<u>Prop. 1.7</u> Suppose G is a Lie group and set $\mathfrak{g} := T_e G$.

1. For any $X \in \mathfrak{g}$,

$$L_X(g) := T_e\lambda_g X \in T_g G \quad (\text{resp. } R_X(g) = T_e\rho^g X \in T_g G)$$

is a left- (resp. right) invariant vector field on G.

2. The maps $G \times \mathfrak{g} \longrightarrow TG$, defined by $(g, X) \longmapsto L_X(g)$ resp. $(g, X) \longmapsto R_X(g)$,

are diffeomorphisms.

(3) The map $X \mapsto L_X$ (resp. $X \mapsto R_X$) define linear isomorphisms with inverse $\xi \mapsto \xi(e)$ between $\mathfrak{g}$ and $\mathfrak{X}_L(G)$ (resp. $\mathfrak{X}_R(G)$).

Proof

(2) $(g, X) \mapsto L_X(g)$ is a diffeom.

Define the map $F: G \times \mathfrak{g} \to \underline{TG} \times TG$ given by

$F(g, X) = (0_g, X)$ $(0_g \in T_g G$ zero tangent vector at $g)$

It is smooth and so is $T\mu \circ F : G \times \mathfrak{g} \to TG$, which equals $(g, X) \mapsto L_X(g)$ by Lemma 1.5.

To show $T\mu \circ F$ is a diffeom. we construct a smooth inverse. Define $\tilde{F} : TG \to TG \times TG$ by

$$\tilde{F}(\underset{\substack{\uparrow \\ T_g G}}{\xi_g}) = (0_{g^{-1}}, \xi_g).$$

It is smooth (by ② of Lemma 1.5)

and so is $T\mu \circ \tilde{F}$, which is given by $\xi_g \mapsto T_g\lambda_{g^{-1}} \xi_g$.

$\Rightarrow \xi_g \mapsto (g, T_g\lambda_{g^{-1}} \xi_g) \in G \times \mathfrak{g}$ is smooth map $TG \to G \times \mathfrak{g}$, which is inverse $(g, X) \mapsto L_X(g) = T_e\lambda_g X$

Similarly, one proves the statement for $(g,X) \mapsto R_X(g)$.

① By ② L_X and R_X are smooth vector fields on G.

$$(\lambda_g^* L_X)(h) = (T\lambda_g)^{-1}\big(L_X(\lambda_g(h))\big) =$$

$$= T_{gh}\lambda_{g^{-1}}\, T_e\lambda_{gh}\, X = T_e\big(\lambda_{g^{-1}} \circ \underbrace{\lambda_{gh}}_{\lambda_g \circ \lambda_h}\big) X = L_X(h)$$

$\forall g, h \in G$.

Similarly, one checks right-inv. of R_X.

③ By ①, $X \longmapsto L_X$ defines a linear map $\mathfrak{g} \to \mathfrak{X}_L(G)$ and $L_X(e) = X$.

If $\xi \in \mathfrak{X}_L(G)$, then $\underline{\xi(g)} = (\lambda_{g^{-1}}^* \xi)(g) = T_e \lambda_g \xi(e)$

$$= \underline{L_{\xi(e)}(g)} \quad \forall g \in G.$$

$\square$

<u>Def. 1.8</u> G Lie group, $T_e G =: \mathfrak{g}$

① The diffeomorphism $G \times \mathfrak{g} \to TG$ given by $(g, X) \longmapsto L_X(g)$ (resp. $(g, X) \longmapsto R_X(g)$) of Prop. 1.7 is called the left- (resp. right) trivialization of tangent bundle $TG \to G$:

$$\begin{tikzcd} TG \arrow[rr, "\sim"] \arrow[dr] & & G \times \mathfrak{g} \arrow[dl, "pr_1"] \\ & G & \end{tikzcd}$$

vector bundle isomorphism.

$pr_1 \to$ natural proj. to first factor.

② For $X \in \mathfrak{g}$, L_X (resp. R_X) is called the left- (resp. right) invariant vector field on G generated by X.

Note that any L_X (resp. R_X) is nowhere vanishing on G and choosing a basis $X_1, \dots, X_n$ of the vector space $\mathfrak{g}$, $L_{X_1}(g), \dots, L_{X_n}(g)$ (resp. $R_{X_1}(g), \dots, R_{X_n}(g)$)

forms a basis of $T_g G$.

For any $g \in G$, $\lambda_g^* [\xi, \eta] = [\lambda_g^* \xi, \lambda_g^* \eta] = [\xi, \eta]$ $\forall \xi, \eta \in \mathfrak{X}_L(G)$.

The subspace $\underline{\mathfrak{X}_L(G)} \subseteq \mathfrak{X}(G)$ is a subalgebra of $(\mathfrak{X}(G), [\,,\,])$. Via isomorphism, $\mathfrak{g} \simeq \mathfrak{X}_L(G)$ of 3 of Prop. 1.7, we can transport $[\,,\,]$ to a bracket on $\mathfrak{g}$.

Def. 1.9 Suppose G is a Lie group. Then the tangent space $\mathfrak{g} := T_e G$ at $e \in G$ together with the map

$$[\,,\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$
$$[X,Y] := [L_X, L_Y](e)$$

is called the Lie algebra of G.

One has $L_{[X,Y]} = [L_X, L_Y]$ by construction.

From the properties of the Lie bracket of vector fields it follows:

Prop. 1.10 The map $[\,,\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ as in Def. 1.9 has the following properties:

(i) it is bilinear (over $\mathbb{R}$)

(ii) skew-symmetric: $[X,Y] = -[Y,X] \quad \forall X,Y \in \mathfrak{g}$

(iii) it satisfies the Jacobi-identity:

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

$\forall X,Y,Z \in \mathfrak{g}$.

Def. 1.11

① A real (resp. complex) Lie algebra is real (resp. complex) vector space $\mathfrak{g}$ equipped with a $\mathbb{R}$-(resp $\mathbb{C}$-) bilinear map $[\,,\,]: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying (ii) and (iii) of Prop. 1.10.

② A Lie algebra homomorphism (resp. isomorphism) between Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ is a linear map (resp. linear isomorphism) $\varphi: \mathfrak{g} \to \mathfrak{h}$ s.t.

$$\varphi([X,Y]) = [\varphi(X), \varphi(Y)] \quad \forall X, Y \in \mathfrak{g}.$$

(3) A subalgebra of a Lie algebra $\mathfrak{g}$ is a subspace $\mathfrak{h}$ of $\mathfrak{g}$ s.t. $[X,Y] \in \mathfrak{h}$ $\forall X,Y \in \mathfrak{h}$.

Examples

(1) Consider a vector space V $(\dim(V) < \infty)$ as Lie group w.r to addition $+$. Then the left-trivialization is just the usual identification of $TV = V \times V$ and the left-inv. vector fields corresp. to constant functions $V \to V$.

In particular, the Lie bracket of two left-inv. vf.

vanishes and hence the Lie algebra is $T_0 V = V$ equipped with the zero bracket $[v, w] = 0 \quad \forall v, w \in V$.

② Let G and H be Lie groups with Lie algebras $(\mathfrak{g}, [\,,]_{\mathfrak{g}})$ and $(\mathfrak{h}, [\,,]_{\mathfrak{h}})$.

Then $G \times H$ is a Lie group with Lie algebra

$$T_{(e,e)}(G \times H) = T_e G \times T_e H = \underline{\mathfrak{g}} \oplus \underline{\mathfrak{h}}$$

with Lie bracket $[(X, Y), (X', Y')] = ([X, X']_{\mathfrak{g}}, [Y, Y']_{\mathfrak{h}})$
$\forall X, X' \in \mathfrak{g}, \; Y, Y' \in \mathfrak{h}$.

Hence, the Lie algebra of $G \times H$ is what one calls the direct sum of the Lie algebras of $\mathfrak{g}$ and $\mathfrak{h}$.

(check this as an exercise)

③ $G = GL(n,\mathbb{R}) \underset{\uparrow \text{open}}{\subseteq} M_{n\times n}(\mathbb{R})$

$\mathfrak{g} = M_{n\times n}(\mathbb{R})$

For $A \in GL(n,\mathbb{R})$, $\lambda_A : GL(n,\mathbb{R}) \longrightarrow GL(n,\mathbb{R})$

is the restriction of the linear map $\lambda_A : M_{n\times n}(\mathbb{R}) \longrightarrow M_{n\times n}(\mathbb{R})$.

$$\Longrightarrow \quad T_B\lambda_A \underset{\in T_B GL(n,\mathbb{R})}{(B,X)} = (AB, AX) \in T_{AB}GL(n,\mathbb{R})$$

$$\Longrightarrow \quad \underline{L_X}(A) = T_{Id}\lambda_A X = \underline{\underline{AX}}$$

Viewing L_X as a function $\underline{GL(n,\mathbb{R})} \to \underline{M_{n\times n}(\mathbb{R})}$, we know that

$$[X,Y] := [L_X, L_Y](Id) = \underline{T_{Id}L_Y}\, \underbrace{L_X(Id)}_{=X} - T_{Id}L_X\, \underbrace{L_Y(Id)}_{=Y}$$

$$= XY - YX$$

$\forall\, X, Y \in \mathfrak{g} = M_{n\times n}(\mathbb{R})$.

Lie algebra of $GL(n,\mathbb{R})$ is $\mathfrak{g} = M_{n\times n}(\mathbb{R}) =: \mathfrak{gl}(n,\mathbb{R})$ with the commutator of matrices as Lie bracket.

<u>Prop. 1.12</u> Suppose G and H are Lie groups and $\varphi : G \to H$ is a Lie group homomorphism.

① Then $\varphi' := T_e\varphi : T_eG = \mathfrak{g} \longrightarrow T_eH = \mathfrak{h}$ $(\varphi(e) = e)$ is a homomorphism of Lie algebras.

② If G is commutative/abelian, then the Lie bracket on $\mathfrak{g}$ is zero (i.e. $(\mathfrak{g}, [\,,\,])$ is an abelian Lie alg. as one says).

Proof Exercise.

Corollary 1.13 Suppose G is a Lie group and $H \leq G$ a Lie subgroup. Then the Lie algebra of H is a subalgebra of $\mathfrak{g}$. In particular, the Lie algebra of any matrix group $H \leq GL(n, \mathbb{R})$ is a subalgebra of $(\mathfrak{gl}(n, \mathbb{R}), [\,,\,])$, where $[\,,\,]$ is the commutator of matrices.

Proof Apply (1) of Prop. 1.12 to the inclusion $i : H \hookrightarrow G$.

Prop. 1.14 Suppose G is a Lie group with Lie alg. $(\mathfrak{g}, [\,,\,])$.

1. $R_X = \nu^* L_{-X} \quad \forall X \in \mathfrak{g}$

2. $[R_X, R_Y] = -R_{[X,Y]} \quad \forall X, Y \in \mathfrak{g}$.

3. $[L_X, R_Y] = 0 \quad \forall X, Y \in \mathfrak{g}$.

Proof Exercise.

Prop. 1.15 Suppose G is a Lie group and ξ is a left- (resp. right)- invariant vector field on G.

① $Fl_t^{\xi}(g) = g\, Fl_t^{\xi}(e)\ \forall g \in G$ (resp. $Fl_t^{\xi}(g) = Fl_t^{\xi}(e)\, g$ $\forall g \in G$)

② ξ is complete

Proof

① If ξ is left-inversant, then ξ is λ_g-related to itself ($\lambda_g^* \xi(h) = \xi(h)$ $\iff$ $T_h \lambda_g \xi(h) = \xi(gh)$

$$T\lambda_{g^{-1}}\xi(gh) = \xi(h)$$

which implies that the flow of ξ commutes with λ_g :

$$Fl_t^{\xi} \circ \lambda_g = \lambda_g \circ Fl_t^{\xi} \qquad \forall g \in G.$$

Evaluating at e shows : $Fl_t^{\xi}(g) = \underline{g}\, Fl_t^{\xi}(e)$,

(2) Follows from criteria we proved in Global analysis :
If $\exists\, \varepsilon > 0$ s.t. any integral curve through any point of a vector field ξ is defined on $(-\varepsilon, \varepsilon)$, then ξ is complete.

<u>Def. 1.16</u> Suppose G is a Lie group.

A one parameter subgroup of G is a Lie group homomorphism $\alpha : (\mathbb{R}, +) \longrightarrow G$ (i.e. a smooth curve $\alpha : \mathbb{R} \to G$ s.t. $\alpha(s+t) = \alpha(t)\alpha(s)$ $\forall s,t \in \mathbb{R}$).

In particular, $\alpha(0) = e$.

<u>Lemma 1.17</u> G Lie group, $\alpha : \mathbb{R} \longrightarrow G$ smooth with $\alpha(0) = e$ and $X \in \mathfrak{g}$. Then the following are equivalent:

(1) α is a one parameter subgroup with $\alpha'(0) = X$.

(2) $\alpha(t) = Fl^{L_X}_t(e) \quad \forall t \in \mathbb{R}$

(3) $\alpha(t) = Fl^{R_X}_t(e) \quad \forall t \in \mathbb{R}$.

Proof

(1) $\Rightarrow$ (2)

$$\alpha'(t) = \frac{d}{ds}\Big|_{s=0} \alpha(s+t) = \frac{d}{ds}\Big|_{s=0} \underset{\lambda_{\alpha(t)}(\alpha(s))}{\underset{\shortparallel}{\alpha(t)\alpha(s)}} = T_e \lambda_{\alpha(t)} X$$

$$= L_X(\alpha(t))$$

$\Rightarrow$ α is an integral curve of L_X and since $\alpha(0) = e$, we must have $\alpha(t) = Fl_t^{L_X}(e)$ by uniqueness.

② $\to$ ① $\quad \underline{\alpha(t) = Fl_t^{L_X}(e)}$ is a smooth curve in G with $\alpha(0) = e$ and $\alpha'(0) = L_X(\alpha(0)) = X$. Since it is a flow we have

$$\alpha(t+s) = Fl_{t+s}^{L_X}(e) = \underline{Fl_t^{L_X}}(\underbrace{Fl_s^{L_X}(e)}_{=\alpha(s)}) \underset{\text{① of Prop. 1.15}}{=} \alpha(s)\alpha(t)$$

By exchanging roles of s and t, one proves similarly that ① $\Longleftrightarrow$ ③.

$\square$