale

$$M_P \sim 10^{19} \text{ GeV} .$$

On the other hand the physics of the Standard Model describes phenomena at energy scales below  $M_S$  where

$$M_S \sim 100 \text{ GeV} .$$

- The connection between the Standard model of particle physics and early Universe cosmology is through Einstein's equations that couple the space-time geometry to the matter-energy content. We study gravity semi-classically at energy scales well below the Planck scale. The Standard model of particle physics is a **quantum field theory** (QFT) thus the space-time is classical but with sources that are quantum fields. Semi classical gravity is defined by the Einstein equations with the expectation value of the energy-momentum tensor  $\hat{T}^{\mu\nu}$  as sources

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = \frac{\langle \hat{T}^{\mu\nu} \rangle}{M_P^2} , \quad (552)$$

where the expectation value  $\langle \hat{T}^{\mu\nu} \rangle$  is taken in given quantum state or density matrix that is compatible with homogeneity and isotropy so that it has to be

translational and rotational invariant. The ground state of the quantum field theory is usually the state that solves the classical equations of motion or the equations of motion with the quantum correction. In this case the vacuum expectation value of the stress energy tensor corresponds to the classical one. The general formula above has important in case we study the properties of the fluctuations above given classical solutions.

As the next step we review basic facts about the Energy scales, time scales and phase transitions

### Energy scales,time scales and phase transitions

In this section we give a brief overview of the main cosmological epochs by focusing on the energy scales of particle, nuclear and atomic physics.

#### Energy scales:

- **Total Unification**

It is expected that Gravitational, strong and electroweak interactions become unified and described by a single quantum theory at the Planck scale  $M_P \sim 10^{19} \text{ GeV}$ . The most promising approach to this unification is in terms of string theory however their theoretical consistency is still studied and experimental confirmation is not available.

- **Grand Unification:**

Strong and electroweak interactions are expected to become unified at an energy scale

$$M_{GUT} \sim 10^{16} \text{ GeV} , T_{GUT} \sim 10^{29} K$$

under large gauge group  $G$ , for example  $SU(5), SO(10)$  that breaks spontaneously

$$G \rightarrow SU(3)_c \otimes SU(2) \otimes U(1)_Y$$

at scale below unification. Main arguments for the existence of GUT theories follow from merging of the running coupling constants of the strong, electromagnetic and weak interactions for the minimal supersymmetric model and also the explanation of the small neutrino masses via see-saw mechanism.

- **Electroweak:**

Weak and electromagnetic interactions are unified in the electroweak theory based on the gauge group

$$SU(2) \otimes U(1)_Y .$$

The weak interactions become short ranged after symmetry breaking phase transition

$$SU(2) \otimes U(1)_Y \rightarrow U(1)_{em}$$



at the energy scale of the order of the mass of the  $Z^0, W^\pm$  vector bosons corresponding to temperature

$$T_{EW} \sim 100 \text{ GeV} \sim 10^{15} \text{ K} .$$

More precisely, at temperature  $T > T_{EW}$  the symmetry is restored as a consequence of the fact that the effective potential of the theory depends on the temperature as well. For temperature  $T > T_{EW}$  the stable minimum of the potential corresponds to the symmetric phase where all vector bosons are massless and hence the symmetry is restored. On the other hand for  $T < T_{EW}$  the stable minimum of the potential corresponds to the situation when the vector bosons  $W^\pm, Z^0$  become massive through Higgs mechanism while photon remains massless corresponding unbroken  $U(1)$  abelian symmetry of quantum electrodynamics. The temperature  $T_{EW}$  determines the temperature scale of the electroweak phase transition in the early Universe.

- **QCD**

The strong interaction has a typical energy scale

$$\Lambda_{QCD} \sim 200 \text{ MeV} .$$

At this coupling the coupling constant becomes strong  $\alpha_s \sim O(1)$  that corresponds to the temperature scale

$$T_{QCD} \sim 10^{12} \text{ K}$$

QCD is asymptotically free theory that means that the coupling between quarks and gluons becomes smaller at large energies but diverges at the scale  $\Lambda_{QCD}$ . For energies below  $\Lambda_{QCD}$  the quantum chromodynamics is strongly interacting theory and quarks and gluons are bound into mesons and baryons. This phenomenon is interpreted in terms of a phase transition at an energy scale  $\Lambda_{QCD}$  or  $T_{QCD}$ . For  $T > T_{QCD}$  the relevant degrees of freedom are weakly interacting quarks and gluons, while below are hadrons. In the limit when we can presume that up and down quarks are massless,  $QCD$  possesses new  $SU(2)_L \otimes SU(2)_R$  chiral symmetry that is spontaneously broken at about the same temperature scale as the scale of QCD transition. Pions are the Goldstone bosons that emerge in the breakdown of the chiral symmetry

$$SU(2)_L \otimes SU(2)_R \rightarrow SU(2)_{R+L} .$$

The high temperature phase above  $T_{QCD}$  where the quarks and gluons are almost free (because the coupling is small by asymptotic freedom) is a *quark-gluon plasma*.

- **Nuclear Physics**

The low energy scales that are relevant in cosmology are determined by the binding energy of light elements. For example, the binding energy of deuterium

is  $\sim 2 \text{ MeV}$  that corresponds to the temperature  $T \sim 10^{10} \text{ K}$ . This is the energy scale that determines the origin of the primordial nucleosynthesis. The first step in the system of the nuclear reactions that yields the primordial elements is the formation of deuteron in the reaction



These nuclear reactions continue and all neutrons end up in nuclei, mainly helium.

- **Atomic physics**

A further important low energy scale relevant for cosmology is the binding energy of hydrogen  $\sim 10 \text{ eV}$ . This is the energy scale at which free protons and electrons combine into neutral hydrogen. The large number of photons per baryons implies that recombination actually takes place at an energy scale of order  $0.3 \text{ eV}$ , at about 400000 years after the beginning of the Universe. At this time when the neutral hydrogen is formed the Universe becomes transparent since then photons no longer scatter and travel freely. These are the photons measured by CMB experiments today.

**Time Scales:**

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- **Inflation epoch**

This is (according to current cosmological scenario) the earliest period in the life of Universe where the scale factor grows exponentially as

$$a(t) = e^{Ht} .$$

Current experiments put upper bound on the energy scale of inflation as

$$H \leq 10^{13} \text{ GeV} .$$

In order to solve the entropy and horizon problems the inflationary stage has to last a time interval  $\delta t$  so that

$$\delta t H \sim 60 \Rightarrow \delta t \sim 10^{-34} \text{ sec} .$$

- **Radiation dominated era**

The inflationary stage is followed by a radiation dominated era after a short period of reheating during which the energy stored in the field that drives inflation decays into quanta of many other fields. These fields reach the state of thermal equilibrium through the scattering processes.

After the thermal equilibrium is reached we obtain a detailed picture of the thermal history of the Universe. This description is based on the combination

of the statistical mechanics with the basic principles of QFT: During the first 1000 years of the Universe and after the inflation stage that lasted  $\sim 10^{-34}$  sec the Universe was radiation dominated. Universe also expands and cools almost adiabatically. The electroweak transition occurred at the energy scale  $T \sim 100 \text{ GeV}$  that corresponds to the time

$$t_{EW} \sim 10^{-12} \text{ sec} .$$

The QCD transition occurs at

$$t_{QCD} \sim 10^{-5} \text{ sec} .$$

### Local Thermal Equilibrium (LTE) and Non equilibrium

Whether some process occurs in or out of a local thermodynamics equilibrium depends on the comparison of two time scales-the expanding rate and the reaction rate. To have a contact with standard thermodynamics note that we can formulate the same problem as the problem of comparing of the cooling rate (the rate how temperature decreases) and the rate of reaction. In fact the rate of cooling is related to the rate of the expanding through the formula

$$\frac{\dot{T}}{T} = -\frac{1}{Ta^2}\dot{a} = -H(t) \quad (553)$$

as follows from the fact that  $T \sim \frac{1}{a}$ . On the other hand collisions as well as non-collisional processes contribute to establish the equilibrium with a rate  $\Gamma$ . The local thermodynamic equilibrium is established when

$$\Gamma > H(t) \quad (554)$$

In this case the evolution is adiabatic in the sense that the thermodynamics functions depend slowly on time through the temperature. On the other hand when the expanding is too fast

$$H(t) \gg \Gamma$$

local thermodynamics equilibrium cannot be established, the temperature drops too fast for the system to have time to relax.

While a detailed understanding of the relaxation dynamics requires an analysis of the quantum Boltzmann equations a simple order of magnitude estimate for a collision rate is given as follows.

The collision rate can be calculated in the standard statistical physics as

$$\Gamma \sim \langle \sigma n v \rangle , \quad (555)$$

where  $\langle \dots \rangle$  means statistical ensemble average and where  $\sigma$  is a scattering cross section,  $n$  is the density of particles that scatter and  $v$  is velocity of given particles. For electromagnetic scattering a typical cross section is of order

$$\sigma_{em} \sim \frac{\alpha^2}{Q^2} ,$$

where  $Q^2$  is transferred momentum and  $\alpha$  is the electromagnetic coupling constant. At high temperature single photon exchange implies the estimate (the transferred momentum is proportional to the momenta of one photon that is proportional to the temperature)

$$\sigma_{em} \sim \frac{\alpha}{T^2} .$$

The density of relativistic degrees of freedom is  $n \sim T^3$  and for  $v \sim 1$  (This estimate follows from the fact that particles are ultra-relativistic) we obtain

$$\Gamma_{em} \sim \alpha^2 T .$$

In QCD that in the high temperature regime can be treated perturbatively the estimate of the single gluon exchange can be performed in the similar way and we get

$$\Gamma_{QCD} \sim \alpha_s^2 T ,$$

where  $\alpha_s$  is corresponding coupling constant. We have to compare these estimates with  $H$ . However  $H^2 \sim \rho$  that in the case of the radiation dominated era we show that  $\rho \sim T^4$ . Then in this case we find that  $H \sim T^2/M_{pl}$  so that

$$\frac{\Gamma_{QCD}}{H} = \frac{\alpha_s^2 M_{pl}}{T} > 1 \quad (556)$$

and we obtain that the strong interactions are in LTE for

$$T \leq 10^{16} \text{ GeV}$$

In the same way we obtain that electromagnetic interactions are in LTE for

$$T \leq 10^{14} \text{ GeV} .$$

It is important to stress for  $T \leq \alpha^2 M_{pl} \sim 10^{16} \text{ GeV}$  all perturbative interactions should be frozen out and are not effective in maintaining thermal equilibrium. In other words all known interactions together with any new interactions that arise from grand unification are not sufficient for maintaining the thermal equilibrium in the Universe at temperatures greater than  $10^{16} \text{ GeV}$  that corresponds to the time earlier than  $10^{-38} \text{ s}$ . In other words Universe is not in thermal equilibrium at its earliest epoch.

The estimate in case of weak interaction is slightly more involved: a typical scattering process with an energy transfer  $E \ll M_W$  has a scattering cross section

$$\sigma \sim G_F^2 E^2 , \quad E \ll M_W$$

whereas if  $E \gg M_W$  we have

$$\sigma \sim \frac{g^4}{E^2} , \quad E \gg M_W .$$

Then in thermal medium with  $E \sim T$  and with a density of relativistic particles  $n \sim T^3$  a typical weak reaction rate is

$$\Gamma_{EW} \sim g^4 T, \quad T \gg M_W$$

and

$$\Gamma_{EW} \sim G_F^2 T^5$$

for  $T \ll M_W$ . In this latter temperature regime the ratio

$$\frac{\Gamma_E}{H} \sim \left( \frac{T}{MeV} \right)^3$$

and hence the weak interactions fall out of LTE for  $T \leq 1 MeV$ .

Even if this analysis provides an intuitive estimate for the relaxation time scales this analysis neglected several important aspects that however have to be studied on a case-by-case basis. One such an example of subtle effects are *Screening and infrared phenomena*: The relaxation rates  $\Gamma$  were calculated on presumption of an exchange of a vector boson of relativistic degrees of freedom. In a medium at a high temperature and a density there are important screening effects that can change these estimates.

## 5.2 Hot Big Bang

We begin this section with the description of the evolution of the Universe in its hot stage.

The basic presumption is that it is plausible to extrapolate the evolution of the Universe back in time using the known microscopic physics (electrodynamics, nuclear physics, QCD and electroweak theory) and General Relativity. This theory is called as **Hot Big Bang Theory**. According to this theory the Universe was hotter at earlier stages (equivalently, at smaller values of  $a(t)$ ) and the temperature scales as  $a(t)^{-3}$  both for non-relativistic and relativistic particles. At high enough temperatures the Universe was in the phase that is completely different from what we observe today. Instead of the almost empty space with galaxies here and there was dense, hot and almost homogeneous plasma that fills the whole Universe. This is the area whose physical laws are governed by microscopic physics. Note that gravity plays the role of the spectators of the theory and it is considered as classical. Of course we consider back-reaction of this matter on the time evolution of the Universe using the Friedmann equations.

More precisely, the hot Universe theory is based on the phenomena of the phase transitions and the symmetry breaking. Let us consider for example the simplest GUT model based on the gauge group  $SU(5)$ . For temperature  $T \geq 10^{15} GeV$  there was no difference between weak, strong and electroweak interactions. The matter in the Universe was in the form of the dense plasma containing quarks, photons, gluons etc. Then there was no problem in the transformation of quarks to leptons. In other

words it does not make sense to speak about baryon conservation. At  $t_1 \sim 10^{-35} \text{ sec}$  when the temperature has dropped to  $T \sim T_{c_1} \sim 10^{14} - 10^{15} \text{ GeV}$  the first symmetry breaking phase transition takes place:  $SU(5)$  breaks to  $SU(3) \times SU(2) \times U(1)$  where  $SU(3)$  is gauge symmetry of the QCD, theory of the strong interactions. In other words string interactions were separated from electroweak and leptons. Then at  $t_2 \sim 10^{-10} \text{ sec}$  when the temperature dropped to  $T_{c_2} \sim 10^2 \text{ GeV}$  there was a second phase transition that broke the symmetry between weak and electromagnetic interactions  $SU(3) \times SU(2) \times U(1) \rightarrow SU(3) \times U(1)$ . As the temperature reduces further to  $T_{c_3} \sim 10^2 \text{ MeV}$  there was another phase transition with the formation of baryons and mesons from quarks.

### 5.3 Review of the study of the expansion of the Universe

Let us again analyze the evolution of the Universe. As we have argued before at early times the Universe was radiation dominated, then matter dominated and presently dark energy dominated while the curvature term  $\frac{k}{a^2}$  was never important.

#### Deceleration to Acceleration

Since the dark energy dominates at present the Universe accelerates. On the other hand when matter was dominating the Universe was decelerating. In order to see when the change in regime occurred we write the Friedmann equations as

$$\dot{a}^2 = \frac{8\pi G}{3} \rho a^2 = \frac{8\pi G}{3} a^2 (\rho_M + \rho_\Lambda) \quad , \quad (557)$$

where we have neglected spatial curvature and also ultra-relativistic matter for the moment. The reason for this simplification is that the relativistic matter dominates an expanding of the Universe at much earlier stage. The time derivative of the equation above implies

$$\begin{aligned} 2\dot{a}\ddot{a} &= \frac{8\pi G}{3} (\dot{\rho}_M a^2 + 2(\rho_M + \rho_\Lambda)\dot{a}a) = \\ &= \frac{8\pi G a}{3} (-\dot{a}a\rho_M + 2\dot{a}a\rho_\Lambda) \end{aligned} \quad (558)$$

where we used  $\dot{\rho}_M = -3\frac{\dot{a}}{a}\rho_M$ . The expression above is zero when (This event defines the turning point between decelerating and accelerating phase)

$$\frac{2\rho_\Lambda}{\rho_M} = 1 \quad (559)$$

or equivalently

$$\frac{a_0^3}{a^3} \equiv (1+z)^3 = \frac{2\Omega_\Lambda}{\Omega_M} \quad , \quad (560)$$

where of course  $\Omega_M$  is time-dependent. For expected values  $\Omega_\Lambda = 0.7, \Omega_M = 0.3$  we have

deceleration  $\rightarrow$  acceleration:  $z \approx 0.7$

In other words, the Universe was decelerating until fairly recently. Before  $z \approx 0.7$  the expansion was dominated by the non-relativistic matter.

## Radiation domination to matter domination

As we know the energy density of ultra-relativistic matter (radiation) scales as  $a^{-4}$  while the energy density of non-relativistic matter scales as  $a^{-3}$ . Then it follows that the dominant contribution to the energy density of the Universe at very small  $a$  (small  $t$ ) came from ultra-relativistic matter. Now we estimate  $z_{eq}$  at which the equilibrium between matter and radiation occurred. In other words we would like estimate  $z_{eq}$  when the expansion regime changed from the dominance of ultra-relativistic particles to the dominance of non-relativistic matter, we write

$$\frac{\rho_M(t)}{\rho_{rad}(t)} = \frac{\rho_{M0} a_0^3 a^{-3}(t)}{\rho_{rad0} a_0^4 a^{-4}(t)} = \left( \frac{\rho_M}{\rho_{rad}} \right)_0 \frac{a(t)}{a_0}, \quad (561)$$

where again the subscript 0 refers to present values. Equilibrium occurs at

$$\frac{\rho_M(t_{eq})}{\rho_{rad}(t_{eq})} \approx 1 \quad (562)$$

that gives

$$\frac{a_0}{a(t_{eq})} \equiv 1 + z_{eq} \approx \left( \frac{\rho_M}{\rho_{rad}} \right)_0 = \frac{\Omega_M}{\Omega_{rad}}. \quad (563)$$

Since  $\Omega_{rad} \approx 10^{-4}$ ,  $\Omega_M \approx 0.3$  we obtain

radiation domination  $\rightarrow$  matter domination :  $z_{eq} \approx 3000$  .

The corresponding temperature is

$$T_{eq} = T_0(1 + z_{eq}) \approx 10^4 K \approx 1eV. \quad (564)$$

At higher temperatures the expansion of the Universe was dominated by ultra-relativistic matter. We must stress that it is important for structure formation that the most of the part of the lifetime of the Universe is dominated by non-relativistic matter. This follows from the fact that the expanding rate at both radiation dominated and vacuum dominated eras is such that gravitational perturbations grow slowly and only during the matter dominated stage their growth is fast enough so that the existing structures of the Universe can arise.

## 5.4 Epochs of the early Universe

There are two important epochs in the evolution of the Universe: **Recombination epoch** that is the transition from plasma to neutral gas. This occurs at temperature

$T \sim 3000K, t \sim 3 \cdot 10^5 \text{years}$  and **nucleosynthesis epoch** that occurs at temperatures  $T = 1MeV$  to a few  $\cdot 10keV$ . Another event is **neutrino decoupling**. Briefly, at high temperatures the neutrino was in thermal equilibrium with the rest of cosmic plasma. The plasma became transparent for neutrinos at temperature about  $1MeV$ . This decoupling of neutrinos is very important for nucleosynthesis since it affects the neutron-proton ratio just before nucleosynthesis (Since neutrinos decouples the reaction that transfers proton into neutrons simply cannot occur) and hence it leads to the abundances of light elements that need neutrinos for their formations. Further, the fact that neutrinos decoupled much earlier than photons implies that the present neutrino-to-photon ration is less than one. This is consequence of the fact that photons are additionally heated, after neutrino decoupling, due to the annihilations of  $e^+$  with  $e^-$ .

If we move further back in time we obtain that the cosmic plasma has more and more components. At temperatures roughly  $0.5MeV$  there are many electrons and positrons that are frequently pair created and annihilate: at  $T > 100MeV$  the plasma contains muons and pions. This plasma remains in thermal equilibrium except possibly for **phase transitions**

- **QCD phase transition**

At temperatures above  $100MeV$  (QCD scale) strongly interacting particles are dissolved into quarks and gluons. This quark-gluon plasma converts into hadronic matter (mostly pions) during the quark-hadron phase transitions. Theoretical estimates suggest that the temperature of this phase transition is about  $170MeV$ .

- **Electroweak transition**

Briefly, at temperatures well above  $100GeV$  electroweak symmetry is unbroken. The consequence of this fact is that  $W$  and  $Z$  bosons are massless. At  $T \sim 100GeV$  the phase transition of the electroweak symmetry breaking takes place.

- **GUT transition**

It is slightly uncertain when we extrapolate back further (equivalently, we go to higher temperatures), but if we do so we come to the Grand Unification epoch. The temperature of this epoch is set by GUT scale,  $T_{GUT} \sim 10^{16}GeV$ . We expect that at this temperature the Grand Unified phase transition occurs. On the other hand many models of inflation suggest that the Universe never had such a high temperature after inflation.

## Expansion rate and life-time at radiation domination

Now we will discuss in more details the expansion of the Universe in radiation dominated stage where we will presume thermal equilibrium of all ultra-relativistic species<sup>9</sup>. In the very early stages of its evolution was filled with an ultra-relativistic

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<sup>9</sup>This presumption is not however valid for neutrinos at temperatures below  $1MeV$ .



gas of photons, electrons, positrons, etc. At that time the excess of baryons over antibaryons small fraction (at most  $10^{-19}$ ) of the total number of particles. The matter could be considered as a gas of free particles where their rest masses are small compared to temperature. In other words the energy density and entropy density corresponds to the massless species

$$\rho = 3p = \frac{\pi^2}{30}g_*(T)T^4 \ , \quad s = \frac{2\pi^2}{45}g_*(T)T^3 \ . \quad (565)$$

where the effective number of particle species  $g_*(T)$  is  $g_*(T) = g_B(T) + \frac{7}{8}g_F(T)$  where  $g_B$  and  $g_F$  are the number of boson and fermions species degrees of freedom with masses  $m \ll T$ . For example, for photons  $g_B = 2$ ,  $g_F = 2$  for neutrinos and  $g_F = 4$  for electrons (Let us sketch the way how to derive the dependence of  $\rho$  on  $T$ . By definition

$$\rho = \int d^3k e(k) f\left(\frac{e}{T}\right)$$

where  $f(\frac{e}{T})$  is distribution functions and  $e(k)$  is an energy. For particles with  $m \ll T$  we can neglect their rest masses so that  $e = k$ . After substitution  $\frac{k}{T} = m$  we obtain  $\rho = T^4 \int d^3m e(m) f(m) \sim T^4$ .)

Generally  $g_*(T)$  increases with increasing  $T$  but rather slowly. This follows from the fact that at higher temperatures more species are ultra-relativistic (say, electrons contribute at  $T > 0.5 MeV$  and do not contribute at lower temperatures.)

Let us now list some time scales that are relevant for the early stage of the evolution of the Universe:

- **Nucleosynthesis**

The temperature relevant for nucleosynthesis ranges from a few  $MeV$  to about  $70 keV$ . This era begins at

$$t \sim 1 \text{ s} \ . \quad (566)$$

and ends at

$$t \sim 200 \text{ s} \sim 3 \text{ min} \ . \quad (567)$$

After this brief introduction we will discuss the properties of the early Universe in brief details.

## 5.5 Describing Matter

We try to describe matter a perfect fluid described by an energy-momentum tensor

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + pg_{\mu\nu} \ , \quad (568)$$

where  $U_\mu$  is the fluid four-velocity,  $\rho$  is the energy density at rest frame of the fluid and  $p$  is the pressure in that same frame. By definition the stress energy tensor is covariantly conserved

$$\nabla_\mu T^{\mu\nu} = 0 \ . \quad (569)$$

In more complicated examples a fluid will be characterized by quantities in addition to the energy and pressure. Many fluids have a conserved quantity associated with them and so we will also introduce a *number flux density*  $N^\mu$  which is also conserved

$$\nabla_\mu N^\mu = 0 . \quad (570)$$

For non-tachyonic matter  $N^\mu$  is a time-like 4-vector and therefore we can write

$$N^\mu = nU^\mu . \quad (571)$$

In the same way we can introduce an *entropy flux density*  $S^\mu$ . This quantity is not conserved but rather obeys a covariant version of the second law of thermodynamics

$$\nabla_\mu S^\mu \geq 0 . \quad (572)$$

It is useful to resolve  $S^\mu$  into components parallel and perpendicular to the fluid 4-velocity

$$S^\mu = sU^\mu + s^\mu , \quad (573)$$

where  $s_\mu U^\mu = 0$ . The scalar  $s$  is the rest-frame entropy density that can be written as

$$s = \frac{\rho + p}{T} . \quad (574)$$

We must also specify an equation of state. Typically we do this in such a way as to treat  $n$  and  $s$  as independent variables.

For adiabatic expanding Universe  $sa^3 \approx \text{const}$  eq. (565) implies

$$T(t) \sim \frac{1}{a(t)} . \quad (575)$$

We see that the temperature cools during the expansion of the Universe. The background radiation is a result of the cooling of the hot photon gas during the expansion of the Universe.

## 5.6 Particles in Equilibrium

The various particles inhabiting the early Universe can be characterized according to three criteria: in equilibrium vs. out of equilibrium (decoupled), bosonic vs fermionic and relativistic (velocities near 1) vs. non-relativistic. In this subsection we will consider species which are in equilibrium with surrounding thermal bath.

Now we must discuss the conditions under which a particle is in equilibrium with the surrounding thermal plasma. The particles will be in thermal equilibrium as long as its interaction rate is larger than the expansion rate of the Universe. In other words, particles have enough time to share the energy among themselves or equivalently, equilibrium requires that it should be possible for the products of a given reaction have the opportunity to recombine in the reverse reaction. If the expanding of the Universe is rapid enough this will not happen. A particle species

for which the interaction rates have fallen below the expanding rate of the Universe is said to have *frozen out or decoupled*. The interaction rate of some particle with the background plasma is  $\Gamma$  where  $\Gamma$  is inverse of the mean time between the reaction of given particle species with the thermal background. Now the particle will be decoupled from the thermal bath when the particle has not time enough to react with thermal bath if

$$\Gamma \ll H , \quad (576)$$

where the Hubble constant  $H$  sets the cosmological timescale.

At the early Universe the particles are in thermal equilibrium (unless they are very weakly coupled). This can be seen from Friedmann equation when the energy density is dominated by plasma with  $\rho \sim T^4$  and we have

$$H^2 \sim \rho \Rightarrow H \sim \sqrt{\rho} \sim \left( \frac{T}{M_P} \right) T \quad (577)$$

so that the Hubble parameter is suppressed with respect to the temperature by a factor of  $T/M_P$ . At extremely early times (near the Planck era) the Universe may be expanding so quickly so that no species are in equilibrium but as the expansion rate slows the equilibrium becomes possible.

At extremely early times near the Planck era, the Universe may be expanding so quickly that no species are in equilibrium; as the expansion rate slows, equilibrium becomes possible. On the other hand the interaction rate  $\Gamma$  for a particle with cross section  $\sigma$  is typically of the form

$$\Gamma = n \langle \sigma v \rangle , \quad (578)$$

where  $n$  is the number density and  $v$  is typical particle velocity. Since  $n \sim a^{-3}$  the density of particles will reduce so that the equilibrium can once again no longer be maintained. In our current Universe no species are in equilibrium with the background plasma (represented by CMB photons).

Now we review some facts about particles at equilibrium. For a gas of weakly-interacting particles we can describe the state in terms of a *distribution function*  $f(\mathbf{p})$  where the three momentum  $\mathbf{p}$  satisfies

$$E(\mathbf{p})^2 = m^2 + |\mathbf{p}|^2 . \quad (579)$$

The distribution function characterizes the density of particles of given momentum. The number density, energy density and pressure of some species labeled  $i$  are given by

$$\begin{aligned} n_i^{eq}(T) &= \frac{g_i}{(2\pi)^3} \int f_i(\mathbf{p}) d^3p = \frac{g_i}{2\pi^2} T^3 I_i^{11}(\mp) , \\ \rho_i^{eq}(T) &= \frac{g_i}{(2\pi)^3} \int E(\mathbf{p}) f_i(\mathbf{p}) d^3p = \frac{g_i}{2\pi^2} T^4 I_i^{21}(\mp) , \\ p_i^{eq}(T) &= \frac{g_i}{(2\pi)^3} \int \frac{|\mathbf{p}|^2}{3E(\mathbf{p})} f_i(\mathbf{p}) d^3p = \frac{g_i}{6\pi^2} T^4 I_i^{03}(\mp) , \end{aligned} \quad (580)$$

where

$$I_i^{mn}(\mp) = \int_{x_i}^{\infty} y^m (y^2 - x_i^2)^{n/2} (e^y \mp 1)^{-1} dy, \quad x_i = \frac{m_i}{T}, \quad (581)$$

and where  $g_i$  is number of spin states of the particles (massless photons,  $g_\gamma = 2$ , massive vector bosons  $Z$ ,  $g_Z = 3$ .) Further,  $-/+$  refers as before to bosons/fermions. As usual, particles and antiparticles are treated as separate, for spin 1/2 electrons and positrons we have  $g_{e^-} = g_{e^+} = 2$ . In thermal equilibrium at a temperature  $T$  the particles will be in either Fermi-Dirac or Bose-Einstein distributions

$$f(\mathbf{p}) = \frac{1}{e^{E(\mathbf{p})/T} \pm 1}, \quad (582)$$

where the plus sign is for fermions while the minus sign for bosons.

We can do the integrals over the distribution functions in two opposite limits, particles which are relativistic  $T \gg m$  and highly non-relativistic  $T \ll m$ . For relativistic (R) particles that are characterized by condition  $x_i = \frac{m_i}{T} \ll 1$  the integrals in (581) are

$$\begin{aligned} \text{bosons : } \quad I_R^{11}(-) &= 2\zeta(3), \quad I_R^{21}(-) = I_R^{03}(-) = \frac{\pi^4}{15}, \\ \text{fermions : } \quad I_R^{11}(+) &= \frac{3\zeta(3)}{2}, \quad I_R^{21}(+) = I_R^{03}(+) = \frac{7\pi^4}{120}, \end{aligned} \quad (583)$$

where  $\zeta$  is Riemann Zeta function and  $\zeta(3) = 1.202$ . Then we obtain, for relativistic bosons, following results:

$$\begin{aligned} n_i^{eq} &= \frac{\zeta(3)}{\pi^2} g_i T^3, \\ \rho_i^{eq} &= \frac{\pi^2}{30} g_i T^4, \\ p_i^{eq} &= \frac{1}{3} \rho_i \end{aligned} \quad (584)$$

and for relativistic fermions

$$\begin{aligned} n_i^{eq} &= \left(\frac{3}{4}\right) \frac{\zeta(3)}{\pi^2} g_i T^3, \\ \rho_i^{eq} &= \left(\frac{7}{8}\right) \frac{\pi^2}{30} g_i T^4, \\ p_i^{eq} &= \frac{1}{3} \rho_i. \end{aligned} \quad (585)$$

On the other hand non-relativistic (NR) limit, where we have  $x \gg 1$  is the same for bosons and fermions and we recover the Boltzmann distribution

$$\begin{aligned} n_i^{eq} &= g_i \left( \frac{m_i T}{2\pi} \right)^{3/2} e^{-m_i/T} \\ \rho_i^{eq} &= m_i n_i , \\ p_i^{eq} &= n_i^{eq} T \ll \rho_i^{eq} . \end{aligned} \tag{586}$$

independently of whether the particle is bosons or fermions. The results given above imply several interesting facts. For example, since the densities of relativistic particles are roughly the same, the relativistic particles remain approximately equal abundances in equilibrium. We also see that once the particles become non-relativistic, they become exponentially suppressed with respect to the relativistic species. This is a result of the fact that it becomes harder for massive particle-antiparticle pairs to be produced in a plasma with  $T \ll m$ .

We would like also mention that although matter is much more dominant than radiation in the Universe today, since their energy densities scale differently, the early Universe was radiation dominated. We can write the ratio of the density parameters in matter and radiation as

$$\frac{\Omega_M}{\Omega_R} = \frac{\Omega_{M0}}{\Omega_{R0}} \frac{a}{a_0} = \frac{\Omega_{M0}}{\Omega_{R0}} (1+z)^{-1} . \tag{587}$$

In the same way as we did above we can determine the redshift of the matter-radiation equality as

$$1 + z_{eq} = \frac{\Omega_{M0}}{\Omega_{R0}} \approx 3 \times 10^3 . \tag{588}$$

From the form of the expression above where we compare the densities that scale as  $a^{-3}$  for matter and  $a^{-4}$  for radiation it is clear that we have made an assumption that particles that are non-relativistic today were also non-relativistic at  $z_{eq}$ . It can be shown that this presumption is safe.

At any given time not all particles will, be in fact in equilibrium at a common temperature  $T$ . A particle will be in kinetic equilibrium with the background thermal plasma, i.e when  $T_i = T$  only while it is interacting. In other words as long as the scattering rate

$$\Gamma = n \langle \sigma v \rangle > H . \tag{589}$$

Here  $\langle \sigma v \rangle$  is the velocity averaged cross-section for  $2 \rightarrow 2$  processes such as

$$i\gamma \rightarrow i\gamma , il^\pm \rightarrow il^\pm \tag{590}$$

that maintain good thermal contact between  $i$ -particles and the particles (that has the particle density  $n$ ) that constitute the background plasma ( $\gamma$ -fotons,  $l^\pm$ -refers to electrons which are abundant down to  $T \sim m_e$  and remain strongly coupled to photons through the Compton scattering through the entire Radiation dominate era

so that  $T_e = T$  always.) We say that  $i$ -particle decouple at the temperature  $T_i$  when the condition

$$\Gamma(T_i) \approx H(T_i) \quad (591)$$

is satisfied. Of course no particle is ever truly decoupled since there are always some residual interactions. On the other hand it can be shown that their effects are generally negligible.

If the particle is relativistic at this time ( $m_i < T_i$ ) then it will also be in the chemical equilibrium with the thermal plasma that is characteristic with the condition for chemical potentials of the particles  $i$   $\mu_i$ , their anti-particles  $\mu_{\bar{i}}$  and the chemical potential of photons  $\mu_\gamma$

$$\mu_i + \mu_{\bar{i}} = \mu_{l^+} + \mu_{l^-} = \mu_\gamma = 0 \quad (592)$$

through processes such as

$$i\bar{i} \leftrightarrow \gamma\gamma, i\bar{i} \leftrightarrow l^+l^- \quad (593)$$

Then its abundance at decoupling will be just the equilibrium value at the temperature of decoupling

$$n_i^{eq}(T_i) = \left(\frac{g_i}{2}\right) n_\gamma(T_i) f_{B.F}, \quad (594)$$

where  $f_B = 1$  if  $i$  is boson and  $f_F = \frac{3}{4}$  if  $i$  is fermion.

Then the decoupled particles  $i$  will expand freely without interactions so that their number in a comoving volume is conserved as  $n_i a^3 = const$  and their pressure and energy density are functions of the scale factor  $a$  alone. Even if these particles do not interact their phase space distribution will retain their equilibrium form (582) with  $T_i$ . As long as the particles remain relativistic,  $E_i$  and  $T_i$  scale as  $a^{-1}$ . Initially the temperature  $T_i$  will track the photon temperature  $T$ . However as the Universe cools below to some mass thresholds (in other words temperature is less than some mass of particles), these massive particles will become non-relativistic and annihilate. The annihilation will produce additional photons and other interacting particles that has an effect of the heating of them. On the other hand  $T_i$  is not affected and hence  $T_i$  will drop below  $T$  and consequently the fraction  $n_i/n_\gamma$  will decrease below its value at decoupling.

It can be shown that decoupled photons maintain a thermal distribution even if they are not in thermal equilibrium. This follows from the fact that the thermal distribution function redshifts into similar distribution function with lower temperature proportional  $1/a$ . Then we can speak about an effective temperature of relativistic species that freezes out at a temperature  $T_f$  and a scale factor  $a_f$  so that

$$a_f T_f = a T(a) \Rightarrow T^{rel}(a) = T_f \left(\frac{a_f}{a}\right). \quad (595)$$

For example, neutrinos decouple at  $T \approx 1MeV$ , shortly thereafter electrons and positrons annihilate into photons and hence transfer energy and entropy into plasma leaving neutrinos decoupled. Consequently we expect a neutrino background and

current Universe with a temperature of approximately  $2K$  while the photon temperature (that arise from the annihilation of electrons and positrons after decoupling of neutrinos) is about  $3K$ .

Similar effect occurs for particles which are non-relativistic at decoupling however there is one important difference. For non-relativistic particles the temperature is proportional to  $\frac{1}{2}mv^2$  that has the redshift as  $1/a^2$  and we therefore have

$$T_i^{non-rel}(a) = T_f \left( \frac{a_f}{a} \right)^2 . \quad (596)$$

The whole picture is as follows: We imagine that the species freeze out while relativistic or non-relativistic and stay this way afterwards.

Now the notion of the effective temperature allows us to define a corresponding notion of an effective number of relativistic degrees of freedom that can be defined as

$$g_* = \sum_{bosons} g_i \left( \frac{T_i}{T} \right)^4 + \frac{7}{8} \sum_{fermions} g_i \left( \frac{T_i}{T} \right)^4 , \quad (597)$$

where the temperature  $T$  is actual temperature of the background plasma assumed to be in equilibrium while we have taken into account that different species  $i$  could have a thermal distribution with a different temperature that of the photons. Then the total energy density in all relativistic species comes from adding the contribution of each species and we obtain a simple formula

$$\rho = \frac{\pi^2}{30} g_* T^4 . \quad (598)$$

We can do the same thing for the entropy density. Since the entropy density of relativistic particles goes as  $T^3$  rather  $T^4$ , we define the effective number of relativistic degrees of freedom for entropy as

$$g_{*S} = \sum_{bosons} g_i \left( \frac{T_i}{T} \right)^3 + \frac{7}{8} \sum_{fermions} g_i \left( \frac{T_i}{T} \right)^3 \quad (599)$$

so that the entropy density of relativistic species is then

$$s = \frac{2\pi}{45} g_{*S} T^3 . \quad (600)$$

For example, in Standard model, we have

$$g_* \approx g_{*S} \begin{cases} 100 & \text{for } T > 300 \text{ MeV} \\ 10 & \text{for } 300 \text{ MeV} > T > 1 \text{ eV} \\ 3 & \text{for } T < 1 \text{ MeV} \end{cases} \quad (601)$$

The events that change the effective number of relativistic degrees of freedom are the QCD phase transition at  $300 \text{ MeV}$  where quarks and gluons start to form bound states, and the annihilation of electron-positron pairs at  $T \approx 1 \text{ MeV}$ .

Thanks to the release of the energy into the background plasma when species annihilate it is only approximation that the temperature goes as  $1/a$ . It is better to say that comoving entropy density is conserved so that

$$s \approx a^{-3} \tag{602}$$

which holds in all forms of adiabatic evolutions, entropy is only produced at a process like a first-order phase transition or out-of-equilibrium decay. It is expected that the entropy production from such processes is very small compared to the total entropy and the adiabatic presumption is excellent approximation for almost the entire early Universe. If we now combine (602) with (600) we obtain a better expression for the evolution of the temperature

$$T \approx g_{*S}^{-1/3} a^{-1} . \tag{603}$$

We see the difference with the naive time dependence  $T \sim 1/a$ . In fact, the temperature will consistently decrease under adiabatic evolution in an expanding Universe but it decreases more slowly when the effective number of relativistic degrees of freedom is diminished.

## 5.7 Thermal relics

As we know particles typically do not stay in equilibrium forever, they density can be so low that the interactions become infrequent and the particle freeze out. Since essentially all of the particles in our current universe belong to this category it is important to study the relic abundance of decoupled species.

We have seen that relativistic or hot particles have a number density that is proportional to  $T^3$  in equilibrium. Thus a species  $X$  that freezes out while still relativistic will have number density at freeze-out  $T_f$  given by

$$n_X(T_f) \sim T_f^3 . \tag{604}$$

Since this is comparable to the number density of photons at that time and since after this freeze-out both photons and species  $X$  have densities that dilute by a factor  $a(t)^{-3}$  as the Universe expands, we see that the abundance of  $X$  particles today should be comparable to the abundance of CMB photons

$$n_{X_0} \sim n_{\gamma_0} \sim 10^2 \text{cm}^{-3} . \tag{605}$$

We express this estimate as  $10^2$  rather than the precise number since the roughness of this estimate does not warrant such misleading precision. For example, neutrinos that are light ( $m_\nu < \text{MeV}$ ) have a number density of  $n_\nu = 115 \text{cm}^{-3}$  for each species. Then a corresponding contribution to the density parameter (if they are heavy enough to be non-relativistic today)

$$\Omega_{0,\nu} = \left( \frac{m_\nu}{92 \text{eV}} \right) h^{-2} . \tag{606}$$



Thus, a neutrino with  $m_\nu \sim 10^{-2} \text{ eV}$  would contribute  $\Omega_\nu \sim 2 \times 10^{-4}$ . We see that this is not large enough to make neutrinos to be dark matter.

Let us now consider species  $X$  that is non-relativistic or cold at the time of decoupling. In this case it is much harder to calculate the relic abundance of a cold relic than a hot one simply because the equilibrium abundance of non-relativistic species is changing rapidly with respect to the background plasma. Then we have to be quite precise following the freeze-out process to obtain a reliable answer. The direct calculation typically involves very complicated procedure. We rather give here reasonable approximate expression. If  $\sigma_0$  is annihilation cross-section of the species  $X$  at temperatures  $T = m_X$ , then the final number density in terms of the photon density can be determined to be equal to

$$n_X(T < T_f) \sim \frac{1}{\sigma_0 m_X M_P} n_\gamma . \quad (607)$$

Since the particles are non-relativistic at the time of decoupling, they are certainly non-relativistic today and their energy density is

$$\rho_X = m_X n_X . \quad (608)$$

Then finally we obtain the density parameter

$$\Omega_X = \frac{\rho_X}{\rho_{cr}} \sim \frac{n_\gamma}{\sigma_0 M_P^3 H_0^2} . \quad (609)$$

Numerically, when  $\hbar = c = 1$  we have  $1 \text{ cm} \sim 2 \times 10^{-14} \text{ GeV}$  so the photon density today is

$$n_\gamma \sim 100 \text{ cm}^{-3} \sim 10^{-39} \text{ GeV}^{-3} . \quad (610)$$

The present value of the Hubble constant is

$$H_0 \sim 10^{-42} \text{ GeV} \quad (611)$$

and the Planck mass is

$$M_P \sim 10^{18} \text{ GeV} . \quad (612)$$

Then finally (609) gives

$$\Omega_X \sim \frac{1}{\sigma_0 (10^9 \text{ GeV}^2)} . \quad (613)$$

We see an interesting fact that  $\Omega_X$  does not depend on  $m_X$  but it depends on the annihilation cross-section. Let us elaborate more about this result and consider some weakly interacting massive particle. The annihilation cross-section of these particles, since they are weakly interacting, should be  $\sigma_0 \sim \alpha_W^2 G_F$ , where  $\alpha_W$  is weak coupling constant and  $G_F$  is the Fermi constant. Using

$$G_F \sim (3000 \text{ GeV})^2 , \quad \alpha_W \sim 10^{-2} \quad (614)$$

and we obtain

$$\sigma_0 \sim 10^{-9} \text{ GeV}^{-2} . \quad (615)$$

Then the density parameter of such particles would be

$$\Omega_X \sim 1 . \quad (616)$$

In other words, a stable particle with weak interaction cross section produces relic density of order of the critical density today and hence provides a perfect candidate for cold dark matter.

After this introduction let us present the simplest possible scenario, that, of course, can be refined by more careful calculations.

Let us again assume that there exists a heavy stable particle  $X$  and its anti-particle  $\bar{X}$ . Let us also presume that the dominant process in which these particles can be destroyed or created is their pair-annihilation or creation with annihilation products being the particles of the Standard Model. Let us also presume that there is no asymmetry between  $X$  and  $\bar{X}$  in the early Universe, in other words the densities  $X$  and  $\bar{X}$  are equal to each other. However we have to mention that this is actually a strong assumption that is valid in many, but not all, realistic extensions of the Standard Model <sup>10</sup>.

Let us outline the overall cosmological behavior of these particles. At high temperatures,  $T \gg M_X$ , the  $X$ - particles are in thermal equilibrium with the rest of cosmic plasma. There are many  $X - \bar{X}$  pairs in the plasma that are continuously created and annihilate. As the temperature drops below  $M_X$ , the equilibrium number density decreases. At some “freeze-out” temperature  $T_f$  the number density becomes so small so that  $X$  and  $\bar{X}$  can no longer meet each other during the Hubble time and their annihilation terminates. After that the number densities of survived  $X$  and  $\bar{X}$  decreases as  $a^{-3}(t)$  and these relic particles contribute to the mass density of the present Universe. The purpose of the following analysis is to estimate the range of properties of  $X$  particles in which their present mass density is of the order of the critical density  $\rho_c$  so that  $X$  may serve as dark matter candidates.

Let us again assume thermal equilibrium. It is well known that the mean free path  $\langle l \rangle$  of a particle in a gas depends on the lifetime  $\tau_{ann}$  of a non-relativistic  $X$ -particle as

$$\sigma_{ann} \cdot v \cdot \tau_{ann} \cdot n_{\bar{X}} = \langle l \rangle , \quad (617)$$

where  $v$  is mean velocity of  $X$  particle,  $\sigma_{ann}$  is the annihilation cross section at velocity  $v$  and  $n_{\bar{X}} = n_X$  is equilibrium number density

$$n_X = g_X \left( \frac{m_X T}{2\pi} \right)^{3/2} e^{-\frac{m_X}{T}} . \quad (618)$$

In order to find the life-time of the non-relativistic particle  $X$  we have to take some reasonable value of  $\langle l \rangle$ . It is natural to presume that it is of order 1 in the natural units  $\langle l \rangle \sim 1$ . Further, it can be also shown that for non-relativistic velocities the annihilation cross section takes the form

$$\sigma_{ann} = \frac{\sigma_0}{v} , \quad (619)$$

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<sup>10</sup>In fact, the alternative scenario with the generation of  $X$  asymmetry is also interesting since it might be related to baryon asymmetric the density of dark matter.

where  $\sigma_0$  is constant. We will discuss its value later. We should now compare the life-time with the Hubble time, or annihilation rate  $\Gamma_{ann} = \tau_{ann}^{-1}$  with the expansion rate  $H$ . At  $T \sim m_X$  the equilibrium density is of order  $n_X \sim T^3$  and  $\Gamma_{ann} \ll H$  for not too small  $\sigma_0$ . Conversely, the life-time is much smaller than Hubble time and consequently the annihilation and creation of  $X - \bar{X}$  pairs is rapid and hence  $X$ -particles are in equilibrium with plasma. On the other hand for very small temperatures  $T \ll m_X$  the number density  $n_X$  is exponentially small and  $\Gamma_{ann} \ll H$  ( $\tau_{ann} \gg H^{-1}$ ). Then it is clear that the thermal equilibrium between  $X$ -particles and background plasma is not maintained. In other words the number density  $n_X$  gets diluted only because of cosmological expansion.

The freeze-out temperature  $T_f$  is determined by the relation

$$\tau_{ann}^{-1} \equiv \Gamma_{ann} \sim H , \quad (620)$$

where we can still use the equilibrium formula as  $X$  particles are in thermal equilibrium (with respect to annihilation and creation) just before freeze-out. Then we find

$$\sigma_0 n_X(T_f) \sim H \sim \frac{T_f^2}{M_P^*} , \quad (621)$$

where we have introduced the effective Planck mass

$$M_P^* = \frac{M_P}{1.66\sqrt{g_*(t)}} , \quad (622)$$

and hence the expansion rate is equal to

$$H(t) = \frac{T^2(t)}{M_P^*} . \quad (623)$$

The solution of the equation (621) gives the freeze-out temperature, up to log terms

$$T_f \approx \frac{m_X}{\ln(m_P^* m_X \sigma_0)} . \quad (624)$$

This temperature is quite bit smaller than  $m_X$  which means that  $X$ -particles freeze out when they are indeed non-relativistic and hence it is natural to call them as *cold dark matter*.

At the freeze-out temperature we use (621) to get

$$n_X(T_f) = \frac{T_f^2}{M_P^* \sigma_0} . \quad (625)$$

It is interesting to note that this density is inversely proportional to the annihilation cross section. The explanation of this fact is that for higher annihilation cross section the creation-annihilation processes are longer in equilibrium and less  $X$  particles survive.

In order to estimate the present density of  $X$ -particles, it is convenient to consider the ratio  $n_X/s$  where  $s$  is the entropy density

$$s = \frac{2\pi^2}{45} g_* T^3 . \quad (626)$$

The point is that during the adiabatic expansion after freeze-out, the entropy density scales as  $s \sim a^{-3}$  since in the adiabatic process  $sa^3 = \text{const}$ . In the same way since we are in the freeze-out regime we have that  $n_X a^3 = \text{const}$  we obtain that  $n_X$  scales in the same way  $n_X \sim a^{-3}$ . Then, up to a factor of order 1, this ratio at freeze-out is

$$\frac{n_X}{s} \sim \frac{1}{g_*(T_f) M_P^* T_f \sigma_0} . \quad (627)$$

At late times, the entropy density, again up to a factor of order 1, is equal to the number density of photons, so the present number density of particles is of order

$$\begin{aligned} \frac{n_{X,0}}{s_0} &= \left( \frac{n_X}{s} \right)_{\text{freeze-out}} \Rightarrow \\ &\Rightarrow n_{X,0} = s_0 \left( \frac{n_X}{s} \right)_{\text{freeze-out}} \sim s_{\gamma,0} \left( \frac{n_X}{s} \right)_{\text{freeze-out}} \end{aligned} \quad (628)$$

and the present mass density is

$$\rho_{X,0} = m_X n_{X,0} \sim n_{\gamma,0} \frac{\ln(M_P^* m_X \sigma_0)}{g_*(T_f) M_P^* \sigma_0} , \quad (629)$$

where we have also used (624). The formula above is very interesting since we see that the present mass density depends mostly on one parameter, the annihilation cross section  $\sigma_0$ . The dependence on the mass of  $X$ -particle is through the logarithm and  $g_*(T_f)$  is very mild. From this formula we derive the condition that ensures that  $X$ -particles are dark energy candidates, i.e. their present mass density is of order  $\rho_c$

$$\sigma_0 \sim \frac{n_{\gamma,0}}{g_*(T_f) M_P^* \rho_c} \ln(M_P^* m_X \sigma_0) \quad (630)$$

that leads to the estimate

$$10^{-11} \sigma_0 < 10^{-9} \text{ GeV}^{-2} , \quad (631)$$

where the uncertainty in the estimate is a consequence of the way we deal with various numerical factors. In any case the estimate given above tells us what the relevant range of mass scales is. To see this note that the annihilation cross section may be parameterized as

$$\sigma_0 = \frac{\alpha^2}{\mathcal{M}^2} , \quad (632)$$

where  $\alpha$  is some coupling constant and  $\mathcal{M}$  is the mass scale (In the calculation above  $\mathcal{M}^2 = G_F$ ). With  $\alpha \sim 10^{-2}$  the estimate of the mass scale for  $\sigma_0 \sim 10^{-11}$  is roughly

$$\mathcal{M} \sim 1 \text{ TeV} . \quad (633)$$

In other words, under very mild assumptions we find that the non-baryonic dark energy matter may naturally originate from the  $TeV$ -scale physics. Then it follows that one natural candidate for the cold dark matter is neutralino. More precisely, in supersymmetric extensions of the Standard Model the neutralino—that is mixture of super-partners of photon,  $Z$ -boson and neutral Higgs bosons—is the lightest supersymmetric particle that is often stable with the suitable value of the annihilation cross section. In fact, the search for both direct and indirect signals from neutralino dark matter is an active area of experimental research.

The mechanism discussed here is of course not the only one mechanism that is able to model cold dark matter. Other dark matter candidates include very heavy relics produced toward the end of inflation, axions, gravitinos, massive gravitons and so on.

## 5.8 Baryogenesis

The symmetry between particles and antiparticles is firmly established in collider physics. However then we lead to the following question; why the observed Universe is composed almost entirely of matter with little or no primordial antimatter.

Outside the particle accelerators the antimatter can be seen in cosmic rays in the form of anti protons where the ratio of these antiprotons to protons is

$$\frac{n_{\bar{p}}}{n_p} \sim 10^{-4} . \quad (634)$$

However this ratio is consistent with secondary anti proton productions through accelerator-like processes

$$p + p \rightarrow 3p + \bar{p} \quad (635)$$

as the cosmic rays stream toward us. In other words there is no evidence for primordial antimatter in our galaxy. Also let us imagine that we have clusters of matter and antimatter galaxies. Then we could expect that we could detect background of  $\gamma$ -radiation from nucleon anti nucleon annihilations with clusters. This background is not observed and so we conclude that there is negligible antimatter on the scale of clusters.

All these considerations put an experimental upper bound on the amount of antimatter in the Universe.

In order to study this problem in more details let us introduce the *baryon to entropy ratio*

$$\eta \equiv \frac{n_B}{s} = \frac{n_b - n_{\bar{b}}}{s} , \quad (636)$$

where  $n_B$  is the difference between the number of baryons and anti-baryons per unit volume. The range of  $\eta$  was determined recently as is equal to

$$\eta = 6.1 \times 10^{-10} \pm 0.210^{-10} . \quad (637)$$

At early times, at temperatures well above  $100 \text{ MeV}$ , cosmic plasma contained many quark-anti quark pairs whose number density was of the order of the entropy density

$$n_q + n_{\bar{q}} \sim s , \quad (638)$$

while baryon number density was related to densities of quarks and antiquarks as follows (baryon number of quarks equals  $1/3$ )

$$\Delta n_b = \frac{1}{3}(n_q - n_{\bar{q}}) . \quad (639)$$

Hence in terms of quantities characterize the very early epoch, the baryon asymmetry may be expressed as

$$\eta \sim \frac{n_q - n_{\bar{q}}}{n_q + n_{\bar{q}}} . \quad (640)$$

We see that there was one extra one extra quark per about 10 billion quark-antiquark pairs. It is this tiny excess that is responsible for entire baryonic matter in the present Universe. Thus the natural question arises, as the Universe cooled from early times to today, what processes, both particle and cosmological, were responsible for the generation of this very specific baryon asymmetry?

Of course there is no logical contradiction to suppose that this tiny excess of quarks to antiquarks was built in as an initial condition. Of course, this is not very satisfactory for physics. Furthermore, inflationary scenario does not provide such an initial condition for Hot Big Bang, rather, inflation theory predicts that the Universe was baryon-symmetric just after inflation. In other words we would like to explain the baryon asymmetry dynamically.

As pointed by Sakharov, a small baryon asymmetry may have been produced in the early Universe from initially symmetric state if three necessary conditions are satisfied:

- Baryon number ( $B$ ) violation,
- Violation of  $C$  (charge conjugation symmetry) and  $CP$  (the composition of parity and  $C$ )
- Departure from thermal equilibrium.

The first condition is clear since when we start from a baryon symmetric Universe, baryon number violation must take case in order the Universe to evolve into the state with baryon number violation. In other words, if the baryon number were conserved that this charge would remain constant during time evolution and hence we would not observe the baryon number asymmetry.

The second Sakharov criterion is required since, when  $C$  and  $CP$  are exact symmetries it can be shown that the total rate for any processes that produces an excess of baryons is equal to the rate of the complementary process which produces an excess of antibaryons and so no net baryon number can be created.  $CP$  violation is present either if there are complex phases in the Lagrangian which cannot be

reabsorbed by field redefinition (explicit symmetry breaking) or if some Higgs scalar field acquires an VEV which is not real (spontaneous symmetry breaking).

Finally, in order to explain the third equilibrium let us calculate the thermal equilibrium average of the baryon number operator  $B$  at temperature  $T = 1/\beta$

$$\begin{aligned} \langle B \rangle_T &= \text{Tr}(e^{-\beta H} B) = \text{Tr}((CPT)(CPT)^{-1} e^{-\beta H} B) = \\ &= \text{Tr}(e^{-\beta H} (CPT)^{-1} B (CPT)) = -\text{Tr}(e^{-\beta H} B) , \end{aligned} \tag{641}$$

using the fact that  $(CPT)$  commutes with  $H$  and cyclicity of the trace. Finally, we have used the fact that  $B$  is odd under  $(PC)$ . Then from the equation above we see that in the thermal equilibrium the baryon number is equal to zero and there is not any generation of baryon number.

The first two Sakharov's conditions may be investigated only within a given particle model, while the third condition the departure from thermal equilibrium may be discussed in a more general way.

## 5.9 Baryon Number Violation

At present there are two well understood mechanisms of baryon number non-conservation. One emerges in Grand-Unified Theories (GUT). Briefly, these GUT describe the fundamental interactions by means of the unique gauge group  $G$  that contains the Standard Model group

$$SU(3)_C \otimes SU(2)_L \otimes U(1)_Y .$$

The fundamental idea of GUT is that at energies higher than a certain energy  $M_{GUT}$  the group symmetry is  $G$  and that, at lower energies, the symmetry is broken down to the SM gauge symmetry, possibly through the chain of symmetry breaking. The motivation for this scenario, whose explanation, however, is beyond the scope of this review, is the fact that in some models, the (running) gauge couplings of the SM unify at the scale  $M_{GUT} \simeq 2 \times 10^{16} \text{ GeV}$ .

The interesting fact considering GUT is that the baryon number violation emerges very naturally in it. Briefly, the mechanisms of the baryon number violation is due to the exchange of super-massive particles. The scale of these new, baryon number violating interacting is of order  $10^{16} \text{ GeV}$ .

Another mechanism of the baryon number violation is related to the triangle anomaly in the baryonic current. It exists already in the Standard Model and possibly it operates in all its extensions. The main feature of this mechanism, as applied to the early Universe, is that it is effective over a wide range of temperatures

$$100 \text{ GeV} < T < 10^{11} \text{ GeV} .$$

In summary, realistic mechanism of baryon number non-conservation are rare, but there are several ways the baryon asymmetry could have been generated. They differ by the characteristic temperature at which the asymmetry is produced.

The GUT mechanisms operates at extremely high temperatures

$$T \sim 10^{15} - 10^{16} \text{ GeV}$$

The most well developed source of the baryon asymmetry in this context are  $B$ - and  $CP$ - violating decays of ultra-heavy particles. At late times the baryon number is violated by anomalous electroweak processes.

*Electroweak baryogenesis* is scenario in which the baryon asymmetry is generated entirely due to the anomalous electroweak processes. Its generation would occur at temperature of order  $100 \text{ GeV}$  which is the energy at which these anomalous processes are switched off. On the other hand the electroweak baryogenesis is still under development.

In summary, the observed asymmetry may be explained by a number of mechanisms all of which, however, exist in *extensions of the Standard Model only*. The problem is that direct proof that any given mechanism is indeed responsible for the baryon asymmetry.

## 5.10 Departure from the Thermal Equilibrium

In some scenarios, such as GUT baryogenesis, the third Sakharov condition is satisfied due to the presence of superheavy decaying particles in a rapidly expanding Universe. These processes are called as out-of-equilibrium decay mechanisms.

The underlying idea is simple. If the decay rate  $\Gamma_X$  of the superheavy particles  $X$  at the time they become non-relativistic (at the temperature  $T \sim M_X$ ) is much smaller than the expansion rate of the Universe, then the  $X$  particles cannot decay on the time scale of the expansion and so they remain as abundant as photons for  $T \leq M_X$ . In other words at some temperature  $T > M_X$  the superheavy particles  $X$  are so weakly interacting so they decouple from the thermal bath while they are still relativistic, so that

$$n_X \sim n_\gamma \sim T^3 \tag{642}$$

at the time of decoupling.

Then we obtain that at temperature  $T \simeq M_X$  they populate the Universe with an abundance which is much larger than the equilibrium one. This abundance is precisely the departure from thermal equilibrium needed to produce a final non-vanishing baryon asymmetry when heavy states  $X$  decay in  $B$  and  $CP$  violating decays.

It can be shown that the out-of-equilibrium condition requires very heavy states

$$M_X \leq (10^{10} - 10^{16}) \text{ GeV} , \tag{643}$$

if these heavy particles decay through renormalizable operators.

A different mechanism of the departure from the thermal equilibrium can be found in the electroweak theory.

A further natural way to depart from equilibrium is provided by the dynamics of the topological defects.



## 5.11 Neutrino background

As an example of the previous discussion let us consider the fate of neutrinos in the expanding Universe. The dynamics of the neutrinos and their reactions with other components of the matter are governed by the Standard model. Then using the rules of standard quantum field theory one can calculate the reaction rate  $\Gamma$  of the neutrinos with the rest of the matter (Roughly speaking the inverse  $\Gamma^{-1}$  is the average time between collision of the neutrinos with all form of the matter). When  $\Gamma^{-1}$  is larger than  $H^{-1}$  (conversely, when  $\Gamma$  is less than  $H$ ) there cannot occur the reactions between the neutrinos and the rest of the matter. We say that in this case neutrinos effectively decouple from the rest of matter. It can be shown that the relevant rasion is given by

$$\frac{\Gamma}{H} \approx \left( \frac{T}{1.4 \text{ MeV}} \right)^3 = \left( \frac{T}{1.6 \times 10^{10} \text{ K}} \right)^3 . \quad (644)$$

This formula implies that for  $T \leq 1.6 \times 10^{10}$  the neutrinos decouple from the rest of the matter. On the other hand electrons and positrons can still annihilate at slightly lower temperature. This process increases the number of the photons. As a result the photon temperature goes up with respect to neutrino temperature (Remember that it is natural to speak about two different temperatures for two different species of particles since they have already decoupled.). We can calculate this increase of temperature as follows. The increase of  $T$  is due to the change of degree of freedom  $g$  and is given by

$$\frac{(aT_\gamma)_{after}^3}{(aT_\gamma)_{before}^3} = \frac{g_{before}}{g_{after}} = \frac{\frac{7}{8}(2+2) + 2}{2} = \frac{11}{4} . \quad (645)$$

Let us explain factors given above. In the numerator, one 2 is for electron, one 2 is for positron and the factor 7/8 arises because of fermions. The remaining 2 in numerator is for photon. In denominator 2 is for photon since they remain after the annihilation of positrons with electrons. Using the relation above we obtain

$$\begin{aligned} (aT_\gamma)_{after} &= \left( \frac{11}{4} \right)^{1/3} (aT_\gamma)_{before} = \left( \frac{11}{4} \right)^{1/3} (aT_\nu)_{before} = \\ &= \left( \frac{11}{4} \right)^{1/3} (aT_\nu)_{after} = 1.4 (aT_\nu)_{after} . \end{aligned} \quad (646)$$

The first equality is from (646), the second follows from the fact that the photons and neutrinos had the same temperature originally. The third equality follows from the fact that for decoupled neutrinos  $aT_\nu$  are constant. The final result leads to the prediction that at present the Universe will contain a bath of neutrinos that has temperature that is lower than of CMBR.

## 5.12 Primordial Nucleosynthesis

Theory of Big Bang Nucleosynthesis and observations of primordial abundances of light elements probe the earliest epoch of the evolution of the Universe that is accessible to observation today. This epoch corresponds to temperatures ranging from 1  $MeV$  to a few 10  $keV$  and age of the Universe from 1  $s$  to 200  $s$ .

Let us briefly review the properties of the matter at this early epoch of the Universe.

At temperatures above 1  $MeV$  there is a thermal equilibrium with respect to reactions

$$p + e \leftrightarrow n + \nu_e . \quad (647)$$

As the Universe cools down below  $T \approx 1 MeV$  neutrons are no longer produced or destroyed, their concentration (relative to protons) "freezes out". Alternatively saying, the weak interactions are frozen out and neutrons and protons cannot interconvert. The equilibrium abundance of neutrinos at this temperature is about 1/6 the abundance of neutrons due to the slightly larger neutron mass.

When we reach a temperature somewhat below 100  $keV$  the Big-Bang Nucleosynthesis (BBN) begins<sup>11</sup>. At that point the neutron/proton ratio is about 1/7. Since it is energetically favorable for nucleons to form  $He$  the most part of the free neutrons are converted into  $He$ . For every two neutrons and fourteen protons we end up with one helium nucleus and twelve protons. In other words 25 % of the baryons are converted to helium. There are also trace amounts of deuterium and lithium. Heavier elements are not synthesized in the Big Bang but require supernova explosions in the later universe. These elements remain in the Universe so their primordial abundance is measurable today.

It is important to stress that Big Bang Nucleosynthesis serves also as a source of constraints on particle physics. The fact that the temperature of the Universe reached at least 1  $MeV$  or so and that the expansion was described by known physics at this stage constrain significantly some extensions of the Standard models.

The most amazing fact about nucleosynthesis is that, given the Universe is radiation dominated during the relevant epoch, the relative abundances of the light elements depend essentially on one parameter, the *baryon to entropy ratio*

$$\eta \equiv \frac{n_B}{s} = \frac{n_b - n_{\bar{b}}}{s} , \quad (648)$$

where  $n_B$  is the difference between the number of baryons and anti-baryons per unit volume. The range of  $\eta$  was determined recently as is equal to

$$\eta = 6.1 \times 10^{-10} \pm 0.210^{-10} . \quad (649)$$

Let us be now more specific. We know that at present the Universe is expanding and filled with radiation that is very cold today ( $T_0 = 2.73K$ ). If we trace the evolution

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<sup>11</sup>Note that the nuclear binding energy per nucleon is typically of order 1  $MeV$  so that one could expect that BBN would occur earlier. However the large number of photons per nucleons at that time prevent BBN to occur until the temperature drops below 100  $keV$ .

of the Universe back in time to earlier epochs that were hotter and denser, the early Universe is a Primordial Nuclear Reactor during its first 20 minutes ( $\approx 1000$ ). In fact, when the temperature of the Universe is higher than the binding energy of nuclei ( $\sim MeV$ ) none of the heavy elements (helium and metals) could have existed in the Universe. The binding energy of the first four light nuclei,  $H^2, H^3, He^3$  and  $He^4$  are  $2.22MeV, 6.92MeV, 7.72MeV$  and  $28.3MeV$  respectively. Since the average energy in the thermal ansamble is proportional to the temperature we obtain that these nuclei could be formed when the temperature of the Universe is in the range  $(1 - 30)MeV$ . Surprisingly, the actual synthesis takes place at much lower temperature  $T_{nuc} = T_n \approx 0.1MeV$ . The reason for this delay is the high entropy of the Universe that implies that the ration of photons to baryons,  $\eta^{-1}$  is high. Numerically

$$\eta = \frac{n_B}{n_\gamma} = 5.5 \times 10^{-10} \left( \frac{\Omega_B h^2}{0.02} \right), \quad \Omega h^2 = 3.65 \times 10^{-3} \left( \frac{T_0}{2.73K} \right)^3 \eta_{10}. \quad (650)$$

Thus, even if the thermal equilibrium is maintained the significant synthesis of nuclei can occur only at  $T \leq 0.3MeV$ . Then we can expect significant production  $X_A \sim 1$  of nuclear species  $A$  at temperature  $T \leq T_A$ . However it turns out that the rate of the nuclear reaction is not high enough to maintain thermal equilibrium between various species. In order to study non equilibrium abundances in an expanding Universe is based on rate equations. Let us now review its general concepts.

### 5.12.1 Rate equations

Consider a reaction in which two particles 1 and 2 interact to form two other particles 3 and 4. For example, let us consider reaction  $n + \nu_e = p + e$  that converts neutrons into protons in the forward direction and proton into neutrinos in the reverse direction. Another example is the reaction  $p + e = H + \gamma$  where the forward reaction describes recombination of electron and proton forming a neutral hydrogen atom with the emission of photon. In general we are interested in how the number density  $n_1$  of particle species 1 changes due to the reaction of the form  $1 + 2 \Leftrightarrow 3 + 4$ . Remember that even in case where there is no reaction the number density changes as  $n_1 \propto a^{-3}$  due to the expansion of the Universe. In other words the quantity that changes due to the reaction is  $n_1 a^3$ . Further, the forward reaction will be clearly proportional to the product of the number densities  $n_1 n_2$  while the reverse reaction will be proportional to  $n_3 n_4$ . Hence we can write the equation for the rate of the change of particle species  $n_1$  in the form

$$\frac{1}{a^3} \frac{d(n_1 a^3)}{dt} = \mu (A n_3 n_4 - n_1 n_2) \quad (651)$$

The left hand side is the relevant rate of change over and above that due to the expansion of the Universe. On the right hand side the two proportionality constants have been written as  $\mu$  and  $A\mu$  that generally are functions of time. Usually  $\mu \simeq \sigma v$  where  $\sigma$  is the cross section for the relevant process and  $v$  is relative velocity. The

left hand side has to vanish for system in thermal equilibrium with  $n_i = n_i^{eq}$  where the superscript  $eq$  denotes the equilibrium densities of the different species labeled with  $i = 1 \dots 4$ . If we insert in the above equation the condition  $n_i = n_i^{eq}$  we can express  $A$  as

$$An_3^{eq}n_4^{eq} - n_1^{eq}n_2^{eq} = 0 \Rightarrow A = \frac{n_1^{eq}n_2^{eq}}{n_3^{eq}n_4^{eq}} \quad (652)$$

and than the rate equation becomes

$$\frac{1}{a^3} \frac{d(n_1 a^3)}{dt} = \mu n_1^{eq} n_2^{eq} \left( \frac{n_3 n_4}{n_3^{eq} n_4^{eq}} - \frac{n_1 n_2}{n_1^{eq} n_2^{eq}} \right). \quad (653)$$

On the left hand side we can write  $\frac{d}{dt} = aH \frac{d}{da}$  that shows that the relevant scale for this processes is  $H^{-1}$ . Clearly when  $\frac{H}{\mu n_i} \ll 1$  the right hand side becomes ineffective because the factor  $\frac{\mu}{H}$  factor. Then we see that the number of particles of species 1 does not change. In other words when the expansion rate of the Universe is large compared to the reaction rate ( $\frac{\mu}{H} \ll 1$ ) the given reaction is ineffective in changing the number of particles. However this result does not mean that the reactions have reached thermal equilibrium and  $n_i = n_i^{eq}$ . In fact, the opposite situation occurs: The reactions are not fast enough to drive the number densities towards equilibrium densities and the number densities "freeze out" at non-equilibrium values. Of course the right hand side in (653) will also vanish when  $n_i = n_i^{eq}$  that is the extreme limit of thermal equilibrium.

Using this general formalism we will now apply it to the process of nucleosynthesis which requires protons and neutrons that combine together to form bound nuclei of heavier elements like deuterium, helium... The abundance of these elements are going to be determined by the relative abundance of neutrons and protons in the Universe. For that reason we should start the discussion with the problem of the thermal equilibrium between protons and the neutrons in the early Universe. As long as the inter-conversion between  $n$  and  $p$  through the weak interaction processes

$$\nu + n \leftrightarrow p + e, \bar{\nu} + n \leftrightarrow p + \bar{\nu} \quad (654)$$

or their decay

$$n \leftrightarrow p + e + \bar{\nu} \quad (655)$$

is rapid with respect to the expansion rate of the Universe thermal equilibrium can be maintained. Then the equilibrium static physics implies that the equilibrium  $n/p$  ration is equal to

$$\left( \frac{n_n}{n_p} \right) = \frac{X_n}{X_p} = \exp(-Q/T), \quad (656)$$

where  $Q = m_n - m_p = 1.293MeV$ . For  $T \gg Q$  the factor in the exponent is approaching zero and we obtain  $X_n \approx X_p$ . However when  $T$  drops below about  $1.3MeV$  the neutron fraction will drop exponentially on condition that the thermal equilibrium is still maintained. However to check weather the thermal equilibrium

is maintained we have to compare the expansion rate with the reaction rate. The expansion rate is

$$H = \sqrt{\frac{8\pi G\rho}{3}}, \quad (657)$$

where

$$\rho = \frac{\pi^2}{30}gT^4, \quad (658)$$

where  $g \approx 10.75$  represents the relativistic degrees of freedom present at these temperatures. At  $T = Q$  this gives  $H \approx 1.1s^{-1}$ . The reaction rate needs to be computed from weak interaction theory. The neutron to proton conversion rate is approximated by

$$\lambda_{np} \approx 0.29s^{-1} \left(\frac{T}{Q}\right)^5 \left[ \left(\frac{Q}{T}\right)^2 + 6\left(\frac{Q}{T}\right) + 12 \right]. \quad (659)$$

At  $Q = T$  this gives  $\lambda \approx 5s^{-1}$  that is more rapid than the expansion rate. But as  $T$  drops below  $Q$  this decreases rapidly and the reaction ceases to be fast enough to maintain thermal equilibrium. Then we have to work out the neutron abundance using the equation (653).

If we denote  $n_1 = n_n$ ,  $n_3 = n_p$  and  $n_2, n_4 = n_l$  where the subscript  $l$  stands for leptons then the equation (653) becomes

$$\frac{1}{a^3} \frac{d(n_n a^3)}{dt} = \mu n_l^{eq} \left( \frac{n_p n_n^{eq}}{n_p^{eq}} - n_n \right). \quad (660)$$

To proceed we use the fact that  $\mu n_l^{eq}$  is equal to the rate of the neutron to proton conversion  $\lambda_{np}$ . We also use the relation

$$\frac{n_n^{eq}}{n_p^{eq}} = \exp(-Q/T) \quad (661)$$

Let us now introduce the fractional abundance

$$X_n = \frac{n_n}{(n_n + n_p)} \quad (662)$$

Then the equation (660) takes the form

$$\frac{dX_n}{dt} = \lambda_{np}((1 - X_n)e^{-Q/T} - X_n), \quad (663)$$

where we have used

$$X_n + X_p = 1, \quad X_p = \frac{n_p}{n_n + n_p} \quad (664)$$

and also the fact

$$\frac{1}{a^3} \frac{d(n_n a^3)}{dt} = \frac{a^3(n_n + n_p)}{a^3} \frac{dX_n}{dt} \quad (665)$$

since  $(n_n + n_p)a^3$  is constant. This equation can be integrated numerically and determine how the neutron abundance changes with time. The neutron fraction falls out of equilibrium when temperature drop below  $1MeV$  and it freezes to about 0.15 at temperature below  $0.5MeV$ . As the temperature decreases further the neutron decays with a half life of  $\tau_n \approx 886.7sec$  becomes important and starts to reduce the neutron number density. Then the only way how the neutrons can survive is through the synthesis of light elements. As the temperature falls further to  $T = T_{He} \approx 0.28MeV$  significant amount of  $He$  could have been produced if the nuclear reaction rates were high enough. These reactions are all based on  $D, He$  and  $H$  and do not occur rapidly enough because the mass fraction of  $D, He$  and  $H$  are still quite small  $[10^{-12}, 10^{-19}, 5 \times 10^{-19}]$  at  $T \simeq 0.3MeV$ . The equilibrium deuterium abundance passes through unity at temperature of about  $0.07MeV$  which is when nucleosynthesis can really begin.

The production of still heavier elements-even those like  $C, O$  which have higher binding energies than  $He$  is suppressed in the early Universe.

### 5.13 Decoupling of matter and radiation

In the early hot phase the radiation will be in thermal equilibrium with matter. As the Universe cools below  $k_B T \simeq (\epsilon_a/10)$  is the binding energy of atoms the electrons and ions will combine to form neutral atoms and radiation will decouple from matter. This occurs at  $T \simeq 3 \times 10^3 K$ . As the Universe expands further these photons will continue to exist without any further interaction. We shall now discuss some details related to the formation of neutral atoms and decoupling of photons.

The relevant reaction is



If the rate of this reaction is faster than the expansion rate then one can calculate the neutral fraction as follows. Introducing the fractional ionization  $X_i$  for each of the particle species and using the facts that  $n_p = n_e$  and  $n_p + n_H = n_B$ . We also have  $X_p = X_e$  and  $X_H = \frac{n_H}{n_B} = 1 - \frac{n_p}{n_B} = 1 - X_e$ . The equation that governs the time evolution of  $X_e$  that expresses the equilibrium situation now takes the form

$$\frac{1 - X_e}{X_e^2} \approx 3.84\eta \left( \frac{T}{m_e} \right)^{3/2} \exp(B/T) , \quad (667)$$

where  $\eta = 2.68 \times 10^{-8}(\Omega_B h^2)$  is the baryon-to-photon ratio. We define  $T_e$  as the temperature at which 90 percent of the electrons have combined with protons. This implies  $n_p = 0.1n_B$  and hence  $X_e = X_p = 0.1$ . This leads to the condition

$$(\Omega_B h^2)^{-1} \tau^{3/2} \exp[-13.6\tau^{-1}] = 3.13 \times 10^{-18} , \quad (668)$$

where  $\tau = (T/1eV)$ . The solution of this equation can be given by iterative procedure. For  $\Omega_B h^2 = 1, 0.1, 0.01$  we then obtain  $T_{atom} = 0.324eV, 0.307eV, 0.292eV$ .

These results were based on the equilibrium densities. Then it is important to check that the rate of the reaction  $p + e \leftrightarrow H + \gamma$  is fast enough to maintain

equilibrium. It turns out however that this is not fully satisfied and hence we have to again use the rate equation. The rate equation (653) for  $n_1 = n_e, n_2 = n_p, n_3 = n_H$  and  $n_4 = n_\gamma$  and for  $X_e = \frac{n_e}{n_e + n_H}$  takes the form

$$\frac{dX_e}{dt} = \alpha \left( \frac{\beta}{\alpha} (1 - X_e) - n_b X_e^2 \right), \quad (669)$$

where the recombination rate  $\alpha$  is the rate is given by

$$\alpha = 9.78 r_0^2 c \left( \frac{B}{T} \right)^{1/2} \ln \left( \frac{B}{T} \right), \quad (670)$$

where  $r_0 = \frac{e^2}{m_e^2 c^2}$  is classical electron radius. In (669) the ration  $\beta/\alpha$  is given as

$$\frac{\beta}{\alpha} = \left( \frac{m_e T}{2\pi} \right)^{3/2} \exp[-B/T] \quad (671)$$

Using this result we obtain that the value of  $T_{atom}$  does not change significantly.

## 5.14 Structure formation and linear perturbation theory

The structure formation is based on the key idea that if there exist small fluctuations in the energy density in the early Universe, then gravitational instability then leads in a well understood manner leading to structures like galaxies today. The most popular model for generating these fluctuations is based on the idea that if the very early Universe went through the inflation phase then the quantum fluctuations of the field driving the inflation can lead to energy density fluctuations.

Let us illustrate this idea on the example of the massless scalar field  $\phi$  minimally coupled to gravity. The action of the scalar field is

$$S_\phi = -\frac{1}{2} \int d^4x \sqrt{-g} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \quad (672)$$

In spatial flat FRW background this action has the form

$$S_\phi = -\frac{1}{2} \int dx dt a^3(t) [-(\partial_t \phi)^2 + \frac{1}{a^2} (\partial_i \phi)^2] \quad (673)$$

so that the equation of motion takes the form

$$\partial_t (a^3 \partial_t \phi) - a \partial_i \partial^i \phi = 0 \quad (674)$$

or equivalently

$$\ddot{\phi} + 3H(t) \dot{\phi} - \frac{1}{a^2} \partial_i \partial^i \phi = 0, \quad (675)$$

where  $\dot{x} = \partial_t x$ ,  $\ddot{x} = \partial_t^2 x$ . Thanks to the homogeneity and isotropy of space it is natural to work in the momentum representation where we search for the solutions in the form

$$e^{i\mathbf{x}\mathbf{k}} \phi_{\mathbf{k}}(t). \quad (676)$$

If we insert (676) into (675) we obtain ordinary differential equation for  $\phi_{\mathbf{k}}$  in the form

$$\ddot{\phi}_{\mathbf{k}} + 3H(t)\dot{\phi}_{\mathbf{k}} + \frac{k^2}{a^2}\phi = 0 . \quad (677)$$

Note that  $\mathbf{k}$  is a *coordinate momentum*. The physical momentum at time  $t$  is

$$\mathbf{p} = \frac{\mathbf{k}}{a} \quad (678)$$

and it depends on time.

Looking on (677) we see that the second term in it acts as a friction term. Then we can consider two regimes with the qualitatively different behavior of the modes  $\phi_{\mathbf{k}}$ : **Subhorizon modes:**

These modes are characterized condition

$$p = \frac{k}{a} \gg H . \quad (679)$$

Modes obeying this property are subhorizon modes since their physical length  $\lambda \sim p^{-1}$  is much shorter than the Hubble distance  $H^{-1}$  that is a horizon size in matter and radion dominated Universe. More precisely, for modes obeying the condition (679) we can neglect the friction term in (677) and hence we get

$$\ddot{\phi} + \omega_{\mathbf{k}}^2(t)\phi = 0 , \omega_{\mathbf{k}}(t) = \frac{k}{a} \quad (680)$$

This equation has the general solution

$$\phi_{\mathbf{k}} = \frac{1}{a} e^{\pm i \int_{t_0}^t dt' \omega_{\mathbf{k}}(t')} \quad (681)$$

since

$$\begin{aligned} \dot{\phi}_{\mathbf{k}} &= -H\phi_{\mathbf{k}} + i\omega_{\mathbf{k}}\phi_{\mathbf{k}} \approx i\omega_{\mathbf{k}}\phi_{\mathbf{k}} , \\ \ddot{\phi}_{\mathbf{k}} &= i\dot{\omega}_{\mathbf{k}}\phi_{\mathbf{k}} - \omega_{\mathbf{k}}^2\phi_{\mathbf{k}} = -iH\omega_{\mathbf{k}}\phi_{\mathbf{k}} - \omega_{\mathbf{k}}^2\phi_{\mathbf{k}} \approx -\omega_{\mathbf{k}}^2\phi_{\mathbf{k}} . \end{aligned} \quad (682)$$

This solution (modulo slowly varying prefactor) describes oscillations with the frequently experiencing redshift (The frequency is lowered with time).

**Superhorizon modes:**

These modes are characterized by condition

$$p = \frac{k}{a} \ll H . \quad (683)$$

In this case the last term in (677) are negligible and the solutions are

$$\begin{aligned} \text{constant mode :} & \quad \phi_{\mathbf{k}} = \text{const} , \\ \text{growing mode :} & \quad \phi_{\mathbf{k}}(t) = K \int_{t_0}^t \frac{dt'}{a^3(t')} . \end{aligned} \quad (684)$$



It is clear that the constant mode is solution of (677). The growing mode is solution as well since

$$\dot{\phi}_{\mathbf{k}} = \frac{K}{a^3}, \ddot{\phi}_{\mathbf{k}} = -3H\dot{\phi}_{\mathbf{k}}. \quad (685)$$

The gravitational waves obey precisely the same equations as (677) so that they have exactly the same behavior, in particular, for given  $k$  one of the superhorizon modes blows up at small  $t$ . It follows that the whole picture of the FRW Universe with small perturbations is thus self-consistent only if this modes vanishes at finite times.

Now recall that for radiation dominated and matter dominated Universe  $H \sim t^{-1}$  while the scale factor behaves as  $a \sim t^{1/2}$  for radion dominated Universe and  $a \sim t^{2/3}$  for matter dominated Universe. Then the ratio of physical momentum to  $H$  behaves as

$$\frac{p(t)}{H(t)} \propto t^{1/2} \quad (686)$$

for radiation dominated Universe and

$$\frac{p(t)}{H(t)} \propto t^{1/3} \quad (687)$$

for matter dominated Universe. These results mean that all modes start as superhorizon and then enter the horizon. In the scalar mode example the requirement that the growing mode vanishes determines the initial date for each  $\mathbf{k}$  up to overall amplitude. Then we have

$$\phi_{\mathbf{k}} = c_{\mathbf{k}}, \frac{k}{a} \ll H, \quad (688)$$

and

$$\phi_{\mathbf{k}} = c_{\mathbf{k}} \cos \left( \int_0^t dt' \omega_{\mathbf{k}}(t') \right), \frac{k}{a} \gg H. \quad (689)$$

For density perturbations the oscillating behavior means that at late enough times there are sound waves in the primordial plasma with the wave-lengths that are shorter than the horizon size at each moment of time. Briefly speaking the fate of the primordial density perturbations is as follows. They stay constant until they enter the horizon at radiation or matter dominate stage. After that they start to oscillate and make the sound waves. The amplitudes of these waves grow during the matter dominated stage due to the gravitational instability. The regions with higher density tend to gravitationally attract matter and become even more overdense. The dense regions collapse and form gravitationally bound structures.

Let us now discuss in more details how the simple description given above is related to the more realistic situation. As long as the fluctuations are small one can study their evolution by linear perturbation theory. The basic idea of linear perturbation theory is well defined and simple. We write the metric as

$$g_{\mu\nu} = g_{\mu\nu}^{FRW} + h_{\mu\nu}, \quad (690)$$

where  $g_{\mu\nu}^{FRW}$  is background FRW metric and  $h_{\mu\nu}$  is small perturbations that propagate on the background characterized with  $g_{\mu\nu}^{FRW}$ . In the same way we perturb the source energy momentum tensor by

$$T_{\mu\nu} = T_{\mu\nu}^{FRW} + \delta T_{\mu\nu} , \quad (691)$$

where again  $T_{\mu\nu}^{FRW}$  is the stress energy tensor for the background matter that solves the FRW equations and  $\delta T_{\mu\nu}$  are perturbations. If we linearize the Einstein's equations one can relate the perturbed quantities by a relation of the form

$$\mathcal{L}(g_{\mu\nu}^{FRW})h_{\mu\nu} = \delta T_{\mu\nu} , \quad (692)$$

where  $\mathcal{L}$  is second order linear differential operator depending on the background metric  $g_{\mu\nu}^{FRW}$ . As we argued above due to the fact that the background is maximally symmetric one can separate out time and space and we can write down the equation for any given mode labeled with the wave vector  $\mathbf{k}$  as

$$\mathcal{L}(a(t), \mathbf{k})h_{\mu\nu}(t, \mathbf{k}) = \delta T_{\mu\nu}(t, \mathbf{k}) . \quad (693)$$

Then careful analysis performed in case of metric perturbations implies that the linearized equations of motion for gravity perturbations take the forms given in the toy example of the massless scalar fields studied above. More precisely, it can be shown, after some simplifications and presumption, that are all well justified, that perturbed metric can be written in the form

$$ds^2 = a^2(\eta)[(1 + 2\Phi)d\eta^2 - (1 - 2\Phi)\delta_{ab}dx^a dx^b] . \quad (694)$$

In other words we obtain one perturbed scalar degree of freedom  $\Phi$ . Then it can be shown that the dynamics of the mode  $\Phi$  is governed by the equations that has the same form as (677).

## 6 Inflation cosmology

### 6.1 Problems of the standard Big-Bang model

The standard Big-Bang model suffers from number of problems. Before we enter in their discussion we review some properties of the Friedmann models at the early stage of the Universe.

The question is what can we say about the Hubble parameter  $H = \frac{\dot{a}}{a}$ , the density  $\rho$  and the quantity  $k$ ?

At the earliest stages of the evolution of the Universe  $H$  and  $\rho$  could be arbitrarily large. On the other hand it is believed that for  $\rho \geq M_P^4$  effects of quantum gravity are significant and the quantum fluctuations of metric exceed the classical value of  $g_{\mu\nu}$ . The standard cosmology where the metric is treated in the classical manner restricts to the region of phenomena where

$$\rho \leq M_P^4 , \quad T \leq M_P \sim 10^{19} GeV, \quad H < M_P . \quad (695)$$

We also have to stress that in the expanding Universe thermodynamics equilibrium cannot be established immediately but only when the temperature  $T$  is sufficiently low. The behavior of the non-equilibrium Universe at densities of order of the Planck density is very important problem.

Now we come to the list of problems of the standard hot Universe theory

## 6.2 Problems of the standard scenario

### The singularity problem

The Friedmann equations imply that the density of matter in the Universe goes to infinity as  $t \rightarrow 0$  and the corresponding solutions cannot be formally continued to the domain  $t < 0$ .

One of the most exciting questions of cosmology is whether anything existed *before*  $t = 0$ . If there is nothing before  $t < 0$  the question is: where did the Universe come from?

Studies of the general structure of space-time near a singularity suggest that it is highly unlikely that this problem could be solved with the framework of the classical gravitation theory. One hope that these questions could be answered in the context of string theory. We will review some string theory inspired models in next sections. However these models are faced with many important and conceptual problems so that the problem of the birth of the Universe is the most challenging un answered question in physics.

### Flatness Problem

The flatness problem concerns with the observation that the real density of the Universe,  $\rho$ , is known to be very close to the critical density  $\rho_c$ . Recall, that in the previous section we have studied the Friedmann equation

$$H^2 = \frac{1}{3M_P^2}\rho - \frac{k}{a^2} , \quad (696)$$

where now  $M_P \equiv \frac{1}{\sqrt{8\pi G}} \sim 2 \cdot 10^{18} GeV$  is the four dimensional Planck mass. Recall also that  $H = \frac{\dot{a}}{a}$  where  $a(t)$  is the scale factor with the spacetime metric on the form

$$ds^2 = -dt^2 + a^2 d\Sigma , \quad (697)$$

where  $d\Sigma$  is comoving volume element of space with  $k = 0, +1, -1$  corresponding to flat, positively curved and negatively curved spaces respectively. As we known we can rewrite the Friedmann equation in the form

$$\Omega - 1 = \frac{k}{a^2 H^2} , \quad (698)$$

where  $\Omega$  means the sum of particular  $\Omega$ 's. Note that for ordinary type of matter,  $\frac{1}{a^2 H^2}$  will increase with time. To see this we use the continuum equation given by

$$\dot{\rho} + 3H(\rho + p) = 0 . \quad (699)$$

If we assume an equation of state of the form

$$p = w\rho , \quad (700)$$

for  $w = \text{const}$  then the continuity equation can be written as

$$\frac{d\rho}{da} \frac{da}{dt} + 3\frac{\dot{a}}{a}(1+w)\rho = \frac{d\rho}{da} + 3(1+w)\frac{\rho}{a} = 0 , \quad (701)$$

that implies

$$\rho \sim a^{-3(1+w)} . \quad (702)$$

If we start with  $\Omega \sim 1$  we obtain that  $k \sim 0$ . Then the Friedman equation is

$$H^2 \sim \rho \Rightarrow \frac{\dot{a}}{a} \sim a^{-3(1+w)/2} \quad (703)$$

that implies

$$daa^{(1+3w)/2} = t \Rightarrow a \sim t^{\frac{2}{3(1+w)}} . \quad (704)$$

As a consequence we get that

$$\frac{1}{a^2 H^2} \sim t^{2 - \frac{4}{3(1+w)}} . \quad (705)$$

This expression grows with time for any  $w > -1/3$ -examples include pressureless dust with  $w = 0$  and radiation with  $w = 1/3$ . Looking on the form of the Friedman equation (698) we see that, unless the Universe is exactly flat ( $k = 0$ ) and, as a consequence  $\Omega = 1$ ,  $\Omega$  will rapidly evolve away from  $\Omega = 1$ . In order to have a value of  $\Omega$  close to 1 today, one would therefore expect to need a value of  $\Omega$  even closer to 1 in the early Universe. This is the famous *Flatness problem*. That is, how can  $\Omega$  be so close to one?

We can argue alternatively as follows. Looking on the form of Friedmann equation we see that the curvature contribution is

$$|\Omega_{curv}| \equiv \frac{\rho_{curv}}{\rho_c} = \frac{3M_P}{a^2 H^2} , \quad (706)$$

where we have defined the curvature contribution to the Friedmann equation as

$$|\rho_{curv}| \frac{3M_P}{a^2} . \quad (707)$$

The present value of the equation (706) is

$$|\Omega_{curv}| < 0.02 . \quad (708)$$

Since  $|\rho|_{curv}$  scales as  $1/a^2$  while the radiation matter and radiation scales as  $1/a^3$  and  $1/a^4$  respectively. This implies that the curvature contribution to the Friedman equations was even smaller in the past, for example

$$\begin{aligned} \text{nucleosynthesis : } & |\Omega_{curv}| < 10^{-16} , \\ \text{electroweak epoch , } & |\Omega_{curv}| < 10^{-26} . \end{aligned} \tag{709}$$

In other words the spatial curvature of the Universe was tiny at the beginning. The question is, why the initial conditions were so flat? This flatness problem cannot be solved within Hot Big Bang theory.

### The total entropy and total mass problem

The question is why the total entropy  $S$  and total mass  $M$  of matter in the observable part of the Universe with  $R_p$  is so large. The total entropy  $S$  of the present Universe can be estimate as follows. The size of the observable part of the Universe is

$$l_{H,0} \sim 2H_0^{-1} \sim 10^{26} \text{ m}$$

The entropy inside a sphere of the size  $l_{H,0}$  is roughly of the order of the number of photons

$$S \sim N_\gamma \sim n_\gamma l_{H,0}^3 . \tag{710}$$

Using also the fact that

$$n_\gamma \sim T_\gamma^3 \sim 2.7 \text{ K}$$

where  $T_\gamma$  is the temperature of the primordial background radiation. Then we finally obtain

$$S = 10^{88} . \tag{711}$$

On the other hand the estimate of the total mass in the observable Universe is

$$M \sim l_{H,0}^3 \rho_c \sim 10^{55} \text{ g} . \tag{712}$$

In the Hot Big Bang theory the expansion of the Universe is almost adiabatic so this huge entropy should be built in as an initial condition. Certainly this initial condition is very special. Moreover, the condition of naturality, which is the statement that all dimensionless quantities should be of order 1 implies that such a initial conditions with huge entropy are rather un-natural.

### Horizon problem

We known that the region of the Universe look very similar even though, assuming normal radiation dominated expansion of the early Universe, they can not have been in causal contact. In fact, the horizon problem steams from the existence of particle horizons in FRW cosmologies. Horizons exist because there is only a finite amount of time since the Big Bang singularity and thus only a finite distance that photons

can travel within the age of the Universe. Consider a photon moving along a radial trajectory in a flat Universe. In a flat, Universe, we can normalize the scale factor to be  $a_0 = 1$ . A radial null path obeys

$$0 = ds^2 = -dt^2 + a^2 dr^2 \quad (713)$$

so the comoving (coordinate) distance traveled by such a photon between times  $t_1$  and  $t_2$  is

$$\Delta r = \int_{t_1}^{t_2} \frac{dt}{a(t)} . \quad (714)$$

To get a physical distance as it would be measured by an observer at any time  $t$  simply multiply by  $a(t)$ . For simplicity, we are in matter dominated Universe for which

$$a = \left( \frac{t}{t_0} \right)^{2/3} . \quad (715)$$

The Hubble parameter is therefore given by

$$H = \frac{\dot{a}}{a} = \frac{2}{3t} = a^{-2/3} H_0 , \quad (716)$$

where  $H_0$  is Hubble parameter of today Universe. Then the photon travels a comoving distance

$$\Delta r = 2H_0^{-1}(\sqrt{a_2} - \sqrt{a_1}) \quad (717)$$

The comoving horizon size when  $a = a_*$  is the distance a photon travels since the Big Bang

$$r_h(a_*) = 2H_0^{-1}\sqrt{a_*} . \quad (718)$$

The physical horizon size, as measured on the spatial hypersurface at  $a_*$  is therefore simply

$$d_h(a_*) = a_* r_h(a_*) = 2H_0^{-1} a_*^{3/2} = 2H_0^{-1} \frac{H_0}{H_*} = 2H_*^{-1} . \quad (719)$$

The horizon problem is simply the fact that CMB is isotropic to high degree of precision even though widely separated points on the last scattering surface are completely outside each other's horizons. When we look at the CMB we see the Universe at a scale factor  $a_{CMB} \approx 1/200$ . The comoving distance between a point on the CMB and an observer on Earth is

$$\Delta r = 2H_0^{-1}(1 - \sqrt{a_{CMB}}) \approx 2H_0^{-1} . \quad (720)$$

However, the comoving horizon distance for such a point is

$$r_h(a_{CMB}) = 2H_0^{-1}\sqrt{a_{CMB}} = 6 \times 10^{-2} H_0^{-1} . \quad (721)$$

Hence if we observe two widely separated parts of the CMB they will have non-overlapping horizons; different patches of the CMB sky were causally disconnected at recombination. On the other hand they are observed to be at the same temperature at high precision. This is the core of the famous horizon problem.

## **Problem of the large-scale homogeneity and isotropy of the Universe**

As we argued in introduction all cosmological models are based on the presumption of absolutely homogeneous and isotropic Universe. Of course Universe is not absolutely homogeneous and isotropic at now at least on small scale and hence there is no reason to believe that it was homogeneous at its beginning. The most natural assumption is that the initial conditions at points that are sufficiently far from one another were chaotic and uncorrelated. On the other hand it was shown by Collins and Hawking that class of the initial conditions for which the Universe tends asymptotically (at large  $t$ ) to Friedmann Universe is one of measure zero among all possible conditions. In other words according to this classical analysis Friedmann model is very improbable. This is the problem of large scale homogeneity and isotropy.

### **The galaxy formation problem**

We know that Universe contains many inhomogeneities as stars, galaxies and so on. In order to explain the origin of galaxies one have to presume an existence of initial inhomogeneities whose spectrum is usually taken to be almost scale invariant. For a long time the origin of such density inhomogeneities remained obscure.

### **The baryon asymmetry problem**

This is the problem why the Universe is added almost entirely of matter with almost no antimatter and why on the other hand the number of baryons is much less than number of photons  $\frac{n_B}{n_\gamma} \sim 10^{-9}$ .

### **The domain wall problem**

It is natural to presume that the symmetry breaking occurs independently in all causally unconnected regions of Universe. Then at all these regions that comprise Universe at the time of symmetry-breaking phase transition, both field  $\phi = +\mu/\sqrt{\lambda}$  and the field  $\phi = -\mu/\sqrt{\lambda}$ . Domains filled by the field  $\phi = +\mu/\sqrt{\lambda}$  are separated from those with the field  $\phi = -\mu/\sqrt{\lambda}$  by domain walls. It can be shown that the energy density of these walls is so high so that their existence is inconsistent with cosmological consequences. Since the theories based on the spontaneously breaking of gauge symmetry are very appealing and since in these theories domain walls arise in natural way we meet *Domain wall problem*. In other words how to deal with such theories in cosmology.

### **The primordial monopole problems**

This problem is closely related to the domain wall problems. Many theories based on symmetry-braking mechanism can produce another nontrivial structures that are nontrivial configurations of the scalar and gauge fields and that are stable. However it can be shown that these objects are very massive. Moreover it can be also shown

that the monopole density at present would be comparable with the baryon density. Thanks to the enormous massivity these objects we obtain that the Universe filled of monopoles is  $10^{15}$  higher than the critical density. This implies that Universe filled with such matter would have collapsed long ago. The explanation of the mechanism how to deal with monopoles is one of the most important problems in cosmology.

## Unwanted Relics

We have argued that for correct description of the early Universe the models of particle physics should be present. However these models contain monopoles and other topological defects. However the energy density of these objects can be very big and hence the monopole abundance in GUT is serious problem for cosmology if GUT have anything to do with reality.

### 6.3 Inflation as a solution

#### 6.3.1 The General Idea of Inflation

The horizon problem is an extremely serious problem for the standard cosmology. Cosmological inflation is mechanism that can solve this problem.

The main idea is that the Universe undergoes a period of accelerated expansion defined as a period when  $\ddot{a} > 0$  at early times. The effect of this acceleration is to quickly expand a small region of space to huge size. At this process the spatial curvature of the Universe is reduced and consequently we make the Universe extremely close to flat. In addition, the horizon size is greatly increased so that distant points on the CMB actually are in causal contact and unwanted relics are diluted, solving the monopole problem. Finally, quantum fluctuations imply that inflation cannot smooth out the Universe with perfect precision, so there is a spectrum of remnant density perturbations.

The general idea of inflation is that before Hot Big Bang (but after Planck era) the Universe was in vacuum-like state and then it went through the era of the exponential expansion

$$a(t) = \text{const} \cdot e^{\int H_{infl} dt} , \quad (722)$$

where  $H_{infl}$  is almost constant in time. Due to the exponential expansion a small patch of the Universe expands to great size. Let us presume that the duration of inflation  $t_{infl}$  exceeds 140 Hubble times

$$t_{infl} > \frac{140}{H_{infl}} . \quad (723)$$

Let us also presume that the size of the patch is initially at the order Planck size  $l_P = \frac{1}{M_P} \sim 10^{-33} \text{cm}$ . Then at the time  $t_{inf}$  the size exceeds the present horizon size  $l_{H,0} \sim 10^{28} \text{cm}$ . It is also clear the Universe flattens out, any initial inhomogeneities are diluted out. In the end of inflation, the Universe becomes spatially flat, homogeneous



and isotropic at exponentially large spatial scales. This solves the horizon and flatness problems.

A natural way to ensure that the Universe expands exponentially is to assume that the matter at inflationary stage is in the vacuum-like state characterized with the energy density  $\rho_{infl}$  that is almost constant in time. At some point this energy density should transform into conventional energy density of hot plasma. This transformation is called reheating and after reheating the Hot Big Bang era begins. During reheating, huge entropy is released and this solves the entropy problems.

## 6.4 Many models of inflation

Before we come to the more detailed study of the question how the inflation works we give summary of some models of the inflation theory. The common property of these model is that the matter with suitable equation of state is in the form of the scalar field(s).

The initial model of inflation (“old inflation model”) was based on idea that the scalar field  $\phi$  was initially in a false vacuum with large potential energy. To end of inflation, a quantum tunneling from the false vacuum to the true vacuum was performed. However this model has the problem that it leads to an initially microscopical bubble of the true vacuum which cannot grow to contain our present observed Universe. Hence the attention shifted to models in which the scalar field  $\phi$  slowly rolls during the inflation.

Models of scalar field-driven inflation can be divided into three groups:

- **Small-field inflation**
- **Large-field inflation**
- **Hybrid inflation**

*Small field inflationary models* are based on ideas from spontaneous symmetry breaking in particle physics. For example, let us consider the scalar field with the potential in the form

$$V(\phi) = \frac{1}{4}(\phi^2 - \sigma^2)^2, \quad (724)$$

where we interpret  $\sigma$  as the symmetry breaking scale and  $\lambda$  as a dimensionless coupling constant. The main idea of the small-field models (“new inflation”) was that the scalar field starts to roll close to its symmetric point  $\phi = 0$ . At sufficient high temperature  $\phi = 0$  is a stable ground state of the one-loop finite temperature effective potential  $V_T(\phi)$ . When the temperature drops below to some value that is smaller than  $T_c$ ,  $\phi = 0$  becomes unstable local minimum of  $V_T(\phi)$  and  $\phi$  can roll towards a ground state of the zero temperature potential (724) with

$$\phi_{gr} = \pm\sigma. \quad (725)$$

The problem of this model is that the slow-roll conditions <sup>12</sup>

$$\left(\frac{V'}{V}\right)^2 M_P^2 \ll 1, \frac{V''}{V} M_P^2 \ll 1 \quad (726)$$

that for the potential (724) take the form

$$\frac{\phi^2}{(\phi^2 - \sigma^2)^2} \ll \frac{1}{M_P^2}, \frac{3\phi^2 - \sigma^2}{(\phi^2 - \sigma^2)^2} \ll \frac{1}{M_P^2} \quad (727)$$

and that have to be valid for inflation to works imply that

$$\sigma \sim M_P. \quad (728)$$

However this is in contradiction with the fact that we have to presume that  $\sigma$  is some symmetry breaking scale of the standard quantum field theory while  $M_P$  is the scale of the quantum gravity regime where the approximation of the quantum field theory in curved space time cannot be valid. The potential (724) can be changed to satisfy the slow-roll conditions however this procedure needs several fine-tuning of the shape of the potential. A further problem of the slow-roll model is that the initial field velocity must be constrained to be small which is again fine-tuned initial condition.

As the alternative to the small-field inflationary models are *large-field inflation models* that are also known as *chaotic inflation*. The simplest example is provided by a massive scalar field with the potential

$$V(\phi) = \frac{1}{2}m^2\phi^2. \quad (729)$$

In the chaotic inflation scenario it is presumed that the scalar field rolls towards the origin from large values of  $|\phi|$ . The slow roll conditions for the potential (??) takes the form <sup>13</sup>

$$|\phi| \gg M_P. \quad (730)$$

Values of  $|\phi|$  comparable or larger than  $M_P$  are also required in other realizations of large-field inflations. The question is whether such a model can consistently be embedded in a realistic particle physics model, as for example supergravity. In many these models  $V(\phi)$  receives supergravity-induced correction terms that destroys the flatness of the potential for  $|\phi| > M_P$ . The value  $m \sim 10^{13} GeV$  is required in order to obtain the observed amplitude of density fluctuations.

With two scalar fields it is possible to construct a class of models which combine some of the nice features of large-field inflation models which is large set of the initial conditions that lead to inflation with the small-field inflation where the inflation takes place at sub-Planckian field values. These models are known as *Hybrid inflation*. For example, let us consider two scalar fields  $\phi$  and  $\xi$  with the potential

$$V(\phi, \xi) = \frac{1}{4}\lambda_\xi(\xi^2 - \sigma^2)^2 + \frac{1}{2}m^2\phi^2 - \frac{1}{2}g^2\phi^2\xi^2. \quad (731)$$

<sup>12</sup>Precise definition of these conditions will be given in next section

<sup>13</sup>Note that the dimensional analysis that implies that  $V$  has dimension  $[V] = 4$  in mass unit implies that  $[\phi] = 1$ .

In the absence of the thermal equilibrium it is natural to assume that  $|\phi|$  begins at large values. For large  $\phi$  the term

$$\frac{1}{2}g^2\phi^2\xi^2$$

that serves as an effective mass term for  $\xi$  is positive and hence  $\xi$  has stable minimum at  $\xi = 0$ . The parameters in (731) are chosen such that  $\phi$  is slowly rolling for values of  $|\phi|$  somewhat smaller than  $M_P$  but the parameters are chosen in such a way that the potential energy for these fields values is dominated by the first term in (731). The field  $\phi$  is slowly rolling whereas the potential energy is determined by the contribution from  $\xi$ . Once  $\phi$  drops to the value

$$|\phi|_c = \frac{\sqrt{\lambda_\xi}}{g}\sigma . \quad (732)$$

For this value the effective potential for  $\xi$  takes the form

$$V(\phi_c, \xi) = \frac{\lambda_\xi}{4}(\phi^2 - 2\sigma^2)^2 \quad (733)$$

that has three extrema

$$\xi_0 = 0 , \quad V(0) = \lambda_\xi\sigma^4 \quad \xi_\pm = \pm\sqrt{2}\sigma , \quad V(\phi_\pm) = 0 \quad (734)$$

that clearly shows that the configuration with  $\xi = 0$  is unstable and decays to the one of the states  $\xi_\pm = \pm\sqrt{2}\sigma$ . Since in this case the ground state is not unique we have a possibility of the formation of topological defects at the end of the inflations.

After the slow-roll conditions break down the period of inflation ends and the inflation begins to oscillate around its ground state. Since the inflation field  $\phi$  couples to other matter fields the energy of the Universe, that at the end of the period of inflation is stored completely in  $\phi$  is transferred to the matter fields of the particle physics Standard model. The description of this process is very complicated,

## 6.5 How does the inflation work

The key property of the laws of physics that makes inflation possible is the existence of states of negative pressure. To recognize the effect negative pressure let us again consider Friedmann equation

$$\begin{aligned} \ddot{a} &= -\frac{4\pi G}{3}(\rho + 3p)a , \\ H^2 &= \frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho - \frac{k}{a^2} , \\ \dot{\rho} &= -3H(\rho + p) . \end{aligned} \quad (735)$$

Once again, the metric is given by Robertson-Walker form

$$ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right] , \quad (736)$$

where  $k = 0, 1, -1$ . From the first equation in (735) we see that positive pressure ( $\rho$  is always positive) contributes to the deceleration of the Universe while the negative pressure can cause acceleration. In other words, negative pressure produces a repulsive form of gravity.

The characteristic property of the inflation is that the physical wavelengths grow *faster* than the size of the Hubble radius

$$d_H = \frac{a(t)}{\dot{a}(t)} = \frac{1}{H}$$

as follows from the fact

$$\frac{\dot{\lambda}_{phys}}{\lambda_{phys}} = \frac{1}{a(t)\lambda_0} \frac{d(a(t)\lambda_0)}{dt} = \frac{\dot{a}}{a} = H = \frac{\dot{d}_H}{d_H} + d_H \frac{\ddot{a}}{a}. \quad (737)$$

This equation shows that during inflation when  $\frac{\ddot{a}}{a} > 0$  the physical wavelengths become larger than the Hubble radius. However when the physical wavelength becomes larger than Hubble radius it is causally disconnected from physical processes. The inflationary era is followed by the radiation dominated and matter dominated stages where the Hubble radius grows faster than the scale factor and the wavelengths that were outside now re-enter Hubble radius. This is the basic mechanism how the inflation explains the generation of temperature fluctuations and also the origin of the emergence of large scale formation: Briefly, quantum fluctuations generated early in the inflationary stage exit the Hubble radius during inflation and then eventually re-enter during the matter dominated era.

Remarkably, we can easily find form of the matter that produces negative pressure.

## 6.6 Slowly-Rolling Scalar Fields

In order the inflation to solve the problems of the standard cosmology it must be active at extremely early times. Thus we would like to study the earliest times in the Universe amenable to classical description. It is expected that this is around the Planck time  $t_P$ . For that reason we will retain values of Planck mass in the equation of this section. As we will see there are many models of inflation. In this section we will restrict ourselves to the study of the model of *chaotic inflation*.

Consider matter in the form of the scalar field  $\phi$  that is described with the action

$$S_{matter} = - \int d^4x \sqrt{-g} \left[ \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + V(\phi) \right]. \quad (738)$$

In field theory the stress energy tensor is defined as

$$T_{\mu\nu} = - \frac{2}{\sqrt{-g}} \frac{\delta S_{matter}}{\delta g^{\mu\nu}} \quad (739)$$

that for the action of the form  $S = - \int d^4x \sqrt{-g} \mathcal{L}$  takes the form

$$T_{\mu\nu} = -g_{\mu\nu} \mathcal{L} + 2 \frac{\delta \mathcal{L}}{\delta g^{\mu\nu}}, \quad (740)$$

where we have used

$$\frac{\delta \sqrt{-g}}{\delta g^{\mu\nu}} = -\frac{1}{2} \sqrt{-g} g_{\mu\nu}. \quad (741)$$

More precisely, for the action (738) the stress energy tensor takes the form

$$T_{\mu\nu} = (\nabla_\mu \phi)(\nabla_\nu \phi) - g_{\mu\nu} \left[ \frac{1}{2} g^{\alpha\beta} (\nabla_\alpha \phi)(\nabla_\beta \phi) + V(\phi) \right], \quad (742)$$

where for the scalar field  $\phi$  we have  $\nabla_\alpha \phi = \partial_\alpha \phi$ . Let us now restrict to the homogenous case in which all quantities depend only on cosmological time  $t$  and we also set  $k = 0$ . A homogenous real scalar field behaves as a perfect fluid with

$$\rho = T_{00} = \frac{\dot{\phi}^2}{2} + V(\phi). \quad (743)$$

The other components of the stress energy tensor take the form

$$T_{ij} = -g_{ij} \left( \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + V \right) + \partial_i \phi \partial_j \phi. \quad (744)$$

If we define pressure as

$$p = \frac{1}{3} \sum_{i=1}^3 T_{ii} \quad (745)$$

we get

$$p = \frac{\dot{\phi}^2}{2} - V(\phi). \quad (746)$$

Thus any state which is dominated by the potential energy of a scalar field will have negative pressure.

Note also that the equation of motion for the scalar field are given by

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0, \quad (747)$$

that can be thought of as a usual equation of motion for a scalar field in Minkowski space but with a friction term due to the expansion of the Universe. The Friedmann equation with such a field as a sole energy source is

$$H^2 = \frac{8\pi G}{3} \left[ \frac{1}{2} \dot{\phi}^2 + V(\phi) \right]. \quad (748)$$

The accelerated expansion occurs if the Universe is dominated by an energy component that approximates a cosmological constant. In that case the associated expansion rate will be exponential. From (743) we see that for  $\dot{\phi}^2 \ll V(\phi)$  the potential energy of the scalar field is the dominant contribution to both the energy

density and pressure and the resulting equation of state is  $p = -\rho$  that has the same form as the state equation for cosmological constant.

More technically, the *slow-roll approximation* for inflation involves neglecting the  $\ddot{\phi}$  term in (747) and neglecting the kinetic energy compared to  $\phi$  compared to the potential energy. In this case the scalar field equation of motion and the Friedmann equation become

$$\begin{aligned}\dot{\phi} &= -\frac{V'}{3H}, \\ H^2 &= \frac{8\pi G}{3}V(\phi).\end{aligned}\tag{749}$$

The slow roll conditions are conveniently characterized with so named *slow roll parameters*

$$\epsilon = \frac{M_P^2}{2} \left( \frac{V'}{V} \right)^2, \eta = M_P^2 \frac{V''}{V},\tag{750}$$

where

$$8\pi G = M_p^{-2}.\tag{751}$$

It is easy to see that the slow-roll conditions yield inflation. Recall that inflation is defined by

$$\frac{\ddot{a}}{a} > 0\tag{752}$$

that using the fact that

$$\dot{H} = \frac{\ddot{a}a - \dot{a}^2}{a^2} \Rightarrow \frac{\ddot{a}}{a} = \dot{H} + \left( \frac{\dot{a}}{a} \right)^2$$

or alternatively

$$\frac{\ddot{a}}{a} = \dot{H} + H^2.\tag{753}$$

Then the inflation occurs when

$$\frac{\dot{H}}{H^2} > -1.\tag{754}$$

But in slow roll

$$2\dot{H}H = \frac{8\pi G}{3}V'\dot{\phi} = -\frac{8\pi G}{9}\frac{V'^2}{H}\tag{755}$$

and hence

$$\frac{\dot{H}}{H^2} = -\frac{4\pi G}{9}\frac{V'^2}{H^4} = -\frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2 = -\epsilon\tag{756}$$

which will be small. Smallness of the second parameter  $\eta$  ensures that inflation will continue for a sufficient period.

It is useful to have a general expression that describes how much inflation occurs once it has begun. Such a quantity is the *number of e-folds* defined by

$$N(t) \equiv \ln \left( \frac{a(t_{end})}{a(t)} \right).\tag{757}$$

Usually we are interested in how many e-folds occur between a given field value  $\phi$  and the field value at the end of inflation  $\phi_{end}$  where  $\epsilon(\phi_{end}) = 1$ . To do this we express  $N(t)$  as

$$\begin{aligned}
N(t) &= \ln \left( \frac{a(t_{end})}{a(t)} \right) = \int_{a(t)}^{a(t_{end})} \frac{da'}{a'} = \\
&= \int_t^{t_{end}} \frac{\dot{a}}{a} dt' = \int_t^{t_{end}} H dt' = \int_{\phi}^{\phi_{end}} H \frac{d\tilde{\phi}}{\dot{\phi}} = \\
&= -3 \int_{\phi}^{\phi_{end}} H^2 \frac{d\tilde{\phi}}{V'} = -\frac{1}{M_P^2} \int_{\phi}^{\phi_{end}} \frac{V}{V'} d\tilde{\phi} .
\end{aligned} \tag{758}$$

The problem of the initial conditions for inflation is very subtle. In case of chaotic inflation in which we assume that the early Universe emerges from the Planck epoch with the scalar field taking different values in different part of the Universe with typically Planckian energies.

Let us now consider some examples of the potential that could lead to inflation. We start with the simple monomial

$$V = \lambda M_P^{4-\alpha} \phi^\alpha . \tag{759}$$

For potential above we obtain following slow roll parameters

$$\epsilon = \frac{\alpha^2 M_P^2}{2\phi^2} , \quad \eta = \alpha(\alpha - 1) \frac{M_P^2}{\phi^2} . \tag{760}$$

Inflation starts at a large value of  $\phi$  and the inflaton then rolls slowly towards the minimum with increasing  $\epsilon$  and  $\eta$ . Inflation ends when the slow roll conditions are saturated,

$$\phi \sim \lambda M_P . \tag{761}$$

The number of e-foldings we obtain before this happens is given by

$$\begin{aligned}
N &= \ln \frac{a(t_e)}{a(t_i)} = \left( H dt = \frac{da}{a} \Rightarrow \int H dt = \ln(a_f) - \ln(a_i) \right) \int_{t_i}^{t_e} H dt = \\
&= \int_{\phi_i}^{\phi_e} H \frac{d\phi}{\dot{\phi}} = - \int_{\phi_i}^{\phi_e} \frac{3H^2}{V'} d\phi = -\frac{1}{M_P^2} \int_{\phi_i}^{\phi_e} \frac{V}{V'} d\phi = -\frac{1}{M_P^2 \alpha} \int_{\phi_i}^{\phi_e} \phi d\phi = \\
&= \frac{\phi_i^2}{2M_P^2 \alpha} - \frac{1}{4} \approx \frac{1}{2\alpha M_P^2} \phi_i^2
\end{aligned} \tag{762}$$

that implies

$$\phi_i = \sqrt{2\alpha N} M_P \gg M_P . \tag{763}$$

Using this initial value  $\phi_i$  we can determine the values of slow roll parameters at  $t_i$

$$\epsilon_i \sim \frac{\alpha}{4N} , \eta \sim \frac{\alpha - 1}{N} . \tag{764}$$

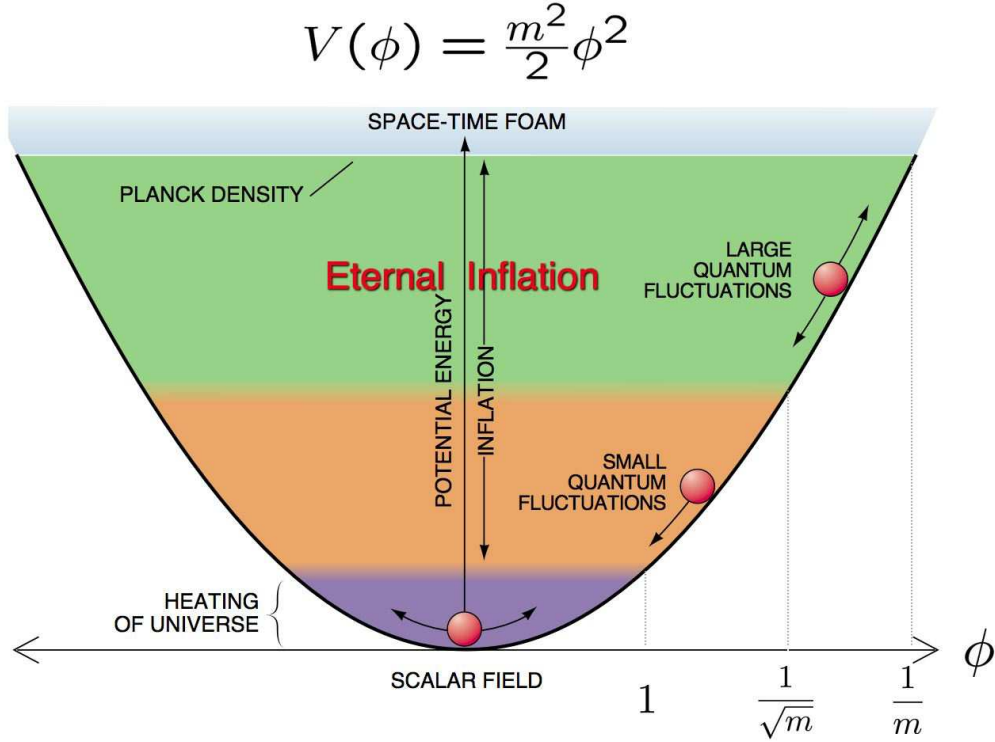


Figure 1: As an example that illustrates the main idea of inflation is motion of the scalar field in the theory with  $V(\phi) = \frac{m^2}{2}\phi^2$ . Several different regimes are possible, depending on the value of the field  $\phi$ . If the potential energy density of the field is greater than the Planck density  $M_p^4 = 1$ ,  $\phi \sim m^{-1}$ , quantum fluctuations of space-time are so strong that one cannot describe it in usual terms. Such a state is called space-time foam. At a somewhat smaller energy density (for  $m \sim V(\phi) \sim 1$ ,  $m^{-1/2} \sim \phi \sim m^{-1}$ ) quantum fluctuations of space-time are small, but quantum fluctuations of the scalar field  $\phi$  may be large. Jumps of the scalar field due to quantum fluctuations lead to a process of eternal self-reproduction of inflationary universe which we are going to discuss later. At even smaller values of  $V(\phi)$  (for  $m^2 \sim V(\phi) \sim m$ ,  $1 \sim \phi \sim m^{-1/2}$ ) fluctuations of the field  $\phi$  are small; it slowly moves down as a ball in a viscous liquid. Inflation occurs for  $1 \sim \phi \sim m^{-1}$ . Finally, near the minimum of  $V(\phi)$  (for  $\phi \sim 1$ ) the scalar field rapidly oscillates, creates pairs of elementary particles, and the universe becomes hot.



Another example of the inflation potential is

$$V = V_0 e^{-\sqrt{\frac{2}{p}} \frac{\phi}{M_P}} \quad (765)$$

with the slow roll parameters

$$\epsilon = \frac{1}{p}, \eta = \frac{2}{p}. \quad (766)$$

Recall that for this potential we can combine the equation of motion to get

$$\dot{\phi} = -\frac{M_P}{\sqrt{3}} \frac{V'}{\sqrt{V}} = \sqrt{\frac{2}{3p}} \sqrt{V} \quad (767)$$

that has the solution

$$V \sim \frac{3M_4^2 p^2}{t^2} \quad (768)$$

and hence

$$H^2 \sim \frac{p^2}{t^2} \Rightarrow \ln a \sim p \ln t \Rightarrow a \sim t^p. \quad (769)$$

To gain more insight in the idea of inflation note that in most inflation models the energy density  $\rho$  is approximately constant leading to exponential expansion of the scale factor. In fact, using  $p = -\rho$  in the Friedmann equation we get

$$\ddot{a} = \frac{8\pi G}{3} \rho a \quad (770)$$

that in the approximation of  $\rho = \text{const}$  can be solved with the ansatz  $a = e^{\lambda t}$  that inserted in the equation above implies

$$\lambda^2 - \frac{8\pi G}{3} \rho_f = 0 \Rightarrow \lambda = \sqrt{\frac{8\pi G}{3} \rho_f}, \quad (771)$$

where  $\rho_f$  is constant energy density.

In the original model of inflation the state that drove the inflation involved a scalar field in a local (but no global) minimum of its potential energy. The scalar field state employed in the original version of inflation is called a *false vacuum* since the state temporally acts as if it were the state of lowest possible energy density. Classically this state is stable that there is no possibility how the scalar field crosses a potential energy barrier that separates it from the states of lower energy. However quantum mechanically this state would decay through tunneling. Initially it was hoped that this tunneling could successfully ends an inflation but it was soon found that the randomness of the bubble formation when the false vacuum decayed would produced large inhomogeneities.

This problem was solved in the *new inflation scenario* proposed by Linde. In this theory the inflation is driven by an scalar field with the potential in the form in the form

$$V = -\frac{A}{2} \phi^2 + \frac{B}{4} \phi^4 \quad (772)$$

that has minima at  $\phi = 0, V(0) = 0$  that is a false vacuum and also minima at  $\phi_{\pm} = \pm\sqrt{\frac{A}{B}}$  with  $V(\phi_{\pm}) = -\frac{A^2}{4B}$ . This scalar field is called *inflaton*. If this theory the inflation is driven by the scalar field on the plateau of the potential energy diagram (region around the point  $\phi = 0$ ). If this plateau is flat enough, such a state can be stable enough for successful inflation. Soon after the introduction of the new inflation scenario it was shown that the inflaton potential need not have either a local minimum or a gentle plateau: This new scenario is known as a *chaotic inflation*.

## 6.7 Solving the problems of standard cosmology

To demonstrate the fact that inflation can solve the problems of the standard cosmology let us again consider the potential with the simplest form

$$V(\phi) = \frac{1}{2}m^2\phi^2 . \quad (773)$$

With this potential the Friedmann equation takes the form

$$\dot{\phi} = -\frac{m^2\phi}{3H} , H = \frac{m}{\sqrt{6}M_P}\phi \quad (774)$$

and we find

$$\phi = \phi_0 - \sqrt{\frac{2}{3}}\frac{m}{M_P}t \quad (775)$$

and

$$a = C \exp\left[\frac{m}{\sqrt{6}M_P}\left(\phi_0 t - \frac{\sqrt{2}M_P}{2\sqrt{3}}t^2\right)\right] = a_0 \exp\left[\frac{1}{4M_P^2}(\phi_0^2 - \phi^2)\right] . \quad (776)$$

The period of time during the solution above is valid ends at  $t \sim \Delta t$  at which

$$a(\Delta t) \sim a(0) \exp\left(\frac{1}{\epsilon^2}\right) . \quad (777)$$

If we take a typical value for  $m$  for which  $\epsilon < 10^{-4}$  we obtain

$$a(\Delta t) \sim a(0) \times 10^{2.7 \times 10^8} . \quad (778)$$

This has remarkable consequence. A proper distance  $L_P$  at  $t = 0$  will inflate to a size  $10^{10^8} cm$  after a time  $\Delta t \sim 5 \times 10^{-36} s$ . As we know the size of observable Universe today is  $H_0^{-1} \sim 10^{28} cm$ . Therefore, only a small fraction of the original Planck length comprises today's entire observable Universe.

### General arguments

Inflation is not really a theory, but instead it is a paradigm, or class of theories. Each specific model of inflation makes definitive predictions but the class of the models as a whole can be tested only by looking for generic features that are common for all models. Nevertheless, there are number of features of the Universe that seem to be characterize consequences of inflation. The basic arguments for inflation are as follows:

- *The Universe is big*

We know that Universe is very large; the visible part of the Universe contains about  $10^{90}$  particles. Most of scientists believe that the creation of Universe can be explained in scientific terms. Thus we think about the theory that could explain how the Universe got so big. Such a theory has to explain the number of particles,  $10^{90}$  or more. Simple way to get such a huge number, with small number as an input, is for the calculation to involve an exponential. The exponential expansion of inflation can explain this huge number. Moreover, inflationary cosmology suggests that, even though the observed Universe is incredible large, it is only a small fraction of the entire Universe.

- *The Hubble Expansion*

In standard FRW cosmology the Hubble expansion is part of the postulates that define the initial conditions. But the inflation offers the possibility of explaining how the Hubble expansion began.

- *Homogeneity and Isotropy*

As we have shown before the degree of uniformity of Universe is startling. The intensity of the cosmic microwave background radiation is the same in all directions. The cosmic background radiation was released 400000 years after big bang after the Universe cooled enough so that the opaque plasma neutralized into a transparent gas. The cosmic background radiation photons have mostly been traveling on straight lines since then so they provide an image of what the Universe looked like at 40000 years after big bang. The observed uniformity of radiation therefore implies that the observed Universe had become uniform in temperature by that time. In standard FRW cosmology a simple calculation shows that the uniformity could be established so quickly if signals could propagate at about 100 times the speed of light a proposition clearly contradicting the known laws of physics.

In inflationary cosmology the uniformity is easily explained. It is created initially on microscopic scales by thermal equilibrium processes and then inflation takes over and stretches the regions of uniformity to become large enough to encompass the observed Universe and more.

- *Flatness problem*

The problem concerns the value of the ration

$$\Omega_{tot} \equiv \frac{\rho_{tot}}{\rho_0} , \quad (779)$$

where  $\rho_{tot}$  is total mass density of the Universe and where  $\rho_0 = \frac{3H^2}{8\pi G}$  is the critical density that would make the Universe spatially flat (In  $\rho_{tot}$  the vacuum energy, it is nonzero, is included.)

There is now general agreement that  $\Omega_{tot}$  lies in the range

$$0.1 \leq \Omega_0 \leq 2 , \quad (780)$$

but it was very hard to pinpoint the value with more precision. Despite this large range the value of  $\Omega$  at early times is highly constrained, since  $\Omega = 1$  is an unstable equilibrium point of the standard model evolutions. Thus, if  $\Omega$  was exactly equal to one, it would remain exactly one forever. On the other hand if  $\Omega$  differs slightly from one in the early Universe, that difference-whether positive or negative, would be amplified with time. More generally, it can be shown that  $\Omega - 1$  grows as

$$\Omega - 1 \begin{cases} t & \text{(during the reaiation - dominated era)} \\ t^{2/3} & \text{(during the matter - dominated era)} \end{cases} \quad (781)$$

It was shown that at  $t = 1s$  when the processes of big bang nucleosynthesis were just beginning,  $\Omega$  must be equal to one to an accuracy of one part of  $10^{15}$ . Classical cosmology cannot explain this fact. In the context of modern particle physics cosmology, where we try to push all thinks all the way back to Planck scale  $10^{-43}sec$  the problem becomes even more severe.

While this extraordinary flatness of the early Universe has o explanation in classical FRW cosmology, it is a natural prediction for inflation cosmology. During the inflationary period, we have following relation

$$\Omega - 1 \approx e^{-2H_{inf}t} , \quad (782)$$

where  $H_{inf}$  is Hubble parameter during inflation. Thus, as long as there is a sufficient period of inflation,  $\Omega$  can start at almost any value and it will be driven to unity by the exponential expansion. Moreover, recent observation favored value of  $\Omega_0$  to be equal to  $\Omega_0 = 1.02 \pm 0.02$  according with recent WMAP results that is in beautiful agreement with inflation.

- *Absence of magnetic monopoles*

All grand unified theories predict that there should be, in the spectrum of possible particles,extremely massive particles carrying a net magnetic charge. It was shown in the context of the standard cosmology that magnetic monopoles would be produced so strongly so that they would overweigh everything else in the Universe by a factor of about  $10^{12}$ . Such a large mass density would cause that the Universe would come to its big crunch in about 30.000 years. Inflation is simplest known mechanism to eliminate monopoles from the visible Universe even though they are still in the spectrum of possible particles. The monopoles are eliminated simply due to the fact that inflation diluted them to a completely negligible level.

- *Anisotropy of the cosmic microwave background radiation*

The process of inflation smooths the Universe completely. On the other hand the density fluctuations are generated as inflation ends by the quantum fluctuations of the inflaton field. The general properties of these fluctuations are that are adiabatic, Gaussian, and nearly scale-invariant.

## 6.8 Reheating and Preheating

The great strength of inflation is its ability to redshift away all unwanted relics, such as topological defects. However during this process radiation and dust-like matter are similarly redshifted away to nothing so that at the end of inflation the Universe contains nothing but the inflationary scalar field condensate. The question is how does the matter arise and how is the Universe reheated?

The problem of reheating is very complicated and complex. In fact, the theory of reheating of the Universe after inflation is the most important application of the quantum theory of particle creation since almost all matter constituting the Universe was created during this process.

Now we sketch the standard picture.

Inflation ends when the slow-roll conditions are violated and the field begins to fall towards the minimum of the potential. Initially all energy density is in the inflation however now this energy is damped by two possible terms. Firstly, the expansion of the Universe naturally damps the energy density. Secondly, the inflation may decay into other particles, such as radiation or massive particles, both fermionic or bosonic. To describe this process one introduces a phenomenological decay term  $\Gamma_\phi$  into the scalar field equation. For example, if we consider the fermions only, then the rough expression for how the energy density evolves is

$$\dot{\rho}_\phi + (3H + \Gamma_\phi)\rho_\phi = 0 . \quad (783)$$

It can be shown that the inflaton undergoes damped oscillations and decays into radiation that equilibrates rapidly at a temperature known as the *reheat temperature*  $T_{RH}$ .

More precisely, early theory of reheating of Universe after inflation were based on the idea that the homogeneous inflation field can be represented as a collection of the particles of the field  $\phi$ . Put differently, we expect that inflation field has the same form as the ordinary quantum field in the flat spacetime. Then we can model reheating as a decay of each particle separately and this process can be studied in the standard perturbative description of particle decay. Typically, it takes thousands of oscillations of the inflaton field until it decays into usual elementary particles by this mechanism.

In case of bosons the situation is more complicated since now inflaton oscillations may give rise to parametric resonance that is characterised by an extremely rapid decay that results into distributions of products that are far from equilibrium and only much later settles down to an equilibrium distribution at energy  $T_{RH}$ . Such a decay due to the parametric resonance is known as *preheating*. The parametric resonance is an example of the coherent field effect that leads to the homogeneous field decay much faster than would be predicted by perturbative effects. These coherent effects produce high energy, nonthermal fluctuations that could have significance for understanding developments at the early Universe, as for example baryogenesis.

## 6.9 Quantum fluctuations

The key problem is how to test an inflation. The answer is the structure formation. As we have seen an important reason to involve an inflation is to make the Universe smooth and flat. However as we observe every day there is a large amount of structure in Universe. This structure can be traced back to subtle variations in the matter distribution during the time when the cosmic microwave background was released. The naive application of inflation in fact excludes such non-uniformity. It is a nice example of the application of the quantum field theory in curved background that explains the emergence of non-uniformity.

The main point is that inflation magnifies microscopic quantum fluctuation to cosmic size and hence provides seeds for structure formations. It is very interesting that then the details of physics at the highest energy scales is therefore reflected in the distribution of galaxies and other structures on large scales. More precisely, the fluctuations start at their smallest scales and grow larger (in wavelength) as the Universe expands. Eventually they become larger than the horizon and free. Intuitively, the different parts of wave can no longer communicate with each other since light can not keep up with the expansion of Universe. This is a consequence of the fact that the scale factor grows faster than the horizon which is a defining property of an accelerating and inflating Universe. At a later time, when inflation stops, the scale factor will start to grow slower than the horizon and the fluctuations will eventually come back within the causal horizon. The fluctuations will then appear as acoustic waves in the plasma and hence they will affect the CMB.

Let us now study this problem in more details. We assume that metric as well as the inflaton can be split into a classical background piece and a piece due to fluctuations according to

$$\begin{aligned} g_{\mu\nu} &= g_{\mu\nu}^{(0)} + h_{\mu\nu}(\tau, \mathbf{x}) , \\ \phi &= \phi^{(0)} + \delta\phi(\tau, \mathbf{x}) , \end{aligned} \tag{784}$$

where for convenience we have introduced *conformal time*  $\tau$  such that the metric is given by

$$ds^2 = a(\tau)^2(d\tau^2 - d\mathbf{x}^2) . \tag{785}$$

Since the background metric is homogenous it is convenient to Fourier transform the fluctuation mode  $\delta\phi$  as

$$\delta\phi(\tau, \mathbf{x}) = \frac{1}{(2\pi)^{3/2}} \int d\mathbf{k} \delta\phi_{\mathbf{k}} e^{i\mathbf{k}\mathbf{x}} . \tag{786}$$

Since we can presume that fluctuation are small in magnitude we can neglect the potential term for the fluctuation mode  $\delta\phi$  so that its equation of motion takes to form

$$\frac{1}{\sqrt{-g}} \partial_\mu [\sqrt{-g} g^{\mu\nu} \partial_\nu \delta\phi] = 0 \tag{787}$$

that using the (785) takes the form

$$\frac{1}{a^2}\delta\phi'' + \frac{2a'}{a}\delta\phi' - \frac{1}{a^2}\partial_i\partial^i\delta\phi = 0 , \quad (788)$$

where  $(\dots)' = \frac{d(\dots)}{d\tau}$ . Finally, using (786) we obtain differential equation for mode  $\delta\phi_{\mathbf{k}}$

$$\delta\phi_{\mathbf{k}}'' + 2\frac{a'}{a}\delta\phi_{\mathbf{k}}' + k^2\delta\phi_{\mathbf{k}} = 0 . \quad (789)$$

If we introduce the rescaled mode  $\mu_{\mathbf{k}} = a\delta\phi_{\mathbf{k}}$  so that

$$\delta\phi_{\mathbf{k}}' = \frac{\mu_{\mathbf{k}}'}{a} - \frac{\mu_{\mathbf{k}}a'}{a^2} , \delta\phi_{\mathbf{k}}'' = \frac{\mu_{\mathbf{k}}''}{a^2} - 2\frac{\mu_{\mathbf{k}}'a'}{a^2} - \frac{\mu_{\mathbf{k}}a''}{a^2} + 2\frac{\mu_{\mathbf{k}}(a')^2}{a^3} \quad (790)$$

the equation (789) can be transformed into

$$\mu_{\mathbf{k}}'' + \left(k^2 - \frac{a''}{a}\right)\mu_{\mathbf{k}} = 0 . \quad (791)$$

It can be shown that the metric fluctuations can be reduced to two polarizations obeying an equation identical to the one for the scalar fluctuations. In what follows we will consider the scalar fluctuations only.

To proceed let us presume that the conformal factor depend on conformal time as

$$a \sim \tau^{1/2-\nu} , \quad (792)$$

where  $\nu$  is a constant. An important example is  $a \sim e^{Ht}$  with  $H = \text{const.}$  where the change of coordinates gives

$$\frac{d\tau}{dt} = \frac{1}{a(t)} = e^{-Ht} \Rightarrow e^{-Ht} = -H\tau \Rightarrow a(\tau) = -\frac{1}{H\tau} . \quad (793)$$

Comparing with (792) we find that  $-1 = 1/2 - \nu \Rightarrow \nu = 3/2$ . Note also that the physical range of  $\tau$  is  $-\infty < \tau < 0$ . Using now (792) the equation for fluctuation (791) takes the form

$$\mu_{\mathbf{k}}'' + \left(k^2 - \frac{1}{\tau^2} \left(\nu^2 - \frac{1}{4}\right)\right)\mu_{\mathbf{k}} = 0 . \quad (794)$$

It is nice that the equation given above has solution known as a Hankel function. The general solution is given by

$$f_{\mathbf{k}}(\tau) = \frac{\sqrt{-\tau\pi}}{2} (C_1(k)H_{\nu}^{(1)}(-k\tau) + C_2(k)H_{\nu}^{(2)}(-k\tau)) , \quad (795)$$

where  $C_1(k)$  and  $C_2(k)$  are to be determined by initial conditions.

When we quantize this system we need to introduce oscillators  $a_{\mathbf{k}}(\tau)$  and  $a_{-\mathbf{k}}^{\dagger}(\tau)$  such that

$$\begin{aligned} \mu_{\mathbf{k}} &= \frac{1}{\sqrt{2k}} \left( a_{\mathbf{k}}(\tau) + a_{-\mathbf{k}}^{\dagger}(\tau) \right) , \\ \pi_{\mathbf{k}} &= \mu_{\mathbf{k}}'(\tau) + \frac{1}{\tau}\mu_{\mathbf{k}}(\tau) = -i\sqrt{\frac{k}{2}} \left( a_{\mathbf{k}}(\tau) - a_{-\mathbf{k}}^{\dagger}(\tau) \right) , \end{aligned} \quad (796)$$

obey standard commutation relation. It is important to stress that these operators are time dependent and can be expressed in terms of oscillators at a specific moment in time using the Bogolubov transformations

$$\begin{aligned} a_{\mathbf{k}}(\tau) &= u_{\mathbf{k}} a_{\mathbf{k}}(\tau_0) + v_{\mathbf{k}}(\tau) a_{-\mathbf{k}}^\dagger(\tau_0) , \\ a_{-\mathbf{k}}^\dagger(\tau) &= u_{\mathbf{k}}^*(\tau) a_{-\mathbf{k}}^\dagger(\tau_0) + v_{\mathbf{k}}^*(\tau) a_{\mathbf{k}}(\tau_0) , \end{aligned} \quad (797)$$

where

$$|u_{\mathbf{k}}(\tau)|^2 - |v_{\mathbf{k}}(\tau)|^2 = 1 \quad (798)$$

Then we can write the quantum field  $\mu_{\mathbf{k}}$  as

$$\mu_{\mathbf{k}}(\tau) = f_{\mathbf{k}}(\tau) a_{\mathbf{k}}(\tau_0) + f_{\mathbf{k}}^*(\tau) a_{-\mathbf{k}}(\tau_0) , \quad (799)$$

where

$$f_{\mathbf{k}}(\tau) = \frac{1}{\sqrt{2k}} (u_{\mathbf{k}}(\tau) + v_{\mathbf{k}}^*(\tau)) \quad (800)$$

is given in (795).

Now we come the key question that is *what are the initial conditions?* The usual choice is to consider the infinite past and choose a state annihilated by the annihilation operator

$$a_{\mathbf{k}}(\tau_0) |0, \tau_0\rangle = 0 , \quad (801)$$

for  $\tau_0 \rightarrow -\infty$ . However there is great debate about this choice in the past and is commonly known as a *Problem of transplanckian physics*. However we will not discuss this issue in this section and we will continue according to common practise. From (796) we get that

$$\pi_{\mathbf{k}}(\tau_0) |0, \tau_0\rangle = -i \sqrt{\frac{k}{2}} a_{-\mathbf{k}}^\dagger |0, \tau_0\rangle = -ik \mu_{\mathbf{k}}(\tau_0) |0, \tau_0\rangle . \quad (802)$$

Since the Henkel functions behave as for  $\tau_0 \rightarrow -\infty$

$$\begin{aligned} H_\nu^{(1)}(-k\tau) &\sim \sqrt{-\frac{2}{k\tau\pi}} e^{-ik\tau} , \\ H_\nu^{(2)}(-k\tau) &\sim H_\nu^{(1)*}(-k\tau) , \end{aligned} \quad (803)$$

we find that the vacuum choice corresponds to  $C_2(k) = 0$  and  $|C_1(k)| = 1$ .

In summary we have determined the quantum fluctuation and now we would like to see how they act on CMB. To do this we compute the size of the fluctuation according to

$$P(k) = \frac{4\pi k^3}{(2\pi)^3} \langle |\delta\phi_{\mathbf{k}}|^2 \rangle = \frac{k^3}{2\pi^2} \frac{1}{a^2} \langle |\mu_{\mathbf{k}}|^2 \rangle = \frac{k^3}{2\pi^2} \frac{1}{a^2} |f_{\mathbf{k}}|^2 = \frac{k^3}{2\pi^2} \frac{1}{a^2} \frac{|-\pi\tau|}{4} |H_\nu^{(1)}(-k\tau)|^2 \quad (804)$$



where  $\langle(\dots)\rangle$  mean the vacuum expectation value with respect to the state  $|0, \tau_0\rangle$ . Note that we are working in Heisenberg representation where the quantum mechanical operators evolve with time while states not.

Now we should calculate (804) at late times, namely  $\tau \rightarrow 0$ . In this limit the Hankel function behaves as

$$H_\nu^{(1)}(-k\tau) \sim \sqrt{\frac{2}{\pi}}(-k\tau)^{-\nu} \quad (805)$$

and hence (804) for  $\tau \rightarrow 0$  takes the form

$$P \sim \frac{1}{4\pi^2} \frac{1}{a^2} (-\tau)^{1-2\nu} k^{3-2\nu} \sim \frac{1}{4\pi^2} H^2 k^{3-2\nu} . \quad (806)$$

For  $\nu = 3/2$  and for slow roll when  $H$  for  $\tau \rightarrow 0$  is almost constant we can set the scale of the fluctuations. In fact, we find the well known scale invariant spectrum for  $\nu = 3/2$

$$P = \frac{1}{4\pi^2} H^2 . \quad (807)$$

It can be shown that this is more or less the whole story in case of the gravitational, or tensor, perturbations. The scalar fluctuations obey similar equation

$$P_s \sim \left(\frac{H}{\dot{\phi}}\right)^2 \frac{1}{4\pi^2} H^2 . \quad (808)$$

Usually we express the deviation from the scale invariance by introducing spectral indices according to

$$\begin{aligned} n_s - 1 &= \frac{d \ln P_s}{d \ln k} = 3 - 2\nu_s , \\ n_T &= \frac{d \ln P_T}{d \ln k} = 3 - 2\nu_T , \end{aligned} \quad (809)$$

where  $\nu_s$  refers to the scalar perturbations and  $\nu_T$  refers to the gravitational, or tensor perturbations. While not clear from our simplified analysis, the  $\nu$ 's need not be the same in the two cases. Observations show that  $n_s$  is very close to 1 consistent with the basic idea of inflation. It is extremely important to find any slight deviation from the scale invariant value which could give important information about the inflationary potential.

In fact, the flatness of the spectrum of density fluctuations, together with flatness of the Universe  $\Omega = 1$  constitute the two most robust predictions of inflationary cosmology. On the other hand there is an important difference between the prediction of flatness of the Universe and the flatness of the spectrum of perturbations of metric. It is difficult (though possible) to construct an inflationary model deviating from the prediction  $\Omega = 1$ . On the other hand the situation with the flatness of the spectrum is opposite: It is very difficult (though possible) to construct a model

with an exactly flat spectrum of perturbations of metric. In this sense, existence of a small deviation of the spectrum of inflationary perturbations from the flat spectrum (i.e. breaking of the scale invariance of the spectrum) represents an additional robust prediction of inflation.

## 6.10 Eternal Inflation

The eternal inflation scenario is based on the discovery of the process of self-reproduction of inflationary Universe. In fact, this process exists in old inflationary theory and in the new one but its significance was appreciated after discovery of eternal inflation in the simplest versions of the chaotic inflation scenario.

In the case of the new inflation, the exponential expansion occurs as the scalar field rolls from the false vacuum state at the peak of the potential energy towards to the true vacuum. Remarkably, it was shown very briefly after introduction of this model that the new inflation scenario is generically eternal. The key point is that, even though classically the field would roll off the hill, quantum mechanically there is always an amplitude for it to remain at the top.

The time scale for the decay of the false vacuum is controlled by

$$m^2 = - \left. \frac{\partial^2 V}{\partial^2 \phi} \right|_{\phi=0}, \quad (810)$$

which is the negative mass-squared of the scalar field when it is at the top of the hill on the potential. This is a free parameter of each model but  $m$  has to be small compared to Hubble constant or else the model does not lead to enough inflation.

In other words, for parameters chosen so that the inflation works, the exponential decay of false vacuum is slower than an exponential expansion. Even if the false vacuum is decaying, the expansion outruns the decay and the total volume of false vacuum actually increases with time rather than decreases. Thus inflation does not end at all places at once, instead it ends at localized patches, in a succession that continues at infinitum. Each patches essentially a whole Universe so that it can be said that inflation produces not just one Universe but an infinite number of Universes.

In the context of the chaotic Universe models the situation is slightly subtle even if it was shown by A. Linde that these models are eternal as well. We know that inflation occurs as the scalar field rolls down a hill of the potential energy diagram. As the field rolls down the hill quantum fluctuations will be superimposed on top of the classical motion. The best way to think about this is to ask what happens during one time interval of duration  $\Delta t = H^{-1}$  (Hubble time) in a region of one Hubble volume  $H^3$ . Suppose that  $\phi_0$  is the average value of  $\phi$  in this region at the start of the interval. By definition of a Hubble time the rate of the expansion is given by

$$a(t + \Delta t)/a(t) = e^{H\Delta t} = e. \quad (811)$$

This means that the change of volume is

$$V(t + \Delta t)/V(t) = a^3(t + \Delta t)H^{-3}/(a^3(t)H^{-3}) = e^3 \quad (812)$$

Since  $e^3 \approx 20$  we see that volume will expand by a factor 20. Since correlations are extended typically over one Hubble length it follows that in the end of the Hubble time the initial Hubble size region grows and breaks up into 20 independent Hubble sized regions.

During the time interval  $\Delta t$  the classical field  $\phi$  is rolling down the hill. On the other hand the classical change in the field  $\Delta\phi_{cl}$  during the time interval  $\Delta t$  is going to be modified by quantum fluctuations  $\Delta\phi_{qu}$  which can drive the field upwards or downward relative to classical trajectory. For any one of the 20 regions at the end of the Hubble time we can describe the change of the field as

$$\Delta\phi = \Delta\phi_{cl} + \Delta\phi_{qu} . \quad (813)$$

In the crude approximation the fluctuation is treated as a free quantum field. This fact implies that  $\Delta\phi_{qu}$  the quantum fluctuation averaged over one of the 20 Hubble volumes at the end, will have a Gaussian probability distribution, with a width of order  $H/2\pi$ . Then there is then a probability that the sum of the two terms on the right hand side will be positive—that the scalar field will fluctuate up instead down. As long as the probability is bigger than 1 in 20 then the number of inflating regions with  $\phi > \phi_{cl}$  will be larger at the end of the interval than at the beginning. This process will then go on forever so inflation will never end.

We see that the condition for an existence of eternal inflation is that the probability for the scalar field to go up must be bigger than  $1/e^3 \approx 1/20$ . It can be shown that criterion implies the relation

$$\frac{H^2}{\dot{\phi}_{cl}} > 3.8 \quad (814)$$

The probability that  $\Delta\phi$  is positive tends to increase as one considers larger and larger values of  $\phi$  so that sooner or later one reaches the point when the inflation becomes eternal. In fact for that reason we think that inflation is almost always eternal.

The eternal inflation follows from the observation that in many models large quantum fluctuations that are produced during inflation may locally increase the value of the energy density in some parts of the Universe. These regions then expand at a greater rate than their parent domains and quantum fluctuations in them lead to production of new inflationary domains which expand even faster. This leads to an eternal process of self-reproduction of the Universe.

In order to understand the process of self-reproduction we should remember that the processes separated by distances  $l$  greater than  $H^{-1}$  proceed independently one another. This is a consequence of the fact that during an exponential expansion the distance between any two objects separated by more than  $H^{-1}$  is growing with speed exceeding the speed of light. Then an observer in the inflationary Universe can see only the processes occurring inside the horizon of radius  $H^{-1}$ . In this sense any inflationary domain of initial radius exceeding  $H^{-1}$  can be considered as a separate mini-Universe.

In order to study the behavior of such a mini-Universe we should take into account the quantum fluctuations. Let us consider an inflationary domain of initial radius  $H^{-1}$  containing sufficient homogeneous field with initial value  $\phi \gg M_p^2$ . From the basic equation of the inflation model

$$H = \frac{m\phi}{\sqrt{6}}, \dot{\phi} = -m\sqrt{\frac{2}{3}} \quad (815)$$

we can deduce that during time interval  $\Delta t = H^{-1}$  the field inside the domain will be reduced by  $\Delta\phi$  that follows from the second equation above

$$\frac{\Delta\phi}{\Delta t} = -m\sqrt{\frac{2}{3}} \Rightarrow \Delta\phi = -m\sqrt{\frac{2}{3}}H^{-1} = -\frac{2}{\phi}, \quad (816)$$

where in the second step we have used the first equation in (815). On the other hand it can be shown that the quantum fluctuation of the field  $\phi$  is

$$|\delta\phi(x)| \approx \frac{H}{2\pi} = \frac{m\phi}{2\pi\sqrt{6}}. \quad (817)$$

Then we see that the magnitude of quantum fluctuation is larger than  $\Delta\phi$  for

$$\frac{m\phi^*}{2\pi\sqrt{6}} \approx \frac{2}{\phi^*} \Rightarrow \phi^* \sim \frac{5}{\sqrt{m}} \quad (818)$$

Then for  $\phi \ll \phi^*$  the decrease of the field  $\phi$  due to the classical motion is much greater than the average amplitude of the quantum fluctuations  $\delta\phi$  generated during the same time. On the other hand for  $\phi \gg \phi^*$  one has  $\delta\phi(x) \gg \Delta\phi$ . Since the typical wave length of the fluctuation mode is  $\sim H^{-1}$  it turns out that the whole domain after the time  $\Delta t = H^{-1}$  divides into following number of domain with almost homogenous field

$$a(\Delta t)H^{-1}/H^{-1} = e^{3HH^{-1}} \sim 20 \quad (819)$$

where the first expression express the physical size of the domain divided wave length. In summary, we get 20 separated domains of size  $H^{-1}$ , each containing almost homogenous field  $\phi - \Delta\phi + \delta\phi$ . In almost half of these domains the field  $\phi$  grows by  $|\delta\phi(x)| - \Delta\phi \approx H/2\pi$  rather than decreases. This means that the total volume of the Universe containing *growing* field  $\phi$  increases 10 times. During the next time interval  $\Delta t = H^{-1}$  this process repeats. Thus, after the two time intervals  $H^{-1}$  the total volume of the Universe containing the growing scalar field increases 100 times. In other words the Universe enters eternal process of self-reproduction.

One should however be careful with interpretation of this result. There is still an ongoing debate of whether eternal inflation is eternal only in the future or also in the past. To see this precisely where is the problem let us consider any particular time-like geodetic line at the stage of inflation. For any given observer following this geodetic the duration  $t_i$  of the stage of inflation on this geodesic will be finite.

On the other hand eternal inflation implies that if one takes all such geodesics and calculate the time  $t_i$  for each of them, then there will be no upper bound for  $t_i$ . In other words for each time  $T$  there will be such geodesic which experience inflation for the time  $t_i > T$ .

Similarly, if we study any particular geodesic in the past time direction, one can prove that it has finite length. In other words, the inflation in any particular point in the Universe should have a beginning at some time  $\tau_i$ . However there is no reason to expect that there is an upper bound for all  $\tau_i$  on all geodesics. If this upper bound does not exist, then eternal inflation is eternal not only in the future but also in the past.

Put differently, there is a beginning for each part of the Universe and there will be an end for inflation at any particular point. But there will be no end for the evolution of Universe as a whole in the eternal inflation scenario and at present we do not have any reason to believe that there was a single beginning of the evolution of the whole Universe at some moment  $t = 0$  which was traditionally associated with Big Bang.

If this scenario is correct, then physics alone cannot provide a complete explanation for all properties of our part of the Universe.

## 6.11 Eternal Inflation: Implications

Even if the other Universes that are created during the eternal inflation are too remote to imagine observing directly we will see that an eternal inflation has real consequences in terms of the way we extract predictions from theoretical models.

Firstly, the eternal inflation implies that all hypothesis about initial conditions for the Universe, such as the Hartle and Hawking no boundary proposal, the tunneling proposals by Vilenkin or Linde become totally divorced from observation. This follows from the presumption of the eternal inflation with its infinite production of pocket Universes. Then one can expect that the statistical properties of inflating region should approach a steady state which is independent on initial condition. Unfortunately there are great problems with the study of this steady state, for example, the properties of this state seems to depend crucially on the super-Planckian physics which we do not understand at present. It is however possible that string theory could be helpful with this study. More precisely, the same quantum fluctuations that make eternal inflation possible tend to drive the scalar field further and further up to potential energy curve so that some attempts that wanted to quantify the steady state require the imposition of some kind of a boundary condition at large  $\phi$ .

Even if the Universe forgets the details of its genesis the question, how the Universe began still remain interesting. To see this note that eternally inflating Universes continue forever once they start they are apparently not eternal into the past.<sup>14</sup>

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<sup>14</sup>This remark implies that the word “eternal” is not technically correct, we should rather speak

The second consequence of the eternal inflation is that the probability of the onset of inflation becomes totally irrelevant provided that the probability is not identically zero. In fact, this observation is slightly in the clash with our previous claim that chaotic inflation gives better result than the new inflation scenario. Even if the initial conditions necessary for the new inflation scenario cannot be justified on the basis of the thermal equilibrium as was proposed in original papers, in the context of the eternal inflation it is sufficient to conclude that the probability for the required initial conditions is nonzero.

The third consequence of the eternal inflation is the possibility that it offers to rescue the predictive power of theoretical physics. Here we mean the status of M-theory. Even if this theory by itself has uniqueness it appears that the vacuum is far from unique. Since the predictions will depend on the properties of the vacuum, the predictive power of M-theory could be limited. Eternal inflation however provides a possible mechanism to remedy this problem since it might help to constrain the vacuum state of the real Universe and hopefully significantly enhance the predictive power of M-theory. We must however stress that this is pure speculation whose validity is not justified but one can hope that recent works in the context of the string theory landscape could bring new light on this conjecture.

## 6.12 Does Inflation Need a Beginning

We know that according to the inflation scenario is eternal in the future. Then a natural question arises: Is it possible that the inflation is eternal into the past? There is a nice theorem by Borde, Guth and Vilenkin (2003) that proves that the answer to this question is no. There is of course no conclusion that an eternally inflating model must have a unique beginning and no conclusion that there is an upper bound on the length of all backwards-going geodesics from a given point. In other words this theorem shows that some new physics would be needed to describe the past boundary of the inflating region.

## 6.13 Inflation and Observations

It is very nice that inflation can make prediction which can be tested by cosmological observations. The inflationary prediction for nearly flat spectrum of density perturbation is in agreement with both our measurements of the CMB anisotropy and observations of structures in the Universe.

Let us also give another example where the inflation cosmology gives very nice explanation of the observation data.

Today, we have three-dimensional map of the distribution of galaxies in space that contain more than one hundred thousand galaxies. They clearly indicate that the luminous matter in the Universe is neither uniformly nor randomly distributed. We see clusters of galaxies, superclusters, filaments and voids that are regions of space

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about “semi-eternal” or “future-eternal” Universe.

empty of galaxies. The distribution can be quantified in terms of the luminosity power spectrum.

As we have also seen another observation window in cosmology is the cosmic microwave background radiation. This radiation is characterised by a surprising isotropy, in other words it looks the same from all different directions on the sky. However this radiation has also fractional level of a bit less than  $10^{-4}$  of anisotropies. These anisotropies can be characterised in terms of their angular power spectrum. The sky map (that is clearly two-dimensional of topology of sphere) of anisotropies is expanded in spherical harmonics  $Y_{lm}$

$$\frac{\Delta T}{T}(\theta, \phi) = \sum_{l=1}^{\infty} \sum_{m=-l}^l a_{lm} Y_{lm}(\phi, \theta) , \quad (820)$$

where  $\theta, \phi$  are the usual angles on the surface of two-sphere. It can be shown that the angular power spectrum of CMB has characteristic pattern of anisotropies. The challenge of cosmology is to explain both the overall isotropy of CMB and the specific patten of anisotropies.

In order to explain these observation structures we have to look to the very early Universe. The reason is that the Standard Big Bang cosmology that describes the cosmological evolution at late times where the notion “late times” means the times that includes period of nucleosynthesis and later implies that the length scales that are currently observed were outside the Hubble radius in the early times and no causal structure formation scenario is possible.

It is great success of inflationary cosmology that can explains all problems we listed above and also provides a causal mechanism for the origin of inhomogeneities in the Universe.

## References

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