**Def. 1.18** $G$ Lie group with Lie algebra $\mathfrak{g}$.

Then the exponential map of $G$ is given by

$$\exp : \mathfrak{g} \longrightarrow G$$
$$\exp(X) := Fl_1^{L_X}(e)$$

By definition, $\exp(0) = e$.

Thm. 1.19 $G$ Lie group with Lie alg. $\mathfrak{g}$ and $\exp : \mathfrak{g} \to G$ the exponential map.

① The map $\exp$ is smooth and $T_0 \exp : \underset{\cong \mathfrak{g}}{T_0 \mathfrak{g}} \to T_e G = \mathfrak{g}$ equals $\mathrm{Id}_{\mathfrak{g}}$ (= identity on $\mathfrak{g}$).

Hence, $\exp$ restricts to a diffeomorphism from an open neighbhd of $0 \in \mathfrak{g}$ in $\mathfrak{g}$ to an open neighbhd of $e \in G$ in $G$.

② For $X \in \mathfrak{g}$ and $g \in G$ one has:

$$Fl_t^{L_X}(g) = g \cdot \exp(tX) \quad \text{and} \quad Fl_t^{R_X}(g) = \exp(tX) \cdot g$$

Proof

$(X,g) \mapsto L_X(g)$ is a smooth map $\mathfrak{g} \times G \to TG$ (by Prop. 1.7).

$\Rightarrow$ $(X,g) \mapsto (0, L_X(g))$ is a smooth v.f. on $\mathfrak{g} \times G$

Its integral curves are $t \mapsto (X, Fl_t^{L_X}(g))$ and smooth

In particular, $(X,t) \mapsto (X, Fl_t^{L_X}(g))$ is smooth and so is $\exp(X) = Fl_1^{L_X}(g)$.

Now, $Fl_t^{L_X}(e) = Fl_1^{L_{tX}}(e) = \exp(tX)$

( $c: I \to G$ integral curve of $L_X$, then $t \mapsto c(at)$ is an integral curve of $aL_X = L_{aX}$ $\forall a \in \mathbb{R}$).

and $Fl_t^{L_X}(g) = g\, Fl_t^{L_X}(e) = g \exp(tX)$

$Fl_t^{R_X}(g) = \exp(tX)\, g$

by Prop. 1.15.

$$(T_0 \exp)(X) = \frac{d}{dt}\Big|_{t=0} \exp(tX) = \frac{d}{dt}\Big|_{t=0} Fl_t^{L_X}(e) = L_X(e) = X$$

$\square$

## Examples

① Consider the commutative Lie group $(\mathbb{R}_{>0}, \cdot)$.

Its Lie algebra is $\mathbb{R}$ with trivial Lie bracket.

The left-inv. vf generated by $x \in \mathbb{R}$ is

$$L_x(a) = a.x$$
$$\shortparallel$$
$$T_1 \lambda_a x$$

$\Longrightarrow$ Integral curve of $L_x$ through $1 \in \mathbb{R}_{>0}$

$$\underset{\overset{\shortparallel}{c(t)x}}{L_x(c(t))} = c'(t) \qquad c(0) = 1$$

solution is $c(t) = e^{tx}$ usual exponential map.

Hence, $\exp : \mathbb{R} \to \mathbb{R}_{>0}$ is the usual exponential map.

(2) $G = GL(n, \mathbb{R})$, $X \in \mathfrak{g} = M_{n\times n}(\mathbb{R}) = \mathfrak{gl}(n, \mathbb{R})$

$$L_X(A) = AX \qquad \left.\begin{aligned} L_X(c(t)) &= c'(t) \\ \| \\ c(t)X & \\ c(0) &= \mathrm{Id} \end{aligned}\right\} (*)$$

Unique solution (*) is the matrix exponential

$$\exp(tX) = \sum_{k=0}^{\infty} \frac{t^k}{k!} X^k$$

(in operator norm $\| \ \|$ on $M_{n\times n}(\mathbb{R})$ & $\|X^k\| \leq \|X\|^k$, which implies that this power series converges absolutely and uniformly on compact sets).

$\exp(X+Y) \neq \exp(X)\exp(Y)$ unless $[X,Y] = XY - YX = 0$.

<u>Def. 1.20</u> (Exponential coordinates)

$G$ is a Lie group with Lie alg. $\mathfrak{g}$, $V \subseteq \mathfrak{g}$ is an open neighbhd of $0 \in \mathfrak{g}$ s.t. $\exp\big|_V : V \to \exp(V) =: U$ is a diffeom. onto an open neighbhd of $e \in G$.

① Then $(U, \exp|_V^{-1})$ is a local chart for $G$ with $e \in U$ and $(\lambda_g(U), \lambda_g \circ \exp|_V^{-1})$ are around $g \in G$.

" Canonical coordinates of the first kind "

② Choose a basis $\{X_1, \ldots, X_n\}$ of $\mathfrak{g}$, then $v: \mathbb{R}^n \to G$ given by

$$v(t_1, \ldots, t_n) = \exp(t_1 X_1) \cdot \ldots \exp(t_n X_n)$$

restricts to a local diffeom. from a neighbhd $V$ of $0 \in \mathbb{R}^n$

onto an open neighbhd $U$ of $e$ in $G$.

Indeed,

$$\frac{\partial v}{\partial t_i}(0) = X_i$$

and so $T_0 v(a_1, \dots, a_n) = a_1 X_1 + \dots + a_n X_n$

$(U, v^{-1}|_U =: u)$ and $(l_g(U), l_g \circ u)$ are coordinates around $e \in G$ and $g \in G$.

" Canonical coordinates of the second kind ".

<u>Prop. 1.21</u> Let $\varphi : H \to G$ continuous group homomorphism between Lie groups $H$ and $G$. Then $\varphi$ is smooth.

<u>Proof</u>

We first show this for $H = (\mathbb{R}, +)$, i.e. $\varphi$ is a continuous 1-parameter subgroup.

<u>Claim</u> If $\alpha : \mathbb{R} \to G$ is a continuous 1-param. subgr., then $\alpha$ is smooth.

By Thm. 1.19, $\exists\, r > 0$ s.t.

open ball of radius $2r$ for some inner product on $\mathfrak{g}$.

$$\exp : \underbrace{B_{2r}(0)}_{\subseteq \mathfrak{g} \simeq \mathbb{R}^n} \to \exp(B_{2r}(0)) =: \underline{B_{2r}(e)}$$

is a diffeom. onto an open neighbhd $B_{2r}(e)$ of $e \in G$.

Since, $\alpha(0) = e$ and $\alpha$ is continuous, $\exists\, \varepsilon > 0$

s.t. $\alpha([-\varepsilon, \varepsilon]) \subseteq B_r(e)$.

Now let us define

$$\beta : [-\varepsilon, \varepsilon] \longrightarrow \underset{\subseteq \mathfrak{g}}{B_r(0)}$$

$$\beta = \exp\big|_{B_r(0)}^{-1} \circ \alpha \, .$$

For $\underline{|t|} < \frac{\varepsilon}{2}$ we have $\underline{\exp(\beta(2t))} = \alpha(2t) = \alpha(t)\,\alpha(t) = \underline{\exp(2\beta(t))}$

$\underline{\exp(\beta(t))} \atop \shortparallel$

$\Longrightarrow \quad \beta(2t) = 2\beta(t) \quad \Longrightarrow \quad \beta\left(\frac{s}{2}\right) = \frac{1}{2}\beta(s) \quad \forall s \in [-\varepsilon, \varepsilon]$

By induction one shows: $\underline{\beta\left(\frac{s}{2^k}\right)} = \frac{1}{2^k}\beta(s)$

$\forall s \in [-\varepsilon, \varepsilon]$
$\forall k \in \mathbb{N}$

$\Longrightarrow$ for $k, n \in \mathbb{N}$

$$\underline{\alpha\left(\frac{n\varepsilon}{2^k}\right) = \alpha\left(\frac{\varepsilon}{2^k}\right)^n} = \exp\left(\beta\left(\frac{\varepsilon}{2^k}\right)\right)^n = \exp\left(\frac{n}{2^k}\beta(\varepsilon)\right)$$

Since $\alpha(t)^{-1} = \alpha(-t)$ and $\exp(-X) = \exp(X)^{-1}$,

$$(*) \quad \underline{\alpha(t)} = \underline{\exp\left(t\,\frac{1}{\varepsilon}\beta(\varepsilon)\right)} \quad \forall t \in \underbrace{\left\{\frac{n\varepsilon}{2^k} : k \in \mathbb{N}, n \in \mathbb{Z}\right\}}_{=:\,S}$$

Since $S \subseteq \mathbb{R}$ is dense and both sides of $(*)$ are continuous, we deduce that

$$\underline{\alpha(t) = \exp\left(t \frac{1}{\varepsilon} \beta(\varepsilon)\right)} \quad \forall t .$$

In particular, $\alpha$ is smooth, because the right-hand side is.

Now consider the general case $\varphi : H \longrightarrow G$.

Take a basis $\{X_1, \dots, X_n\}$ of $\mathfrak{h}$. Then

$u^{-1}(t_1, \dots, t_n) = \exp(t_1 X_1) \dots \exp(t_n X_n)$ defines a diffeom. from a open neighbhd $0 \in \mathbb{R}^n$ to an open neighbhd of $e \in H$ and its inverse $u$ is a chart.

Then

$$(\varphi \circ u^{-1})(t_1, \dots, t_n) = \varphi\big(\exp(t_1 X_1) \cdots \exp(t_1 X_n)\big)$$

$$= \varphi\big(\exp(t_1 X_1)\big) \cdots \varphi\big(\exp(t_n X_n)\big)$$

continuous
1-parameter subgr., hence
smooth

$\Longrightarrow$ $\varphi \circ u^{-1}$ is smooth and therefore $\varphi$ is smooth locally around $e$.

Therefore, $\underline{\lambda_{\varphi(h)}} \circ \underline{\varphi} = \underline{\varphi} \circ \lambda_{h^{-1}}$ is smooth locally around $e \in H$ $\forall h \in H$.

This shows that $\varphi$ is smooth locally around any $h \in H$

$\square$

<u>Prop. 1.22</u> For a Lie group $G$ we denote by $G_0 \subseteq G$ the connected component of $G$ containing $e \in G$, which is called the connected component of the identity of $G$.

1. $G_0$ is a submfld (it is open subset) of the same dimension as $G$.

2. $G_0$ is a subgroup. In fact, it is a normal subgroup of $G$.

In particular, $G_0 \subseteq G$ is a Lie subgroup of $G$ and $G/G_0$ is also a (discret) group, called the component group of $G$.

Proof.

① ✓

② $g, h \in G_0 \implies \exists$ $C^\infty$-curves $c_g, c_h : [0,1] \to G$ s.t. $c_g(0) = e = c_h(0)$ and $c_g(1) = g$ and $c_h(1) = h$.

$\implies t \mapsto c_g(t) c_h(t)$ is a $C^\infty$-curve connecting $e$ with $gh$

$\Longrightarrow gh \in G_0$

Since $t \mapsto \nu(c_g(t))$ is a $C^\infty$-curve connecting $e$ with $g^{-1}$, also $g^{-1} \in G_0$ for any $g \in G_0$.

$\Longrightarrow G_0 \subseteq G$ is a subgroup.

It is a normal subgroup of $G$: for $g \in G_0$, $k \in G$
$t \mapsto k\, c_g(t)\, k^{-1}$ is a $C^\infty$-curve connecting $e$ with $kgk^{-1} \in G_0$.

$\square$

Thm. 1.23 $G$ and $H$ Lie groups with Lie alg. $\mathfrak{g}$ and $\mathfrak{h}$. Then:

① If $\varphi : G \to H$ is a Lie group homomorphism, then

$$\varphi \circ \underline{\exp_G} = \underline{\exp_H} \circ \underline{\varphi'}$$

where $\varphi' = T_e\varphi : \mathfrak{g} \to \mathfrak{h}$.

② $G_0$ coincides with the subgroup generated by $\exp(\mathfrak{g}) \subseteq G$.

③ If $\varphi, \psi : G \longrightarrow H$ are Lie group homomorphisms s.t. $\varphi' = \psi'$, then $\varphi|_{G_0} = \psi|_{G_0}$.

In particular, if $G$ is connected, then $\varphi = \psi$.

<u>Proof</u>

① Recall that $T_g \varphi L_x(g) = L_{\varphi'(x)}(\varphi(g))$ (in the proof of Prop. 1.12)

which implies $\varphi \circ Fl_t^{L_x} = Fl_t^{L_{\varphi'(x)}} \circ \varphi \quad \forall x \in \mathfrak{g}$.

$\Longrightarrow \underline{\varphi(\exp(X))} = \varphi(Fl_1^{L_X}(e)) = Fl_1^{L_{\varphi'(X)}}(e) = \underline{\exp(\varphi'(X))}$

(2) If $\tilde{G}$ is the subgr. generated by $\exp(\mathfrak{g})$, then

$\tilde{G} \subseteq \underline{G_0}$, since $t \mapsto \exp(tX)$ is a $C^\infty$-curve connecting $e$ to $\exp(X)$.

To see the converse, note that, since exp is a local diffeom. around $0 \in \mathfrak{g}$, $\exp(\mathfrak{g})$ and hence $\tilde{G}$, contains an open neighbhd $U \subseteq G$ of $e \in G$.

$\Rightarrow$ for $g \in \tilde{G}$, $\underline{l_g(U)}$ is an open neighbhd of $g$ contained in $\tilde{G}$. (we used that $\tilde{G}$ is a subgroup).

Hence, $\tilde{G} \subseteq G$ is open.

But $\tilde{G} \subseteq G$ is also closed, since for $g \in G \setminus \tilde{G}$ $\lambda_g(v(U))$ is an open neighbhd of $g$ contained in $G \setminus \tilde{G}$. Hence, $G \setminus \tilde{G}$ is open and therefore $\tilde{G}$ is closed.

$\Longrightarrow \tilde{G} = G_0$.

3 By 1, $\varphi$ and $\psi$ coincide on $\exp(\mathfrak{g})$.

Since $\varphi$ and $\psi$ are group homom., also on $\tilde{G}$.
So the result follows from 2 $\square$

## Example

$\det : GL(n, \mathbb{R}) \to (\mathbb{R}\setminus\{0\}, \cdot)$ is a Lie group homomorphism

$(\det(AB) = \det(A)\det(B))$

$\det' = T_{Id}\det = \mathrm{trace} : \mathfrak{gl}(n, \mathbb{R}) \to \mathbb{R}$

By ① of Thm. 1.23:

$$\boxed{\det(e^X) = e^{tr(X)}}$$

$\exp = e : \mathfrak{sl}(n, \mathbb{R}) \to SL(n, \mathbb{R})$