

# A COURSE IN FIELD THEORY

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# 0 Introduction

Field theory is most successful in describing the process of scattering of particles in the context of the Standard Model, and in particular in the Electromagnetic and Weak Interactions. The Large Electron Positron (LEP) collider operated 1989 till 2000. In a ring of 27 km in diameter, electrons and positrons were accelerated in opposite directions to energies of approximately 45 GeV. This energy is equivalent to half the mass (expressed as energy through  $E = mc^2$ ) of the neutral  $Z^0$  vector boson mass, which mediates part of the weak interactions. The  $Z^0$  particle can thus be created in electron-positron annihilation, at the regions where the electron and positron beams intersect. As a  $Z^0$  can be formed out of an electron and its antiparticle, the positron, it can also decay into these particles. Likewise it can decay in a muon-antimuon pair and other combinations (like hadrons). The cross-section for the formation of  $Z^0$  particles shows a resonance peak around the energy where the  $Z^0$  particle can be formed. The width of this peak is a measure for the probability of the decay of this particle. By the time you have worked yourself through this course, you should be able to understand how to calculate this cross-section, which in a good approximation is given by

$$\sigma = \frac{4\pi\alpha_e^2 E^2 / 27}{(E^2 - M_{Z^0}^2)^2 + M_{Z^0}^2 \Gamma_{Z^0}^2} \quad , \quad (0.1)$$

expressed in units where  $\hbar = c = 1$ .  $\alpha_e = \frac{e^2}{4\pi} \sim 1/137.037$ , is the fine-structure constant,  $E$  is twice the beam energy,  $M_{Z^0}$  the mass and  $\Gamma_{Z^0}$  the decay rate (or width) of the  $Z^0$  vector boson. The latter gets a contribution from all particles in which the  $Z^0$  can decay, in particular from the decay in a neutrino and antineutrino of the three known types (electron, muon and tau neutrinos). Any other unknown neutrino type (assuming their mass to be smaller than half the  $Z^0$  mass) would contribute likewise. Neutrinos are very hard to detect directly, as they have no charge and only interact through the weak interactions (and gravity) with other matter. With the data obtained from the LEP collider (the figure is from the ALEPH collaboration) one has been able to establish that there are *no* unknown types of light neutrinos, i.e.  $N_\nu = 3$ , which has important consequences (also for cosmology).

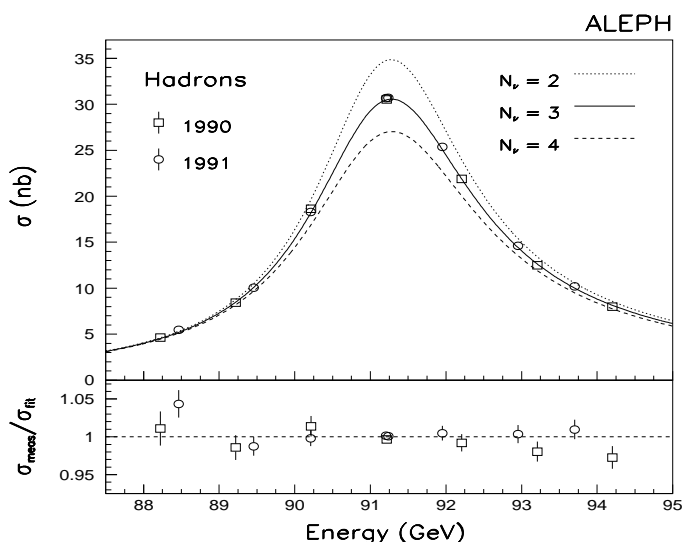


fig. 1

The main aim of this field theory course is to give the student a working knowledge *and understanding* of the theory of particles and fields, with a description of the Standard Model towards the end. We feel that an essential ingredient of any field theory course has

to be to teach the student how Feynman rules are derived from first principles. With the path integral approach this is feasible. Nevertheless, it is equally essential that the student learns how to use these rules. This is why the problems form an integral part of this course. As Julius Wess put it during his course as a Lorentz professor at our Institute “you won’t become a good pianist by listening to good concerts”.

These lecture notes reflect the field theory courses I taught in the fall of 1992 at Utrecht, and 1993, 1994, 1996, 1998 and 2000 at Leiden. I owe much to my teachers in this field, Martinus Veltman and Gerard ’t Hooft. As I taught in Utrecht from ’t Hooft’s lecture notes “Inleiding in de gequantiseerde veldentheorie” (Utrecht, 1990) it is inevitable that there is some overlap. In Leiden I spent roughly 25% longer in front of the classroom (3 lectures of 45 minutes each for 14 weeks), which allowed me to spend more time and detail on certain aspects. The set of problems, 40 in total, were initially compiled by Karel-Jan Schoutens with some additions by myself. In their present form, they were edited by Jeroen Snippe.

Of the many books on field theory that exist by now, I recommend the student to consider using “Quantum Field Theory” by C. Itzykson and J.-B. Zuber (McGraw-Hill, New York, 1980) in addition to these lecture notes, because it offers material substantially beyond the content of these notes. I will follow to a large extent their conventions. I also recommend “Diagrammatica: The path to Feynman diagrams”, by M. Veltman (Cambridge University Press, 1994), for its unique style. The discussion on unitarity is very informative and it has an appendix comparing different conventions. For more emphasis on the phenomenological aspects of field theory, which are as important as the theoretical aspects (a point Veltman often emphasised forcefully) I can recommend “Field Theory in Particle Physics” by B. de Wit and J. Smith (North-Holland, Amsterdam, 1986). For path integrals, which form a crucial ingredient of these lectures, the book “Quantum mechanics and path integrals” by R.P. Feynman and A.R. Hibbs (McGraw-Hill, New York, 1978) is a must. Finally, for an introduction to the Standard Model, useful towards the end of this course, the book “Gauge theories of weak interactions”, by J.C. Taylor (Cambridge Univ.Press, 1976), is very valuable.

## 1 Motivation

Field theory is the ultimate consequence of the attempts to reconcile the principles of relativistic invariance with those of quantum mechanics. It is not too difficult, with a lot of hindsight, to understand why a field needs to be introduced. Although this is not an attempt to do justice to history - and perhaps one should spare the student the long struggle to arrive at a consistent formulation, which most likely has not completely crystallised yet either - but the traditional approach of introducing the concept is not very inspiring and most often lacks physical motivation. In the following discussion I was inspired by “Relativistic Quantum Theory” from V.B. Berestetskii, E.M. Lifshitz and L.P. Pitaevskii (Pergamon Press, Oxford, 1971). The argument goes back to L.D. Landau and R.E. Peierls (1930).

An important consequence of relativistic invariance is that no information should propagate at a speed greater than that of light. Information can only propagate inside the future light cone. Consider the Schrödinger equation

$$i\hbar \frac{\partial \Psi(\vec{x}, t)}{\partial t} = H\Psi(\vec{x}, t) \quad . \quad (1.1)$$

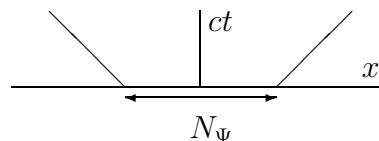


fig. 2

Relativistic invariance should require that  $\Psi(\vec{x}, t) = 0$  for all  $(\vec{x}, t)$  outside the light cone of the support  $N_\Psi = \{\vec{x} | \Psi(\vec{x}, 0) \neq 0\}$  of the wave function at  $t = 0$ .

Naturally, a first requirement should be that the Schrödinger equation itself is relativistically invariant. For ordinary quantum mechanics, formulated in terms of a potential

$$H = \frac{\vec{p}^2}{2m} + V(\vec{x}) \quad , \quad (1.1)$$

this is clearly not the case. Using the relation  $E^2 = \vec{p}^2 c^2 + m^2 c^4$  the most obvious attempt for a relativistically invariant wave equation would be the Klein-Gordon equation

$$-\hbar^2 \frac{\partial^2 \Psi(\vec{x}, t)}{\partial t^2} = -\hbar^2 c^2 \frac{\partial^2 \Psi(\vec{x}, t)}{\partial \vec{x}^2} + m^2 c^4 \Psi(\vec{x}, t) \quad . \quad (1.2)$$

However, for this equation the usual definition of probability density is not conserved

$$\partial_t \int d_3 \vec{x} \Psi^*(\vec{x}, t) \Psi(\vec{x}, t) \neq 0 \quad . \quad (1.3)$$

As this is a consequence of the fact that the equation is second order in time, this can be easily remedied, it seems, by taking the “square root” of the Klein-Gordon equation

$$i\hbar \frac{\partial \Psi(\vec{x}, t)}{\partial t} = \sqrt{\left( -\hbar^2 c^2 \frac{\partial^2}{\partial \vec{x}^2} + m^2 c^4 \right)} \Psi(\vec{x}, t) \quad (1.4)$$

We shall show that this, however, violates the principle of causality, i.e. the wave function propagates outside of its light cone, which is unacceptable. Nevertheless, we will learn something important from that computation, namely that negative energies seem unavoidable when trying to localise wave functions within the light cone of  $N_\Psi$ . But first we will provide a simple heuristic argument based on the uncertainty relation.

From the uncertainty principle  $\Delta x \Delta p > \hbar/2$  and the bound on the speed involved in any measurement of the position, it follows that precision of a measurement of the momentum is limited by the available time  $\Delta t \Delta p > \hbar/c$ . Only for a free particle, where momentum is conserved, such a measurement would be possible, but in that case, of course, the position is completely undetermined, consistent with the plane wave description of such a free particle (the light cone of  $N_\Psi$  would in that case indeed give us no constraint). More instructive is to look at how accurately we can determine the position of a particle. As the momentum is bounded by the (*positive*) energy ( $p \leq E/c$ ) and as the maximal change in the momentum is of the order of  $p$  itself, we find that  $\Delta x > \hbar/p \geq \hbar c/E$ , which coincides with the limit set by the De Broglie wavelength.

If we take this serious, that is position can in principle not be measured with arbitrary accuracy, the notion of a wave function loses its meaning. On the other hand, if we would like to localise the particle more accurate than within its De Broglie wavelength, it seems to require an uncertainty in momentum that can only be achieved by allowing for negative energy states. But negative energy states will be interpreted as antiparticles, and once antiparticles are introduced, which can annihilate with particles, particle number is no longer conserved and we likewise lose the notion of position of a particle. Only a free particle, as a plane wave, seems to be compatible with relativistic invariance.

We will now verify by direct computation that localising the wave function within the light cone will indeed require negative energy states. We consider first the positive square root of the Klein-Gordon equation and solve the Schrödinger equation for the initial condition  $\Psi(\vec{x}, 0) = \delta_3(\vec{x})$ . From this we can solve any initial condition by convolution. As the Schrödinger equation is first order in time, the initial condition uniquely fixes the wave

function for all later times and there will be a unique answer to the question whether the wave function vanishes outside the light cone (i.e. for  $t > |\vec{x}|$ ). Problem 1 asks you to investigate this in the simpler case of one, instead of three, spatial dimensions. For the latter we simply give the result here, using the fact that in Fourier space the solution is trivial. Computing  $\Psi(\vec{x}, t)$  thus requires just some skills in performing Fourier integrals.

$$\begin{aligned}
\Psi(\vec{x}, t) &= \int \frac{d^3\vec{p}}{(2\pi\hbar)^3} e^{i\vec{p}\cdot\vec{x}/\hbar} e^{-it\sqrt{p^2c^2+m^2c^4}/\hbar} \\
&= \int \frac{p^2 dp \sin(\theta) d\theta}{(2\pi)^2 \hbar^3} e^{ipr \cos(\theta)/\hbar} e^{-it\sqrt{p^2c^2+m^2c^4}/\hbar} \\
&= \frac{1}{2\pi^2 r \hbar^2} \int p dp \sin(pr/\hbar) e^{-it\sqrt{p^2c^2+m^2c^4}/\hbar} \\
&= \frac{-i}{2\pi^2 r} \frac{\partial^2}{\partial r \partial t} \int_0^\infty dp \frac{\cos(pr/\hbar)}{\sqrt{p^2c^2+m^2c^4}} e^{-it\sqrt{p^2c^2+m^2c^4}/\hbar} .
\end{aligned} \tag{1.5}$$

We now introduce

$$\begin{aligned}
p &= mc \sinh(u) , \quad mcr/\hbar = z \cosh(v) , \quad mc^2t/\hbar = z \sinh(v) , \\
z^2 &= m^2c^2(r^2 - c^2t^2)/\hbar^2 ,
\end{aligned} \tag{1.6}$$

such that (the last identity simply being the definition of the modified Bessel function  $K_o$ )

$$\begin{aligned}
\Psi(\vec{x}, t) &= \frac{-i}{4\pi^2 rc} \frac{\partial^2}{\partial r \partial t} \int_{-\infty}^\infty du \cos(z \sinh(u) \cosh(v)) e^{-iz \sinh(v) \cosh(u)} \\
&= \frac{-i}{8\pi^2 rc} \frac{\partial^2}{\partial r \partial t} \int_{-\infty}^\infty du \left( e^{-iz \sinh(u+v)} + e^{-iz \sinh(u-v)} \right) \\
&= \frac{-i}{2\pi^2 rc} \frac{\partial^2}{\partial r \partial t} \int_0^\infty du \cos(z \sinh(u)) \equiv \frac{-i}{2\pi^2 rc} \frac{\partial^2}{\partial r \partial t} K_o(z) .
\end{aligned} \tag{1.7}$$

Outside of the light cone,  $z$  is real ( $r^2 > c^2t^2$ ) and  $\Psi(\vec{x}, t)$  is purely imaginary. It decays exponentially, but does *not* vanish! Inside the light cone we find by analytic continuation (see e.g. appendix C of “Relativistic Quantum Fields” by J.D. Björken and S.D. Drell (McGraw Hill, New York, 1965)) the following explicit expression

$$\Psi(\vec{x}, t) = \frac{1}{4\pi rc} \frac{\partial^2}{\partial r \partial t} \left( iY_o(mc\sqrt{c^2t^2 - r^2}/\hbar) - \text{sign}(t)J_o(mc\sqrt{c^2t^2 - r^2}/\hbar) \right) , \quad r^2 < c^2t^2 . \tag{1.8}$$

If we want to insist on locality, i.e.  $\Psi(\vec{x}, t) = 0$  for  $|\vec{x}| > ct$  and want to stay as close as possible to the solutions of the Schrödinger equation, we could take the real part of  $\Psi$  as the wave function. It satisfies the Klein-Gordon equation, but not its positive square root.  $\Psi^*$  is a solution of the negative square root of the Klein-Gordon equation, and corresponds to a *negative* energy solution. Apparently, localisation is only possible if we allow for negative energy solutions.

## 2 Quantisation of fields

As position is no longer a quantum observable, but free particles do not seem to be in contradiction with relativistic invariance, we can try to introduce such a free particle as a quantum observable. This observable is hence described by a plane wave

$$\varphi_{\vec{k}}(\vec{x}, t) = e^{-i(k_o t - \vec{x}\cdot\vec{k})/\hbar} , \tag{2.1}$$

which satisfies the Klein-Gordon equation

$$-\hbar^2 \frac{\partial^2 \varphi(\vec{x}, t)}{\partial t^2} = -\hbar^2 c^2 \frac{\partial^2 \varphi(\vec{x}, t)}{\partial \vec{x}^2} + m^2 c^4 \varphi(\vec{x}, t) \quad , \quad (2.2)$$

where  $k_0 = \sqrt{c^2 \vec{k}^2 + m^2 c^4}$  is the energy of the free particle. By superposition of these plane waves we can make a superposition of free particles, which is therefore described by a *field*

$$\varphi(\vec{x}, t) = (2\pi\hbar)^{-\frac{3}{2}} \int d_3 \vec{k} \, \tilde{\varphi}(\vec{k}, t) e^{i\vec{k} \cdot \vec{x} / \hbar} \quad . \quad (2.3)$$

It satisfies the Klein-Gordon equation if the Fourier components  $\tilde{\varphi}(\vec{k}, t)$  satisfy the harmonic equation

$$-\hbar^2 \frac{\partial^2 \tilde{\varphi}(\vec{k}, t)}{\partial t^2} = (c^2 \vec{k}^2 + m^2 c^4) \tilde{\varphi}(\vec{k}, t) \equiv k_o^2(\vec{k}) \tilde{\varphi}(\vec{k}, t) \quad . \quad (2.4)$$

Its solutions split in positive and negative frequency components

$$\tilde{\varphi}(\vec{k}, t) = \tilde{\varphi}_+(\vec{k}) e^{-ik_o t / \hbar} + \tilde{\varphi}_-(\vec{k}) e^{ik_o t / \hbar} \quad (2.5)$$

The wave function, or rather the wave functional  $\Psi(\varphi)$ , describes the distribution over the various free particle states. The basic dynamical variables are  $\tilde{\varphi}(\vec{k})$ . These play the role the coordinates used to play in ordinary quantum mechanics and will require quantisation. As they satisfy a simple harmonic equation in time, it is natural to quantise them as harmonic oscillators. The Hamiltonian is then simply the sum of the harmonic oscillator Hamiltonian for each  $\vec{k}$ , with frequency  $\omega(\vec{k}) \equiv k_o(\vec{k}) / \hbar$ .

$$i\hbar \frac{\partial \Psi(\varphi)}{\partial t} = H \Psi(\varphi) = \sum_{\vec{k}} \mathcal{H}(\vec{k}) \Psi(\varphi) \quad ,$$

$$\mathcal{H}(\vec{k}) = \frac{1}{2} |\tilde{\pi}(\vec{k})|^2 + \frac{1}{2} \omega(\vec{k})^2 |\tilde{\varphi}(\vec{k})|^2 \quad , \quad \tilde{\pi}(\vec{k}) \equiv \frac{\hbar}{i} \frac{\partial}{\partial \tilde{\varphi}(\vec{k})} \quad . \quad (2.6)$$

In a finite volume with periodic boundary conditions, the integral over the momenta is replaced by a sum as the momenta are in that case discrete,  $\vec{k} = 2\pi \vec{n} \hbar / L$ ,  $\vec{n} \in \mathbb{Z}^3$ . Like for the harmonic oscillator, we can introduce creation and annihilation operators

$$a(\vec{k}) = \frac{1}{\sqrt{2\hbar\omega(\vec{k})}} \left( \omega(\vec{k}) \tilde{\varphi}(\vec{k}) + i\tilde{\pi}^*(\vec{k}) \right) \quad ,$$

$$a^\dagger(\vec{k}) = \frac{1}{\sqrt{2\hbar\omega(\vec{k})}} \left( \omega(\vec{k}) \tilde{\varphi}^*(\vec{k}) - i\tilde{\pi}(\vec{k}) \right) \quad , \quad (2.7)$$

and express the field operator (the equivalent of the coordinates) in terms of these creation and annihilation operators. To give the field operator its time dependence we have to invoke the Heisenberg picture, which gives  $\varphi(\vec{x}, t) = e^{iHt/\hbar} \varphi(\vec{x}, 0) e^{-iHt/\hbar}$ . Using the well known fact that  $e^{iHt/\hbar} a(\vec{k}) e^{-iHt/\hbar} = e^{-i\omega(\vec{k})t} a(\vec{k})$  and  $e^{iHt/\hbar} a^\dagger(\vec{k}) e^{-iHt/\hbar} = e^{i\omega(\vec{k})t} a^\dagger(\vec{k})$ , which is a consequence of  $[a(\vec{k}), H] = \hbar\omega(\vec{k}) a(\vec{k})$  and  $[a^\dagger(\vec{k}), H] = -\hbar\omega(\vec{k}) a^\dagger(\vec{k})$ , we find

$$\varphi(\vec{x}, t) = L^{-\frac{3}{2}} \sum_{\vec{k}} \frac{\hbar}{\sqrt{2k_o(\vec{k})}} \left( a^\dagger(\vec{k}) e^{-i(\vec{k} \cdot \vec{x} - k_o t) / \hbar} + a(\vec{k}) e^{i(\vec{k} \cdot \vec{x} - k_o t) / \hbar} \right) \quad . \quad (2.8)$$

In an infinite volume we replace  $L^{-\frac{3}{2}} \sum_{\vec{k}}$  by  $(2\pi\hbar)^{-\frac{3}{2}} \int d_3\vec{k}$ . Note that in the Heisenberg picture positive energy modes behave in time as  $e^{iEt/\hbar}$ . Apparently we can identify (up to a factor)  $\tilde{\varphi}_-(\vec{k})$  with  $a^\dagger(-\vec{k})$  and  $\tilde{\varphi}_+(\vec{k})$  with  $a(\vec{k})$ , which is compatible with  $\tilde{\varphi}^*(\vec{k}) = \tilde{\varphi}(-\vec{k})$ , required to describe a real field (complex fields will be discussed in problem 5).

The Hilbert space is now given by the product of the Hilbert spaces of each  $\vec{k}$  separately

$$|\{n_{\vec{k}}\}\rangle = \prod_{\vec{k}} |n_{\vec{k}}\rangle = \prod_{\vec{k}} \frac{a^\dagger(\vec{k})^{n_{\vec{k}}}}{\sqrt{n_{\vec{k}}!}} |0_{\vec{k}}\rangle \quad , \quad (2.9)$$

with  $n_{\vec{k}}$  the occupation number, which in field theory is now interpreted as the *number* of free particles of momentum  $\vec{k}$ , a definition that makes sense as the energy of such a state is  $n_{\vec{k}} k_o(\vec{k})$  above the state with zero occupation number (the “vacuum”). It is the property of the harmonic oscillator, that its energy is linear in the occupation number, which makes the field theory interpretation in terms of particles possible. The annihilation operator in this language therefore removes a particle (lowering the energy by the appropriate amount), which consequently can be interpreted as the annihilation of the removed particle with an antiparticle (described by the annihilation operator). For a real scalar field, a particle is its own antiparticle and this description is perhaps somewhat unfamiliar. But for the complex field of problem 5, the Fourier component with negative energy is independent of the one with positive energy, hence describing a separate degree of freedom, namely that of an antiparticle with opposite charge.

Interactions between the particles are simply introduced by modifying the Klein-Gordon equation to have non-linear terms, after which in general the different Fourier components no longer decouple. Field theory thus seems to be nothing but the quantum mechanics of an infinite number of degrees of freedom. It is, however, its physical interpretation that crucially differs from that of ordinary quantum mechanics. It is this interpretation that is known as *second quantisation*. We were forced to introduce the notion of fields and the interpretation involving antiparticles, when combining quantum mechanics with relativistic invariance. We should therefore verify that indeed it does not give rise to propagation of information with a speed larger than the speed of light. This is implied by the following identity, which for the free scalar field will be verified in problem 6

$$[\varphi(\vec{x}, t), \varphi(\vec{x}', t')] = 0, \quad \text{for} \quad (\vec{x} - \vec{x}')^2 > (t - t')^2 c^2 \quad . \quad (2.10)$$

It states that the action of an operator on the wave functional at a given space-time point is independent of the action of the operator at an other space-time point, as long as these two points are not causally connected. Due to the description of the time evolution with a Hamiltonian, which requires the choice of a time coordinate it remains to be established, however, that these equations are covariant under Lorentz transformations. We will resolve this by using the path integral approach, in which the Lorentz invariance is intrinsic, but which can also be shown to be equivalent to the Hamiltonian formulation.

Before preparing for path integrals by discussing the action principle, we would first like to address a simple physical consequence of the introduction and subsequent quantisation of fields. It states that empty space (all occupation numbers equal to zero) has nevertheless a non-trivial structure, in the same way that the ground state of a Hydrogen atom is non-trivial. Put differently, empty space is still full of zero-point fluctuations, which are, however, only visible if we probe that empty space in one way or another. Also, formally, as each



zero-point energy is non-zero, the energy of the vacuum in field theory seems to be infinite

$$E_0 = \sum_{\vec{k}} \sqrt{\vec{k}^2 c^2 + m^2 c^4} = \dots? \quad (2.11)$$

However, (as long as gravity is left out of our considerations) one is only sensitive to differences in energy. If we probe the vacuum, its energy can *only* be put to zero for one particular value of the probe. The *dependence* of the vacuum energy on the probe can be used to discover the non-trivial structure of the vacuum.

A famous and elegant method for probing the vacuum was introduced by Casimir (Proc. Kon. Ned. Acad. Wet., ser. B51 (1948) 793), who considered using two conducting plates in empty space. The energy of the vacuum is a function of the distance between the two plates, which gives a force. Strictly speaking we should discuss this in the situation of the quantised electromagnetic field (see Itzykson and Zuber, par. 3-2-4), but the essential ingredient is that Fourier components of the field are affected by the presence of the conducting plates. We can also discuss this in the context of the simple scalar field we have introduced before, by assuming that the field has to vanish at the plates. For simplicity we will also take the mass of the scalar particles to vanish. If furthermore we use periodic boundary conditions in the two other perpendicular directions over a distance  $L$ , then one easily verifies that the force per unit area on the conducting plates is given by

$$F_L(x) = -dE_o(x)/dx = -\frac{1}{2L^2} \frac{d}{dx} \sum_{\vec{n} \in \mathbb{Z}^2} \sum_{k=1}^{\infty} \sqrt{\left(\frac{2\pi\hbar c \vec{n}}{L}\right)^2 + \left(\frac{\pi\hbar c k}{x}\right)^2}, \quad (2.12)$$

where due to the vanishing boundary conditions the Fourier modes in the  $x_1$  direction, perpendicular to the conducting walls, are given by  $\sin(\pi k x_1/x)$  with  $k$  a positive integer, whereas the quantisation of the momenta in the other two directions is as usual.

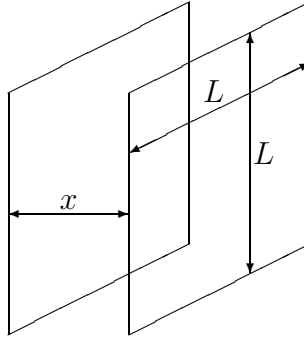


fig. 3

One can now formally take the infinite volume limit

$$F(x) = \lim_{L \rightarrow \infty} F_L(x) = -\frac{c}{8\pi^2 \hbar^2} \frac{d}{dx} \sum_{k=1}^{\infty} \int d_2 \vec{p} \sqrt{\vec{p}^2 + \left(\frac{\pi\hbar k}{x}\right)^2}. \quad (2.13)$$

The integral and the sum are clearly divergent, but as Casimir observed, in practise no conducting plate can shield a field perfectly and especially for high frequency the boundary conditions should be modified. One can mimic this by artificially cutting off the integral and sum at high momenta. We would not expect the physical result to depend on the details of how we do this, as otherwise we could use this experiment in an ingenious way to learn how nature behaves at arbitrarily high energies. Indeed Casimir's careful analysis showed that

the result is independent of the cutoff function chosen. It is an important example of what we will later recognise as renormalisability of field theory. Since the result is insensitive to the method of regularisation (only an overall constant contribution to  $E_o(x)$  depends on it, but that is not observable, as we argued before), we can choose a convenient way to perform the calculation. Details of this will be provided in problem 2. The method of calculation is known as dimensional regularisation, where one works in an arbitrary dimension ( $n \neq 2$ ) and then analytically extends the result to  $n = 2$ . We will find that

$$\begin{aligned} F(x) &= \lim_{n \rightarrow 2} -\frac{c}{8\pi^2 \hbar^2} \frac{d}{dx} \sum_{k=1}^{\infty} \int d_n \vec{p} \sqrt{\vec{p}^2 + \left(\frac{\pi \hbar k}{x}\right)^2} \\ &= \lim_{n \rightarrow 2} (n+1) \zeta(-n-1) \pi^{3n/2} \frac{\hbar c}{8\pi} \frac{\Gamma(-(n+1)/2)}{\Gamma(-1/2)} x^{-(n+2)} \quad . \end{aligned} \quad (2.14)$$

in which  $\zeta(i) \equiv \sum_{k=1}^{\infty} k^{-i}$  is the Riemann  $\zeta$  function. It can be analytically extended to odd negative arguments, where in terms of Bernoulli coefficients  $\zeta(1-2i) = -B_{2i}/(2i)$ . Also  $\Gamma(-\frac{1}{2}) = -\frac{3}{2}\Gamma(-\frac{3}{2})$  is finite, and we simply find that

$$F(x) = -\frac{\pi^2 \hbar c}{480 x^4} \quad . \quad (2.15)$$

Please note that we have disregarded the space *outside* the conducting plates. Imposing also periodic boundary conditions in that direction, one easily finds that the region outside the plates contributes with  $F(L-x)$  to the force, and vanishes when  $L \rightarrow \infty$ . Therefore, the effect of the zero-point fluctuations in the vacuum leads to a (very small) attractive force, which was ten years later experimentally measured by Sparnaay (Physica, 24 (1958) 751). Another famous example of the influence of zero-point fluctuations is the Lamb shift in atomic spectra (hyperfine splittings), to be discussed at the end of the last section.

### 3 Euler-Lagrange equations

The Klein-Gordon equation in Lorentz covariant form ( $x \equiv (ct, x, y, z) \equiv (x_0; \vec{x})$ )

$$g^{\mu\nu} \partial_\mu \partial_\nu \varphi(x) + m^2 \varphi(x) = 0 \quad , \quad g^{\mu\nu} = \begin{pmatrix} 1 & & & \ominus \\ & -1 & & \\ & & -1 & \\ \ominus & & & -1 \end{pmatrix} \quad , \quad (3.1)$$

can be derived by variational calculus from an action principle

$$\begin{aligned} S &= \int d_4 x \mathcal{L}(\varphi, \partial_\mu \varphi, x) \quad , \quad \mathcal{L}(\varphi, \partial_\mu \varphi, x) = \frac{1}{2} (\partial_\mu \varphi)^2 - V(\varphi) \quad , \\ (\partial_\mu \varphi)^2 &\equiv \partial_\mu \varphi \partial^\mu \varphi = g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi \quad , \quad V(\varphi) = \frac{1}{2} m^2 \varphi^2 \quad . \end{aligned} \quad (3.2)$$

We assume the field to be given at the boundary of the domain  $M$  of integration (typically assuming the field vanishes at infinity) and demand the action to be stationary with respect to any variation  $\varphi(x) \rightarrow \varphi(x) + \delta\varphi(x)$  of the field,

$$\begin{aligned} \delta S(\varphi) &\equiv S(\varphi + \delta\varphi) - S(\varphi) = \int_M d_4 x \left( \partial^\mu \varphi \partial_\mu \delta\varphi - \frac{\partial V(\varphi)}{\partial \varphi} \delta\varphi \right) \\ &= \int_M d_4 x \left( -\delta\varphi (\partial_\mu \partial^\mu \varphi + \frac{\partial V(\varphi)}{\partial \varphi}) \right) + \int_{\partial M} d_\mu \sigma (\delta\varphi \partial^\mu \varphi) = 0 \quad , \end{aligned} \quad (3.3)$$

where  $d_\mu\sigma$  is the integration measure on the boundary  $\partial M$ . The variation  $\delta\varphi$  is arbitrary, except at  $\partial M$ , where we assume  $\delta\varphi$  vanishes, and this implies the Euler-Lagrange equation

$$\partial_\mu\partial^\mu\varphi + \frac{\partial V(\varphi)}{\partial\varphi} = 0 \quad , \quad (3.4)$$

which coincides with the Klein-Gordon equation. We can also write the Euler-Lagrange equations for arbitrary action  $S(\varphi)$  in terms of functional derivatives

$$\frac{\delta S}{\delta\varphi(x)} = \frac{\delta S}{\delta\varphi(x)} - \partial_\mu \frac{\delta S}{\delta\partial_\mu\varphi(x)} = 0 \quad , \quad (3.5)$$

where  $\delta$  stands for the total functional derivative, which is then split according to the explicit dependence of the action on the field and its derivatives (usually an action will not contain higher than first order space-time derivatives). Please note that a functional derivative has the property  $\delta\varphi(x)/\delta\varphi(y) = \delta_4(x-y)$ , which is why in the above equation we take functional derivatives of the action  $S$  and not, as one sees often, of the Lagrangian density  $\mathcal{L}$ .

The big advantage of using an action principle is that  $S$  is a Lorentz scalar, which makes it much easier to guarantee Lorentz covariance. As the action will be the starting point of the path integral formulation of field theory, Lorentz covariance is much more easy to establish within this framework (there are instances where the regularisation, required to make sense of the path integral destroys the Lorentz invariance, like in string theory. Examples of these *anomalies* will be discussed later for the breaking of scale invariance and gauge invariance). It is now simple to add interactions to the Klein-Gordon equation, by generalising the dependence of the “potential”  $V(\varphi)$  to include higher order terms, like

$$V(\varphi) = \tfrac{1}{2}m^2\varphi^2 + \frac{1}{4!}\varphi^4 \quad , \quad (3.6)$$

which is known as a scalar  $\varphi^4$  field theory. Later we will see that one can not add arbitrary powers of the field to this potential, except in two dimensions.

As in classical field theory, we can derive from a Lagrangian with  $\varphi(x)$  and  $\dot{\varphi}(x) \equiv \partial\varphi(x)/\partial t$  as its independent variables, the Hamiltonian through a Legendre transformation to the canonical pair of variables  $\pi(x)$  (the “momentum”) and  $\varphi(x)$  (the “coordinate”)

$$\pi(x) = \frac{\delta S}{\delta\dot{\varphi}(x)} \quad , \quad H = \int \mathcal{H}(x)d_3\vec{x} = \int (\pi(x)\dot{\varphi}(x) - \mathcal{L}(x)) d_3\vec{x} \quad . \quad (3.7)$$

The classical Hamilton equations of motion are given by

$$\dot{\varphi}(x) = \frac{\delta H}{\delta\pi(x)} \quad , \quad \dot{\pi}(x) = -\frac{\delta H}{\delta\varphi(x)} + \partial_i \frac{\delta H}{\delta\partial_i\varphi(x)} \quad . \quad (3.8)$$

For the Klein-Gordon field we simply find

$$\mathcal{H} = \tfrac{1}{2}\pi(x)^2 + \tfrac{1}{2}(\partial_i\varphi(x))^2 + \tfrac{1}{2}m^2\varphi^2(x) \quad , \quad (3.9)$$

and in problem 5 one will see that this Hamiltonian coincides with eq. (2.6), if we substitute for  $\varphi(x)$  eq. (2.8). For an interacting scalar field one finds

$$\mathcal{H} = \tfrac{1}{2}\pi(x)^2 + \tfrac{1}{2}(\partial_i\varphi(x))^2 + V(\varphi(x)) \quad , \quad (3.10)$$

which perhaps explains why  $V$  is called the potential.

It is well known that the Hamiltonian equations imply that  $H$  itself is conserved with time, provided the Lagrangian (or Hamiltonian) has no explicit time-dependence

$$\frac{dH}{dt} = \int d_3\vec{x} \left( \dot{\pi}(x) \frac{\delta H}{\delta \pi(x)} + \dot{\varphi}(x) \frac{\delta H}{\delta \varphi(x)} \right) = 0 \quad . \quad (3.11)$$

Conservation of energy is one of the most important laws of nature and it is instructive to derive it more directly from the fact that  $\mathcal{L}$  does not depend explicitly on time. We define the Lagrangian  $L$  as an integral of the Lagrange density  $\mathcal{L}$  over space,  $L \equiv \int d_3\vec{x} \mathcal{L}$ , such that

$$\frac{dL}{dt} = \int d_3\vec{x} \left( \partial_t \varphi(x) \frac{\delta S}{\delta \varphi(x)} + \partial_t (\partial_\mu \varphi(x)) \frac{\delta S}{\delta \partial_\mu \varphi(x)} \right) = \int d_3\vec{x} \partial_\mu \left( \partial_t \varphi(x) \frac{\delta S}{\delta \partial_\mu \varphi(x)} \right) \quad . \quad (3.12)$$

The last term contains a total derivative, which vanishes if we assume that the field is time independent (or vanishes) at the boundary of the spatial integration domain. The above equation becomes now

$$\frac{dL}{dt} = \frac{d}{dt} \int d_3\vec{x} \dot{\varphi}(x) \frac{\delta S}{\delta \dot{\varphi}(x)} = \frac{d}{dt} \int d_3\vec{x} \dot{\varphi}(x) \pi(x) \quad , \quad (3.13)$$

which can also be expressed as

$$\frac{d}{dt} \int d_3\vec{x} (\dot{\varphi}(x) \pi(x) - \mathcal{L}) \equiv \frac{dH}{dt} = 0 \quad . \quad (3.14)$$

In the same fashion one proves conservation of momentum in case the Lagrangian does not explicitly depend on space ( $\partial \mathcal{L} / \partial x_i = 0$ )

$$0 = \frac{dL}{dx_i} = \int d_3\vec{x} \partial_\mu \left( \partial_i \varphi(x) \frac{\delta S}{\delta \partial_\mu \varphi(x)} \right) = \frac{d}{dt} \int d_3\vec{x} \pi(x) \partial_i \varphi(x) \quad . \quad (3.15)$$

The conserved momentum is hence given by

$$P_i = \int d_3\vec{x} \pi(x) \partial_i \varphi(x) \quad . \quad (3.16)$$

Both conservation of momentum and energy are examples of conservation laws that are consequence of symmetries (translation and time invariance). They can be derived as the space integral of the time component of a conserved current or tensor

$$\partial_\mu J^\mu(x) = 0 \quad , \quad \partial_\mu T^{\mu\nu}(x) = 0 \quad . \quad (3.17)$$

In problem 3 these quantities will be defined for a charged scalar field, where  $J_\mu(x)$  can be identified with the current, whose time component is the charge density. Indeed the total charge is conserved. Assuming the current to vanish at spatial infinity one easily finds

$$\frac{d}{dt} \int d_3\vec{x} J_0(x) = \int d_3\vec{x} \partial_i J_i(x) = 0 \quad (3.18)$$

The underlying principle is described by the Noether theorem, which implies that if the Lagrangian  $\mathcal{L}$  is invariant under  $\varphi \rightarrow \varphi_\Lambda$ , where  $\Lambda$  is a parameter (such as a shift of the coordinates or a phase rotation of a complex field), then the following current is conserved

$$J^\mu(x) = \frac{\delta S}{\delta(\partial_\mu \varphi(x))} \frac{\partial \varphi_\Lambda(x)}{\partial \Lambda} \quad . \quad (3.19)$$

The proof is simple and uses the Euler-Lagrange equations to substitute  $\partial_\mu\{\delta S/\delta(\partial_\mu\varphi(x))\}$  for  $\delta S/\delta\varphi(x)$

$$\begin{aligned} 0 &= \frac{d\mathcal{L}(\varphi_\Lambda)}{d\Lambda} = \frac{\delta S}{\delta\varphi(x)} \frac{\partial\varphi_\Lambda(x)}{\partial\Lambda} + \frac{\delta S}{\delta(\partial_\mu\varphi(x))} \frac{\partial(\partial_\mu\varphi_\Lambda(x))}{\partial\Lambda} \\ &= \partial_\mu \left( \frac{\delta S}{\delta(\partial_\mu\varphi(x))} \right) \frac{\partial\varphi_\Lambda(x)}{\partial\Lambda} + \frac{\delta S}{\delta(\partial_\mu\varphi(x))} \frac{\partial(\partial_\mu\varphi_\Lambda(x))}{\partial\Lambda} = \partial_\mu J^\mu(x) \end{aligned} \quad (3.20)$$

We here considered the invariance under a *global* symmetry, but important in nature are also the *local* symmetries, like the gauge invariance related to local changes of phase and the general coordinate invariance in general relativity. Particularly with the latter in mind we demand therefore that the action  $S$  (and not just  $\mathcal{L}$ ) is invariant under  $\varphi(x) \rightarrow \varphi_{\Lambda(x)}(x)$ , with  $\Lambda$  an arbitrary function of space-time. This actually leads to the same conserved currents in case  $\mathcal{L}$  is also invariant. The same computation as above, *still using the Euler-Lagrange equations*, shows that

$$0 = \frac{\delta S}{\delta\Lambda(x)} = \partial_\mu J^\mu(x) \quad . \quad (3.21)$$

As an important example we will discuss how this construction leads to conservation of the energy-momentum tensor, using general coordinate invariance, which is the local version of translation invariance. For this we have to make the action invariant under such local coordinate redefinitions. As long as indices are contracted with the metric tensor  $g$ ,  $\mathcal{L}$  will be invariant under general coordinate transformations, due to the transformation property

$$\bar{x}^\mu = x^\mu + \varepsilon^\mu(x) \quad , \quad \bar{g}^{\mu\nu}(\bar{x}) = \frac{\partial\bar{x}^\mu}{\partial x^\alpha} \frac{\partial\bar{x}^\nu}{\partial x^\beta} g^{\alpha\beta}(x) \quad . \quad (3.22)$$

For global translation invariance,  $\varepsilon^\mu$  is constant and eqs. (3.14) and (3.15) can be easily generalised to show that the energy-momentum tensor,  $T_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi - g_{\mu\nu}\mathcal{L}$ , is conserved (eq. (3.17)). For  $\varepsilon^\mu$  not constant, we note that the integration measure  $d_4x$  is not a scalar under general coordinate transformations, but the associated Jacobian can be easily absorbed by  $\sqrt{-\det g}$ , where the determinant is applied to the  $4 \times 4$  matrix  $g_{\mu\nu}$ . For a scalar field this leads to the following invariant action

$$S = \int d_4x \sqrt{-\det g} \left( \frac{1}{2} g^{\mu\nu} \partial_\mu\varphi\partial_\nu\varphi - V(\varphi) \right) \quad (3.23)$$

For the original coordinates  $x$  of Minkowski space-time the metric is given as in eq. (3.1), in particular  $\sqrt{-\det g(x)} = 1$ , and by expanding  $\bar{g}$  to first order in  $\varepsilon^\mu(x)$  we find

$$S = \int d_4\bar{x} \left( 1 - \partial_\lambda\varepsilon^\lambda(x) \right) \left[ \frac{1}{2} \bar{g}^{\mu\nu} \partial_\mu\varphi(\bar{x})\partial_\nu\varphi(\bar{x}) - V(\varphi(\bar{x})) \right] + \partial_\alpha\varepsilon^\mu(x) \partial_\mu\varphi(\bar{x}) \partial^\alpha\varphi(\bar{x}) \quad . \quad (3.24)$$

Now observe that  $g^{\mu\nu}(x)$  is constant, such that the term independent of  $\varepsilon$  is a function of  $\bar{x}$ , integrated over  $\bar{x}$ , which is simply the action itself, as  $\bar{x}$  *now* plays the role of a dummy integration variable. The term linear in  $\varepsilon$  therefore has to vanish, but note that it *only* involved the variation of the metric under the general coordinate transformation. Hence,

$$0 = \int d_4x \varepsilon_\mu(x) \partial_\alpha (g^{\mu\alpha} \mathcal{L}(x) - \partial^\mu\varphi(x) \partial^\alpha\varphi(x)) \equiv - \int d_4x \varepsilon_\mu(x) \partial_\alpha T^{\alpha\mu}(x) \quad , \quad (3.25)$$

which implies conservation of the energy-momentum tensor ( $T_{oo} = \mathcal{H}$ ). From the fact that  $\delta g_{\mu\nu} = -\partial_\mu\varepsilon_\nu - \partial_\nu\varepsilon_\mu$ ,  $\delta g^{\mu\nu} = -g^{\mu\alpha}\delta g_{\alpha\beta}g^{\beta\nu}$  and  $\delta\sqrt{-\det g} = \frac{1}{2}g^{\mu\nu}\sqrt{-\det g}\delta g_{\mu\nu}$ , we derive the identity

$$T^{\mu\nu}(x) \equiv -2 \frac{\delta S}{\delta g_{\mu\nu}(x)} \quad . \quad (3.26)$$

In taking the derivative with respect to the metric, it is important that any Lorentz vector (like the derivative  $\partial_\mu \varphi$ ) or tensor appears in the Lagrangian with its indices down. Furthermore, the result is to be evaluated for Minkowski space. Eq. (3.26) always gives a symmetric energy-momentum tensor and from the derivation it is clear that the result holds not only for a simple scalar field, but for any other bosonic field theory (fermions form an exception, see problem 23) like the one for the electromagnetic field, which we discuss now.

The field is given by the tensor  $F_{\mu\nu}(x)$ , with  $E_i(x) = F_{0i}(x)$  its electric and  $B_i(x) = \frac{1}{2}\varepsilon_{ijk}F_{jk}(x)$  its magnetic components. In terms of the vector potential  $A_\mu(x)$  one has

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) \quad . \quad (3.27)$$

This already implies one of the Maxwell equations (through the so-called Jacobi or integrability conditions)

$$\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0 \quad . \quad (3.28)$$

Written as  $\varepsilon^{\mu\nu\lambda\sigma}\partial_\nu F_{\lambda\sigma} = 0$  they are easily seen (resp. for  $\mu = 0$  and  $\mu = i$ ) to give

$$\text{div} \vec{B} = 0 \quad , \quad \partial_0 \vec{B} + \text{rot} \vec{E} = \vec{0} \quad . \quad (3.29)$$

The dynamical equations determining the fields in terms of the currents, or the sources,  $J^\mu = (c\rho; \vec{J})$  are given by

$$\partial_\mu F^{\mu\nu} = \frac{1}{c} J^\nu \quad \text{or} \quad \text{div} \vec{E} = \rho \quad , \quad \text{rot} \vec{B} - \partial_0 \vec{E} = \vec{J} \quad . \quad (3.30)$$

We have chosen Heaviside-Lorentz units and in the future we will also often choose units such that  $\hbar = c = 1$ .

These Maxwell equations follow from the following action

$$S_{\text{em}}(J) = \int d_4x \left( -\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x) - A_\mu(x) J^\mu(x) \right) \quad . \quad (3.31)$$

We note, as is well known, that the equations of motion imply that the current is conserved. With Noether's theorem this makes us suspect that this is caused by a symmetry and indeed it is known that under the gauge transformation

$$A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu \Lambda(x) \quad (3.32)$$

the theory does not change. Our action is invariant under this symmetry if and only if the current is conserved. This gauge symmetry will play a crucial role in the quantisation of the electromagnetic field.

An example of a conserved current can be defined for a complex scalar field. Its action for a free particle is given by

$$S_0 = \int d_4x \left( \partial_\mu \varphi^*(x) \partial^\mu \varphi(x) - m^2 \varphi^*(x) \varphi(x) \right) \quad . \quad (3.33)$$

It is invariant under a phase rotation  $\varphi(x) \rightarrow \exp(i\Lambda)\varphi(x)$  and from Noether's theorem we deduce that

$$J_\mu(x) \equiv ie(\varphi(x) \partial_\mu \varphi^*(x) - \varphi^*(x) \partial_\mu \varphi(x)) \quad (3.34)$$

is conserved, see problem 3. We can extend this global phase symmetry to a local symmetry if we couple the scalar field *minimally* to the vector potential

$$S = \int d_4x \left( -\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x) + (D_\mu \varphi)^*(x) D^\mu \varphi(x) - m^2 \varphi^*(x) \varphi(x) \right) \quad , \quad (3.35)$$

$$D_\mu \varphi(x) \equiv \partial_\mu \varphi(x) - ie A_\mu(x) \varphi(x) \quad .$$

This guarantees the combined invariance under a local gauge transformation

$$\varphi(x) \rightarrow \exp(ie\Lambda(x))\varphi(x) , \quad A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu\Lambda(x) \quad . \quad (3.36)$$

which makes the *covariant* derivative  $D_\mu\varphi(x)$  of the scalar field transform as the scalar field itself, even for local phase rotations. Note that we can write this action also as

$$S = S_{\text{em}}(J) + S_0 + \int d_4x \, e^2 A_\mu(x) A^\mu(x) |\varphi(x)|^2 \quad , \quad (3.37)$$

with  $J$  as given in eq. (3.34). We leave it as an exercise to show how the action of the electromagnetic field can be generalised to be invariant under general coordinate transformations and to derive from this the energy-momentum tensor. The result is given by

$$S_{\text{em}}(J=0) = -\frac{1}{4} \int d_4x \, g^{\mu\lambda} g^{\nu\sigma} F_{\mu\nu} F_{\lambda\sigma} \sqrt{-\det g} \quad , \quad T^{\mu\nu} = \frac{1}{4} g^{\mu\nu} F^{\lambda\sigma} F_{\lambda\sigma} - F^{\mu\lambda} F^\nu{}_\lambda \quad . \quad (3.38)$$

## 4 Tree-level diagrams

In general, in the presence of interactions, the equations of motions can not be solved exactly and one has to resort to a perturbative expansion in a small parameter. We discuss the scalar case first, as it is as always the simplest. We add to the Lagrangian density  $\mathcal{L}$  a so-called source term, which couples linearly to the field  $\varphi$  (compare the driving force term for a harmonic oscillator)

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi)^2 - V(\varphi) - J(x)\varphi(x) \quad . \quad (4.1)$$

For sake of explicitness, we will take the following expression for the potential

$$V(\varphi) = \frac{1}{2}m^2\varphi^2(x) + \frac{g}{3!}\varphi^3(x) \quad . \quad (4.2)$$

The Euler-Lagrange equations are now given by

$$\partial_\mu\partial^\mu\varphi(x) + m^2\varphi(x) + \frac{1}{2}g\varphi^2(x) + J(x) = 0 \quad . \quad (4.3)$$

If  $g = 0$  it is easy to solve the equation (describing a free particle interacting with a given source) in Fourier space. Introducing the Fourier coefficients

$$\tilde{J}(k) = \frac{1}{(2\pi)^2} \int d_4x \, e^{ikx} J(x) \quad , \quad \tilde{\varphi}(k) = \frac{1}{(2\pi)^2} \int d_4x \, e^{ikx} \varphi(x) \quad (4.4)$$

it follows that

$$(-k^2 + m^2)\tilde{\varphi}(k) + \tilde{J}(k) = 0 \quad \text{or} \quad \varphi(x) = \int d_4y \, G(x-y)J(y) \quad , \quad (4.5)$$

where  $G$  is called the Green's function, as it is the solution of the equation

$$(\partial_\mu\partial^\mu + m^2)G(x-y) = -\delta_4(x-y) \quad (4.6)$$

Explicitly, it is given by the following Fourier integral

$$G(x-y) = \int \frac{d_4k}{(2\pi)^4} \frac{e^{-ik(x-y)}}{k^2 - m^2 + i\varepsilon} \quad . \quad (4.7)$$

Please note our short-hand notation of  $k^2$  for  $k_\mu k^\mu$  and  $k(x-y)$  for  $k_\mu(x^\mu - y^\mu)$ . A Green's function is not uniquely specified by its 2nd order equations, but also requires boundary conditions. These boundary conditions are, as we will see, specified by the term  $i\varepsilon$ . Because of the interpretation of the negative energy states as antiparticles, which travel “backwards” in time, the quantum theory will require that the positive energy part vanishes for past infinity, whereas the negative energy part will be required to vanish for future infinity. Classically this would not make sense, and we would require the solution to vanish outside the future light cone. The effect of the  $i\varepsilon$  prescription is, to shift the poles on the real axes to the complex  $k_0$  plane at  $k_0 = \pm((\vec{k}^2 + m^2)^{\frac{1}{2}} - i\varepsilon)$ . In section 5 we will see that this will imply the appropriate behaviour required by the quantum theory.

Now we have found the solution for the free field coupled to a source, we can do perturbation in the strength of the *coupling constant*  $g$ .

$$\partial_\mu \partial^\mu \varphi(x) + m^2 \varphi(x) + J(x) = -\frac{1}{2} g \varphi^2(x) \quad (4.8)$$

can be solved iteratively by substituting a series expansion for  $\varphi(x)$ ,

$$\varphi(x) = \varphi_0(x) + g \varphi_1(x) + g^2 \varphi_2(x) + \dots \quad (4.9)$$

Obviously we have

$$\varphi_0(x) = \int d_4 y \, G(x-y) J(y) \quad , \quad (4.10)$$

whereas  $\varphi_1(x)$  will be determined by the equation

$$\partial_\mu \partial^\mu \varphi_1(x) + m^2 \varphi_1(x) = -\frac{1}{2} \varphi_0^2(x) \quad . \quad (4.11)$$

We can therefore interpret the right-hand side as a source (up to a minus sign) and this allows us to solve  $\varphi_1(x)$  using the Green's function

$$\varphi_1(x) = \frac{1}{2} \int d_4 y \, G(x-y) \varphi_0^2(y) = \frac{1}{2} \int G(x-y) d_4 y \int G(y-z) G(y-w) J(z) J(w) d_4 z d_4 w \quad . \quad (4.12)$$

This looks particularly simple in Fourier space

$$\tilde{\varphi}_1(k) = \frac{1}{2(2\pi)^2} \frac{1}{k^2 - m^2 + i\varepsilon} \int d_4 p \, \frac{\tilde{J}(p) \tilde{J}(k-p)}{(p^2 - m^2 + i\varepsilon)((k-p)^2 - m^2 + i\varepsilon)} \quad . \quad (4.13)$$

It is clear that this can be continued iteratively, e.g.

$$\partial_\mu \partial^\mu \varphi_n(x) + m^2 \varphi_n(x) = -\frac{1}{2} \sum_{i=0}^{n-1} \varphi_i(y) \varphi_{n-1-i}(y) \quad . \quad (4.14)$$

which is solved by

$$\varphi_n(x) = \frac{1}{2} \int d_4 y \, G(x-y) (2\varphi_0(y) \varphi_{n-1}(y) + 2\varphi_1(y) \varphi_{n-2}(y) + \dots) \quad . \quad (4.15)$$

Here we have written out the terms in the sum explicitly to indicate that all terms occur twice and are the product of two different terms, except for the term  $\varphi_{\frac{1}{2}(n-1)}(y)^2$  at  $n$  odd, which occurs once. In Fourier space one finds

$$\tilde{\varphi}_n(k) = \frac{1}{2(2\pi)^2} \frac{1}{k^2 - m^2 + i\varepsilon} \int d_4 p \, (2\tilde{\varphi}_0(p) \tilde{\varphi}_{n-1}(k-p) + 2\tilde{\varphi}_1(p) \tilde{\varphi}_{n-2}(k-p) + \dots) \quad . \quad (4.16)$$


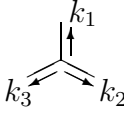
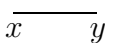





By induction it is now easy to prove that

$$\begin{aligned}
\varphi(x) &= \sum_{\text{diagrams}} \left( \text{---}\times + \frac{1}{2} \text{---}\begin{array}{c} \times \\ \diagup \diagdown \\ \times \end{array} + \frac{1}{2} \text{---}\begin{array}{c} \times \\ \diagup \diagdown \\ \times \end{array} + \frac{1}{8} \text{---}\begin{array}{c} \times \\ \diagup \diagdown \\ \times \end{array} + \dots \right) \\
&= \sum_{\text{diagrams}} \frac{g^{\#\text{vertices}}}{N(\text{diagram})} \int \prod_{\{i_v\}} dx_{i_v} \prod_{\langle i,j \rangle} G(x_i - x_j) \prod_{\{k_s\}} J(x_{k_s}) . \quad (4.17)
\end{aligned}$$

Here the index  $i_v$  runs over all vertices, sources and the point  $x$ , (so it does *not* label the four space-time components of a single point, frequently it will be assumed that it is clear from the context what is meant), whereas  $k_s$  runs only over positions of the sources. The expression  $\langle i, j \rangle$  stands for the pairs of points in a diagram connected by a line (called *propagator*).

The Feynman rules to convert a diagram to the solution are apparently that each line (propagator) between points  $x$  and  $y$  contributes  $G(x - y)$  and each cross (source) at a point  $x$  contributes  $J(x)$ . Furthermore, for each vertex at a point  $x$  we insert  $\int d_4x$  and a power of the coupling constant  $g$ . Finally each diagram comes with an overall factor  $1/N(\text{diagram})$ , being the inverse of the order of the permutation group (interchange of lines and vertices) that leaves the diagram invariant (which is also the number of ways the diagram can be constructed out of its building blocks). We have derived these rules for the case that  $\lambda = 0$ , such that only three point vertices appear. All that is required to generalise this to the arbitrary case with  $n$ -point vertices, is that each of these come with their own coupling constant (i.e.  $\lambda$  for a four-point vertex). This is the reason why these vertices are weighed by a factor  $1/n!$  in the potential and hence by a factor  $1/(n - 1)!$  in the equations of motion (to be precise, if  $V(\varphi) = g_n \varphi^n / n!$ , the equation of motion gives  $\partial_\mu^2 \varphi(x) + J(x) = -g_n \varphi^{(n-1)}(x) / (n - 1)!$ , and the factor  $(n - 1)!$  is part of the combinatorics involved in interchanging each of the  $n - 1$  factors  $\varphi$  in the interaction term).

coordinate space	momentum space	table 1
 $\equiv g \int d_4x$	 $\equiv \frac{g}{(2\pi)^2} \delta_4(\sum_i k_i)$	vertex
 $\equiv G(x - y)$	 $\equiv \int d_4k \frac{1}{k^2 - m^2 + i\varepsilon}$	propagator
 $\equiv \int d_4x J(x)$	 $\equiv \tilde{J}(k)$	source

It is straightforward to translate these Feynman rules to momentum space, by inserting the Fourier expansion of each of the terms that occur. Each propagator, which carries a momentum  $k$  is replaced by a factor  $1/(k^2 - m^2 + i\varepsilon)$  and  $\int d_4k$ , each source with momentum  $k$  flowing in the source by a factor  $\tilde{J}(k)$ , each vertex by a factor of the coupling constant (i.e.  $g_n$  for an  $n$ -point function), a factor  $1/(2\pi)^2$  (for an  $n$ -point function a factor  $(2\pi)^{4-2n}$ ) and a momentum conserving delta function. To understand why momentum is conserved at each vertex we use that in the coordinate formulation each vertex comes with an integration over its position. As each line entering the vertex carries a Green's function that depends

on that position (this being the only dependence), we see that a vertex at the point  $x$  gives rise to

$$\begin{aligned} \int d_4x \prod_{\alpha} G(x - x_{\alpha}) &\rightarrow \int d_4x \prod_{\alpha} \int \frac{d_4k_{\alpha} e^{-ik_{\alpha}(x-x_{\alpha})}}{(2\pi)^4(k_{\alpha}^2 - m^2 + i\varepsilon)} \\ &= (2\pi)^4 \prod_{\alpha} \int \frac{d_4k_{\alpha} e^{ik_{\alpha}x_{\alpha}}}{(2\pi)^4(k_{\alpha}^2 - m^2 + i\varepsilon)} \delta_4(\sum_{\alpha} k_{\alpha}) \quad . \end{aligned} \quad (4.18)$$

Conventions in the literature can differ on how the factors of  $i$  (which will appear in the quantum theory) and  $2\pi$  are distributed over the vertices and propagators. Needless to say that the final answers have to be independent of the chosen conventions.

As a last example in this section we will look again at the electromagnetic field (whose particles are called photons). In Fourier space the equations of motion are given by

$$(-k^2\delta_{\mu}^{\nu} + k_{\mu}k^{\nu})\tilde{A}^{\mu}(k) = \tilde{J}^{\nu}(k) \quad . \quad (4.19)$$

Unfortunately the matrix  $-k^2\delta_{\mu}^{\nu} + k_{\mu}k^{\nu}$  has no inverse as  $k^{\mu}$  is an eigenvector with zero eigenvalue. This is a direct consequence of the gauge invariance as the gauge transformation of eq. (3.32) in Fourier language reads

$$\tilde{A}_{\mu}(k) \rightarrow \tilde{A}_{\mu}(k) + ik_{\mu}\tilde{\Lambda}(k) \quad . \quad (4.20)$$

The component of  $A_{\mu}$  in the direction of  $k_{\mu}$  is for obvious reasons called the longitudinal component, which can be fixed to a particular value by a gauge transformation. Fixing the longitudinal component of the electromagnetic field (also called *photon field*) is called gauge fixing, and the gauge choice is prescribed by the gauge condition. An important example is the so-called *Lorentz gauge*

$$\partial_{\mu}A^{\mu}(x) = 0 \quad \text{or} \quad k_{\mu}\tilde{A}^{\mu}(k) = 0 \quad . \quad (4.21)$$

Because of the gauge invariance, the choice of gauge has no effect on the equations of motion, because the current is conserved, or  $k^{\mu}\tilde{J}_{\mu}(k) = 0$ . The current (i.e. the source) does not couple to the unphysical longitudinal component of the photon field. It stresses again the importance of gauge invariance and its associated conservation of currents.

To impose the gauge fixing, we can add a term to the Lagrangian which enforces the gauge condition. Without such a term the action is stationary under any longitudinal variation  $\delta A_{\mu}(x) = \partial_{\mu}\Lambda(x)$  of the vector field, and the added term should be such that stationarity in that direction imposes the gauge condition. For any choice of the parameter  $\alpha \neq 0$  this is achieved by the action

$$S = \int d_4x \left( -\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) - \frac{1}{2}\alpha(\partial_{\mu}A^{\mu}(x))^2 - A_{\mu}(x)J^{\mu}(x) \right) \quad . \quad (4.22)$$

Indeed, the variation  $\delta A_{\mu}(x) = \partial_{\mu}\Lambda(x)$  in the longitudinal direction leads to the equation

$$-\alpha \int d_4x \partial_{\mu}\partial^{\mu}\Lambda(x)\partial_{\nu}A^{\nu}(x) = 0 \quad , \quad (4.23)$$

which implies the Lorentz-gauge (assuming vanishing boundary conditions for the vector potential at infinity).

The equations of motion for this action now yields

$$\partial_{\mu}F^{\mu\nu}(x) + \alpha\partial^{\nu}\partial_{\mu}A^{\mu}(x) = J^{\nu}(x) \quad , \quad (4.24)$$

or in Fourier space

$$(-k^2\delta_\mu^\nu + (1-\alpha)k_\mu k^\nu)\tilde{A}^\mu(k) = \tilde{J}^\nu(k) \quad , \quad (4.25)$$

which is invertible, as long as  $\alpha \neq 0$ . The result is given by

$$\tilde{A}^\mu(k) = -\frac{\delta_\nu^\mu - \frac{k^\mu k_\nu(1-\alpha^{-1})}{k^2+i\varepsilon}}{k^2+i\varepsilon}\tilde{J}^\nu(k) \quad . \quad (4.26)$$

This is consequently the propagator of the electromagnetic field (in the Lorentz gauge), also simply called the photon propagator. Like in the scalar case it can be used to perform a perturbative expansion for the classical equations of motion.

Note that the photon propagator simplifies dramatically if we choose  $\alpha = 1$ , but all final results should be independent of the choice of  $\alpha$  and even of the choice of gauge fixing all together. This is the hard part in gauge theories. One needs to fix the gauge to perform perturbation theory and then one has to prove that the result does not depend on the choice of gauge fixing. In the quantum theory this is not entirely trivial, as the regularisation can break the gauge invariance explicitly. Fortunately, there are regularisations that preserve the gauge invariance, like dimensional regularisation which we already encountered in section 2 (in discussing the Casimir effect). In the presence of fermions the situation can, however, be considerably more tricky. Some different choices of gauge fixing will be explored in problems 8 and 9.

## 5 Hamiltonian perturbation theory

We consider the Hamiltonian for a free scalar particle coupled to a source. We will see that the source can be used to create particles from the vacuum in the quantum theory, and it forms an important ingredient, like for the derivation of the classical perturbation theory of the previous section, in deriving scattering amplitudes and cross sections. Also the Green's function will reappear, but now with a unique specification of the required boundary conditions following from the time ordering in the quantum evolution equations.

For the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 - \bar{\varepsilon}J\varphi \quad , \quad (5.1)$$

the Hamiltonian is given by

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\partial_i\varphi)^2 + \frac{1}{2}m^2\varphi^2 + \bar{\varepsilon}J\varphi \quad , \quad (5.2)$$

where  $\bar{\varepsilon}$  is a small expansion parameter. We will quantise the theory in a finite volume  $V = [0, L]^3$  with periodic boundary conditions, such that the momenta are discrete,  $\vec{k} = 2\pi\vec{n}/L$ .

$$\begin{aligned} \varphi(\vec{x}, t=0) &= \sum_{\vec{k}} \frac{1}{\sqrt{2Vk_o(\vec{k})}} \left( a(\vec{k})e^{i\vec{k}\cdot\vec{x}} + a^\dagger(\vec{k})e^{-i\vec{k}\cdot\vec{x}} \right) \quad , \\ \pi(\vec{x}, t=0) &= -i \sum_{\vec{k}} \sqrt{\frac{k_o(\vec{k})}{2V}} \left( a(\vec{k})e^{i\vec{k}\cdot\vec{x}} - a^\dagger(\vec{k})e^{-i\vec{k}\cdot\vec{x}} \right) \quad . \end{aligned} \quad (5.3)$$

The Hamiltonian is now given by  $H(t) = H_0 + \bar{\varepsilon}H_1(t)$ , and we work out the perturbation theory in the Schrödinger representation. We have

$$\begin{aligned} H_0 &= \sum_{\vec{k}} k_o(\vec{k}) (a^\dagger(\vec{k})a(\vec{k}) + \frac{1}{2}) \quad , \\ H_1(t) &= \int_V d_3\vec{x} J(\vec{x}, t)\varphi(\vec{x}) = \sum_{\vec{k}} \frac{1}{\sqrt{2k_o(\vec{k})}} \tilde{J}(\vec{k}, t) \left( a(-\vec{k}) + a^\dagger(\vec{k}) \right) \quad , \end{aligned} \quad (5.4)$$

here  $\tilde{J}(\vec{k}, t)$  is the Fourier coefficient of  $J(\vec{x}, t)$ , or

$$J(\vec{x}, t) = \frac{1}{\sqrt{V}} \sum_{\vec{k}} \tilde{J}(\vec{k}, t) e^{i\vec{k} \cdot \vec{x}} . \quad (5.5)$$

Let us start at  $t = 0$  with the vacuum state  $|0\rangle$ , which has the property that  $a(\vec{k})|0\rangle = 0$  for all momenta, then it follows that

$$\frac{d}{dt} |\Psi(t)\rangle = -iH(t) |\Psi(t)\rangle , \quad (5.6)$$

which can be evaluated by perturbing in  $\bar{\varepsilon}$ .

$$\begin{aligned} |\Psi(t)\rangle &\equiv e^{-iH_0 t} |\hat{\Psi}(t)\rangle , \quad |\hat{\Psi}(t)\rangle = \sum_{n=0}^{\infty} \bar{\varepsilon}^n |\hat{\Psi}_n(t)\rangle , \\ \frac{d}{dt} |\hat{\Psi}_n(t)\rangle &= -ie^{iH_0 t} H_1(t) e^{-iH_0 t} |\hat{\Psi}_{n-1}(t)\rangle . \end{aligned} \quad (5.7)$$

Actually, by transforming to  $|\hat{\Psi}(t)\rangle$  we are using the interaction picture, which is the usual way of performing Hamiltonian perturbation theory known from ordinary quantum mechanics. These equations can be solved iteratively as follows

$$\begin{aligned} |\hat{\Psi}_1(t)\rangle &= -i \int_0^t dt_1 e^{iH_0 t_1} H_1(t_1) e^{-iH_0 t_1} |0\rangle , \\ |\hat{\Psi}_2(t)\rangle &= -i \int_0^t dt_1 e^{iH_0 t_1} H_1(t_1) e^{-iH_0 t_1} |\hat{\Psi}_1(t_1)\rangle \\ &= - \int_0^t dt_1 e^{iH_0 t_1} H_1(t_1) \int_0^{t_1} dt_2 e^{iH_0(t_2-t_1)} H_1(t_2) e^{-iH_0 t_2} |0\rangle , \\ |\hat{\Psi}_n(t)\rangle &= -i \int_0^t dt_1 e^{iH_0 t_1} H_1(t_1) e^{-iH_0 t_1} |\hat{\Psi}_{n-1}(t_1)\rangle = \dots . \end{aligned} \quad (5.8)$$

Please note the time ordering, which is essential as  $H_1(t)$  does not commute with  $H_1(t')$  for different  $t$  and  $t'$ . We can for example compute the probability that at time  $t$   $|\Psi(t)\rangle$  is still in the ground state (whose energy we denote by  $E_0$ , which will often be assumed to vanish)

$$\begin{aligned} \langle 0 | \Psi(t) \rangle &= e^{-iE_0 t} \langle 0 | \hat{\Psi}(t) \rangle = e^{-iE_0 t} \{ 1 - i\bar{\varepsilon} \int_0^t dt_1 \langle 0 | H_1(t_1) | 0 \rangle \\ &\quad - \bar{\varepsilon}^2 \int_0^t dt_1 \int_0^{t_1} dt_2 \langle 0 | H_1(t_1) e^{i(H_0-E_0)(t_2-t_1)} H_1(t_2) | 0 \rangle + \mathcal{O}(\bar{\varepsilon}^3) \} . \end{aligned} \quad (5.9)$$

It is simple to see that the term linear in  $\bar{\varepsilon}$  will vanish, as the vacuum expectation values of the creation and annihilation operators vanish, i.e.  $\langle 0 | a | 0 \rangle = \langle 0 | a^\dagger | 0 \rangle = 0$ . To evaluate the remaining expectation value in the above equation we substitute  $H_1$  in terms of the creation and annihilation operators (see eq. (5.4))

$$\begin{aligned} \langle 0 | H_1(t_1) e^{i(H_0-E_0)(t_2-t_1)} H_1(t_2) | 0 \rangle &= \\ \sum_{\vec{k}, \vec{p}} \frac{\tilde{J}(\vec{p}, t_1) \tilde{J}(\vec{k}, t_2)}{\sqrt{4k_0(\vec{p})k_0(\vec{k})}} \langle 0 | (a(-\vec{p}) + a^\dagger(\vec{p})) e^{i(H_0-E_0)(t_2-t_1)} (a(-\vec{k}) + a^\dagger(\vec{k})) | 0 \rangle \\ &= \sum_{\vec{k}} \frac{\tilde{J}(-\vec{k}, t_1) \tilde{J}(\vec{k}, t_2)}{2k_0(\vec{k})} e^{-ik_0(\vec{k})(t_1-t_2)} . \end{aligned} \quad (5.10)$$

Combining these results we find

$$\begin{aligned}
\langle 0|\Psi(t)\rangle e^{iE_0 t} &= 1 - \bar{\varepsilon}^2 \sum_{\vec{k}} \int_0^t dt_1 \int_0^{t_1} dt_2 \frac{\tilde{J}(-\vec{k}, t_1) \tilde{J}(\vec{k}, t_2)}{2k_0(\vec{k})} e^{-ik_0(\vec{k})(t_1-t_2)} + \mathcal{O}(\bar{\varepsilon}^3) \\
&= 1 - \frac{1}{2} \bar{\varepsilon}^2 \sum_{\vec{k}} \int_0^t dt_1 \int_0^t dt_2 \frac{\tilde{J}(-\vec{k}, t_1) \tilde{J}(\vec{k}, t_2)}{2k_0(\vec{k})} e^{-ik_0(\vec{k})|t_1-t_2|} + \mathcal{O}(\bar{\varepsilon}^3). \quad (5.11)
\end{aligned}$$

Especially the last identity is useful to relate this to the Green's function we introduced in the previous section. Using contour deformation in the complex  $\omega$  plane we find

$$\int_{-\infty}^{\infty} d\omega \frac{e^{i\omega t}}{\omega^2 - k_0^2(\vec{k}) + i\varepsilon} = -\frac{2\pi i}{2k_0(\vec{k})} e^{-ik_0(\vec{k})|t|}. \quad (5.12)$$

This can be shown as follows. When  $t > 0$ , we can deform the contour of integration to the upper half-plane (where  $e^{i\omega t}$  decays exponentially) and only the pole at  $\omega = \omega_- \equiv -k_0(\vec{k}) + i\varepsilon$  contributes, with a residue  $2\pi i e^{-ik_0(\vec{k})t} / (-2k_0(\vec{k}))$  (see the figure). Instead, for  $t < 0$  the contour needs to be deformed to the lower half-plane and the pole at  $\omega = \omega_+ \equiv k_0(\vec{k}) - i\varepsilon$  contributes with the residue  $2\pi i e^{ik_0(\vec{k})t} / (-2k_0(\vec{k}))$  (note that the contour now runs clockwise, giving an extra minus sign).

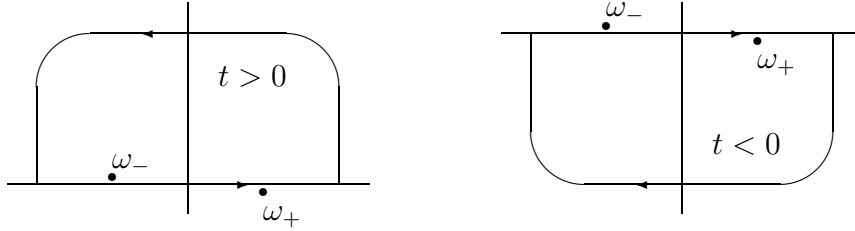


fig. 4

This means that we can rewrite eq. (5.11) as

$$\lim_{t \rightarrow \infty} \langle 0|\Psi(t)\rangle e^{iE_0 t} = 1 - \frac{i}{2} \bar{\varepsilon}^2 \sum_{\vec{k}} \int_{-\infty}^{\infty} d\omega \frac{\tilde{J}(-\omega, -\vec{k}) \tilde{J}(\omega, \vec{k})}{\omega^2 - \vec{k}^2 - m^2 + i\varepsilon} + \mathcal{O}(\bar{\varepsilon}^3), \quad (5.13)$$

where

$$\tilde{J}(\omega, \vec{k}) = \frac{1}{\sqrt{2\pi}} \int dt \tilde{J}(\vec{k}, t) e^{i\omega t} = \frac{1}{\sqrt{2\pi V}} \int_{\mathbf{R} \times V} d_3 \vec{x} dt J(x) e^{ikx}. \quad (5.14)$$

The last expression should be replaced by  $(2\pi)^{-2} \int_{\mathbf{R}^4} d_4 x J(x) e^{ikx}$  in case the volume is infinite. It is important to note that we have chosen  $J(x) = 0$  for  $t < 0$ . Equivalently we can start at  $t = -\infty$  and integrate the quantum equation of motion up to  $t = \infty$ . We have to require that  $J(x)$  vanishes sufficiently rapidly at infinity.

In an infinite volume we therefore find for what is known as the vacuum to vacuum amplitude of the scattering matrix

$$\begin{aligned}
\lim_{t \rightarrow \infty} \langle 0|\Psi(t)\rangle e^{iE_0 t} &= 1 - \frac{i}{2} \bar{\varepsilon}^2 \int d_4 k \frac{\tilde{J}(-k) \tilde{J}(k)}{k^2 - m^2 + i\varepsilon} + \mathcal{O}(\bar{\varepsilon}^3) \\
&= 1 - \frac{i}{2} \bar{\varepsilon}^2 \int d_4 x d_4 y J(x) G(x-y) J(y) + \mathcal{O}(\bar{\varepsilon}^3). \quad (5.15)
\end{aligned}$$

where  $G(x-y)$  is exactly the Green's function we introduced in the previous section. The so-called  $i\varepsilon$  prescription, which is equivalent with specifying the boundary conditions, has

therefore been derived from the time-ordering in the Hamiltonian evolution of the system and is thus prescribed by the requirement of *causality*. Note that we can use the diagrams introduced in the previous section to express this result (taking  $E_0 = 0$  from now on) as

$$\langle 0|\Psi(t)\rangle = 1 - \frac{i}{2} \begin{array}{c} \times \text{---} \times \\ \bar{\varepsilon}J \quad \bar{\varepsilon}J \end{array} + \mathcal{O}(\bar{\varepsilon}^3) \quad , \quad (5.16)$$

where the factor of a half is a consequence of the symmetry under interchanging the two sources.

For a complex scalar field,  $\varphi$  and  $\varphi^*$  are independent and we need to introduce two sources by adding to the Lagrangian  $-\varphi J^* - \varphi^* J$  (see problem 17). It is not too difficult to show that in this case

$$\langle 0|\Psi(t)\rangle = 1 - i \begin{array}{c} \times \text{---} \times \\ \bar{\varepsilon}J \quad \bar{\varepsilon}J^* \end{array} + \mathcal{O}(\bar{\varepsilon}^3) \quad , \quad (5.17)$$

without a factor of one half because the sources  $J$  and  $J^*$  are independent and cannot be interchanged. This is why in this case the propagator has a direction.

## 6 Path integrals in quantum mechanics

For simplicity we will start with a one-dimensional Hamiltonian

$$H = \frac{\hat{p}^2}{2m} + V(\hat{x}) \quad , \quad \hat{p} = \frac{\hbar}{i} \frac{\partial}{\partial x} \quad , \quad (6.1)$$

where we have indicated a hat on top of operators, to distinguish them from number-valued coordinate and momentum. We wish to study the time-evolution operator  $\exp(-iHT/\hbar)$ . In the coordinate representation its matrix is given by

$$\langle x'|e^{-iHT/\hbar}|x\rangle \quad , \quad (6.2)$$

where  $|x\rangle$  is the position eigenfunction. We will also need the momentum eigenfunction  $|p\rangle$ , i.e.  $\hat{p}|p\rangle = p|p\rangle$ , whose wave function in the coordinate space is given by

$$\langle x|p\rangle = \frac{e^{ipx/\hbar}}{\sqrt{2\pi\hbar}} \quad . \quad (6.3)$$

Indeed, one verifies that

$$\frac{\hbar}{i} \frac{\partial}{\partial x} \langle x|p\rangle = p \langle x|p\rangle \quad . \quad (6.4)$$

An important ingredient in deriving the path integral expression will be the completeness relations

$$\hat{1} = \int dx |x\rangle\langle x| \quad \text{and} \quad \hat{1} = \int dp |p\rangle\langle p| \quad . \quad (6.5)$$

For arbitrary  $N$  we can use this to write

$$\begin{aligned} \langle x'|e^{-iHT/\hbar}|x\rangle &= \langle x'| \left( e^{-iHT/N\hbar} \right)^N |x\rangle = \int \cdots \int \langle x'|e^{-iHT/N\hbar}|x_{N-1}\rangle dx_{N-1} \\ &\quad \langle x_{N-1}|e^{-iHT/N\hbar}|x_{N-2}\rangle dx_{N-2} \cdots \langle x_2|e^{-iHT/N\hbar}|x_1\rangle dx_1 \\ &\quad \langle x_1|e^{-iHT/N\hbar}|x\rangle \quad . \end{aligned} \quad (6.6)$$

We will now use the so-called Trotter formula

$$e^{-i(A+B)/N} = e^{-iA/N} e^{-iB/N} (1 + \mathcal{O}(N^{-2})) \quad (6.7)$$

for two operators  $A$  and  $B$ . This can be seen by expanding the exponents, and the error term is actually of the form  $[A, B]/N^2$ . (One can also use the Campbell-Baker-Hausdorff formula, which will be introduced later). With the Hamiltonian of eq. (6.1) this can be used to write for  $N \rightarrow \infty$

$$e^{-iHT/N\hbar} = e^{-i\hat{p}^2 T/2mN\hbar} e^{-iV(\hat{x})T/N\hbar} \quad (6.8)$$

By inserting the completeness relation for the momentum we can eliminate the operators

$$\begin{aligned} \langle x_{i+1} | e^{-iHT/N\hbar} | x_i \rangle &= \int dp_i \langle x_{i+1} | p_i \rangle \langle p_i | e^{-iHT/N\hbar} | x_i \rangle \\ &\approx \int dp_i \langle x_{i+1} | p_i \rangle \langle p_i | e^{-i\hat{p}^2 T/2mN\hbar} e^{-iV(\hat{x})T/N\hbar} | x_i \rangle \\ &= \int dp_i \langle x_{i+1} | p_i \rangle \langle p_i | e^{-ip_i^2 T/2mN\hbar} e^{-iV(x_i)T/N\hbar} | x_i \rangle \\ &= \int dp_i \frac{e^{ip_i(x_{i+1}-x_i)/\hbar}}{2\pi\hbar} e^{-i(\frac{p_i^2}{2m} + V(x_i))T/N\hbar} \quad (6.9) \end{aligned}$$

This can be done for each matrix element occurring in eq. (6.6). Writing  $\Delta t = T/N$ ,  $x_N = x'$  and  $x_0 = x$  we find

$$\begin{aligned} \langle x' | e^{-iHT/\hbar} | x \rangle &= \lim_{N \rightarrow \infty} \int \cdots \int \prod_{i=1}^{N-1} dx_i \prod_{j=0}^{N-1} dp_j \prod_{i=0}^{N-1} \langle x_{i+1} | p_i \rangle \langle p_i | e^{-iH\Delta t/\hbar} | x_i \rangle \\ &= \lim_{N \rightarrow \infty} \int \frac{dp_0}{2\pi\hbar} \prod_{i=1}^{N-1} \int \frac{dx_i dp_i}{2\pi\hbar} \exp \left[ \frac{i\Delta t}{\hbar} \sum_{i=0}^{N-1} \left( \frac{p_i(x_{i+1} - x_i)}{\Delta t} - \frac{p_i^2}{2m} - V(x_i) \right) \right] \quad (6.10) \end{aligned}$$

It is important to observe that there is one more  $p$  integration than the number of  $x$  integrations.

The integrals in the path integral are strongly oscillating and can only be defined by analytic continuation. As parameter for this analytic continuation one chooses the time  $t$ . For  $\Delta t = T/N \equiv -iT/N = -i\Delta\tau$  the Gaussian integral over the momenta is easily evaluated

$$\int_{-\infty}^{\infty} dp_i \exp \left[ \frac{ip_i(x_{i+1} - x_i)}{\hbar} - \frac{p_i^2 \Delta\tau}{2m\hbar} \right] = \sqrt{2\pi m\hbar/\Delta\tau} \exp \left[ -\frac{1}{2}m \frac{(x_{i+1} - x_i)^2}{\Delta\tau\hbar} \right] \quad (6.11)$$

which leads to

$$\langle x' | e^{-iHT/\hbar} | x \rangle = \lim_{N \rightarrow \infty} \left( \frac{mN}{2\pi\mathcal{T}\hbar} \right)^{\frac{1}{2}N} \int \prod_{i=1}^{N-1} dx_i \exp \left[ -\frac{\Delta\tau}{\hbar} \sum_{i=0}^{N-1} \left( \frac{m(x_{i+1} - x_i)^2}{2\Delta\tau^2} + V(x_i) \right) \right] \quad (6.12)$$

or after substituting  $\mathcal{T} = iT$  we find

$$\langle x' | e^{-iHT/\hbar} | x \rangle = \lim_{N \rightarrow \infty} \left( \frac{mN}{2\pi iT\hbar} \right)^{\frac{1}{2}N} \int \prod_{i=1}^{N-1} dx_i \exp \left[ i\frac{\Delta t}{\hbar} \sum_{i=0}^{N-1} \left( \frac{m(x_{i+1} - x_i)^2}{2\Delta t^2} - V(x_i) \right) \right] \quad (6.13)$$

This is the definition of the path integral, but formally it will often be written as

$$\langle x' | e^{-iHT/\hbar} | x \rangle = \int_{x(0)=x}^{x(T)=x'} \mathcal{D}x(t) \exp[iS/\hbar], \quad S = \int_0^T dt \left\{ \frac{1}{2}m\dot{x}^2(t) - V(x(t)) \right\} \quad (6.14)$$

since the discretised version of the action with  $x_j \equiv x(t = j\Delta t)$  is precisely

$$S_{\text{discrete}} = \Delta t \sum_{i=0}^{N-1} \left( \frac{1}{2} m \frac{(x_{i+1} - x_i)^2}{\Delta t^2} - V(x_i) \right) . \quad (6.15)$$

It is important to note that the continuous expression is just a notation for the discrete version of the path integral, but formal manipulations will be much easier to perform in this continuous formulation. Furthermore, the integral is only defined through the analytic continuation in time.

However, if we integrate over  $x_N = x_0$  this analytically continued path integral, with  $T = -i\mathcal{T}$ , has an important physical interpretation

$$\int dx \langle x | e^{-HT/\hbar} | x \rangle = \text{Tr}(e^{-\beta H})_{\beta=T/\hbar} . \quad (6.16)$$

It is the quantum thermal partition function (the Boltzmann distribution) with a temperature of  $\hbar/kT$ . In the continuous formulation we therefore have

$$\text{Tr}(e^{-T H/\hbar}) = \int_{x(T)=x(0)} \mathcal{D}x(\tau) \exp[-S_E/\hbar] \quad (6.17)$$

in which  $S_E$  is the so-called Euclidean action

$$S_E = \int_0^T d\tau \left\{ \frac{1}{2} m (dx(\tau)/d\tau)^2 + V(x(\tau)) \right\} . \quad (6.18)$$

It is only in this Euclidean case that one can define the path integral in a mathematically rigorous fashion on the class of piecewise continuous functions in terms of the so-called Wiener measure

$$\int dW_{x,x'}(\mathcal{T}) \equiv \int_{x(0)=x}^{x(T)=x'} \mathcal{D}x(\tau) \exp\left(-\int_0^T \frac{1}{2} m \dot{x}(\tau)^2 d\tau/\hbar\right) , \quad (6.19)$$

meaning that this measure is independent of the way the path is discretised.

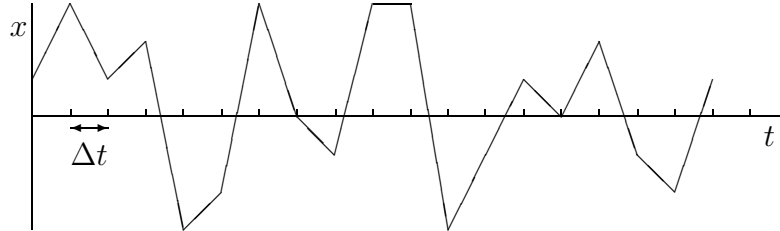


fig. 5

For more details on this see “Quantum Physics: A functional integral point of view”, by J. Glimm and A. Jaffe (2nd ed., Springer, New York, 1987).

We will now do an exact computation to give us some confidence in the formalism. To be specific, what we will compute is the quantum partition function for the harmonic oscillator, where  $V(x) = \frac{1}{2} m \omega^2 x^2$ .

$$Z_N \equiv \left( \frac{mN}{2\pi\mathcal{T}\hbar} \right)^{\frac{1}{2}N} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=0}^{N-1} dx_i \exp \left[ -\frac{\Delta\tau}{\hbar} \sum_{i=0}^{N-1} \frac{1}{2} m \left( \frac{x_{i+1} - x_i}{\Delta\tau} \right)^2 + \frac{1}{2} m \omega^2 x_i^2 \right] . \quad (6.20)$$



Note that we have now  $N$  integrations, because we also integrate over  $x_0 = x(0) = x(\mathcal{T}) = x_N$  to implement the trace. The path involved is thus periodic in time, a general feature of the expression for the quantum partition function in terms of a path integral. We now rescale

$$y_i = x_i \left( \frac{m}{\Delta\tau\hbar} \right)^{\frac{1}{2}} , \quad \tilde{\omega} = \omega\Delta\tau \quad , \quad (6.21)$$

to obtain the simple result

$$Z_N = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=0}^{N-1} \frac{dy_i}{\sqrt{2\pi}} \exp \left[ - \sum_{i=0}^{N-1} \frac{1}{2} (y_{i+1} - y_i)^2 + \frac{1}{2} \tilde{\omega}^2 y_i^2 \right] \quad . \quad (6.22)$$

We can diagonalise the quadratic term by using Fourier transformation

$$y_k = \frac{1}{\sqrt{N}} \sum_{\ell=0}^{N-1} b_{\ell} e^{2\pi i k \ell / N} , \quad b_{\ell}^* = b_{N-\ell} , \quad b_0^* = b_0 \quad . \quad (6.23)$$

It is easy to verify that the Jacobian for the change of variables  $y_i \rightarrow b_{\ell}$  is 1, and one obtains a result that must look familiar from the classical small oscillations problem for a finite number of weights connected by strings,

$$Z_N = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{\ell=0}^{N-1} \frac{db_{\ell}}{\sqrt{2\pi}} \exp \left[ - \frac{1}{2} \sum_{\ell=0}^{N-1} \left( 4 \sin^2(\pi \ell / N) + \tilde{\omega}^2 \right) |b_{\ell}|^2 \right] \quad . \quad (6.24)$$

Note that if  $b_{\ell}$  is complex, we mean by  $db_{\ell} = d\text{Re}b_{\ell} d\text{Im}b_{\ell}$ . The integral can now easily be evaluated

$$Z_N = \prod_{\ell=0}^{N-1} \left( 4 \sin^2(\pi \ell / N) + \tilde{\omega}^2 \right)^{-\frac{1}{2}} \quad . \quad (6.25)$$

We can convert the product to a sum using a Laplace transform. We start with the identity

$$\log(\lambda/\mu) = - \lim_{a \searrow 0} \int_0^{\infty} ds \, s^{a-1} (e^{-s\lambda} - e^{-s\mu}) \quad , \quad (6.26)$$

such that

$$\log[Z_N(\tilde{\omega})/Z_N(\tilde{\omega}_0)] = \frac{1}{2} \lim_{a \searrow 0} \int_0^{\infty} ds \, s^{-1+a} (Q(s, \tilde{\omega}) - Q(s, \tilde{\omega}_0)) \quad . \quad (6.27)$$

We read off, from the definition of  $Z_N$ , that  $Q$  is a sum of exponentials

$$Q(s, \tilde{\omega}) \equiv \sum_{\ell=0}^{N-1} \exp \left[ -s \left( 4 \sin^2(\pi \ell / N) + \tilde{\omega}^2 \right) \right] \equiv N e^{-s(\tilde{\omega}^2 + 2)} f_s(0) \quad , \quad (6.28)$$

where

$$f_s(x) = \frac{1}{N} \sum_{\ell=0}^{N-1} e^{2s \cos(2\pi(\ell+x)/N)} \quad . \quad (6.29)$$

This is a periodic function with period 1 ( $f_s(x+1) = f_s(x)$ ) and its discrete Fourier coefficients can be computed exactly

$$\begin{aligned} \tilde{f}_s(k) &= \int_0^1 dx \, e^{2\pi i k x} f_s(x) = \frac{1}{N} \int_0^N dx \, e^{2\pi i k x} e^{2s \cos(2\pi x/N)} \\ &= \frac{1}{2\pi} \int_0^{2\pi} d\theta \, e^{i N k \theta} e^{2s \cos(\theta)} \equiv I_{Nk}(2s) \quad . \end{aligned} \quad (6.30)$$

Please note that we have exchanged the sum over  $\ell$  with extending the integration of  $x$  to the interval  $[0, N]$ . The last identity is one of the definitions of the modified Besselfunction, see e.g. “Handbook of mathematical functions”, by M. Abramowitz and I. Stegun (Dover, New York, 1978). The advantage of these manipulations is that the Laplace transform of this Besselfunction is known (see the same reference)

$$\int_0^\infty ds e^{-\lambda s} I_\nu(s) = \frac{[\lambda + \sqrt{\lambda^2 - 1}]^{-|\nu|}}{\sqrt{\lambda^2 - 1}} \quad , \quad (6.31)$$

and as we can express  $f_s(x)$  as a sum over these Besselfunctions

$$f_s(x) = \sum_{k \in \mathbb{Z}} e^{-2\pi i k x} I_{Nk}(2s) \quad , \quad (6.32)$$

this allows us to evaluate eq. (6.27). For technical reasons it is easier to compute the variation of the free energy with the frequency, where the free energy  $F$  is defined as

$$Z_N(\tilde{\omega}) = \exp[-\beta F(\tilde{\omega})] \quad . \quad (6.33)$$

We therefore find (using  $\beta = \mathcal{T}/\hbar = N\Delta\tau/\hbar$ )

$$\begin{aligned} \frac{\partial}{\partial \tilde{\omega}} F(\tilde{\omega}) &= \hbar \omega \sum_{k \in \mathbb{Z}} \int_0^\infty ds e^{-(2+\tilde{\omega}^2)s} I_{Nk}(2s) \\ &= \frac{\hbar \omega}{2\sqrt{(\frac{1}{2}\tilde{\omega}^2 + 1)^2 - 1}} \sum_{k \in \mathbb{Z}} \left[ \frac{1}{2}\tilde{\omega}^2 + 1 + \sqrt{(\frac{1}{2}\tilde{\omega}^2 + 1)^2 - 1} \right]^{-N|k|} . \end{aligned} \quad (6.34)$$

The geometric series is of course easily summed, but to make the result more transparent we introduce the scaled *effective* frequency  $\Omega$

$$\omega \mathcal{T}/N = \omega \Delta\tau = \tilde{\omega} \equiv 2 \sinh(\frac{1}{2}\Omega) \quad (6.35)$$

and using the identity  $\frac{1}{2}\tilde{\omega}^2 + 1 = \cosh(\Omega)$ , one easily finds that

$$\frac{\Delta\tau}{\hbar} \frac{\partial}{\partial \Omega} F(\Omega) = \frac{1}{2} \sum_{k \in \mathbb{Z}} e^{-N\Omega|k|} = \frac{1}{1 - e^{-N\Omega}} - \frac{1}{2} = -\frac{1}{N} \frac{\partial}{\partial \Omega} \log \left( \frac{e^{-\Omega N/2}}{1 - e^{-N\Omega}} \right) \quad . \quad (6.36)$$

The last identity can be seen as the free energy of the harmonic oscillator with the frequency  $\Omega N/\mathcal{T} = \Omega/\Delta\tau$ , as it can also be written as

$$\frac{\Delta\tau}{\hbar} \frac{\partial}{\partial \Omega} F(\Omega) = -\frac{1}{N} \frac{\partial}{\partial \Omega} \log \left( \sum_{n=0}^{\infty} e^{-(n+\frac{1}{2})\Omega N} \right) \quad . \quad (6.37)$$

Amazingly, even at finite  $N$  the euclidean path integral agrees with the quantum partition function of a harmonic oscillator, but with a frequency that is modified by the discretisation, see eqs. (6.21) and (6.35). It is trivial to check now that the limit  $N \rightarrow \infty$  is well defined and gives the required result, since

$$\lim_{N \rightarrow \infty} \Omega N/\mathcal{T} = \omega \quad . \quad (6.38)$$

In general the exact finite  $N$  path integral is no longer of a simple form. Nevertheless, one can evaluate this exact expression in relatively simple terms (which will verify the above

result along a different route, see also problem 10). So from now on we will take the potential arbitrary and in a sense we follow the derivation of the path integral in the reverse order.

$$\begin{aligned}
Z_N(x', x; \mathcal{T}) &= \int \frac{dp_0}{2\pi\hbar} \prod_{i=1}^{N-1} \int \frac{dx_i dp_i}{2\pi\hbar} \exp \left[ \frac{\Delta\tau}{\hbar} \sum_{i=0}^{N-1} \left( \frac{ip_i(x_{i+1} - x_i)}{\Delta\tau} - \frac{p_i^2}{2m} - V(x_i) \right) \right] \\
&= \int dp_0 \prod_{i=1}^{N-1} \int dx_i dp_i \prod_{j=0}^{N-1} \langle x_{j+1} | p_j \rangle \langle p_j | \exp \left( -\frac{\Delta\tau \hat{p}^2}{2m\hbar} \right) \exp \left( -\frac{\Delta\tau V(\hat{x})}{\hbar} \right) | x_j \rangle \\
&= \langle x' | \left\{ \exp \left( -\frac{\Delta\tau \hat{p}^2}{2m\hbar} \right) \exp \left( -\frac{\Delta\tau V(\hat{x})}{\hbar} \right) \right\}^N | x \rangle .
\end{aligned} \tag{6.39}$$

This means that we can define an *effective* Hamiltonian by

$$e^{-H(N)\mathcal{T}/\hbar} \equiv \left\{ \exp \left( -\frac{\Delta\tau \hat{p}^2}{2m\hbar} \right) \exp \left( -\frac{\Delta\tau V(\hat{x})}{\hbar} \right) \right\}^N . \tag{6.40}$$

But this Hamiltonian is not hermitian as one easily checks from the above expression, since under conjugation the order of the exponents containing the kinetic and potential terms is reversed. This can be corrected in two ways

$$\begin{aligned}
e^{-H_1(N)\Delta\tau/\hbar} &\equiv \exp \left( -\frac{\Delta\tau \hat{p}^2}{4m\hbar} \right) \exp \left( -\frac{\Delta\tau V(\hat{x})}{\hbar} \right) \exp \left( -\frac{\Delta\tau \hat{p}^2}{4m\hbar} \right) , \\
e^{-H_2(N)\Delta\tau/\hbar} &\equiv \exp \left( -\frac{\Delta\tau V(\hat{x})}{2\hbar} \right) \exp \left( -\frac{\Delta\tau \hat{p}^2}{2m\hbar} \right) \exp \left( -\frac{\Delta\tau V(\hat{x})}{2\hbar} \right) ,
\end{aligned} \tag{6.41}$$

leading to two equivalent expressions for the finite  $N$  path integral

$$\begin{aligned}
Z_N(x', x; \mathcal{T}) &= \langle x' | \exp \left( -\frac{\Delta\tau \hat{p}^2}{4m\hbar} \right) \exp(-H_1(N)\mathcal{T}/\hbar) \exp \left( \frac{\Delta\tau \hat{p}^2}{4m\hbar} \right) | x \rangle \\
Z_N(x', x; \mathcal{T}) &= \langle x' | \exp \left( \frac{\Delta\tau V(\hat{x})}{2\hbar} \right) \exp(-H_2(N)\mathcal{T}/\hbar) \exp \left( -\frac{\Delta\tau V(\hat{x})}{2\hbar} \right) | x \rangle
\end{aligned} \tag{6.42}$$

In particular the partition function is given by

$$Z_N = \int dx Z_N(x, x, \mathcal{T}) = \text{Tr}(e^{-H_1(N)\mathcal{T}/\hbar}) = \text{Tr}(e^{-H_2(N)\mathcal{T}/\hbar}) . \tag{6.43}$$

It is actually not too difficult to show that there exists a unitary transformation  $U$ , such that  $UH_1U^\dagger = H_2$ , which shows that both choices are indeed physically equivalent.

In principle we can now compute  $H_i(N)$  for finite  $N$  as an expansion in  $1/N$ , by using the so-called Campbell-Baker-Hausdorff formula

$$e^A e^B = e^{F(A,B)} , \quad F(A,B) = A + B + \frac{1}{2}[A, B] + \frac{1}{12}[A, [A, B]] + \frac{1}{12}[B, [B, A]] + \dots , \tag{6.44}$$

which is a series in multiple commutators of the, in general non-commuting, operators  $A$  and  $B$ . It can be derived by expanding the exponentials, but in the mathematics literature more elegant constructions are known, based on properties of Lie groups and Lie algebras. These objects will be discussed in sect. 18. For the harmonic oscillator, working out the products of the exponential can be done to all orders and one finds (see problem 10 for details)

$$\Delta\tau H_i(N) = \frac{\hat{p}^2}{2M_i} + \frac{1}{2}M_i\Omega^2\hat{x}^2 , \tag{6.45}$$

with  $\Omega$  defined as in eq. (6.35) and the *effective* masses  $M_i$  defined by

$$M_1 = \frac{2m \tanh(\frac{1}{2}\Omega)}{\Delta\tau\Omega}, \quad M_2 = \frac{m \sinh(\Omega)}{\Delta\tau\Omega} \quad . \quad (6.46)$$

One can now explicitly verify that (cmp. eq. (6.37))

$$\text{Tr}(\exp(-H_1(N)\mathcal{T}/\hbar)) = \text{Tr}(\exp(-H_2(N)\mathcal{T}/\hbar)) = \exp(-F(\Omega)\mathcal{T}/\hbar) \quad . \quad (6.47)$$

Now we have seen that, at least for some examples, the limit of increasingly finer discretisation is in principle well defined, we can think of generalisation to an arbitrary number of dimensions ( $n$ ) (for field theory even to an infinite number of dimensions).

$$\begin{aligned} \langle \vec{x}' | e^{-iHT/\hbar} | \vec{x} \rangle &\equiv \int \mathcal{D}\vec{x}(t) \exp \left[ \frac{i}{\hbar} \int_0^T dt L(\vec{x}(t), \dot{\vec{x}}(t)) \right] \\ &= \lim_{N \rightarrow \infty} \int \frac{d_n p_0}{(2\pi\hbar)^n} \int \prod_{i=1}^{N-1} \frac{d_n x_i d_n p_i}{(2\pi\hbar)^n} \exp \left[ i \frac{\Delta t}{\hbar} \sum_{i=0}^{N-1} \left( \vec{p}_i \cdot \frac{(\vec{x}_{i+1} - \vec{x}_i)}{\Delta t} - H(\vec{p}_i, \vec{x}_i) \right) \right] \\ &= \lim_{N \rightarrow \infty} \left( \frac{mN}{2\pi i T \hbar} \right)^{\frac{1}{2}nN} \int \prod_{i=1}^{N-1} d_n x_i \exp \left[ i \frac{\Delta t}{\hbar} \sum_{i=0}^{N-1} L(\vec{x}_i, (\vec{x}_{i+1} - \vec{x}_i)/\Delta t) \right] \quad . \quad (6.48) \end{aligned}$$

We have purposely also given the expression that involves the path integral as an integral over phase space, as it shows that the Gaussian integration over the momenta effectuates the Legendre transform

$$i\vec{p} \cdot \dot{\vec{x}} - i \frac{\vec{p}^2}{2m} - iV(\vec{x}) = -i \frac{(\vec{p} - m\dot{\vec{x}})^2}{2m} + i \frac{m\dot{\vec{x}}^2}{2} - iV(\vec{x}) \quad , \quad (6.49)$$

which is equivalent to the stationary phase approximation for the momentum integration

$$\frac{\delta}{\delta \vec{p}} (\vec{p} \cdot \dot{\vec{x}} - H(\vec{p}, \vec{x})) = \dot{\vec{x}} - \frac{\partial H}{\partial \vec{p}} = \dot{\vec{x}} - \frac{\vec{p}}{m} = 0 \quad . \quad (6.50)$$

An interesting other example of the path integral is the case of the interaction of a charged particle with a magnetic field. In that case one has for the Hamiltonian

$$H(\hat{\vec{p}}, \hat{\vec{x}}) = \frac{(\hat{\vec{p}} - e\vec{A}(\hat{\vec{x}}))^2}{2m} + V(\hat{\vec{x}}) \quad . \quad (6.51)$$

Now, however, the matrix element  $\langle \vec{p}_i | \exp(-iH\Delta t/\hbar) | \vec{x}_i \rangle$  will depend on the specific ordering for the position and momentum operators in  $H$ . Different orderings differ by terms linear in  $\hbar$ , or

$$\vec{A}(\hat{\vec{x}}) \cdot \hat{\vec{p}} = \hat{\vec{p}} \cdot \vec{A}(\hat{\vec{x}}) + i\hbar \partial_i A_i(\hat{\vec{x}}) \quad . \quad (6.52)$$

So, by choosing the so-called Coulomb gauge  $\partial_i A_i(\vec{x}) = 0$ , the problem of the operator ordering disappears. We leave it as an exercise to verify that the action, obtained from the Legendre transform, is given by

$$S = \int_0^T dt \left( \frac{1}{2} m \dot{\vec{x}}^2 - V(\vec{x}) + e \dot{\vec{x}} \cdot \vec{A}(\vec{x}) \right) \quad . \quad (6.53)$$

Under a gauge transformation  $A_i(\vec{x}) \rightarrow A_i(\vec{x}) + \partial_i \Lambda(\vec{x})$ , one finds that the action changes to  $S_\Lambda = S + e\{\Lambda(\vec{x}(T)) - \Lambda(\vec{x}(0))\}$ . Using the path integral this means that

$$\langle \vec{x}' | \exp(-iH_\Lambda T/\hbar) | \vec{x} \rangle = \exp(ie\Lambda(\vec{x}')/\hbar) \langle \vec{x}' | \exp(-iHT/\hbar) | \vec{x} \rangle \exp(-ie\Lambda(\vec{x})/\hbar) \quad . \quad (6.54)$$

Since it is easily shown that  $H_\Lambda = \exp(i e \Lambda(\vec{x})/\hbar) H \exp(-i e \Lambda(\vec{x})/\hbar)$ , this proves that the path integral derived from eq. (6.53) has the correct properties under gauge transformations, despite the fact that the derivation was performed by first going to the Coulomb gauge.

As long as the Hamiltonian is quadratic in the momenta, the stationary phase approximation for the momentum integral is exact. However, also for the coordinate integrals we can use the stationary phase *approximation* (exact for a harmonic oscillator), which is related to the WKB approximation in quantum mechanics. It gives a way of defining an expansion in  $\hbar$ , where in accordance to the correspondence principle, the lowest order term reproduces the classical time evolution. Indeed the stationary phase condition

$$\frac{\delta S}{\delta x^i(t)} = \frac{\delta S}{\delta x^i(t)} - \frac{d}{dt} \frac{\delta S}{\delta \dot{x}^i(t)} = 0 \quad (6.55)$$

is precisely solved by the classical solutions,  $\vec{x}_{cl}(t)$ , with  $\vec{x}_{cl}(0) = \vec{x}$  and  $\vec{x}_{cl}(T) = \vec{x}'$ . We expand around these solutions by writing

$$\vec{x}(t) = \vec{x}_{cl}(t) + \vec{q}(t) \ , \quad \vec{q}(0) = \vec{q}(T) = \vec{0} \ , \quad (6.56)$$

such that

$$S(\vec{x}) = S(\vec{x}_{cl}) + \frac{1}{2} \int dt' dt \ q^i(t) \frac{\delta^2 S(\vec{x}_{cl})}{\delta q^i(t) \delta q^j(t')} q^j(t') + \mathcal{O}(q^3) \ . \quad (6.57)$$

There is no term linear in  $q^i(t)$ , as this term is proportional to the equations of motion, or equivalently to the stationary phase condition. For the simple Lagrangian  $L = \frac{1}{2} m \dot{\vec{x}}^2 - V(\vec{x})$  one has

$$\frac{\delta^2 S(\vec{x}_{cl})}{\delta q^i(t) \delta q^j(t')} = -\delta(t-t') \left( m \delta_{ij} \frac{d^2}{dt^2} + M_{ij}(t) \right) \ , \quad M_{ij}(t) \equiv \frac{\partial^2 V(\vec{x})}{\partial x^i \partial x^j} \Big|_{\vec{x}=\vec{x}_{cl}(t)} \ . \quad (6.58)$$

For the harmonic potential,  $V = \frac{1}{2} m \omega^2 \vec{x}^2$ , where the stationary phase approximation is exact, i.e. there are no  $\mathcal{O}(q^3)$  corrections. Introducing  $\vec{q}(t)$ , however, splits the action in a classical piece, that depends on the boundary conditions, and a quantum piece described by a harmonic oscillator action for the fluctuations around the classical path, that is independent of the boundary conditions *and* the classical path

$$S(\vec{x}) = S(\vec{x}_{cl}) + \int_0^T dt \left( \frac{1}{2} m \dot{\vec{q}}^2 - \frac{1}{2} m \omega^2 \vec{q}^2 \right) \ . \quad (6.59)$$

In practical situations one splits from the action the part quadratic in the coordinates and velocities, and considers the rest as a perturbation. In that case  $x_{cl}$  is the classical solution of the quadratic part only. As this can always be solved exactly, and as non-quadratic path integrals can rarely be computed explicitly, this will be the way in which we will derive the Feynman rules for the quantum theory, in terms of which one can efficiently perform the perturbative computations.

## 7 Path integrals in field theory

For a scalar field in a finite volume  $V = [0, L]^3$  the Hamiltonian is given in the Fourier representation by (see eq. (2.6))

$$H = \sum_{\vec{k}=2\pi\vec{n}/L} \left( \frac{1}{2} |\tilde{\pi}(\vec{k})|^2 + \frac{1}{2} (\vec{k}^2 + m^2) |\tilde{\varphi}(\vec{k})|^2 + V(\tilde{\varphi}) + \tilde{\varphi}(\vec{k}) \tilde{J}(-\vec{k}, t) \right) \ . \quad (7.1)$$

As  $\varphi(x)$  is real we have  $\tilde{\varphi}^*(\vec{k}) = \tilde{\varphi}(-\vec{k})$ . It is customary to write the term quadratic in the fields (the mass term) explicitly, such that the potential  $V(\varphi)$  only contains the interaction terms. If we like we could split the Fourier modes in their real and imaginary components (the  $\cos(\vec{x} \cdot \vec{k})$  and  $\sin(\vec{x} \cdot \vec{k})$  modes). Or even simpler is to use Dirichlet boundary conditions, i.e.  $\varphi(x) = 0$  at the boundaries of the volume, such that the Fourier modes are given by  $\prod_j \sin(\pi n_j x_j / L)$  (with  $n_j > 0$ ), with real coefficients. In either case, for  $V(\varphi) = 0$  the Hamiltonian simply describes an infinite set of decoupled harmonic oscillators, which can be truncated to a finite set by introducing a so-called momentum cutoff  $|\vec{k}| \leq \Lambda$ . In this case we know how to write the path integral, even in the presence of interactions. The introduction of a cutoff is called a regularisation. The field theory is called renormalisable if the limit  $\Lambda \rightarrow \infty$  can be defined in a suitable way, often by varying the parameters in a suitable way with the cutoff. The class of renormalisable field theory is relatively small. For a finite momentum cutoff the path integral is nothing but a simple generalisation of the one we defined for quantum mechanics in  $n$  dimensions, or in the absence of interactions

$$\begin{aligned}
Z &= \lim_{N \rightarrow \infty} \prod_{\vec{k}} (2\pi i \Delta t)^{-N/2} \int \prod_{j=1}^{N-1} \prod_{\vec{k}} d\tilde{\varphi}_j(\vec{k}) \exp \left[ i \Delta t \sum_{j=0}^{N-1} \sum_{\vec{k}} \frac{|\tilde{\varphi}_{j+1}(\vec{k}) - \tilde{\varphi}_j(\vec{k})|^2}{2\Delta t^2} \right. \\
&\quad \left. - \frac{1}{2}(\vec{k}^2 + m^2)|\tilde{\varphi}_j(\vec{k})|^2 - \tilde{\varphi}_j(\vec{k})\tilde{J}(-\vec{k}, j\Delta t) \right] \\
&\equiv \int \mathcal{D}\tilde{\varphi}(\vec{k}, t) \exp \left( i \int_0^T \sum_{\vec{k}} \left\{ \frac{1}{2}|\dot{\tilde{\varphi}}(\vec{k}, t)|^2 - \frac{1}{2}(\vec{k}^2 + m^2)|\tilde{\varphi}(\vec{k}, t)|^2 - \tilde{\varphi}(\vec{k}, t)\tilde{J}(-\vec{k}, t) \right\} dt \right).
\end{aligned} \tag{7.2}$$

One of course identifies  $\tilde{\varphi}_j(\vec{k}) = \tilde{\varphi}(\vec{k}, t = j\Delta t)$  and performing the Fourier transformation once more, we can write

$$\begin{aligned}
&\int_0^T dt \sum_{\vec{k}} \left\{ \frac{1}{2}|\dot{\tilde{\varphi}}(\vec{k}, t)|^2 - \frac{1}{2}(\vec{k}^2 + m^2)|\tilde{\varphi}(\vec{k}, t)|^2 - \tilde{\varphi}(\vec{k}, t)\tilde{J}(-\vec{k}, t) \right\} \\
&= \int_0^T dt \int_V d_3\vec{x} \left\{ \frac{1}{2}(\partial_t \varphi(\vec{x}, t))^2 - \frac{1}{2}(\partial_i \varphi(\vec{x}, t))^2 - \frac{1}{2}m^2 \varphi^2(\vec{x}, t) - \varphi(\vec{x}, t)J(\vec{x}, t) \right\} \\
&= \int_{V \times [0, T]} d_4x \left\{ \frac{1}{2}\partial_\mu \varphi(x)\partial^\mu \varphi(x) - \frac{1}{2}m^2 \varphi^2(x) - \varphi(x)J(x) \right\} \quad .
\end{aligned} \tag{7.3}$$

The last expression is manifestly Lorentz invariant apart from the dependence on the boundary conditions on the fields (which should disappear once we take  $L$  and  $T$  to infinity). This will allow us to perform perturbation theory in a Lorentz covariant way (things are somewhat subtle as any finite choice of the momentum cutoff does break the Lorentz invariance, and there are some theories where this is not restored when removing the cutoff, i.e. taking the limit  $\Lambda \rightarrow \infty$ ). This achieves a substantial simplification over Hamiltonian perturbation theory. It is now also trivial to reintroduce the interactions, by adding the potential term to the Lagrange density, and we find in yet another shorthand notation for the measure the following expression for the path integral (implicitly assuming that the boundary values  $\varphi(\vec{x}, 0)$  and  $\varphi(\vec{x}, T)$  are fixed, prescribed functions)

$$Z = \int \mathcal{D}\varphi(x) \exp \left( i \int_{V \times [0, T]} d_4x \left\{ \frac{1}{2}\partial_\mu \varphi(x)\partial^\mu \varphi(x) - \frac{1}{2}m^2 \varphi^2(x) - V(\varphi) - \varphi(x)J(x) \right\} \right). \tag{7.4}$$

In principle a path integral should be independent of the discretisation used in order to define it. For the euclidean path integral, one particular way that is used quite often is

the lattice discretisation, where instead of a momentum cutoff one makes not only time but also space discrete. This means that the field now lives on a lattice and its argument takes the values  $ja$  where  $j \in \mathbb{Z}^4$  and  $a$  is the so-called lattice spacing, which in the end should be taken to zero. By suitably restricting the components of  $j$ , with appropriate boundary conditions on the fields, one keeps space and time finite,  $V = a^3 M^3$  and  $\mathcal{T} = aN$ . This leads to an integral of the form

$$(2\pi a)^{-NM^3/2} \int \prod_j d\varphi_j \exp \left( -a^4 \sum_j \left\{ \frac{1}{2} \sum_\mu \frac{(\varphi_{j+e_\mu} - \varphi_j)^2}{a^2} + \frac{1}{2} m^2 \varphi_j^2 + V(\varphi_j) + J_j \varphi_j \right\} \right). \quad (7.5)$$

where  $e_\mu$  is a unit vector in the  $\mu$  direction, and  $\varphi_j$  is identified with  $\varphi(aj)$ . In a sense, the momentum cutoff  $\Lambda$  is similar to the space-time cutoff  $1/a$ . The lattice formulation is very suitable for numerically evaluating the path integral, whereas the momentum cutoff is suitable for performing perturbation theory around the quadratic approximation of the action. For the latter we will compute, using the path integral, the same quantity as was calculated in section 5, using Hamiltonian perturbation theory.

In the presence of a source, the Hamiltonian depends on time and the evolution operator has to be written in a way that takes the time ordering into account. As the time evolution operator  $U(t)$  satisfies the Schrödinger equation

$$i \frac{d}{dt} U(t) = H(t) U(t), \quad U(0) = 1, \quad (7.6)$$

its solution can be written as (note the absence of  $1/n!$ )

$$U(t) = \text{Texp} \left( -i \int_0^t H(t) dt \right) \equiv \sum_{n=0}^{\infty} (-i)^n \int_0^t dt_1 \int_0^{t_1} \cdots \int_0^{t_{n-1}} dt_n H(t_1) \cdots H(t_n). \quad (7.7)$$

For convenience we introduce the notation

$$U(t_2, t_1) \equiv \text{Texp} \left( -i \int_{t_1}^{t_2} H(t) dt \right), \quad t_2 > t_1, \quad (7.8)$$

which satisfies the property that

$$U(t_3, t_2) U(t_2, t_1) = U(t_3, t_1), \quad t_3 > t_2 > t_1. \quad (7.9)$$

In section 5 we calculated the matrix element  $\langle 0 | U(T) | 0 \rangle$  to second order in the source (from now on we put  $\bar{\varepsilon} = 1$ ). The Lagrangian relevant for the path integral evaluation is given by  $\mathcal{L} = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2} m^2 \varphi^2 - J\varphi$ , with  $J(x) = 0$  for  $t < 0$  and  $t > T$  (and for  $\vec{x} \notin [0, L]^3$ ). The vacuum  $|0\rangle$  is the state where all  $\vec{k}$  oscillators are in their ground state. It turns out that we do not need an explicit expression for this vacuum wave functional, denoted by  $\Psi_0(\{\tilde{\varphi}(\vec{k})\}) = \langle \{\tilde{\varphi}(\vec{k})\} | 0 \rangle$ . We have

$$\langle 0 | U(T) | 0 \rangle = \int \prod_{\vec{p}} d\tilde{\varphi}(\vec{p}) d\tilde{\varphi}'(\vec{p}) \langle 0 | \{\tilde{\varphi}'(\vec{p})\} \rangle \langle \{\tilde{\varphi}'(\vec{p})\} | U(T) | \{\tilde{\varphi}(\vec{p})\} \rangle \langle \{\tilde{\varphi}(\vec{p})\} | 0 \rangle, \quad (7.10)$$

where  $\{\tilde{\varphi}(\vec{p})\}$  plays the role of  $x$  and  $\{\tilde{\varphi}'(\vec{p})\}$  the role of  $x'$ . The path integral expression for the evolution operator therefore becomes

$$\begin{aligned} \langle \{\tilde{\varphi}'(\vec{p})\} | U(T) | \{\tilde{\varphi}(\vec{p})\} \rangle &= \int \mathcal{D}\varphi(x) \exp \left( i \int_0^T \int_V \mathcal{L}(\varphi(x)) d_4 x \right) \\ &= \int \mathcal{D}\tilde{\varphi}(k) \exp \left( i \sum_{\vec{k}, k_0} \left\{ \frac{1}{2} (k^2 - m^2) |\tilde{\varphi}(k)|^2 - \tilde{\varphi}(k) \tilde{J}(-k) \right\} \right). \end{aligned} \quad (7.11)$$

We have here also performed the Fourier transformation with respect to time, thereby converting the path integral measure  $\mathcal{D}\tilde{\varphi}(\vec{k}, t)$  to the multiple integral over the (discrete temporal) Fourier components  $\mathcal{D}\tilde{\varphi}(k)$ , exactly as was done in one dimension, see eq.(6.23), hence we also find a unit Jacobian for this change of variables. Obviously our notations are such that  $k \equiv (k_0, \vec{k})$  and  $k^2 = k_0^2 - \vec{k}^2$ . If we take the limit of space and time to infinity ( $L \rightarrow \infty$  and  $T \rightarrow \infty$ ), the sums over  $k$  can be converted into integrals. Finally we note that the oscillatory integrals occurring in the path integral can be dampened by replacing  $m^2$  by  $m^2 - i\varepsilon$  as this leads to replacement  $\exp(i \int \mathcal{L}(\varphi) d_4x) \rightarrow \exp(i \int \mathcal{L}(\varphi) d_4x - \varepsilon \int \varphi^2 d_4x)$ . This prescription also allows us to make the analytic continuation to imaginary time, and coincides with the prescription derived for the propagator in the Hamiltonian formulation, so that causality is also properly implemented in the path integral approach. To recover the result obtained in the Hamiltonian approach, we simply split off a square

$$\begin{aligned} \langle \{\tilde{\varphi}'(\vec{p})\} | U(T) | \{\tilde{\varphi}(\vec{p})\} \rangle &= \int \mathcal{D}\tilde{\varphi}(k) \exp \left( i \sum_{\vec{k}, k_0} \frac{1}{2} (k^2 - m^2 + i\varepsilon) \left| \tilde{\varphi}(k) - \frac{\tilde{J}(k)}{k^2 - m^2 + i\varepsilon} \right|^2 \right) \\ &\quad \times \exp \left( -\frac{i}{2} \sum_k \frac{|\tilde{J}(k)|^2}{k^2 - m^2 + i\varepsilon} \right) \quad . \quad (7.12) \end{aligned}$$

We can now shift the integration of  $\tilde{\varphi}(k)$  over  $\tilde{J}(k)/(k^2 - m^2 + i\varepsilon)$  and introduce the Green's function in coordinate space (eq. (4.7)) to get

$$\begin{aligned} \langle \{\tilde{\varphi}'(\vec{p})\} | U(T) | \{\tilde{\varphi}(\vec{p})\} \rangle &= \\ &= \exp \left( -\frac{i}{2} \int d_4x d_4y J(x) G(x-y) J(y) \right) \int \mathcal{D}\tilde{\varphi}(k) \exp \left( i \sum_{\vec{k}, k_0} \frac{1}{2} (k^2 - m^2 + i\varepsilon) |\tilde{\varphi}(k)|^2 \right) \\ &= \exp \left( -\frac{i}{2} \int d_4x d_4y J(x) G(x-y) J(y) \right) \langle \{\tilde{\varphi}'(\vec{p})\} | e^{-iH(J=0)T} | \{\tilde{\varphi}(\vec{p})\} \rangle \quad . \quad (7.13) \end{aligned}$$

In the last step it is crucial to note that the source is taken to vanish for  $t \leq 0$  and for  $t \geq T$ , as otherwise the shift we performed in the field would have changed the boundary values. Using eq. (7.10) we now obtain the remarkable result that

$$\langle 0 | U(T) | 0 \rangle = \langle 0 | e^{-iH(J=0)T} | 0 \rangle \exp \left( -\frac{i}{2} \int d_4x d_4y J(x) G(x-y) J(y) \right) \quad (7.14)$$

to *all* orders in the source  $J$ . Even at the level of a non-interacting scalar field theory, this demonstrates the dramatic simplifications that can arise from using the path integral method for calculating quantum amplitudes. One particularly feature that is noteworthy in the path integral calculation, is that the part of the Lagrangian that is quadratic in the fields represents the inverse propagator. This is no accident and is in general the way the (lowest order) propagator is directly read off from the Lagrangian, since the quadratic part of the action is the starting point of the perturbative expansion. But before we will derive the Feynman rules from the perturbative expansion, it will be useful to emphasise that the time ordering, playing such an important role in the Hamiltonian formulation, is automatically implemented by the path integral. Furthermore, it will be helpful to understand in more detail how the source can be used to create and annihilate particles, as this will be our tool to write down the matrix elements of the evolution operator (the so-called scattering matrix, or for short  $S$ -matrix) with respect to the basis specified by particle number and momentum (the so-called Fock space), see eq. (2.9).



In the Hamiltonian formulation we consider

$$< 0 | \hat{\varphi}(\vec{x}', t_2) \hat{\varphi}(\vec{x}, t_1) | 0 >, \quad t_2 > t_1, \quad (7.15)$$

where the field operator (cmp. eq. (2.8)) is given by

$$\hat{\varphi}(\vec{x}, t) = e^{iHt} \hat{\varphi}(\vec{x}) e^{-iHt}, \quad (7.16)$$

such that

$$< 0 | \hat{\varphi}(\vec{x}', t_2) \hat{\varphi}(\vec{x}, t_1) | 0 > = \frac{< 0 | e^{-iH(T-t_2)} \hat{\varphi}(\vec{x}') e^{-iH(t_2-t_1)} \hat{\varphi}(\vec{x}) e^{-iHt_1} | 0 >}{< 0 | e^{-iHT} | 0 >}, \quad (7.17)$$

where we have made use of the fact that the vacuum  $|0\rangle$  is assumed to be an eigenstate of the Hamiltonian  $H$  (without a source term). The normalisation by  $< 0 | e^{-iHT} | 0 >$  is hence a rather trivial factor. We can even write a similar expression in the presence of the source. In perturbation theory this would not be needed, but it is useful from a general point of view to consider this situation too.

In the presence of a time dependent source one can write

$$< 0 | U(T) \hat{\varphi}(\vec{x}', t_2) \hat{\varphi}(\vec{x}, t_1) | 0 > = < 0 | U(T, t_2) \hat{\varphi}(\vec{x}') U(t_2, t_1) \hat{\varphi}(\vec{x}) U(t_1) | 0 >. \quad (7.18)$$

In this case the field operators are of course given by

$$\hat{\varphi}(\vec{x}, t) = U(t)^\dagger \hat{\varphi}(\vec{x}) U(t), \quad (7.19)$$

where in general  $U(t)$  depends on the source  $J$ . It is now trivial, but a bit tedious, to convert this matrix element to a path integral. One first writes the product of the operators as a product of matrices in a suitable representation (e.g. the field representation  $|\{\tilde{\varphi}(\vec{p})\}\rangle$ ). Each of the matrices for the three evolution operators can be written as a path integral, excluding the integral over the initial and final field components. The matrix product involves an integral over the final field component of the matrix to the right, which is also the initial field component for the matrix to the left. Without the insertion of the field operators  $\hat{\varphi}(x)$ , this would describe the fact that  $U(t_3, t_2)U(t_2, t_1) = U(t_3, t_1)$  in the path integral formulation simply means that one *glues* the paths in  $U(t_3, t_2)$  and  $U(t_2, t_1)$  together by integrating over  $\tilde{\varphi}(\vec{k}, t_2)$ . With the field operator sandwiched between the two  $U$  matrices one simply includes its eigenvalue in the integrand over the paths, since the field operator (or its Fourier components) is diagonal on the states  $|\{\tilde{\varphi}(\vec{p})\}\rangle$ . The final result can be written as

$$\int \prod_{\vec{p}} d\tilde{\varphi}(\vec{p}) d\tilde{\varphi}'(\vec{p}) < 0 | \{\tilde{\varphi}'(\vec{p})\} > \{ \int \mathcal{D}\varphi(x) \exp(i \int_{t_2}^T \mathcal{L} d_4x) \varphi(\vec{x}', t_2) \exp(i \int_{t_1}^{t_2} \mathcal{L} d_4x) \times \\ \varphi(\vec{x}, t_1) \exp(i \int_0^{t_1} \mathcal{L} d_4x) \} < \{\tilde{\varphi}(\vec{p})\} | 0 >, \quad (7.20)$$

where we implicitly assumed that the boundary conditions for the field  $\varphi$  in the path integral is in momentum space given by  $\tilde{\varphi}(\vec{p}, t=0) = \tilde{\varphi}(\vec{p})$  and  $\tilde{\varphi}(\vec{p}, t=T) = \tilde{\varphi}'(\vec{p})$ . The way the time ordering in the path integral is manifest is now obvious. Note that the field expectation values can also be written in terms of derivatives with respect to the sources, which is particularly simple to derive in the path integral formulation

$$< 0 | U(T) \hat{\varphi}(\vec{x}', t_2) \hat{\varphi}(\vec{x}, t_1) | 0 > = - \frac{\delta^2}{\delta J(\vec{x}', t_2) \delta J(\vec{x}, t_1)} < 0 | U(T) | 0 >. \quad (7.21)$$

Since we assumed that  $|0\rangle$  is an eigenstate of  $H$  (i.e. at  $J = 0$ ),  $\langle 0|U_{J=0}(T) = \langle 0|e^{-iE_0T}$  (with  $E_0$  the vacuum energy) and we can bring the trivial phase factor  $e^{-iE_0T}$  to the other side by normalising with  $\langle 0|U_{J=0}(T)|0\rangle$ , as in eq. (7.17).

$$\langle 0|\hat{\varphi}(\vec{x}', t_2)\hat{\varphi}(\vec{x}, t_1)|0\rangle_{J=0} = \left[ -\langle 0|U(T)|0\rangle^{-1} \frac{\delta^2 \langle 0|U(T)|0\rangle}{\delta J(\vec{x}', t_2)\delta J(\vec{x}, t_1)} \right]_{J=0} . \quad (7.22)$$

where in the path integral formulation one has

$$\langle 0|U(T)|0\rangle = \int \prod_{\vec{p}} d\tilde{\varphi}(\vec{p}) d\tilde{\varphi}'(\vec{p}) \langle 0|\{\tilde{\varphi}'(\vec{p})\} \rangle \int \mathcal{D}\varphi(x) \exp(i \int_0^T \mathcal{L} d_4x) \langle \{\tilde{\varphi}(\vec{p})\} |0\rangle , \quad (7.23)$$

with the boundary conditions as listed below eq. (7.20).

To study the role the source plays in creating and annihilating particles, we will calculate both in the Hamiltonian and in the path integral formulations the matrix element

$$\langle \vec{p}|U(T)|0\rangle , \quad |\vec{p}\rangle \equiv a^\dagger(\vec{p})|0\rangle . \quad (7.24)$$

Hence  $|\vec{p}\rangle$  is the one-particle state with momentum  $\vec{p}$ . Using the result of eqs. (5.7) and (5.8), which is equivalent to the result  $|\Psi(t)\rangle = U(t)|0\rangle$ , we find for this matrix element in lowest non-trivial order

$$\begin{aligned} \langle \vec{p}|U(T)|0\rangle &= -i \int_0^T dt \langle 0|a(\vec{p})e^{i(t-T)H_0} H_1(t)e^{-itH_0}|0\rangle \\ &= -i \sum_{\vec{k}} \int_0^T dt \langle 0|a(\vec{p})e^{i(p_0(\vec{p})+E_0)(t-T)} \tilde{J}(\vec{k}, t) \frac{(a(-\vec{k}) + a^\dagger(\vec{k}))}{\sqrt{2k_0(\vec{k})}} e^{-iE_0t}|0\rangle \\ &= -i \frac{e^{-i(E_0+p_0(\vec{p}))T}}{\sqrt{2p_0(\vec{p})}} \int_0^T dt \tilde{J}(\vec{p}, t) e^{ip_0(\vec{p})t} = -i \frac{\sqrt{\pi} e^{-i(E_0+p_0(\vec{p}))T}}{\sqrt{p_0(\vec{p})}} \tilde{J}(\vec{p}) . \end{aligned} \quad (7.25)$$

To write down the path integral result, we first express the annihilation operator in terms of the field. From eq. (5.3) we find

$$a(\vec{p}) + a^\dagger(-\vec{p}) = \sqrt{\frac{2p_0(\vec{p})}{V}} \int d_3\vec{x} \hat{\varphi}(\vec{x}) e^{-i\vec{p}\cdot\vec{x}} , \quad (7.26)$$

such that (using  $\langle 0|a^\dagger(-\vec{p}) = 0$ )

$$\langle \vec{p}|U(T)|0\rangle = \sqrt{\frac{2p_0(\vec{p})}{V}} \int d_3\vec{x} e^{-i\vec{p}\cdot\vec{x}} \langle 0|\hat{\varphi}(\vec{x})U(T)|0\rangle . \quad (7.27)$$

We now use eqs. (7.14), (7.19) and an obvious generalisation of eq. (7.21), such that

$$\begin{aligned} \langle 0|\hat{\varphi}(\vec{x})U(T)|0\rangle &= \langle 0|U(T)\hat{\varphi}(\vec{x}, T)|0\rangle = i \frac{\delta}{\delta J(\vec{x}, T)} \langle 0|U(T)|0\rangle \\ &= ie^{-iE_0T} \frac{\delta}{\delta J(\vec{x}, T)} \exp\left(-\frac{i}{2} \int d_4x d_4y J(x)G(x-y)J(y)\right) . \end{aligned} \quad (7.28)$$

We evaluate this to linear order in the source  $J$ , using that in a finite volume the Green's function is given by

$$G(x-y) = \frac{1}{V} \sum_{\vec{k}} \int \frac{dk_0}{2\pi} \frac{e^{-ik(x-y)}}{k^2 - m^2 + i\varepsilon} , \quad (7.29)$$

such that

$$\begin{aligned}
\langle \vec{p} | U(T) | 0 \rangle &= e^{-iE_0 T} \sqrt{\frac{2p_0(\vec{p})}{V}} \int d_3 \vec{x} e^{-i\vec{p} \cdot \vec{x}} \int d_4 y G(x-y) J(y) + \mathcal{O}(J^3) \\
&= e^{-iE_0 T} \sqrt{\frac{p_0(\vec{p})}{\pi}} \int dp_0 \frac{\tilde{J}(p) e^{-ip_0 T}}{p^2 - m^2 + i\varepsilon} + \mathcal{O}(J^3) .
\end{aligned} \tag{7.30}$$

Note that in the last step we integrate over  $p_0$  as a dummy variable, which in the expression for the Green's function above is called  $k_0$  - this renaming is just for ease of notation. Also,  $x_0 = T$  is assumed. For the  $p_0$  integration we need the analytic behaviour of  $\tilde{J}(p)$  for imaginary  $p_0$  in order to see if we are allowed to deform the integration contour such that only one of the poles in the integrand contribute. Since  $\tilde{J}(p) = \int_0^T dt e^{ip_0 t} \tilde{J}(\vec{p}, t) / \sqrt{2\pi}$ ,  $\tilde{J}(p)$  will vanish for  $\text{Im} p_0 \rightarrow \infty$ , whereas  $e^{-ip_0 T} \tilde{J}(p)$  will vanish for  $\text{Im} p_0 \rightarrow -\infty$ . In eq. (7.30) the  $p_0$  integration can therefore be deformed to the lower half-plane in a clockwise fashion giving a minus sign and a residue from the pole at  $p_0 = p_0(\vec{p}) \equiv \sqrt{\vec{p}^2 + m^2}$ , which yields the result

$$\langle \vec{p} | U(T) | 0 \rangle = -i \frac{\sqrt{\pi} e^{-i(E_0 + p_0(\vec{p})T)}}{\sqrt{p_0(\vec{p})}} \tilde{J}(p) \exp\left(-\frac{i}{2} \int d_4 x d_4 y J(x) G(x-y) J(y)\right) . \tag{7.31}$$

To linear order in the source  $J$  this coincides with the result of eq.(7.25). Again, the path integral trivially allows an extension to arbitrary order in the source, as indicated.

For later use, we will also consider the matrix element

$$\begin{aligned}
\langle 0 | U(T) | \vec{p} \rangle &= \sqrt{\frac{2p_0(\vec{p})}{V}} \int d_3 \vec{x} e^{i\vec{p} \cdot \vec{x}} \langle 0 | U(T) \hat{\varphi}(\vec{x}, 0) | 0 \rangle \\
&= i \sqrt{\frac{2p_0(\vec{p})}{V}} \int d_3 \vec{x} e^{i\vec{p} \cdot \vec{x}} \frac{\delta \langle 0 | U(T) | 0 \rangle}{\delta J(\vec{x}, 0)} .
\end{aligned} \tag{7.32}$$

The analogue of eq. (7.30) becomes

$$\begin{aligned}
\langle 0 | U(T) | \vec{p} \rangle &= e^{-iE_0 T} \sqrt{\frac{2p_0(\vec{p})}{V}} \int d_3 \vec{x} e^{i\vec{p} \cdot \vec{x}} \int d_4 y G(x-y) J(y) + \mathcal{O}(J^3) \\
&= e^{-iE_0 T} \sqrt{\frac{p_0(\vec{p})}{\pi}} \int dp_0 \frac{\tilde{J}(-\vec{p}, p_0)}{p^2 - m^2 + i\varepsilon} + \mathcal{O}(J^3) ,
\end{aligned} \tag{7.33}$$

in which case  $x_0 = 0$  is assumed. Now we must deform the contour for the  $p_0$  integration to the upper half-plane in a counter-clockwise fashion such that the residue at the pole  $p_0 = -p_0(\vec{p})$  contributes. This gives, analogously to eq. (7.31),

$$\langle 0 | U(T) | \vec{p} \rangle = -i \frac{\sqrt{\pi} e^{-iE_0 T}}{\sqrt{p_0(\vec{p})}} \tilde{J}(-p) \exp\left(-\frac{i}{2} \int d_4 x d_4 y J(x) G(x-y) J(y)\right) . \tag{7.34}$$

Apart from the trivial difference of the factor  $\exp(-ip_0 T)$ , we see that the amplitude for the annihilation of a one-particle state is proportional to  $\tilde{J}(p)$  whereas it is proportional (exactly with the same factor) to  $\tilde{J}(-p)$  for the creation of a one-particle state (in both cases  $p_0 = p_0(\vec{p})$ ).

## 8 Perturbative expansion in field theory

As we have seen in the previous section ( $\int \mathcal{D}\varphi(x)$  where relevant includes groundstate factors)

$$Z(J, g_n) \equiv \langle 0|U(T)|0 \rangle = \int \mathcal{D}\varphi(x) \exp(i \int d_4x \mathcal{L}(\varphi)) \quad (8.1)$$

will play the role of a generating functional for calculating expectation values of products of field operators, which will now be studied in more detail. In general the Lagrange density for a scalar field theory is given by

$$\mathcal{L}(\varphi) = \mathcal{L}_2(\varphi) - V(\varphi) - J(x)\varphi(x) \quad , \quad (8.2)$$

where  $\mathcal{L}_2(\varphi)$  is quadratic in the fields, hence for a scalar field

$$\mathcal{L}_2(\varphi) = \frac{1}{2} \left( \partial_\mu \varphi(x) \partial^\mu \varphi(x) - m^2 \varphi^2(x) \right) \quad , \quad V(\varphi) = \frac{g_3}{3!} \varphi^3(x) + \frac{g_4}{4!} \varphi^4(x) + \dots \quad . \quad (8.3)$$

As mentioned before, it is customary to not include the mass term in the potential  $V$ , such that  $V$  describes the interactions. We can add the interaction as an operator, when evaluating the path integral for the quadratic approximation  $\mathcal{L}(\varphi) = \mathcal{L}_2(\varphi) - \varphi(x)J(x)$ ,

$$\begin{aligned} Z(J, g_n) &= \int \mathcal{D}\varphi(x) \exp \left( i \int d_4x \{ \mathcal{L}_2(\varphi) - \varphi(x)J(x) \} \right) \exp \left( -i \int d_4x V(\varphi) \right) \\ &= \langle 0|U_2(T) \text{Texp} \left( -i \int d_4x V(\hat{\varphi}) \right) |0 \rangle \quad . \end{aligned} \quad (8.4)$$

We can now use the fact that

$$\begin{aligned} &\int \mathcal{D}\varphi(x) \prod_j \varphi(x_{(j)}) \exp \left( i \int d_4x (\mathcal{L}_2(\varphi) - \varphi(x)J(x)) \right) \\ &= \prod_j \left( \frac{i\delta}{\delta J(x_{(j)})} \right) \int \mathcal{D}\varphi(x) \exp \left( i \int d_4x (\mathcal{L}_2(\varphi) - \varphi(x)J(x)) \right) \\ &= \prod_j \left( \frac{i\delta}{\delta J(x_{(j)})} \right) \exp \left( -\frac{i}{2} \int d_4x d_4y J(x)G(x-y)J(y) \right) \quad , \end{aligned} \quad (8.5)$$

to find a somewhat formal, but in an expansion with respect to the coupling constants  $g_n$  well defined expression for the fully interacting path integral

$$\begin{aligned} Z(J, g_n) &= \exp \left( -i \int d_4x V \left( \frac{i\delta}{\delta J(x)} \right) \right) \exp \left( -\frac{i}{2} \int d_4x d_4y J(x)G(x-y)J(y) \right) \\ &\equiv \exp \left( -i \int d_4x V \left( \frac{i\delta}{\delta J(x)} \right) \right) Z_2(J) \quad , \end{aligned} \quad (8.6)$$

We have assumed the vacuum energy to be normalised to zero, in absence of interactions, such that  $Z(J = g_n = 0) = 1$ . Equivalently,  $Z(J, g_n)$  is synonymous with  $Z(J, g_n)/Z(J = g_n = 0)$ . We now define  $G_J$  as

$$G_J \equiv \log Z(J, g_n) \quad , \quad (8.7)$$

where the dependence on the coupling constants in  $G_J$  is implicit. We will show that  $G_J$  can be seen as the sum of all connected diagrams. A diagram is connected if it can not be decomposed in the product of two diagrams that are not connected. Note that at

$J = 0$ ,  $iG_J/T$  equals the energy of the ground state as a function of the coupling constants, normalised so as to vanish at zero couplings.

The different diagrams arise from the expansion of

$$\exp \left( -i \int d_4x \, V \left( \frac{i\delta}{\delta J(x)} \right) \right) = \exp \left( -i \int d_4x \sum_{\ell=3} \frac{g_\ell}{\ell!} \left( \frac{i\delta}{\delta J(x)} \right)^\ell \right) \quad (8.8)$$

in powers of  $g_\ell$ . Each factor  $\frac{g_\ell}{\ell!} (i\delta/\delta J(x))^\ell$  will represent an  $\ell$ -point vertex, with coordinate  $x$ , which is to be integrated over. As we saw in the derivation of the classical equations of motion, the integral over  $x$  in the Fourier representation gives rise to conservation of momentum at the vertex. Using

$$\varphi(x) = \frac{1}{(2\pi)^2} \int d_4k \, e^{-ikx} \tilde{\varphi}(k) \, , \quad J(x) = \frac{1}{(2\pi)^2} \int d_4k \, e^{-ikx} \tilde{J}(k) \quad , \quad (8.9)$$


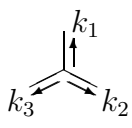
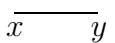
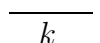

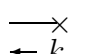
we have for each vertex in the Fourier representation

$$-ig_\ell \int d_4x \left( \frac{i\delta}{\delta J(x)} \right)^\ell = -ig_\ell (2\pi)^4 \delta_4 \left( \sum_{j=1}^\ell k_{(j)} \right) \int \prod_{j=1}^\ell \left( \frac{d_4k_{(j)}}{(2\pi)^2} \frac{i\delta}{\delta \tilde{J}(k_{(j)})} \right) \quad . \quad (8.10)$$

Note that factors of  $2\pi$  are dropping out in the identities

$$\begin{aligned} \exp \left( -\frac{i}{2} \int d_4x d_4y \, J(x) G(x-y) J(y) \right) &= \exp \left( -\frac{i}{2} \int d_4k \, \frac{\tilde{J}(k) \tilde{J}(-k)}{k^2 - m^2 + i\varepsilon} \right) \quad , \\ \int d_4x \, \varphi(x) J(x) &= \int d_4k \, \tilde{\varphi}(k) \tilde{J}(-k) \quad . \end{aligned} \quad (8.11)$$

In the quantum theory we have to keep track of the factors  $i$ . Compared to the Feynman rules of section 4, the propagator will come with an extra factor  $-i$ . A vertex will now carry a factor  $ig_\ell((2\pi)^2/i)^{2-\ell}$  (in a finite volume this becomes  $ig_\ell(\sqrt{2\pi V}/i)^{2-\ell}$ ), see the table below. To compute the vacuum energy there is an overall factor  $i$ , since  $E_0 T = iG_{J=0} = i \log Z(J=0, g_n)$ . The same factor of  $i$  applies for using the tree-level diagrams to solve the classical equations of motion. It is easy to see that these Feynman rules give identical results for these tree-level diagrams, as compared to the Feynman rules introduced in section 4. The factors of  $i$  exactly cancel each other.

coordinate space		momentum space		table 2
	$\equiv i^{\ell-1} g_\ell \int d_4x$		$\equiv i^{\ell-1} (2\pi)^{4-2\ell} g_\ell \delta_4(\sum_i k_i)$	vertex
	$\equiv -iG(x-y)$		$\equiv \int d_4k \frac{-i}{k^2 - m^2 + i\varepsilon}$	propagator
	$\equiv \int d_4x J(x)$		$\equiv \tilde{J}(k)$	source

We note that the propagator connected to a source comes down whenever a derivative in the source acts on  $Z_2(J)$ , see eq. (8.7). When this derivative acts on terms that have already come down from previous derivatives, one of the sources connected to a propagator

is removed and this connects that propagator to the vertex associated to  $\delta/\delta J(x)$ . As any derivative is connected to a vertex, the propagator either runs between a vertex and a source  $\text{---}\times$ , between two vertices  $\text{---}\times\text{---}\times$ , or it connects two legs of the *same* vertex  $\text{---}\bigcirc$ . The possibility of closed loops did not occur in solving the classical equations of motion, and is specific to the quantum theory.

To prove that  $G_J$  only contains connected diagrams we write

$$G_J = \log \{ \exp(X(i\delta/\delta J)) \exp(Y(J)) \} 1 \quad , \quad (8.12)$$

with

$$X(i\delta/\delta J(x)) \equiv -i \int d_4x V(\delta/\delta J(x)) \quad , \quad Y(J) \equiv -\frac{i}{2} \int d_4x d_4y J(x) G(x-y) J(y) \quad . \quad (8.13)$$

As  $J(x)$  and  $\delta/\delta J(x)$  form an algebra (similar to the algebra of  $\hat{x}$  and  $\hat{p}$  in quantum mechanics, however, generalised to infinite dimensions), also  $X$  and  $Y$  are elements from the algebra, and we can express  $G_J$  in a sum of multiple commutators using the Campbell-Baker-Hausdorff formula (see eq. (6.44)).

$$G_J = \left( X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] + \frac{1}{12}[Y, [Y, X]] + \dots \right) 1 \quad . \quad (8.14)$$

Due to the multiple commutators, all components that do not commute are connected. However, if the components would commute they would not contribute to the commutators. This is even true if we do not put the derivatives with respect to the source to zero, once they have been moved to the right (this is why we consider the action on the identity).

The only thing that remains to be discussed is with which combinatorial factor each diagram should contribute. This is, as in section 4, with the inverse of the order of the permutation group that leaves the topology of the diagram unchanged. These combinatorial factors are clearly independent of the space-time integrations and possible contractions of vector or other indices. We can check them by reducing the path integral to zero dimensions, or  $\varphi(x) \rightarrow \varphi$  and  $\mathcal{D}\varphi(x) \rightarrow d\varphi$ . In other words, we replace the path integral by an ordinary integral. As an example consider

$$\begin{aligned} Z(J, g) &= C \int d\varphi \exp \left( i \left\{ \frac{1}{2} \varphi M \varphi - \frac{g}{3!} \varphi^3 - \varphi J \right\} \right) \\ &= \exp \left( -i \frac{g}{3!} \left( \frac{i\partial}{\partial J} \right)^3 \right) \exp \left( -\frac{i}{2} J M^{-1} J \right) \quad . \end{aligned} \quad (8.15)$$

The constant  $C$  is simply to normalise  $Z(J = g_n = 0) = 1$ . Expanding the exponents we get in lowest non-trivial order

$$\begin{aligned} Z(J = 0) &= 1 - \frac{g^2}{2(3!)^3} \left( \frac{i\partial}{\partial J} \right)^6 \left( -\frac{i}{2} J M^{-1} J \right)^3 \dots \\ &= \exp \left( i \frac{g^2}{2^4(3!)^3} \left( \frac{\partial}{\partial J} \right)^6 (J M^{-1} J)^3 + \mathcal{O}(g^3) \right) = \exp \left( i \frac{5g^2}{24M^3} + \mathcal{O}(g^3) \right) \\ &= \exp \left( \frac{1}{12} \text{---}\bigcirc\text{---}\bigcirc + \frac{1}{8} \bigcirc\text{---}\bigcirc + \mathcal{O}(g^3) \right) \quad . \end{aligned} \quad (8.16)$$

In the last term, the numerical factors in front of the diagrams indicate the combinatorial factors (for the first diagram a factor 2 from interchanging the two vertices and a factor 3!

from interchanging the three propagators, for the second diagram the latter factor is replaced by 4 as we can only interchange for each vertex the two legs that do not interconnect the two vertices). The Feynman rules for this simple case are that each vertex gets a factor  $-g$  (in zero dimensions there are no factors  $2\pi$ ) and each propagator gets a factor  $-i/M$ . In problem 13 the exponentiation is checked for  $Z(J)$  to  $\mathcal{O}(g^2)$  and  $\mathcal{O}(J^5)$  (giving the simplest non-trivial check).

We will now show how the number of loops in a diagram is related to the expansion in  $\hbar$ . We can expect such a relation, as we have shown at the end of section 6 that the  $\hbar \rightarrow 0$  limit is related to the classical equations of motion, whereas we have shown in section 4 that these classical equations are solved by tree diagrams. If we call  $L$  the number of loops of a diagram, we will show that

$$Z(J, g_n) = \int \mathcal{D}\varphi \exp\left(\frac{i}{\hbar} \int (\mathcal{L}(\varphi) - \varphi J)\right) \equiv \exp(G_J/\hbar), \quad G_J = \sum_{L=0}^{\infty} \hbar^L G_{L,J}, \quad (8.17)$$

where  $G_{L,J}$  is the sum of all connected diagrams with exactly  $L$  loops. This means that a loop expansion is equivalent with an expansion in  $\hbar$ . To prove this we first note that due to reinstating  $\hbar$  the source term will get an extra factor  $1/\hbar$ , the propagator a factor  $\hbar$  and the coupling constants  $g_n$  are replaced by  $g_n/\hbar$ . A diagram with  $V_n$   $n$ -point vertices,  $E$  external lines (connected to a source) and  $P$  propagators has therefore an extra overall factor of

$$\hbar^{-E} \hbar^P \prod_{n=3} \hbar^{-V_n}. \quad (8.18)$$

We can relate this to the number of loops by noting that the number of momentum integrations (i.e. the number of independent momenta) in a diagram equals the number of loops plus the number of external lines, minus one for the overall conservation of energy and momentum, i.e.  $L + E - 1$ . On the other hand, the number of momentum integrations is also the number of propagators minus the number of delta functions coming from the vertices, i.e.  $P - \sum_{n=3} V_n$ . Hence

$$L = 1 + P - E - \sum_{n=3} V_n, \quad (8.19)$$

which implies that the total number of  $\hbar$  factors in a diagram is given by  $L - 1$ . In the next sections we will often consider so-called amputated diagrams, where the external propagators connected to a source are taken off from the expressions for the diagram. If we do not count these external propagators, eq. (8.19) has to be replaced by  $L = 1 + P - \sum_{n=3} V_n$ , as there are exactly  $E$  such external propagators.

## 9 The scattering matrix

We would like to compute the amplitude for the transition of  $n$  incoming particles at  $t = T_{\text{in}}$  to  $\ell$  outgoing particles at  $t = T_{\text{out}}$  in the limit where  $T_{\text{out}} \rightarrow \infty$  and  $T_{\text{in}} \rightarrow -\infty$ . The difference with quantum mechanics is that the particle number is no longer conserved.

$${}_{\text{out}} \langle \vec{p}_1, \vec{p}_2, \dots, \vec{p}_\ell | \vec{k}_1, \vec{k}_2, \dots, \vec{k}_n \rangle_{\text{in}} \equiv \langle \vec{p}_1, \vec{p}_2, \dots, \vec{p}_\ell | U(T_{\text{out}}, T_{\text{in}}) | \vec{k}_1, \vec{k}_2, \dots, \vec{k}_n \rangle. \quad (9.1)$$

In terms of creation and annihilation operators this can be written as

$$\begin{aligned} {}_{\text{out}} \langle \vec{p}_1, \vec{p}_2, \dots, \vec{p}_\ell | \vec{k}_1, \vec{k}_2, \dots, \vec{k}_n \rangle_{\text{in}} = \\ \langle 0 | a(\vec{p}_1) a(\vec{p}_2) \dots a(\vec{p}_\ell) U(T_{\text{out}}, T_{\text{in}}) a^\dagger(\vec{k}_1) a^\dagger(\vec{k}_2) \dots a^\dagger(\vec{k}_n) | 0 \rangle. \end{aligned} \quad (9.2)$$

From eqs. (7.26), (7.28) and (7.32) we know how to implement these creation and annihilation operators on the generating functional  $Z(J, g_n)$

$$\begin{aligned}\hat{a}^\dagger(\vec{p}) &= i\sqrt{\frac{2p_0(\vec{p})}{V}} \int d_3\vec{x} \ e^{i\vec{p}\cdot\vec{x}} \frac{\delta}{\delta J(\vec{x}, t = T_{\text{in}})} = i\sqrt{2p_0(\vec{p})} \frac{\delta}{\delta \tilde{J}(\vec{p}, t = T_{\text{in}})} , \\ \hat{a}(\vec{p}) &= i\sqrt{\frac{2p_0(\vec{p})}{V}} \int d_3\vec{x} \ e^{-i\vec{p}\cdot\vec{x}} \frac{\delta}{\delta J(\vec{x}, t = T_{\text{out}})} = i\sqrt{2p_0(\vec{p})} \frac{\delta}{\delta \tilde{J}(-\vec{p}, t = T_{\text{out}})} .\end{aligned}\quad (9.3)$$

This implies the following identity for the scattering matrix

$${}_{\text{out}}\langle \vec{p}_1, \vec{p}_2, \dots, \vec{p}_\ell | \vec{k}_1, \vec{k}_2, \dots, \vec{k}_n \rangle_{\text{in}} = \prod_{i=1}^{\ell} \hat{a}(\vec{p}_i) \prod_{j=1}^n \hat{a}^\dagger(\vec{k}_j) \exp(G_J)|_{J=0} . \quad (9.4)$$

In principle, this allows us to calculate the scattering, taking  $T_{\text{in}} \rightarrow -\infty$  and  $T_{\text{out}} \rightarrow \infty$ .

There is, however, a problem to associate the particle states in the presence of interactions with the ones we have derived from the non-interacting theory. The problem is that particles can have self-interactions long before and after the different particles have scattered off each other. We have to reconsider our notion of particle states, as in experiments we are unable to switch off these self-interactions. For simplicity we assume that the one-particle states are stable, as in the simple scalar theory we have been considering. This implies from conservation of probability that

$$\int d_3\vec{k} \ | {}_{\text{out}}\langle \vec{p} | \vec{k} \rangle_{\text{in}}|^2 = 1 \quad , \quad (9.5)$$

independently of  $\vec{p}$ . In general, conservation of probability implies that the  $S$ -matrix is unitary. Formally, unitarity of a  $S$ -matrix is guaranteed as soon as the Hamiltonian is hermitian. Because of the necessity to regulate the quantum theory, e.g. by introducing a cut-off, this is generally no longer true and one has to show that unitarity is restored when the cut-off is removed. If this is not possible, the theory is ill-defined, or at best does not make sense above the energies where unitarity is violated.

For the free theory, unitarity is of course satisfied. In this case, the only diagram contributing to  $G_J$  is the one with a single propagator connecting two sources, which is also called the connected two-point function  $G_c^{(2)}(J)$

$$G_J = G_c^{(2)}(J) = -\frac{i}{2} \int \int dt ds \sum_{\vec{p}} \int \frac{dp_0}{2\pi} e^{-ip_0(t-s)} \frac{\tilde{J}(\vec{p}, t) \tilde{J}(-\vec{p}, s)}{p^2 - m^2 + i\varepsilon} . \quad (9.6)$$

This implies that ( $T \equiv T_{\text{out}} - T_{\text{in}}$ )

$${}_{\text{out}}\langle \vec{p} | \vec{k} \rangle_{\text{in}} = \hat{a}(\vec{p}) \hat{a}^\dagger(\vec{k}) \exp(G_c^{(2)}(J))|_{J=0} = 2ip_0(\vec{p}) \delta_{\vec{k}, \vec{p}} \int \frac{dp_0}{2\pi} \frac{e^{ip_0 T}}{p^2 - m^2 + i\varepsilon} = e^{-ip_0(\vec{p})T} \delta_{\vec{k}, \vec{p}} , \quad (9.7)$$

where the  $p_0$  integration is performed by deforming the contour to the upper half-plane, giving a contribution from the pole at  $p_0 = -p_0(\vec{p})$  only. It is trivial to see that this is the same as what can be obtained within the Hamiltonian formulation.

In the presence of interactions, this result is no longer true, since the connected two-point function will deviate from the one in the free theory. In this case we can write (from now on

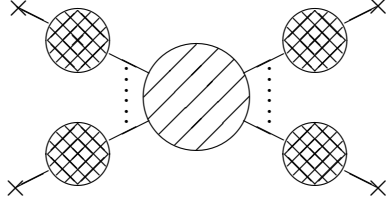




As for the free theory, each of these operators  $\hat{a}_\pm(\vec{p})$  will replace one external line (propagator plus source) by an appropriate wavefunction factor and puts these external lines on the mass-shell. We will call the connected  $n$ -point function with amputated external lines the amputated connected  $n$ -point function  $G_c^{(\text{amp})}(p_1, p_2, \dots, p_n)$  (in general not one-particle irreducible), i.e.

$$G_c^{(n)}(J) \equiv \prod_{i=1}^n \left\{ \int d_4 p_i \frac{-i\tilde{J}(p_i)}{p_i^2 - m^2 - \Sigma(p_i) + i\varepsilon} \right\} G_c^{(\text{amp})}(p_1, p_2, \dots, p_n) \quad . \quad (9.15)$$

Diagrammatically this looks as follows


(9.16)

To get the  $S$ -matrix we have to compute (see eq. (9.13))

$$\left( \prod_{i=1}^{\ell} \hat{a}_-(\vec{p}_i) \prod_{j=1}^n \hat{a}_+(\vec{k}_j) e^{G_J} \right)_{J=0} = e^{G_{J=0}} \prod_{i=1}^{\ell} \hat{a}_-(\vec{p}_i) \prod_{j=1}^n \hat{a}_+(\vec{k}_j) \left( \sum_{\Sigma r q_r = \ell + n} \prod_r \frac{1}{r!} \{G_c^{(q_r)}(J)\}^r \right) \quad . \quad (9.17)$$

The sum is over all possible partitions of  $\ell + n$ . Let us first consider the most important term, corresponding to connected graphs, where  $q_1 = \ell + n$

$$\begin{aligned} \prod_{i=1}^{\ell} \hat{a}_-(\vec{p}_i) \prod_{j=1}^n \hat{a}_+(\vec{k}_j) G_c^{(\ell+n)}(J) &= \prod_{j=1}^n -i\sqrt{\pi Z/k_0^{(j)}} \prod_{i=1}^{\ell} -i\sqrt{\pi Z/p_0^{(i)}} G_c^{(\text{amp})}(\{-p_i\}, \{k_j\}) e^{-i\Sigma p_0^{(i)} T} \\ &\equiv \frac{-2\pi i V \delta_4(\sum_{i=1}^{\ell} p_i - \sum_{j=1}^n k_j) \mathcal{M}_{\ell+n}(\{-p_i\}, \{k_j\})}{\sqrt{\prod_{i=1}^{\ell} 2p_0^{(i)} V \prod_{j=1}^n 2k_0^{(j)} V}} e^{-i\Sigma p_0^{(i)} T} \quad . \end{aligned} \quad (9.18)$$

Here  $\mathcal{M}_{\ell}$  (a Lorentz scalar as we will see later) is the so-called reduced matrix element with  $\ell$  external lines, all on the mass-shell. Note that we have extracted the trivial energy factors (remember that  $\sum p_0^{(i)} T_{\text{out}} - \sum k_0^{(j)} T_{\text{in}} = \sum p_0^{(i)} (T_{\text{out}} - T_{\text{in}}) \equiv \sum p_0^{(i)} T$ ), such that the limits  $T_{\text{in}} \rightarrow -\infty$  and  $T_{\text{out}} \rightarrow \infty$  can be taken. Each  $\hat{a}(\vec{p})_{\pm}$  will act on one of the factors between curly brackets in eq. (9.15). Concentrating on one such a factor we have

$$\int d_4 p \frac{-i\tilde{J}(p) G_c^{(\text{amp})}(p, p_2, \dots, p_n)}{p^2 - m^2 - \Sigma(p) + i\varepsilon} = \int \frac{d_4 p}{\sqrt{2\pi}} \int dt \frac{-i\tilde{J}(\vec{p}, t) e^{ip_0 t} G_c^{(\text{amp})}(p, p_2, \dots, p_n)}{p^2 - m^2 - \Sigma(p) + i\varepsilon} \quad . \quad (9.19)$$

such that, using eq. (9.14)

$$\begin{aligned} \hat{a}_-(\vec{p}) \int d_4 p \frac{-i\tilde{J}(p) G_c^{(\text{amp})}(p, p_2, \dots, p_n)}{p^2 - m^2 - \Sigma(p) + i\varepsilon} &= \sqrt{\frac{p_0(\vec{p})}{\pi Z}} \int dp_0 \frac{e^{ip_0 T_{\text{out}}} G_c^{(\text{amp})}((p_0, -\vec{p}), \dots, p_n)}{p^2 - m^2 - \Sigma(p) + i\varepsilon} \\ &= -i\sqrt{2\pi V Z} \frac{G_c^{(\text{amp})}(-p, p_2, \dots, p_n)}{\sqrt{2p_0(\vec{p}) V}} e^{-ip_0(\vec{p}) T_{\text{out}}} \quad . \end{aligned} \quad (9.20)$$

Since  $T_{\text{out}} \rightarrow \infty$ , we can extend the  $p_0$  contour integration to the upper half-plane, under mild regularity conditions for  $G_c^{(\text{amp})}(p, p_2, \dots, p_n)$  as  $\text{Im} p_0 \rightarrow -\infty$  (that can easily be shown

to be satisfied at any finite order in perturbation theory). Thus, the integral over  $p_0$  only gets a contribution from the pole at  $p_0 = -p_0(\vec{p}) = -\sqrt{\vec{p}^2 + \tilde{m}^2}$ , with residue  $-Z/2p_0(\vec{p})$  (cmp. eq. (9.12)). For the creation operators one similarly finds

$$\begin{aligned} \hat{a}_+(\vec{p}) \int d_4p \frac{-i\tilde{J}(p)G_c^{(\text{amp})}(p, p_2, \dots, p_n)}{p^2 - m^2 - \Sigma(p) + i\varepsilon} &= \sqrt{\frac{p_0(\vec{p})}{\pi Z}} \int dp_0 \frac{e^{ip_0 T_{\text{in}}} G_c^{(\text{amp})}((p_0, \vec{p}), \dots, p_n)}{p^2 - m^2 - \Sigma(p) + i\varepsilon} \\ &= -i\sqrt{2\pi V Z} \frac{G_c^{(\text{amp})}(p, p_2, \dots, p_n)}{\sqrt{2p_0(\vec{p})V}} e^{ip_0(\vec{p})T_{\text{in}}} \quad . \end{aligned} \quad (9.21)$$

Likewise, as  $T_{\text{in}} \rightarrow -\infty$ , we can now extend the  $p_0$  contour integration to the lower half-plane (under the same regularity conditions for the amputated  $n$ -point functions as for eq. (9.20) to be valid), such that we pick up the contribution of the pole at  $p_0 = p_0(\vec{p})$  with residue  $Z/2p_0(\vec{p})$ . Combining these results proves the first identity in eq. (9.18), the second is merely a definition.

Note that the derivation is not valid for the case  $\ell + n = 2$ , where the generalisation of eq. (9.7) to the interacting case implies

$${}_{\text{out}} \langle \vec{p} | \vec{k} \rangle_{\text{in}} = \hat{a}_-(\vec{p}) \hat{a}_+(\vec{k}) \exp(G_J)|_{J=0} = \delta_3(\vec{k} - \vec{p}) e^{-i[E_0 + p_0(\vec{p})]T} \quad . \quad (9.22)$$

Here we used eq. (9.12) and the fact that  $\exp(G_{J=0}) = \exp(-iE_0 T)$ , which is often also normalised to 1, but till now we had only required this to be the case at zero couplings. The reason this case is special is because eq. (9.15) requires us to define for the amputated two-point function

$$G_c^{(\text{amp})}(p_1, p_2) \equiv i\delta_4(p_1 + p_2)(p_1^2 - m^2 - \Sigma(p_1) + i\varepsilon) \quad , \quad (9.23)$$

which vanishes on the mass-shell. In using eq. (9.20) and eq. (9.21) it was implicitly assumed that the amputated Green's function has no zero that will cancel the pole.

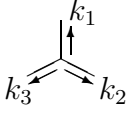
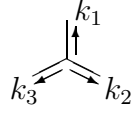




The Feynman rules in momentum space for computing the reduced matrix elements will obviously have to be modified for the external lines to a factor  $-i\sqrt{2\pi V Z}$  and an overall factor  $i/(2\pi V)$  (as always, in an infinite volume one replaces  $V$  by  $(2\pi)^3$ ). If we associate a momentum delta function to a vertex and a momentum integration to a propagator (as was done up to now), the delta function for overall energy and momentum conservation should not be written explicitly in eq. (9.18), since it is contained in the reduced matrix element. Instead, if we choose to integrate over the independent loop momenta, implementing energy and momentum conservation at each vertex (as will be done from now on, see the table below), the definition of eq. (9.18) is the appropriate one. The overall factors of  $i$ ,  $2\pi$  and  $V$  can be determined with the help of the two identities

$$L = P + 1 - \sum_n V_n \quad , \quad E + 2P = \sum_n nV_n \quad . \quad (9.24)$$

The proof for the first identity was discussed below eq. (8.19). For the second identity we put a dot on each end of a propagator (  $\text{---}\cdot$  ) and one dot on each external line (  $\cdot\text{---}$  ), giving a total of  $2P + E$  dots. The same dots can also be associated to each line of a vertex (  $\cdot\text{---}\cdot$  ), giving  $\sum nV_n$  dots, thus proving the second identity (see also problem 15). To keep the derivation general, we evaluate the overall factor in a finite volume

$$i^{-P} i^{-E+1} i^{\sum (n-1)V_n} (2\pi V)^{-\frac{1}{2}\sum (n-2)V_n} (2\pi V)^{\frac{1}{2}E-1} = \left( \frac{i}{2\pi V} \right)^L \quad . \quad (9.25)$$

This implies that we can shift all numerical factors from the propagators, vertices and external lines to a factor  $i/(2\pi V)$  (or  $i/(2\pi)^4$  in an infinite volume) for each loop, giving the Feynman rules listed in the table for an infinite volume. Note that the extraction of the factor  $-2\pi iV$  in the definition of  $\mathcal{M}$  is merely a convention (such that in lowest order  $\mathcal{M}_n = g_n$ ).

momentum space	Itzykson and Zuber	table 3
 $\equiv g_\ell \text{ and } \sum k_i = 0$	 $\equiv -i(2\pi)^4 g_\ell \delta_4(\sum_i k_i)$	vertex
 $\equiv \frac{1}{k^2 - m^2 + i\varepsilon}$	 $\equiv \int \frac{d_4 k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\varepsilon}$	propagator
 $\equiv \sqrt{Z}$	 $\equiv \sqrt{Z}$	external line
$i \int \frac{d_4 k}{(2\pi)^4}$	1	loop factor

In the literature many different conventions are being used. As an example, the table compares our Feynman rules with those of Itzykson and Zuber. Their convention for  $\mathcal{M}_n$  is likewise to make it coincide to lowest order with the  $n$ -point vertex. However, as the latter does already contain a factor  $-i(2\pi)^4$  (in a finite volume  $-2\pi iV$ ), that factor should be absent in relating the reduced matrix element to the amputated  $n$ -point function. Combining the extra factors of  $i$  and  $2\pi$  in the Feynman rules of Itzykson and Zuber gives  $i(2\pi)^{-4} i^{P-\Sigma V_n} (2\pi)^{4(\Sigma V_n - P)} = [i/(2\pi)^4]^L$ , guaranteeing equivalence of the two sets of Feynman rules.

Concerning the symmetry factor associated to a particular diagram we note the following. As we have generally fixed the external momenta, interchanging external lines is no longer allowed. But from eq. (9.15) we see that the symmetry factor  $n!$ , to be taken into account for  $G_c^{(n)}(J)$ , will be compensated by the  $n$  derivatives on  $n$  sources. Hence, in computing the reduced matrix elements the symmetry factors are determined without allowing for permutations on the external lines.

To conclude this section we return to eq. (9.16), and discuss the contributions that will be associated to the diagrams that are not connected. Each factor of  $r!$  is compensated for by the differentiations on  $\{G_c^{(q_r)}(J)\}^r$ . For the corresponding connected components the rules are identical to the ones specified above. In particular each connected component will carry its own factor  $-i(2\pi)^4 \delta_4(\sum p_i)$  for the conservation of energy and momentum ( $p_i$  is now assumed to run over a subset of both the incoming and the negative of the outgoing four-momenta). In a physical picture the disconnected parts correspond to situations where only a subset of the incoming particles will interact with each other (the ones connected by a particular diagram). Quite often, the experimental situation is such that the energy-momentum conservation will only be compatible with the fully connected part. We just have to avoid the incoming momenta to coincide with any of the outgoing momenta. In a collider, this means one excludes particles that escape in the direction of the beams, where indeed it is not possible to put a detector. As an illustration we will give the situation for  $n = \ell = 2$  and all momenta non-zero (to avoid tadpole diagrams,  $-\bigcirc$ ) to second order in

the three-point coupling  $g_3$ , putting all other couplings to zero

$$\begin{aligned} \text{out} < \vec{p}_1, \vec{p}_2 | \vec{k}_1, \vec{k}_2 >_{\text{in}} = \exp \left( -i[E_0 + p_0^{(1)} + p_0^{(2)}]T \right) \times \\ \left[ \delta_3(\vec{p}_1 - \vec{k}_1) \delta_3(\vec{p}_2 - \vec{k}_2) \left( \text{---} \text{---} \right) + \delta_3(\vec{p}_1 - \vec{k}_2) \delta_3(\vec{p}_2 - \vec{k}_1) \left( \text{---} \text{---} \right) \right. \\ \left. -i \frac{(2\pi)^4 \delta_4(p_1 + p_2 - k_1 - k_2)}{\sqrt{2p_0^{(1)}(2\pi)^3 2p_0^{(2)}(2\pi)^3 2k_0^{(1)}(2\pi)^3 2k_0^{(2)}(2\pi)^3}} \left( \text{---} \text{---} + \text{---} \text{---} + \mathcal{O}(g_3^3) \right) \right]. \end{aligned} \quad (9.26)$$

The first two diagrams, which have to be treated with special care (see eqs. (9.22) and (9.23)), represent the situation without scattering. By definition they have no higher order corrections.

## 10 Cross sections

In many experimental situations we are interested in the scattering of two particles with momenta  $k_1$  and  $k_2$  to a state with  $n$  particles with momenta  $p_1, p_2, \dots, p_n$ . We denote by  $\int d_3 \vec{k}_1 \tilde{\Psi}_1(\vec{k}_1) |\vec{k}_1 >$  and  $\int d_3 \vec{k}_2 \tilde{\Psi}_2(\vec{k}_2) |\vec{k}_2 >$  the wave functionals of the incoming particles. This is to describe the more realistic case of a wave packet. The amplitude for scattering to take place is hence given by

$$\mathcal{A} = -i(2\pi)^4 \int d_3 \vec{k}_1 d_3 \vec{k}_2 \frac{\delta_4(\sum_{i=1}^n p_i - k_1 - k_2) \mathcal{M}_{2+n}(\{-p_i\}, \{k_j\})}{\sqrt{\prod_{i=1}^n 2p_0^{(i)}(\vec{p}_i)(2\pi)^3} \prod_{j=1}^2 2k_0^{(j)}(\vec{k}_j)(2\pi)^3} \tilde{\Psi}_1(\vec{k}_1) \tilde{\Psi}_2(\vec{k}_2) e^{-i[E_0 + \Sigma p_0^{(i)}]T}. \quad (10.1)$$

If we define the wave function in coordinate space as usual

$$\Psi_j(x) = \int \frac{d_3 \vec{k}}{\sqrt{2k_0(\vec{k})(2\pi)^3}} e^{-ikx} \tilde{\Psi}_j(\vec{k}) \quad , \quad (10.2)$$

we can compute the overlap of the two wave functions

$$\int d_4 x \Psi_1(x) \Psi_2(x) e^{ipx} = (2\pi)^4 \int \frac{d_3 \vec{k}_1}{\sqrt{2k_0(\vec{k}_1)(2\pi)^3}} \frac{d_3 \vec{k}_2}{\sqrt{2k_0(\vec{k}_2)(2\pi)^3}} \delta_4(k_1 + k_2 - p) \tilde{\Psi}_1(\vec{k}_1) \tilde{\Psi}_2(\vec{k}_2). \quad (10.3)$$

We assume that over the range of momenta in the wave packets, the reduced matrix elements are constant (which can be achieved with arbitrary precision for arbitrarily narrow wave packets in momentum space). This allows us to write for the scattering probability of two particles into  $n$  particles, with momenta in between  $p_i$  and  $p_i + dp_i$ ,

$$\begin{aligned} dW &= |\mathcal{M}(\{-p_i\}, \{\bar{k}_j\})|^2 f(p) \prod_{i=1}^n \frac{d_3 \vec{p}_i}{2p_0(\vec{p}_i)(2\pi)^3} \quad , \\ f(p) &\equiv \int d_4 x d_4 y \Psi_1(x) \Psi_2(x) \Psi_1^*(y) \Psi_2^*(y) e^{ip(x-y)} \quad , \end{aligned} \quad (10.4)$$

where  $p = \sum_{i=1}^n p_i = k_1 + k_2$ . The momenta  $\bar{k}_i$  in the reduced matrix element are the central values of the wave packet in momentum space for the two incoming particle beams. Under the same assumption that the momentum spread in the beams is very small, the function  $f(p)$  will be highly peaked around  $p = \bar{k}_1 + \bar{k}_2$ , such that

$$f(p) \approx \delta_4(p - \bar{k}_1 - \bar{k}_2) \int d_4 p f(p) = (2\pi)^4 \delta_4(p - \bar{k}_1 - \bar{k}_2) \int d_4 x |\Psi_1(x)|^2 |\Psi_2(x)|^2. \quad (10.5)$$

The quantities  $|\Psi_j(x)|^2$  are of course related to the probability densities of the two particles in their respective beams,

$$\rho_j(x) \equiv i(\Psi_j^*(x)\partial_t\Psi_j(x) - \Psi_j(x)\partial_t\Psi_j^*(x)) \approx 2\bar{k}_0^{(j)}|\Psi_j(x)|^2 \quad , \quad (10.6)$$

again using the fact that the wave packet is highly peaked in momentum space. Putting these results together we find

$$dW = (2\pi)^4 \delta_4(\sum p_i - \bar{k}_1 - \bar{k}_2) |\mathcal{M}(\{-p_i\}, \{\bar{k}_j\})|^2 \prod_{i=1}^n \frac{d_3\vec{p}_i}{2p_0(\vec{p}_i)(2\pi)^3} \int d_4x \frac{\rho_1(x)\rho_2(x)}{4\bar{k}_0^{(1)}\bar{k}_0^{(2)}}. \quad (10.7)$$

Since  $\rho_j(x)$  will depend on the experimental situation, we should normalise with respect to the total number of possible interactions in the experimental setup, also called the integrated luminosity  $L$ .

$$\int dt L(t) \equiv \int d_4x \rho_1(x)\rho_2(x)|\vec{v}_1 - \vec{v}_2| \quad . \quad (10.8)$$

Here  $\int d_3\vec{x} \rho_1(x)\rho_2(x)$  is the number of possible interactions per unit volume at a given time and  $|\vec{v}_1 - \vec{v}_2|$  is the relative velocity of the two beams. We have assumed that either, one of the velocities is zero (fixed target), or that the two velocities are parallel (colliding beams). Hence,  $L(t) = \int d_3\vec{x} \rho_1(\vec{x}, t)\rho_2(\vec{x}, t)|\vec{v}_1 - \vec{v}_2|$  is a flux, typically of the order of  $10^{28} - 10^{33}\text{cm}^{-2}\text{s}^{-1}$ . To consider the general case we note that we can also write

$$\int dt L(t) = \sqrt{(\bar{k}_1 \cdot \bar{k}_2)^2 - m_1^2 m_2^2} \int d_4x \frac{\rho_1(x)\rho_2(x)}{\bar{k}_0^{(1)}\bar{k}_0^{(2)}} \quad . \quad (10.9)$$

After all, for a fixed target situation  $\bar{k}_2 = (m_2, \vec{0})$ , such that

$$\frac{\sqrt{(\bar{k}_1 \cdot \bar{k}_2)^2 - m_1^2 m_2^2}}{\bar{k}_0^{(1)}\bar{k}_0^{(2)}} = \frac{\sqrt{(\bar{k}_0^{(1)})^2 - m_1^2}}{\bar{k}_0^{(1)}} = |\vec{v}_1| \quad , \quad (10.10)$$

whereas for colliding beams of particles and antiparticles with mass  $m$ , where  $\bar{k}_1 = (E, \vec{k}) = E(1, \vec{v})$  and  $\bar{k}_2 = (E, -\vec{k}) = E(1, -\vec{v})$ , one finds

$$\frac{\sqrt{(\bar{k}_1 \cdot \bar{k}_2)^2 - m_1^2 m_2^2}}{\bar{k}_0^{(1)}\bar{k}_0^{(2)}} = \frac{\sqrt{(E^2 + \vec{k}^2)^2 - m^4}}{E^2} = 2|\vec{k}|/E = 2|\vec{v}| \quad , \quad (10.11)$$

such that both expressions reduce to  $|\vec{v}_1 - \vec{v}_2|$ . We leave it as an exercise to prove the result for the general case of parallel beams.

We can therefore define a machine independent differential cross section  $d\sigma$  by normalising the scattering probability by the total luminosity,

$$d\sigma = (2\pi)^4 \delta_4(\sum_{i=1}^n p_i - \bar{k}_1 - \bar{k}_2) \frac{|\mathcal{M}(\{-p_i\}, \{\bar{k}_j\})|^2}{4\sqrt{(\bar{k}_1 \cdot \bar{k}_2)^2 - m_1^2 m_2^2}} S \prod_{i=1}^n \frac{d_3\vec{p}_i}{2p_0(\vec{p}_i)(2\pi)^3} \quad . \quad (10.12)$$

The parameter  $S$  is the inverse of the permutation factor for identical particles in the final state, as a detector will not be able to distinguish them. This will avoid double counting when performing the phase space integrals. If there are  $n_r$  identical particles of sort  $r$  in the final state,  $S = \prod_r 1/n_r!$  (in the present case  $S = 1/n!$ ).

Typical electromagnetic cross sections, as we will compute later, are of the order of nanobarns ( $1 \text{ nb} = 10^{-33}\text{cm}^2$ ). With a luminosity of  $10^{33}\text{cm}^{-2}\text{s}^{-1}$ , approximately one collision event per second will take place. In the weak interaction the cross sections are typically 5 to 6 orders of magnitude smaller, such that not more than one event per day will take place in that case.

## 11 Decay rates

The definition of the decay rate (also called decay width) of an unstable particle is best defined by considering its self-energy,

$$\Sigma(p) = \frac{1}{Z} \text{---} \textcircled{\diagup\diagdown} \text{---}, \quad (11.1)$$

where the diagram for the  $1PI$  two-point function is now to be evaluated using the Feynman rules in table 3 (pg. 42). The relation with the self-energy follows from the fact that, apart from the overall factor  $i/(2\pi)^4$ , one has for each of the two external lines an extra factor  $-i(2\pi)^2\sqrt{Z}$  as compared to the amputated  $1PI$  2-point function, in total one therefore has  $-iZ$  times the amputated 2-point function. The latter indeed equals  $i\Sigma(p)$ , see eq. (9.9). We will now consider a simple example of a scalar field theory with two types of fields, a field  $\varphi(x)$  associated with a light particle (mass  $m$ ) and a field  $\sigma(x)$  associated with a heavy particle (mass  $M > 2m$ ), which can decay in the lighter particles if we allow for a coupling between one  $\sigma$  and two  $\varphi$  fields,

$$V(\sigma, \varphi) = \frac{1}{2} g \sigma \varphi^2, \quad \text{---}\sigma \begin{array}{c} \nearrow \varphi \\ \searrow \varphi \end{array} \equiv g, \quad (11.2)$$

For the  $\sigma$  two-point function in lowest order we find

$$\text{Diagram with diagonal lines} = \text{Diagram with } p-k \text{ and } k \text{ labels} + \text{Diagram with vertical lines} + \dots \quad (11.3)$$

If  $\sigma$  is a stable particle (i.e.  $M < 2m$ ), the loop in the first diagram corresponds to virtual  $\varphi$  particles moving between the vertices, since always  $k^2 \neq m^2$  and  $(k-p)^2 \neq m^2$ . However, as soon as  $M > 2m$ , the loop integral will contain contributions where the  $\varphi$  particles can be on the mass-shell and behave as real particles, e.g.  $k^2 = m^2$  and  $(k-p)^2 = m^2$  in the first diagram. The real  $\varphi$  particles can escape to infinity, thereby describing the decay of the  $\sigma$  particle. Its number will reduce as a function of time.

Indeed we will see that only if  $M > 2m$ , the self-energy will be able to develop a non-zero imaginary part,  $\gamma \equiv \text{Im}(-Z_\sigma \Sigma_\sigma(p))|_{p^2=M^2} \neq 0$ . On the mass-shell the  $\sigma$  propagator is in that case modified for  $t > 0$  to

$$\int \frac{d_4 p}{(2\pi)^4} \frac{Z_\sigma e^{-ipx}}{p^2 - M^2 + i\gamma + i\varepsilon} = -\frac{i}{2} \int \frac{d_3 \vec{p}}{(2\pi)^3} \frac{Z_\sigma e^{i\vec{p} \cdot \vec{x}}}{\sqrt{\vec{p}^2 + M^2 - i\gamma}} e^{-it\sqrt{\vec{p}^2 + M^2 - i\gamma}}. \quad (11.4)$$

The poles  $p_0 = \pm\sqrt{i\vec{p}^2 + M^2 - i\gamma}$  are now complex and for  $\gamma \ll M^2$  one has in a good approximation

$$e^{-it\sqrt{\vec{p}^2+M^2-i\gamma}} = e^{-it\sqrt{\vec{p}^2+M^2}} e^{-\frac{1}{2}\gamma t/\sqrt{\vec{p}^2+M^2}} \quad . \quad (11.5)$$

The amplitude of the wave function for the  $\sigma$  particle consequently decays with a decay rate of  $\Gamma(p) = \gamma/\sqrt{p^2 + M^2}$ , the life-time of this particle is hence  $\tau(p) = 1/\Gamma(p)$ .

We will evaluate the imaginary part of self-energy for the  $\sigma$  2-point function in eq. (11.3) first to lowest order in  $g$

$$\begin{aligned}\mathrm{Im}(-\Sigma_\sigma(p)) &= \mathrm{Im}\left(-\frac{ig^2}{2}\int\frac{d_4k}{(2\pi)^4}\frac{1}{(k^2-m^2+i\varepsilon)((k-p)^2-m^2+i\varepsilon)}\right) \\ &= \mathrm{Im}\left(-\frac{ig^2}{2}\int d_4x\,G^2(x)e^{ipx}\right)\quad,\end{aligned}\tag{11.6}$$

where we used the definition of the Green's function, eq. (4.7). With the help of eq. (5.12) we can write

$$G(x) = -i \int \frac{d_3 \vec{k}}{(2\pi)^3} \frac{e^{i\vec{k}\cdot\vec{x}} e^{-ik_0(\vec{k})|t|}}{2k_0(\vec{k})} , \quad (11.7)$$

which yields (after changing  $x$  to  $-x$  at the right place)

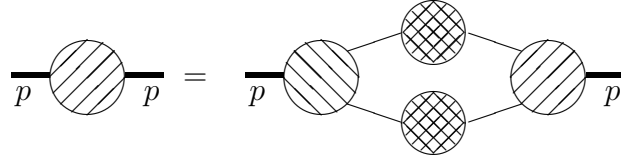
$$\begin{aligned} \text{Im}(-\Sigma_\sigma(p)) &= -\frac{1}{4}g^2 \int d_4x (G(x)^2 + G^*(-x)^2) e^{ipx} = \\ &\frac{1}{4}g^2(2\pi)^4 \int \frac{d_3 \vec{k}_1}{2k_0^{(1)}(2\pi)^3} \frac{d_3 \vec{k}_2}{2k_0^{(2)}(2\pi)^3} \delta_3(\vec{k}_1 + \vec{k}_2 - \vec{p}) \left( \delta(k_0^{(1)} + k_0^{(2)} - p_0) + \delta(k_0^{(1)} + k_0^{(2)} + p_0) \right) , \end{aligned} \quad (11.8)$$

where we have implicitly defined  $k_0^{(j)} \equiv \sqrt{\vec{k}_j^2 + m^2}$ . As we wish to study the  $\sigma$  2-point function at  $t > 0$  near the mass-shell, we can put  $p_0 \equiv \sqrt{\vec{p}^2 + m^2}$  as well. In particular, restricting ourselves to  $p_0 > 0$ , gives

$$\Gamma(p) \equiv \frac{\text{Im}(-Z_\sigma \Sigma_\sigma(p))}{p_0} = \frac{g^2(2\pi)^4}{4p_0} \int \frac{d_3 \vec{k}_1}{2k_0^{(1)}(2\pi)^3} \frac{d_3 \vec{k}_2}{2k_0^{(2)}(2\pi)^3} \delta_4(k_1 + k_2 - p) , \quad (11.9)$$

which is the result to lowest non-trivial order in  $g$  (to this order we can take  $Z_\sigma = 1$ ).

To any order we can, however, decompose the part of the  $\sigma$  self-energy, with *two*  $\varphi$  particles as an intermediate state, in its *1PI* components as follows



$$(11.10)$$

where the full  $\varphi$  2-point function is given by  $1/(p^2 - m^2 - \Sigma_\varphi(p) + i\varepsilon)$ , which on the mass-shell reduces to  $Z_\varphi/(p^2 - \tilde{m}^2 + i\varepsilon)$ , the only part that actually contributes to  $\text{Im}(-Z_\sigma \Sigma_\sigma(p))$ . The *1PI*  $\varphi\varphi\sigma$  3-point function can easily be seen to be equal to  $\mathcal{M}_{\varphi\varphi\sigma}(\{-k_i\}, p)/(Z_\varphi \sqrt{Z_\sigma})$  (or its complex conjugate if in- and outgoing lines are interchanged), since in lowest order it should coincide with the  $\varphi\varphi\sigma$  3-point vertex. In this way we easily find the partial decay rate  $d\Gamma(p)$  to be

$$d\Gamma(p) = (2\pi)^4 \delta_4(\Sigma_i k_i - p) \frac{|\mathcal{M}_{\varphi\varphi\sigma}(\{-k_i\}, p)|^2}{2p_0} S \prod_i \frac{d_3 \vec{k}_i}{2k_0^{(i)}(2\pi)^3} , \quad (11.11)$$

which, as it should be, is always positive. The symmetry factor  $S$  is the same as for the cross section in eq. (10.12). The total decay rate is found by integrating over the phase space of the outgoing particles  $\Gamma(p) \equiv \int d\Gamma(p)$ . The large resemblance with the formula for the cross section is no coincidence, as in both cases we have to calculate the probability for something to happen (respectively a decay or a scattering). In its present form the formula for  $\Gamma(p)$  is also valid for the decay of a particle in  $n$  other particles. The derivation is almost identical, e.g. in eq. (11.8) one now encounters  $G^n(x)$  instead of  $G^2(x)$  and  $g$  now stands of course for the coupling constant of  $n$   $\varphi$  fields to the  $\sigma$  field. It is not necessary for this coupling to occur in the Lagrangian; at higher orders one can generate it from the lower couplings that do occur in the Lagrangian.



## 12 The Dirac equation

To obtain a Lorentz invariant Schrödinger equation, we considered the square root of the Klein-Gordon equation. This had the disadvantage that the Hamiltonian  $H = \sqrt{\vec{p}^2 + m^2}$  contains an infinite number of powers of  $\vec{p}^2/m^2$ , the parameter in which the square root should be expanded. It would have been better to treat space and time on a more equal footing in the Schrödinger equation. This is what Dirac took as his starting point. As the Schrödinger equation is linear in  $p_0 = -i\partial/\partial t$ , one is looking for a Hamiltonian that is linear in the momenta  $p_j = -i\partial/\partial x^j$ .

$$i\frac{\partial\Psi}{\partial t} = H\Psi = -i\alpha_k\frac{\partial\Psi}{\partial x^k} + \beta m\Psi \quad . \quad (12.1)$$

The question Dirac posed himself was to find the simplest choice for  $\alpha_k$  and  $\beta$ , such that the square of the Schrödinger equation gives the Klein-Gordon equation

$$p_0^2 = (\vec{p} \cdot \vec{\alpha} + \beta m)^2 = \vec{p}^2 + m^2 \quad . \quad (12.2)$$

Dirac noted that only in case we allow  $\alpha_k$  and  $\beta$  to be non-commuting objects (i.e. matrices), one can satisfy these equations. The above equation is equivalent to

$$\beta^2 = 1 \quad , \quad \frac{1}{2}(\alpha_j\alpha_k + \alpha_k\alpha_j) = 1\delta_{jk} \quad \text{and} \quad \alpha_j\beta + \beta\alpha_k = 0 \quad . \quad (12.3)$$

Historically, Dirac first considered  $m \neq 0$ , but the massless case ( $m = 0$ ) is somewhat simpler, as it allows one to use  $\beta = 0$  and  $\alpha_k = \sigma_k$  for a solution of eq. (12.3). Here  $\sigma_k$  are the Pauli-matrices, familiar from describing spin one-half particles.

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad . \quad (12.4)$$

It is clear that two will be the smallest matrix dimension for which one can solve the equation  $\frac{1}{2}(\alpha_j\alpha_k + \alpha_k\alpha_j) \equiv \frac{1}{2}\{\alpha_j, \alpha_k\} = 1\delta_{jk}$ . It is not hard to prove that in a two dimensional representation all solutions to this equation are given by

$$\alpha_j = \pm U\sigma_jU^{-1} \quad , \quad (12.5)$$

where  $U$  is an arbitrary non-singular complex  $2 \times 2$  matrix. We should, however, require that  $H$  (and hence  $\alpha_j$ ) is hermitian. This narrows  $U$  down to a unitary matrix, since

$$\alpha_j\alpha_j^\dagger \equiv U\sigma_jU^{-1}U^{\dagger-1}\sigma_j^\dagger U^\dagger = U([U^\dagger U\sigma_j]^{-1}\sigma_jU^\dagger U)U^{-1} = 1 \quad , \quad (12.6)$$

such that  $U^\dagger U\sigma_j = \sigma_jU^\dagger U$  for each  $j$ . The only  $2 \times 2$  matrix that commutes with all Pauli-matrices is a multiple of the identity, which proves that  $U$  is unitary (up to an irrelevant overall complex factor, which does not affect  $\alpha_j$ ).

Since the Hamiltonian is now a  $2 \times 2$  matrix, the wave function  $\Psi(x)$  becomes a complex two dimensional vector, also called a *spinor*, which describes particles with spin  $\hbar/2$

$$-p_0\Psi(x) = \pm \vec{p} \cdot \vec{\sigma}\Psi(x) \quad . \quad (12.7)$$

We have to demonstrate that the Dirac equation is covariant under Lorentz transformations. We first put the boosts to zero, because we already know from quantum mechanics how a spinor transforms under rotations

$$\vec{p} \rightarrow \vec{p}' = \exp(\vec{\omega} \cdot \vec{L})\vec{p} \quad , \quad \Psi(x) \rightarrow \Psi'(x') = \exp(\frac{i}{2}\vec{\omega} \cdot \vec{\sigma})\Psi(x) \quad . \quad (12.8)$$

Here  $L^i$  are real  $3 \times 3$  matrices that generate the rotations in  $\mathbb{R}^3$

$$L_{jk}^i = \varepsilon_{ijk} \quad , \quad (12.9)$$

such that  $[L^i, L^j] \equiv L^i L^j - L^j L^i = -\varepsilon_{ijk} L^k$ . These reflect the commutation relations of the generators  $i\sigma_j/2$ . We will later, in the context of non-Abelian gauge theories, show that this describes the fact that  $SU(2)$  (the group of unitary transformations acting on the spinors) is a representation of  $SO(3)$  (the group of rotations in  $\mathbb{R}^3$ ). To show the covariance of the Dirac equation under rotations, i.e.

$$-p_0 \Psi(x) = \pm \vec{p} \cdot \vec{\sigma} \Psi(x) \rightarrow -p_0 \Psi'(x') = \pm \vec{p}' \cdot \vec{\sigma} \Psi'(x') \quad , \quad (12.10)$$

we work out the Dirac equation in the rotated frame. Using eq. (12.8) we get

$$-p_0 \Psi(x) = \pm \vec{p}' \cdot \exp(-\frac{i}{2} \vec{\omega} \cdot \vec{\sigma}) \vec{\sigma} \exp(\frac{i}{2} \vec{\omega} \cdot \vec{\sigma}) \Psi(x) \quad , \quad (12.11)$$

which should reduce to eq. (12.7). To prove this, we use the following general result for matrices  $X$  and  $Y$

$$e^X Y e^{-X} = \exp(\text{ad} X)(Y) \quad , \quad \text{ad} X(Y) \equiv [X, Y] \quad , \quad (12.12)$$

which is derived from the fact that  $f_1(t) = e^{tX} Y e^{-tX}$  and  $f_2(t) = \exp(\text{ad}(tX)) Y$  satisfy the same differential equation,  $df_i(t)/dt = [X, f_i(t)]$ . Since also  $f_1(0) = f_2(0)$  it follows that  $f_1(1) = f_2(1)$ , being the above equation. Applying this result to  $Y = \sigma_k$  and  $X = -i\vec{\omega} \cdot \vec{\sigma}/2$ , using the fact that

$$\text{ad}(-\frac{i}{2} \vec{\omega} \cdot \vec{\sigma}) \sigma_k = [-\frac{i}{2} \vec{\omega} \cdot \vec{\sigma}, \sigma_k] = (\vec{\omega} \cdot \vec{L})_{kj} \sigma_j \quad , \quad (12.13)$$

the r.h.s. of eq.(12.11) becomes  $\pm \vec{p}' \cdot \vec{\sigma}' \Psi(x) = \pm \vec{p} \cdot \vec{\sigma} \Psi(x)$ , where  $\vec{\sigma}' = \exp(\vec{\omega} \cdot \vec{L}) \vec{\sigma}$ .

The interpretation of this Schrödinger equation caused Dirac quite some trouble, as its eigenvalues are  $\pm |\vec{p}|$ , and it is not bounded from below. In the scalar theory we could avoid this by just considering the positive root of the Klein-Gordon equation. Only when we required localisation of the wave function inside the light-cone, we were forced to consider negative energy states. In the present case, restricting to one of the eigenstates would break the rotational invariance of the theory. For the massive case Dirac first incorrectly thought that the positive energy states describe the electron and the negative energy states the proton. At that time antiparticles were unknown. Antiparticles were predicted by Dirac because the only way he could make the theory consistent was to invoke the Pauli principle and to fill all the negative energy states. A hole in this sea of negative energy states, the so-called Dirac sea, then corresponds to a state of positive energy. These holes describe the antiparticle with the same mass as the particle. Obviously particle number will no longer be conserved and also the Dirac equation will require “second quantisation” and the introduction of a field, which will be discussed later.

For the massive Dirac equation we need to find a matrix  $\beta$  that anticommutes with all  $\alpha_i$ . For  $2 \times 2$  matrices this is impossible, since the Pauli matrices form a complete set of anticommuting matrices. The smallest size turns out to be a  $4 \times 4$  matrix. The following representation is usually chosen

$$\alpha_i = \begin{pmatrix} \otimes & \sigma_i \\ \sigma_i & \otimes \end{pmatrix} \quad , \quad \beta = \begin{pmatrix} 1_2 & \otimes \\ \otimes & -1_2 \end{pmatrix} \quad , \quad (12.14)$$

for which the non-relativistic limit has a simple form. For the massless case it is often more convenient to use the so-called Weyl representation

$$\tilde{\alpha}_i = \begin{pmatrix} \sigma_i & \mathcal{O} \\ \mathcal{O} & -\sigma_i \end{pmatrix} \quad , \quad \tilde{\beta} = \begin{pmatrix} \mathcal{O} & -1_2 \\ -1_2 & \mathcal{O} \end{pmatrix} \quad . \quad (12.15)$$

We leave it as an exercise for the student to show that these two representations are related by a  $4 \times 4$  unitary transformation  $U$ , i.e.  $\tilde{\gamma}^\mu = U\gamma^\mu U^{-1}$ .

To study the covariance of the Dirac equation

$$-p_0\Psi(x) = (\alpha_i p_i + \beta m)\Psi(x) \quad (12.16)$$

under Lorentz transformations (note that now  $\Psi(x)$  has four complex components), it will be profitable to introduce a “four-vector”  $\gamma^\mu$  of  $4 \times 4$  matrices

$$\gamma^\mu \equiv (\gamma^0, \gamma^i) = (\beta, \beta\alpha_i) \quad , \quad (12.17)$$

such that the Dirac equation becomes

$$(-i\gamma^\mu\partial_\mu + m)\Psi(x) = (\gamma^\mu p_\mu + m)\Psi(x) \equiv (\not{p} + m)\Psi(x) = 0 \quad . \quad (12.18)$$

The Dirac gamma matrices satisfy anticommuting relations

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} \quad . \quad (12.19)$$

As for the covariance under rotations, this equation is covariant under Lorentz boosts if there exists a non-singular complex  $4 \times 4$  matrix  $S$ , such that  $\Psi(x) \rightarrow \Psi'(x') = S\Psi(x)$  and

$$S^{-1}\gamma^\mu S = K^\mu{}_\nu\gamma^\nu \quad , \quad (12.20)$$

where  $K^\mu{}_\nu$  is the Lorentz transformation acting on the momenta as  $p'_\mu = K^\mu{}_\nu p_\nu$  and on the coordinates as  $x'^\mu = K^\mu{}_\nu x^\nu$ . Like for the rotations,  $K$  can be written as an exponent

$$K = \exp(\omega) \quad . \quad (12.21)$$

Here  $\omega^\mu{}_\nu$  is a  $4 \times 4$  matrix, which is antisymmetric when one of its indices is raised or lowered by the metric,  $\omega_{\mu\nu} = g_{\mu\lambda}\omega^\lambda{}_\nu = -\omega_{\nu\mu}$ .

We will now prove that

$$S = S(\omega) = \exp\left(-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right) \quad , \quad \sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu] \quad (12.22)$$

satisfies eq. (12.20). Using the antisymmetry of  $\omega_{\mu\nu}$  and eq. (12.19) we find

$$\begin{aligned} \left[\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}, \gamma^\lambda\right] &= -\frac{1}{4}\omega_{\mu\nu}[\gamma^\mu\gamma^\nu, \gamma^\lambda] \\ &= -\frac{1}{4}\omega_{\mu\nu}(\gamma^\mu\gamma^\nu\gamma^\lambda - \gamma^\lambda\gamma^\mu\gamma^\nu) \\ &= \frac{1}{4}\omega_{\mu\nu}(\gamma^\lambda\gamma^\mu\gamma^\nu + \gamma^\mu\gamma^\lambda\gamma^\nu - 2\gamma^\mu g^{\lambda\nu}) \\ &= \frac{1}{2}\omega_{\mu\nu}(g^{\lambda\mu}\gamma^\nu - \gamma^\mu g^{\lambda\nu}) = \omega^\lambda{}_\mu\gamma^\mu \quad . \end{aligned} \quad (12.23)$$

Applying eq. (12.12) gives the proof for eq. (12.20). One says that  $S(\omega)$  is a representation of  $K(\omega)$ . Note that in general  $S(\omega)$  is not a unitary transformation. This is because the

boosts form a non-compact part of the Lorentz group. There is, however, a relation between  $S^\dagger$  and  $S^{-1}$ ,

$$\gamma^0 S^\dagger \gamma^0 = S^{-1} \quad , \quad (12.24)$$

which is most easily proven in the Weyl representation, since  $\tilde{\gamma}^0 = \tilde{\beta}$  commutes with  $\tilde{\sigma}_{ij}$  but *anti*-commutes with  $\tilde{\sigma}_{k0}$  (as always roman indices run from 1 to 3 and greek indices run from 0 to 3), whereas  $\tilde{\sigma}_{ij}$  is hermitian and  $\tilde{\sigma}_{k0}$  is *anti*-hermitian, as follows from the explicit expressions obtained from eqs. (12.15), (12.17) and (12.22)

$$\tilde{\sigma}_{ij} = \tilde{\sigma}^{ij} = \frac{i}{2}[\tilde{\gamma}^i, \tilde{\gamma}^j] = \varepsilon_{ijk} \begin{pmatrix} \sigma_k & \otimes \\ \otimes & \sigma_k \end{pmatrix} \quad , \quad \tilde{\sigma}_{0k} = \tilde{\sigma}^{k0} = \frac{i}{2}[\tilde{\gamma}^k, \tilde{\gamma}^0] = -i \begin{pmatrix} \sigma_k & \otimes \\ \otimes & -\sigma_k \end{pmatrix} \quad . \quad (12.25)$$

Note that eqs. (12.20) and (12.22) are independent of the representation in which we give the gamma matrices, as any two such representations have to be related by a unitary transformation. In the Weyl representation  $\tilde{S}(\omega)$  is block diagonal, like the Dirac equation for  $m = 0$  (as  $\tilde{\alpha}_i$  is block diagonal). The upper block corresponds to eq. (12.7) with the plus sign and the lower block corresponds to the minus sign. We can verify eq. (12.8) by using the fact that the Lorentz transformations contain the rotations through the identification  $\omega_k = -\frac{1}{2}\varepsilon_{ijk}\omega_{ij}$ . With  $\omega_{0k} = 0$ , one finds  $\tilde{S}(\omega) = 1_2 \otimes \exp(i\vec{\omega} \cdot \vec{\sigma}/2)$ , i.e. it acts on each  $2 \times 2$  block by the same unitary transformation.

The boost parameters are described by  $\omega_{0i}$ . For a boost in the  $x$  direction we have that  $\chi = \omega_{01}$  is related to the boost velocity by  $v_1 = -\tanh(\chi)$ . For  $K$  we find in this case

$$K = \begin{pmatrix} \cosh(\chi) & \sinh(\chi) & 0 & 0 \\ \sinh(\chi) & \cosh(\chi) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad . \quad (12.26)$$

In the Weyl representation,  $\tilde{S}$  splits again in two blocks, but one is the inverse of the other (and neither is unitary). To be precise,  $\tilde{S}$  restricted to the upper-left  $2 \times 2$  block equals  $\exp(\frac{1}{2}\omega_{0k}\sigma_k)$ , whereas for the lower-right  $2 \times 2$  block we find  $\exp(-\frac{1}{2}\omega_{0k}\sigma_k)$ .

As  $S$  is not unitary  $\Psi^\dagger(x)\Psi(x)$  is no longer invariant under Lorentz transformations. But we claim it is nevertheless a probability density, namely the time component  $j^0(x)$  of a conserved current

$$j^\mu(x) = \Psi^\dagger(x)\gamma^0\gamma^\mu\Psi(x) \quad . \quad (12.27)$$

We leave it as an exercise to show that the Dirac equation implies that the current is conserved,  $\partial_\mu j^\mu(x) = 0$ . The combination  $\Psi^\dagger(x)\gamma^0$  will occur so often, that it has acquired its own symbol

$$\overline{\Psi}(x) \equiv \Psi^\dagger(x)\gamma^0 \quad . \quad (12.28)$$

It transforms under a Lorentz transformation as

$$\Psi(x) \rightarrow S\Psi(x) \quad \text{and} \quad \overline{\Psi}(x) \rightarrow \overline{\Psi}(x)S^{-1} \quad . \quad (12.29)$$

We can use this to build the required Lorentz scalars, vectors and tensors

$$\begin{aligned} \text{scalar:} \quad & \overline{\Psi}'(x')\Psi'(x') &= & \overline{\Psi}(x)S^{-1}S\Psi(x) &= & \overline{\Psi}(x)\Psi(x) \\ \text{vector:} \quad & \overline{\Psi}'(x')\gamma^\mu\Psi'(x') &= & \overline{\Psi}(x)S^{-1}\gamma^\mu S\Psi(x) &= & K^\mu{}_\nu\overline{\Psi}(x)\gamma^\nu\Psi(x) \\ \text{tensor:} \quad & \overline{\Psi}'(x')\sigma^{\mu\nu}\Psi'(x') &= & \overline{\Psi}(x)S^{-1}\sigma^{\mu\nu}S\Psi(x) &= & K^\mu{}_\lambda K^\nu{}_\kappa\overline{\Psi}(x)\sigma^{\lambda\kappa}\Psi(x) \end{aligned}$$

The Lagrangian is a Lorentz scalar, which we chose such that its equations of motion reproduce the Dirac equation. As  $\Psi(x)$  is complex, it can be considered independent of  $\bar{\Psi}(x)$  and the following Lagrangian

$$S_{\text{Dirac}} = \int d_4x \mathcal{L}_{\text{Dirac}} = \int d_4x \bar{\Psi}(x)(i\gamma^\mu \partial_\mu - m)\Psi(x) \quad . \quad (12.30)$$

gives the Euler-Lagrange equations

$$\frac{\delta S}{\delta \bar{\Psi}(x)} = (i\gamma^\mu \partial_\mu - m)\Psi(x) = 0 \quad , \quad \frac{\delta S}{\delta \Psi(x)} = \bar{\Psi}(x)(-i\gamma^\mu \overleftarrow{\partial}_\mu - m) = 0 \quad . \quad (12.31)$$

The second equation is the complex conjugate of the first,  $\Psi^\dagger(x)(i(\gamma^\mu)^\dagger \overleftarrow{\partial}_\mu + m) = 0$ , because the gamma matrices satisfy

$$(\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0 \quad , \quad (12.32)$$

which follows from the fact that  $(\gamma^0)^\dagger = \beta^\dagger = \beta = \gamma^0$  and  $(\gamma^i)^\dagger = (\beta \alpha_i)^\dagger = \alpha_i \beta = -\gamma^i$ , or from the explicit representation of the gamma matrices

$$\gamma^0 = \begin{pmatrix} 1_2 & \oslash \\ \oslash & -1_2 \end{pmatrix} \quad , \quad \gamma^i = \begin{pmatrix} \oslash & \sigma_i \\ -\sigma_i & \oslash \end{pmatrix} \quad . \quad (12.33)$$

Hence

$$0 = \Psi^\dagger(x)(i(\gamma^\mu)^\dagger \overleftarrow{\partial}_\mu + m)\gamma^0 = \bar{\Psi}(i\gamma^0(\gamma^\mu)^\dagger \gamma^0 \overleftarrow{\partial}_\mu + m) = \bar{\Psi}(i\gamma^\mu \overleftarrow{\partial}_\mu + m) \quad . \quad (12.34)$$

An important role will be played by a fifth gamma matrix

$$\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} \oslash & 1_2 \\ 1_2 & \oslash \end{pmatrix} \quad , \quad (12.35)$$

which anticommutes with all  $\gamma^\mu$  (see problem 21)

$$\gamma_5 \gamma^\mu = -\gamma^\mu \gamma_5 \quad \text{and} \quad (\gamma_5)^2 = 1_4 \quad . \quad (12.36)$$

This implies that we can introduce projection operators

$$P_\pm \equiv \frac{1}{2}(1 \pm \gamma_5) \quad , \quad (12.37)$$

which satisfy  $(P_\pm)^2 = P_\pm$  and  $P_\pm P_\mp = 0$ . Their role is best described in the Weyl representation, where

$$\tilde{\gamma}_5 = i\tilde{\gamma}_0\tilde{\gamma}_1\tilde{\gamma}_2\tilde{\gamma}_3 = -i\tilde{\alpha}_1\tilde{\alpha}_2\tilde{\alpha}_3 = \begin{pmatrix} 1_2 & \oslash \\ \oslash & -1_2 \end{pmatrix} \quad , \quad (12.38)$$

such that  $\tilde{P}_+ = \frac{1}{2}(1 + \tilde{\gamma}_5)$  projects on the two upper and  $\tilde{P}_- = \frac{1}{2}(1 - \tilde{\gamma}_5)$  on the two lower components of the four-spinors. In the massless case these two components are decoupled,

$$-p_0\Psi(x) = (\vec{p} \cdot \tilde{\vec{\alpha}} + m\tilde{\beta})\Psi(x) = \begin{pmatrix} \vec{p} \cdot \vec{\sigma} & m1_2 \\ m1_2 & -\vec{p} \cdot \vec{\sigma} \end{pmatrix} \Psi \quad . \quad (12.39)$$

Hence, for  $m = 0$  we have

$$\Psi(x) = \begin{pmatrix} \Psi_+(x) \\ \Psi_-(x) \end{pmatrix} \quad , \quad -p_0\Psi_\pm(x) = \pm \vec{p} \cdot \vec{\sigma} \Psi_\pm(x) \quad . \quad (12.40)$$

which is identical to eq. (12.7). The eigenstates of the projection operators  $P_{\pm}$  are called helicity eigenstates. As long as  $m \neq 0$ , helicity is not conserved. But as we saw, for  $m = 0$  the two helicity eigenstates decouple. One can define in that case consistently a particle with a fixed helicity, whose opposite helicity state does not occur (although its antiparticle has opposite helicity). A very important example of such a particle is the neutrino, although experiment has not yet been able to rule out a (tiny) mass for this particle ( $m_{\nu_e} < 10\text{eV}$ ). See problem 22 for more details.

Apart from the invariance of the Dirac equation under Lorentz transformations and translations (which are obvious symmetries of  $\mathcal{L}_{\text{Dirac}}$ ) we also often want invariance under parity ( $\vec{x} \rightarrow -\vec{x}$ ) and time-reversal ( $t \rightarrow -t$ ). One easily checks that

$$P\Psi(x) \equiv \Psi'(t, -\vec{x}) = \gamma_0\Psi(t, \vec{x}) \quad \text{and} \quad T\Psi(x) \equiv \Psi'(-t, \vec{x}) = \gamma_5\gamma_0\gamma^2\Psi^*(t, \vec{x}) \quad (12.41)$$

satisfy the Dirac equation, where  $P$  stands for parity and  $T$  for time-reversal. This implies that the Lorentz covariant combinations  $\bar{\Psi}(x)\gamma_5\Psi(x)$  and  $\bar{\Psi}(x)\gamma_5\gamma^\mu\Psi(x)$  are not invariant under parity and time-reversal. They are called pseudoscalars and pseudovectors. These combinations play an important role in the weak interactions, where parity is not a symmetry. A third discrete symmetry, charge conjugation  $C$ , will be discussed in section 17.

## 13 Plane wave solutions of the Dirac equation

As usual, the Dirac equation can be solved by Fourier decomposition in plane waves,

$$\Psi(x) = \int \frac{d_3\vec{k}}{\sqrt{2k_0(\vec{k})(2\pi)^3}} \tilde{\Psi}(\vec{k})e^{-ikx} \quad . \quad (13.1)$$

where  $\tilde{\Psi}(\vec{k})$  are complex four-vectors that satisfy

$$(\not{k} - m)\tilde{\Psi}(\vec{k}) \equiv (k_\mu\gamma^\mu - m)\tilde{\Psi}(\vec{k}) = 0 \quad , \quad k_0^2 = \vec{k}^2 + m^2 \quad . \quad (13.2)$$

The Lorentz invariance implies

$$\tilde{\Psi}'(\vec{k}') = S(\omega)\tilde{\Psi}(\vec{k}) \quad , \quad k'_\mu = K(\omega)_\mu{}^\nu k_\nu \quad . \quad (13.3)$$

As any  $\vec{k}$  can be obtained from a boost to  $\vec{k} = \vec{0}$ , all solutions of the Dirac equation can be obtained from the ones at rest with  $\vec{k} = \vec{0}$  (see problem 19)

$$(\gamma^0 k_0 - m)\tilde{\Psi}(\vec{0}) = \begin{pmatrix} (k_0 - m)1_2 & \oslash \\ \oslash & -(k_0 + m)1_2 \end{pmatrix} \tilde{\Psi}(\vec{0}) \quad . \quad (13.4)$$

We see that for  $k_0 > 0$  ( $k_0 = m$ ), there are two independent solutions both of the form

$$\tilde{\Psi}_+(\vec{0}) = \begin{pmatrix} \Psi_A \\ 0 \end{pmatrix} \quad , \quad (13.5)$$

where  $\Psi_A$  is a spin one-half two-spinor of which  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$  is the spin-up and  $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$  is the spin-down state. The identification of the spin degrees of freedom follows from the behaviour of

$\tilde{\Psi}$  under rotations (which leave  $\vec{k} = \vec{0}$ ). We leave it as an exercise to verify that also in the Dirac representation of the gamma matrices, like in the Weyl representation (see eq. (12.25)),

$$\sigma_{ij} = \varepsilon_{ijk} \begin{pmatrix} \sigma_k & \oslash \\ \oslash & \sigma_k \end{pmatrix} \quad , \quad (13.6)$$

such that  $\Psi_A$  is easily seen to transform under a rotation as in eq. (12.8) (cmp. the discussion below eq. (12.25)). There are also two solutions for  $k_0 < 0$  ( $k_0 = -m$ ) of the form

$$\tilde{\Psi}_-(\vec{0}) = \begin{pmatrix} 0 \\ \Psi_B \end{pmatrix} \quad , \quad (13.7)$$

where likewise  $\Psi_B$  is a spin one-half two-spinor of which  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$  is the spin-up and  $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$  is the spin-down state, which transform under rotations as  $\Psi_A$ .

For any frame, i.e. for any value of  $\vec{k}$ , we will define the four independent solutions of the Dirac equation as

$$\begin{aligned} k_0 = \sqrt{\vec{k}^2 + m^2} : \quad u^{(\alpha)}(\vec{k}) &= \frac{\not{k} + m}{\sqrt{m + |k_0|}} u_0^{(\alpha)} \quad , \quad u_0^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad , \quad u_0^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad , \\ k_0 = -\sqrt{\vec{k}^2 + m^2} : \quad v^{(\alpha)}(-\vec{k}) &= \frac{\not{k} + m}{\sqrt{m + |k_0|}} v_0^{(\alpha)} \quad , \quad v_0^{(1)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad , \quad v_0^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad . \end{aligned} \quad (13.8)$$

These solutions naturally split in positive energy ( $u^{(\alpha)}(\vec{k})$  with  $\alpha = 1, 2$ ) and negative energy ( $v^{(\alpha)}(-\vec{k})$  with  $\alpha = 1, 2$ ) solutions; for  $\vec{k} = \vec{0}$  and  $m \neq 0$  easily seen to be proportional to the solutions  $\tilde{\Psi}_{\pm}(\vec{0})$ , to be precise  $u^{(\alpha)}(\vec{0}) = u_0^{(\alpha)}/\sqrt{2m}$  and  $v^{(\alpha)}(\vec{0}) = v_0^{(\alpha)}/\sqrt{2m}$ . The normalisation we have chosen allows us to treat massless fermions at the same footing. In that case we can, however, not transform to the restframe. This normalisation also implies that

$$\overline{u^{(\alpha)}(\vec{k})} u^{(\beta)}(\vec{k}) = -\overline{v^{(\alpha)}(-\vec{k})} v^{(\beta)}(-\vec{k}) = 2m\delta_{\alpha\beta} \quad . \quad (13.9)$$

For example

$$\begin{aligned} \overline{u^{(\alpha)}(\vec{k})} u^{(\beta)}(\vec{k}) &= \frac{\overline{u_0^{(\alpha)}} \gamma^0 (\not{k} + m)^\dagger \gamma^0 (\not{k} + m) u_0^{(\beta)}}{(m + k_0)} = \frac{\overline{u_0^{(\alpha)}} (\not{k} + m)^2 u_0^{(\beta)}}{(m + k_0)} = \\ &= \frac{\overline{u_0^{(\alpha)}} (k^2 + m^2 + 2\not{k}m) u_0^{(\beta)}}{(m + k_0)} = \frac{\overline{u_0^{(\alpha)}} (k^2 + m^2 + 2k_0m) u_0^{(\beta)}}{(m + k_0)} = 2m\delta_{\alpha\beta} \quad , \end{aligned} \quad (13.10)$$

where we used  $k^2 = m^2$  and  $\overline{u_0^{(\alpha)}} \gamma^\mu u_0^{(\beta)} = \delta_{\alpha\beta} \delta_{\mu 0}$  (see eq. (12.33)). The computation for  $v^{(\alpha)}(\vec{k})$  is left as an exercise. The fact that we find a result that is independent of  $\vec{k}$  is consistent with our claim that these spinors can also be obtained by applying the appropriate boost to  $\vec{k} = \vec{0}$ , since  $\tilde{\Psi}(\vec{k}) \tilde{\Psi}(\vec{k})$  is a Lorentz scalar.

That these spinors indeed satisfy eq. (13.2) follows from the fact that

$$(\not{k} - m)(\not{k} + m) = k^2 - m^2 = 0 \quad . \quad (13.11)$$

Note that  $v^{(\alpha)}(\vec{k})$  can also be viewed as a positive energy solution for the complex conjugate of the Dirac equation

$$k_0 = \sqrt{\vec{k}^2 + m^2} : \quad (\not{k} + m)v^{(\alpha)}(\vec{k}) = 0 \quad . \quad (13.12)$$

It will play the role of the wave function for the antiparticles (the holes) in the Dirac field theory. To see that our plane waves have the correct amplitude, we use the fact that the probability density can be defined in terms of the zero-component  $\rho(x)$  of the conserved four-vector  $j^\mu(x) = \bar{\Psi}(x)\gamma^\mu\Psi(x)$  (see eq. (12.27)), or

$$\rho(x) = \Psi^\dagger(x)\Psi(x) \quad . \quad (13.13)$$

Indeed,  $\tilde{\Psi}^\dagger(\vec{k})\tilde{\Psi}(\vec{k})$  transforms as a density, i.e. as the energy  $k_0$ , and we find

$$u^{(\alpha)}(\vec{k})^\dagger u^{(\beta)}(\vec{k}) = v^{(\alpha)}(\vec{k})^\dagger v^{(\beta)}(\vec{k}) = 2k_0\delta_{\alpha\beta} \quad , \quad u^{(\alpha)}(\vec{k})^\dagger v^{(\beta)}(\vec{k}) = 0 \quad . \quad (13.14)$$

This can be verified by a direct computation, e.g. ( $k_0 > 0$ )

$$\begin{aligned} v^{(\alpha)}(\vec{k})^\dagger v^{(\beta)}(\vec{k}) &= \frac{v_0^{(\alpha)\dagger}(m - \not{k})^\dagger(m - \not{k})v_0^{(\beta)}}{m + k_0} = \frac{v_0^{(\alpha)\dagger}(k_0^2 + \vec{k}^2 + m^2 - 2mk_0\gamma^0)v_0^{(\beta)}}{m + k_0} = \\ &= \frac{v_0^{(\alpha)\dagger}(k_0^2 + \vec{k}^2 + m^2 + 2mk_0)v_0^{(\beta)}}{m + k_0} = 2k_0\delta_{\alpha\beta} \quad , \end{aligned} \quad (13.15)$$

using  $v_0^{(\alpha)\dagger}\gamma^\mu v_0^{(\beta)} = -\delta_{\mu 0}\delta_{\alpha\beta}$  (see eq. (12.33)). We leave the other identities as an exercise.

This implies that the plane wave solutions ( $k_0 = (\vec{k}^2 + m^2)^{\frac{1}{2}}$ )

$$\frac{u^{(\alpha)}(\vec{k})}{\sqrt{2k_0}(2\pi)^3}e^{-ikx} \quad , \quad \frac{v^{(\alpha)}(\vec{k})}{\sqrt{2k_0}(2\pi)^3}e^{ikx} \quad , \quad \alpha = 1, 2 \quad , \quad (13.16)$$

are normalised to one (of course, in a finite volume we replace  $(2\pi)^3$  by  $V$  and  $\int d_3\vec{k}$  by  $\sum_{\vec{k}}$ ). As for the scalar field we can introduce a Dirac field

$$\Psi(x) = \int \frac{d_3\vec{k}}{\sqrt{2k_0(\vec{k})}(2\pi)^3} \sum_{\alpha=1}^2 \left( b_\alpha(\vec{k})u^{(\alpha)}(\vec{k})e^{-ikx} + d_\alpha^\dagger(\vec{k})v^{(\alpha)}(\vec{k})e^{ikx} \right) \quad . \quad (13.17)$$

Since  $\Psi(x)$  is complex, there is no relation between  $d_\alpha$  and  $b_\alpha$ . In the quantum field theory these will play the role of the annihilation operators for the (anti-)particles.

Finally we note that, for  $m \neq 0$ , we can define projection operators (which are  $4 \times 4$  matrices)

$$\Lambda_+(\vec{k}) = \sum_{\alpha=1}^2 \frac{u^{(\alpha)}(\vec{k}) \otimes \overline{u^{(\alpha)}(\vec{k})}}{2m} = \frac{\not{k} + m}{2m} \quad , \quad \Lambda_-(\vec{k}) = -\sum_{\alpha=1}^2 \frac{v^{(\alpha)}(\vec{k}) \otimes \overline{v^{(\alpha)}(\vec{k})}}{2m} = \frac{-\not{k} + m}{2m} \quad , \quad (13.18)$$

which can be verified from the explicit form of  $u$  and  $v$ , or by first computing them in the restframe where  $\Lambda_\pm(\vec{0}) = \frac{1}{2}(1 \pm \gamma^0)$ . In that case we find

$$\Lambda_\pm(\vec{k}) = \frac{(\pm\not{k} + m)(1 \pm \gamma^0)(\pm\not{k} + m)}{4m(m + k_0)} = \frac{\pm\not{k} + m}{2m} \quad , \quad (13.19)$$

whose proof requires some gamma matrix gymnastics. Independent of the frame  $\Lambda_\pm$  satisfy

$$\Lambda_\pm^2(\vec{k}) = \Lambda_\pm(\vec{k}) \quad , \quad \Lambda_\mp(\vec{k})\Lambda_\pm(\vec{k}) = 0 \quad . \quad (13.20)$$

Note that  $\text{Tr}(\Lambda_\pm(\vec{k})) = 2$ , a result that can also be derived from eq. (13.9).



## 14 The Dirac Hamiltonian

In the Dirac equation we encountered an additional difficulty, namely that the negative energy solutions even arise at the level of the classical theory. In field theory the negative energy solutions had an interpretation in terms of antiparticles, and the field theory Hamiltonian was still positive, and most importantly, bounded from below (see section 2 and problem 5). The field theory Hamiltonian for the Dirac field no longer has this property. The Hamiltonian can again be derived through a Legendre transform of the Lagrangian

$$S = \int d_4x \mathcal{L} = \int d_4x \bar{\Psi}(x)(i\gamma^\mu \partial_\mu - m)\Psi(x) \quad . \quad (14.1)$$

The canonical momentum is hence

$$\pi_a(x) = \frac{\delta S}{\delta \dot{\Psi}_a(x)} = (\bar{\Psi}(x)i\gamma^0)_a = i\Psi_a^*(x) \quad . \quad (14.2)$$

such that

$$\begin{aligned} H &= \int d_3\vec{x} (\pi_a(x)\dot{\Psi}_a(x) - \mathcal{L}) = \int d_3\vec{x} \bar{\Psi}(x)(-i\gamma^j \partial_j + m)\Psi(x) \\ &= \int d_3\vec{x} \Psi^\dagger(x)(-i\alpha^j \partial_j + m\beta)\Psi(x) \\ &= \int d_3\vec{k} k_0(\vec{k}) \sum_\alpha \left( b_\alpha^\dagger(\vec{k}) b_\alpha(\vec{k}) - d_\alpha(\vec{k}) d_\alpha^\dagger(\vec{k}) \right) \quad . \end{aligned} \quad (14.3)$$

Note the resemblance with eq. (12.1) for the middle term. We used eq. (13.17) for the expansion of the Dirac field in plane waves. From this result it is clear that the Hamiltonian is not bounded from below, and this would make the vacuum unstable, as the negative energy states, described by the  $d_\alpha(\vec{k})$ , can lower the energy by an arbitrary amount. It is well-known how Dirac repaired this problem. He postulated that all negative energy states are occupied, and that the states satisfy the Pauli principle, i.e. two particles can not occupy the same quantum state (it is only in that case that we can make sense of what is meant with filling all negative energy states). This implies that one should use anticommuting relations for the creation and annihilation operators

$$\begin{aligned} \{b_\alpha(\vec{k}), b_\beta(\vec{p})\} &= 0 \quad , \quad \{b_\alpha(\vec{k}), b_\beta^\dagger(\vec{p})\} = \delta_{\alpha\beta} \delta_3(\vec{k} - \vec{p}) \quad , \\ \{d_\alpha(\vec{k}), d_\beta(\vec{p})\} &= 0 \quad , \quad \{d_\alpha(\vec{k}), d_\beta^\dagger(\vec{p})\} = \delta_{\alpha\beta} \delta_3(\vec{k} - \vec{p}) \quad , \\ \{d_\alpha(\vec{k}), b_\beta(\vec{p})\} &= 0 \quad , \quad \{d_\alpha(\vec{k}), b_\beta^\dagger(\vec{p})\} = 0 \quad . \end{aligned} \quad (14.4)$$

Indeed, if we define a two particle state as  $|\vec{k}, \vec{p}\rangle \equiv b^\dagger(\vec{k})b^\dagger(\vec{p})|0\rangle$  (suppressing the spinor indices), the anticommutation relations imply that  $|\vec{k}, \vec{p}\rangle = -|\vec{p}, \vec{k}\rangle$ .

A hole in the Dirac sea is by definition the state that is obtained by annihilating a negative energy state in the Dirac sea. As annihilation lowers the total energy by the energy of the annihilated state, which in this case is negative, the net energy is raised. The wave function for the negative energy state is given by  $\exp(ikx)v^{(\alpha)}(\vec{k})/\sqrt{k_0}$  and has momentum  $-\vec{k}$  ( $k_0 \equiv (\vec{k}^2 + m^2)^{\frac{1}{2}}$ ), see eq. (13.8). The reason to associate its Fourier coefficient in eq. (13.17) with a creation operator  $d_\alpha^\dagger(\vec{k})$ , is that conservation of energy and momentum implies that it creates an antiparticle as a hole in the Dirac sea, with momentum  $\vec{k}$  and helicity  $\frac{1}{2}$  for  $\alpha = 2$ , whereas for  $\alpha = 1$  the helicity is  $-\frac{1}{2}$  (hence the helicity and momentum are opposite to the

negative energy state it annihilates). The wave function of an antiparticle with momentum  $\vec{k}$  is hence given by  $\exp(-ikx)v^{(\alpha)}(\vec{k})^\dagger/\sqrt{k_0}$ . If we now use the anticommutation relations of creation and annihilation operators (as a consequence of the Pauli exclusion principle), we see that with the present interpretation the Dirac Hamiltonian is bounded from below

$$H = \int d_3\vec{k} \, k_0(\vec{k}) \sum_{\alpha} \left( b_{\alpha}^{\dagger}(\vec{k}) b_{\alpha}(\vec{k}) + d_{\alpha}^{\dagger}(\vec{k}) d_{\alpha}(\vec{k}) - 1 \right) \quad (14.5)$$

As in the case for scalar field theory we can normalise the energy of the vacuum state to zero by adding a (infinite) constant. Note that this constant has its sign opposite to the scalar case (and is in magnitude *four* times as large). In so-called supersymmetric field theories this is no longer an accident as the Dirac fields will be related to the scalar fields by a symmetry, which is however outside the scope of these lectures.

Important is also to note that the anticommuting relations are crucial to guarantee locality of the Dirac field. In this case Dirac fields specified in different regions of space-time that are space-like separated should anticommute. And indeed, in problem 24 you are asked to prove that

$$\{\Psi_a(x), \Psi_b^{\dagger}(x')\} = 0 \quad \text{for} \quad (x - x')^2 < 0 \quad (14.6)$$

Also, as in the scalar theory, we can couple a source to the free Dirac field (we again introduce  $\bar{\varepsilon}$  as the expansion parameter for taking the interactions due to the source into account in Hamiltonian perturbation theory)

$$\begin{aligned} \mathcal{L}(x) &= \bar{\Psi}(x)(i\gamma^{\mu}\partial_{\mu} - m)\Psi(x) - \bar{\varepsilon}\bar{\mathcal{J}}(x)\Psi(x) - \bar{\varepsilon}\bar{\Psi}(x)\mathcal{J}(x) \quad , \\ \mathcal{H}(x) &= \bar{\Psi}(x)(-i\gamma^k\partial_k + m)\Psi(x) + \bar{\varepsilon}\bar{\mathcal{J}}(x)\Psi(x) + \bar{\varepsilon}\bar{\Psi}(x)\mathcal{J}(x) \quad , \end{aligned} \quad (14.7)$$

where as before the Hamiltonian  $H$  is the spatial integral over the Hamiltonian density  $\mathcal{H}(x)$ ,  $H = \int d_3\vec{x} \, \mathcal{H}(x)$ . Note that the sources  $\mathcal{J}(x)$  and  $\bar{\mathcal{J}}(x)$  are independent, as for a complex scalar field, see problem 17. In problem 24 you are asked to prove that in Hamiltonian perturbation theory one obtains

$$\begin{aligned} \langle 0 | \text{Texp}(-i \int_0^T H(t) dt) | 0 \rangle e^{iE_0 T} &= 1 - i\bar{\varepsilon}^2 \int d_4x d_4y \, \bar{\mathcal{J}}(x) G_F(x-y) \mathcal{J}(y) + \mathcal{O}(\bar{\varepsilon}^3) \\ &= 1 - i \overbrace{\bar{\varepsilon} \mathcal{J} \quad \times \quad \bar{\varepsilon} \bar{\mathcal{J}}}^{\times} + \mathcal{O}(\bar{\varepsilon}^3) \quad . \end{aligned} \quad (14.8)$$

where

$$G_F(x-y) = \int \frac{d_4p}{(2\pi)^4} \frac{e^{-ip(x-y)}}{\not{p} - m + i\varepsilon} = \int \frac{d_4p}{(2\pi)^4} \frac{(\not{p} + m)e^{-ip(x-y)}}{p^2 - m^2 + i\varepsilon} \quad (14.9)$$

Hence, the Green's function for fermions is in Fourier space given by  $G_F(p) = (\not{p} - m + i\varepsilon)^{-1}$ , which is the inverse of the quadratic part of the Lagrangian, as for the scalar fields. In problem 24 it will be evident that, nevertheless, the anticommuting properties of the Dirac field play a crucial role (cmp. section 5). It becomes, however, plausible there is *also* for the fermions a path integral formulation, as splitting of a square (cmp. eqs. (7.11) and (7.12)) is independent of the details of the path integrals. This will be the subject of the next section.

To conclude, we note that the coupling to an electromagnetic field  $A_{\mu}(x)$  should be achieved through the current  $j^{\mu}(x)$ , defined in eq. (12.27)

$$\int d_4x \, j^{\mu}(x) A_{\mu}(x) = \int d_4x \, \bar{\Psi}(x) \gamma^{\mu} A_{\mu}(x) \Psi(x) \quad (14.10)$$

Since this current is conserved, which can also be seen as a consequence of the Noether theorem applied to the invariance of eq. (14.1) (the free Dirac Lagrangian) under global phase transformations, the coupling is gauge invariant. Using the minimal coupling defined as in eq. (3.35),

$$D_\mu \Psi(x) \equiv (\partial_\mu - ieA_\mu(x))\Psi(x) \quad , \quad (14.11)$$

one can fix the normalisation of the electromagnetic current to be  $J^\mu(x) = -ej^\mu(x)$ , since (cmp. eq. (3.31))

$$\bar{\Psi}(x)(i\gamma^\mu D_\mu - m)\Psi(x) = \bar{\Psi}(x)(i\gamma^\mu \partial_\mu - m)\Psi(x) + ej^\mu A_\mu(x) \quad . \quad (14.12)$$

The current  $j^\mu(x)$  is the charge density current, whose time component  $\rho(x)$  at the quantum level is no longer positive definite. Using the anticommuting properties of eq. (14.4) one finds

$$\begin{aligned} \int d_3\vec{x} \hat{\rho}(\vec{x}) &= \int d_3\vec{x} \Psi^\dagger(x)\Psi(x) \\ &= \int d_3\vec{k} \sum_\alpha \left( b_\alpha^\dagger(\vec{k})b_\alpha(\vec{k}) + d_\alpha(\vec{k})d_\alpha^\dagger(\vec{k}) \right) \\ &= \int d_3\vec{k} \sum_\alpha \left( b_\alpha^\dagger(\vec{k})b_\alpha(\vec{k}) - d_\alpha^\dagger(\vec{k})d_\alpha(\vec{k}) + 1 \right) \\ &= Q_0/e + \int d_3\vec{k} \sum_\alpha \left( b_\alpha^\dagger(\vec{k})b_\alpha(\vec{k}) - d_\alpha^\dagger(\vec{k})d_\alpha(\vec{k}) \right) \quad . \end{aligned} \quad (14.13)$$

The vacuum value of this operator is indicated by the (generally infinite) constant  $Q_0/e$ , which can be normalised to zero. After all we want the state with all negative energy states occupied to have zero charge. With the above normalisation of the electric current,  $J^\mu(x) = -ej^\mu(x)$ , we see that the  $b$  modes can be identified with the electrons with charge  $-e$  and the  $d$  modes with their antiparticles, the positrons, with opposite electric charge  $+e$ . To summarise,  $b_\alpha^\dagger(\vec{k})$  corresponds with the creation operator of a spin-up ( $\alpha = 1$ ) or a spin-down ( $\alpha = 2$ ) electron of momentum  $\vec{k}$ , whereas  $d_\alpha^\dagger(\vec{k})$  corresponds with the creation operator of a spin-up ( $\alpha = 2$ ) or a spin-down ( $\alpha = 1$ ) positron of momentum  $\vec{k}$ .

## 15 Path integrals for fermions

For scalar fields, which describe bosons, we used real or complex numbers (the eigenvalues of the operators), in order to perform the path integral. For fermionic fields it is essential to build the anticommuting properties into the path integral.

To this end we introduce a so-called Grassmann algebra, which exists of Grassmann variables  $\theta_i$  that mutually anticommute

$$\{\theta_i, \theta_j\} \equiv \theta_i\theta_j + \theta_j\theta_i = 0 \quad . \quad (15.1)$$

In particular, a Grassmann variable squares to zero

$$\theta^2 = 0 \quad (15.2)$$

A Grassmann variable can be multiplied by a complex number, with which it commutes. A function of a single Grassmann variable has a finite Taylor series

$$f(\theta) = a_0 + a_1\theta \quad , \quad (15.3)$$

and spans a two dimensional (real or) complex vector space. This is exactly what we need to describe a spin one-half particle. Let us introduce the following notation

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad . \quad (15.4)$$

With respect to the Hamiltonian

$$H_0 = \frac{1}{2} \begin{pmatrix} -W & 0 \\ 0 & W \end{pmatrix} \quad (15.5)$$

$|0\rangle$  is the vacuum state (i.e. the state with lowest energy) and we interpret  $|1\rangle$  as the one-particle state (with energy  $W$  above the vacuum). An arbitrary spinor can be written as a linear combination of these two states

$$|\Psi\rangle = a_0|0\rangle + a_1|1\rangle = a_0|0\rangle + a_1 b^\dagger|0\rangle \quad , \quad (15.6)$$

where  $b$  is the fermionic annihilation operator, which in the spinor representation is given by a  $2 \times 2$  matrix

$$b = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad , \quad b^\dagger = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad . \quad (15.7)$$

We note that  $b^2 = (b^\dagger)^2 = 0$ , a property it has in common with a Grassmann variable. We will now look for properties of  $\theta$  such that

$$\Psi(\theta) \equiv \langle \theta | \Psi \rangle = a_0 + a_1 \theta \quad (15.8)$$

is a representation of the state  $|\Psi\rangle$ , similar to  $\Psi(x) = \langle x | \Psi \rangle$  for the case of a single bosonic (i.e. commuting, as opposed to anticommuting) degree of freedom. In the latter case, the normalisation

$$\int dx \Psi^*(x) \Psi(x) = 1 \quad (15.9)$$

is an important property we would like to impose here too (keeping in mind that for path integrals we need to insert completeness relations). As  $\langle 0| = (1, 0)$  and  $\langle 1| = (0, 1)$ , we have

$$\langle \Psi| = \langle 0| a_0^* + \langle 1| a_1^* = \langle 1| b^\dagger a_0^* + \langle 1| a_1^* \quad . \quad (15.10)$$

As in eq. (15.8) (i.e.  $b^\dagger \rightarrow \theta$ ), we anticipate

$$\Psi^*(\theta) \equiv \langle \Psi | \theta \rangle = a_0^* \theta + a_1^* \quad . \quad (15.11)$$

We wish to define integration over Grassmann variables, such that the normalisation of the wave function is as usual

$$\langle \Psi | \Psi \rangle = |a_0|^2 + |a_1|^2 = \int d\theta \Psi^*(\theta) \Psi(\theta) \quad . \quad (15.12)$$

Since the norm should be a number, and as  $\int d\theta 1$  is itself a Grassmann variable, the latter should vanish. For the same reason  $\int d\theta \theta$  (which is itself a commuting object, as is any even product of Grassmann variables) can be seen as a number. Demanding the so-called Grassmann integration to be linear in the integrand, and using  $\theta^2 = 0$ , all possible ingredients have been discussed. Indeed

$$\int d\theta \theta = 1 \quad , \quad \int d\theta 1 = 0 \quad (15.13)$$

is easily seen to give the desired result. Note that  $d\theta$  is considered as an independent Grassmann variable (which is important to realise when multiple Grassmann integrations are involved).

We can now study the action of an operator (like a Hamiltonian  $H$ ) on a state, which in the spinor representation is given by  $2 \times 2$  matrices.

$$M|\Psi\rangle = |\Psi'\rangle \quad \text{or} \quad \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} a'_0 \\ a'_1 \end{pmatrix} \quad . \quad (15.14)$$

Translated to Grassmann variables, this gives

$$\Psi'(\theta) = a'_0 + a'_1\theta = M_{11}a_0 + M_{12}a_1 + (M_{21}a_0 + M_{22}a_1)\theta \equiv \int d\theta' M(\theta, \theta')\Psi(\theta') \quad , \quad (15.15)$$

provided we define

$$M(\theta, \theta') \equiv M_{11}\theta' + M_{12} + M_{21}\theta'\theta - M_{22}\theta = M_{11}\theta' + M_{12} - M_{21}\theta\theta' - M_{22}\theta \quad . \quad (15.16)$$

Indeed,

$$\begin{aligned} \int d\theta' M(\theta, \theta')\Psi(\theta') &= \int d\theta' (M_{11}\theta' + M_{12} - M_{21}\theta\theta' - M_{22}\theta)(a_0 + a_1\theta') \\ &= \int d\theta' \{a_0M_{12} - a_0M_{22}\theta + (a_1M_{12} + a_0M_{11})\theta' - (a_0M_{21} + a_1M_{22})\theta\theta'\} \\ &= \int d\theta' \theta' \{(a_1M_{12} + a_0M_{11}) + (a_0M_{21} + a_1M_{22})\theta\} \\ &= M_{12}a_1 + M_{11}a_0 + (M_{21}a_0 + M_{22}a_1)\theta \quad . \end{aligned} \quad (15.17)$$

The  $2 \times 2$  identity matrix is hence represented by (note the sign)

$$1_2(\theta, \theta') = \theta' - \theta \quad , \quad (15.18)$$

which can be used to write the infinitesimal evolution operator

$$\exp(-iH\Delta t) = 1_2 - iH\Delta t + \mathcal{O}(\Delta t^2) \quad , \quad (15.19)$$

where  $H$  is a (possibly time-dependent)  $2 \times 2$  matrix. In the Grassmann representation this reads

$$\begin{aligned} 1_2(\theta, \theta') - iH(\theta, \theta')\Delta t &= \theta' - \theta - i\Delta t(H_{11}\theta' + H_{12} - H_{21}\theta\theta' - H_{22}\theta) \\ &= \int d\tilde{\theta} \exp(\tilde{\theta}[1_2(\theta, \theta') - iH(\theta, \theta')\Delta t]) \quad . \end{aligned} \quad (15.20)$$

The last identity is exact, and a consequence of the fact that the Taylor series of any function of a Grassmann variable truncates

$$\exp(\tilde{\theta}x) = 1 + \tilde{\theta}x \quad , \quad \int d\tilde{\theta} \exp(\tilde{\theta}x) = x \quad . \quad (15.21)$$

This is valid both for  $x$  a Grassmann variable (in which case the ordering of  $x$  with respect to  $\tilde{\theta}$  is important, with the opposite ordering the result is  $-x$ ), or for  $x$  a complex number.

Another useful property of Grassmann integration is that ( $y$  is a number)

$$\int d\theta \exp(\theta x + y) = x \exp(y) \quad . \quad (15.22)$$

To prove this, we use that

$$\exp(\theta x + y) = \sum_{n=0}^{\infty} \frac{1}{n!} (\theta x + y)^n = \sum_{n=0}^{\infty} \frac{1}{n!} y^n + \theta x \sum_{n=1}^{\infty} \frac{1}{(n-1)!} y^{n-1} = (1 + \theta x) \exp(y) \quad . \quad (15.23)$$

In general it is not true that the exponential function retains the property  $\exp(x + y) = \exp(x) \exp(y)$ , for  $x$  and  $y$  arbitrary elements of the Grassmann algebra. It behaves as if  $x$  and  $y$  are matrices, as it should since the Grassmann representation originates from a  $2 \times 2$  matrix representation. More precisely

$$\begin{aligned} \exp(\theta x) \exp(\theta' y) &= (1 + \theta x)(1 + \theta' y) = 1 + (\theta x + \theta' y) + \frac{1}{2}(\theta x + \theta' y)^2 + \frac{1}{2}[\theta x, \theta' y] \\ &= \exp(\theta x + \theta' y + \frac{1}{2}[\theta x, \theta' y]) \quad . \end{aligned} \quad (15.24)$$

This means that the Campbell-Baker-Hausdorff formula (eq. (6.44)) can be extended to this case. It truncates after the single commutator term as neither  $\theta x$  nor  $\theta y$  can appear more than once.

Let us apply this to the evolution operator, which in the Grassmann representation is given by (see eqs. (15.19) and (15.20))

$$< \theta_{i+1} | U(t_{i+1}, t_i) | \theta_i > \equiv U_i(\theta_{i+1}, \theta_i) = \int d\tilde{\theta}_i \exp(\tilde{\theta}_i [\theta_i - \theta_{i+1} - iH(t_i; \theta_{i+1}, \theta_i) \Delta t]) + \mathcal{O}(\Delta t^2) , \quad (15.25)$$

such that

$$\begin{aligned} \int d\theta_i < \theta_{i+1} | U(t_{i+1}, t_i) | \theta_i > < \theta_i | \Psi > = \\ \int d\theta_i d\tilde{\theta}_i \exp(\tilde{\theta}_i [\theta_i - \theta_{i+1} - iH(t_i; \theta_{i+1}, \theta_i) \Delta t]) \Psi(\theta_i) + \mathcal{O}(\Delta t^2) \quad . \end{aligned} \quad (15.26)$$

which can be iterated, first by one step, to

$$\begin{aligned} \int d\theta_i < \theta_{i+2} | U(t_{i+2}, t_i) | \theta_i > < \theta_i | \Psi > &\equiv \int d\theta_{i+1} U_{i+1}(\theta_{i+2}, \theta_{i+1}) \int d\theta_i U_i(\theta_{i+1}, \theta_i) \Psi(\theta_i) \\ &= \int d\theta_{i+1} d\tilde{\theta}_{i+1} \exp(\tilde{\theta}_{i+1} [\theta_{i+1} - \theta_{i+2} - iH(t_{i+1}; \theta_{i+2}, \theta_{i+1}) \Delta t]) \times \\ &\quad \int d\theta_i d\tilde{\theta}_i \exp(\tilde{\theta}_i [\theta_i - \theta_{i+1} - iH(t_i; \theta_{i+1}, \theta_i) \Delta t]) \Psi(\theta_i) + \mathcal{O}(\Delta t^2) \quad . \end{aligned} \quad (15.27)$$

Note that we have to be careful where we put the differentials, as they are Grassmann variables themselves. If  $H$  is diagonal, as will often be the case for the application we have in mind,  $\exp(\tilde{\theta}[1_2(\theta, \theta') - iH(\theta, \theta') \Delta t])$  will be a commuting object (so-called Grassmann *even*) and it does not matter if we put one of the differentials on one or the other side of the exponential. The combination  $d\theta_i d\tilde{\theta}_i$  is likewise Grassmann even, and the pair can be shifted to any place in the expression for the path integral. Hence, provided of course  $H$  is diagonal, any change in the ordering can at most given an additional minus sign. We now apply eq. (15.24) to the above product of exponentials,

$$\begin{aligned} &\exp(\tilde{\theta}_{i+1} [\theta_{i+1} - \theta_{i+2} - iH(t_{i+1}; \theta_{i+2}, \theta_{i+1}) \Delta t]) \exp(\tilde{\theta}_i [\theta_i - \theta_{i+1} - iH(t_i; \theta_{i+1}, \theta_i) \Delta t]) \quad (15.28) \\ &= \exp\left(\sum_{j=i}^{i+1} \tilde{\theta}_j \{\theta_j - \theta_{j+1} - iH(t_j; \theta_{j+1}, \theta_j) \Delta t\} - \frac{1}{2} \Delta t^2 [\tilde{\theta}_{i+1} H(t_{i+1}; \theta_{i+2}, \theta_{i+1}), \tilde{\theta}_i H(t_i; \theta_{i+1}, \theta_i)]\right) \end{aligned}$$

and evaluate the commutator that appears in the exponent. With the explicit expression for  $H$  (see eq. (15.20)) we find

$$\int d\theta_{i+1} [\tilde{\theta}_{i+1} H(t_{i+1}; \theta_{i+2}, \theta_{i+1}), \tilde{\theta}_i H(t_i; \theta_{i+1}, \theta_i)] = 2\tilde{\theta}_{i+1}\tilde{\theta}_i (\theta_{i+2} H_{21}(t_{i+1}) H_{12}(t_i) - \theta_i H_{12}(t_{i+1}) H_{21}(t_i)) . \quad (15.29)$$

For  $H$  diagonal the commutator term vanishes as was to be expected. In that case, the Campbell-Baker-Hausdorff formula truncates to the trivial term both in the matrix and in the Grassmann representations. This does not mean that there are no discretisation errors in the fermionic path integral when  $H$  is diagonal, as can be seen from eq. (15.19). To be precise, assuming for simplicity that  $H = -\frac{1}{2}W\sigma_3$ , one has

$$\exp(-iH\Delta t) = 1_2 - i\left\{H - \frac{i}{2}W \tan(\frac{1}{4}W\Delta t)1_2\right\} \frac{2\sin(\frac{1}{2}W\Delta t)}{W} , \quad (15.30)$$

which shows that  $\Delta t$  is effectively modified to  $2\sin(\frac{1}{2}W\Delta t)/W$  (it is interesting to contrast this with the result we found for the harmonic oscillator in section 6 and problem 10), whereas  $H$  is shifted by a multiple of the identity that vanishes linearly in  $\Delta t$ .

The generalisation of eqs. (15.27) and (15.28) to  $N$  steps is now obvious and for  $H$  diagonal one easily proves that the limit  $N \rightarrow \infty$  can be taken:

$$\begin{aligned} \int d\theta <\theta'|U(T,0)|\theta><\theta|\Psi> = \lim_{N \rightarrow \infty} \int d\theta_0 U_N(\theta_N = \theta', \theta_0 = \theta) \Psi(\theta_0) \\ = \lim_{N \rightarrow \infty} \prod_{j=0}^{N-1} \int d\theta_j d\tilde{\theta}_j \exp\left(\sum_{j=0}^{N-1} \tilde{\theta}_j [\theta_j - \theta_{j+1} - iH(t_j; \theta_{j+1}, \theta_j)\Delta t]\right) \Psi(\theta_0) , \end{aligned} \quad (15.31)$$

where as usual one has  $\Delta t = T/N$ . In complete analogy with eq. (6.10), reinstating the dependence on Planck's constant, we can write

$$\begin{aligned} <\theta'|T \exp(-i \int H(t)dt/\hbar)|\theta> \\ = \lim_{N \rightarrow \infty} \int d\tilde{\theta}_0 \prod_{j=1}^{N-1} \int d\theta_j d\tilde{\theta}_j \exp\left[\frac{i\Delta t}{\hbar} \sum_{j=0}^{N-1} \left(\frac{i\tilde{\theta}_j(\theta_{j+1} - \theta_j)}{\Delta t} - \tilde{\theta}_j H(t_j; \theta_{j+1}, \theta_j)\right)\right] \\ = \lim_{N \rightarrow \infty} \int d\tilde{\theta}_0 \prod_{j=1}^{N-1} \int d\theta_j d\tilde{\theta}_j \exp\left[\frac{i\Delta t}{\hbar} \sum_{j=0}^{N-1} \left(\frac{i\tilde{\theta}_j(\theta_{j+1} - \theta_j)}{\Delta t} + \tilde{\theta}_j W(t_j) \frac{(\theta_{j+1} + \theta_j)}{2}\right)\right] \\ \equiv \int \mathcal{D}\bar{\Psi}(t) \mathcal{D}\Psi(t) \exp\left[i \int_0^T dt \Psi^\dagger(t)(i\partial_t + W(t))\Psi(t)\right] . \end{aligned} \quad (15.32)$$

Here we have identified  $\tilde{\theta}$  with  $\Psi^\dagger$  (which in this case agrees with  $\bar{\Psi}$ , since in one time and no space dimensions  $\gamma_0 \equiv 1$ ) and  $\theta$  with  $\Psi$ . We have indicated the general case where  $W$  can depend on time, but in absence of this time dependence,  $W$  is the energy of the one-particle state, created from the vacuum. Since we identified  $\theta$  also with  $b^\dagger$ , the creation operator for the one-particle state, we see that  $W\Psi^\dagger\Psi = Wbb^\dagger = -H + \frac{1}{2}W$ , what is to be expected from a relation between the Lagrangian and Hamiltonian (the term  $\frac{1}{2}W$  is of course irrelevant) .

As for scalar field theories, we will be mainly interested in vacuum expectation values of the evolution operator. In the presence of a source term this will allow us to derive all required matrix elements. For the present case we easily find the vacuum wave function to be

$$<\theta|0> = 1 \quad , \quad <0|\theta> = \theta \quad , \quad (15.33)$$

such that

$$\langle 0|U(T)|0 \rangle = \int d\theta' \langle 0|\theta' \rangle \int d\theta \langle \theta'|U(T)|\theta \rangle \langle \theta|0 \rangle = \int d\theta d\theta' \langle \theta'|U(T)|\theta \rangle \quad (15.34)$$

The order of the differentials is important here. However, as in the case of scalar field theories, we do not require the precise form of the vacuum wave function(al) for performing perturbation theory.

Up to now we have described a spin one-half particle pinned-down at a fixed position. It is obvious how this can be generalised to include the quantum mechanical description of a moving spin one-half particle in a one-dimensional potential  $V(x)$ . If also  $W$  depends on the particle position ( $W(x)$ ), the Hamiltonian becomes

$$H = \left( \frac{\hat{p}^2}{2m} + V(\hat{x}) \right) 1_2 - \frac{1}{2} W(\hat{x}) \sigma_3 \quad (15.35)$$

If we write  $\langle x, \theta | \Psi \rangle \equiv \Psi(x, \theta) \equiv a_0(x) + \theta a_1(x)$ , the path integral is easily found to be

$$\begin{aligned} \langle x', \theta' | \exp(-iHT/\hbar) | x, \theta \rangle &= \lim_{N \rightarrow \infty} \int \frac{dp_0 d\tilde{\theta}_0}{2\pi\hbar} \prod_{j=1}^{N-1} \int \frac{dp_j dx_j}{2\pi\hbar} d\theta_j d\tilde{\theta}_j \times \\ &\exp \left[ \frac{i\Delta t}{\hbar} \sum_{j=0}^{N-1} \left( \frac{p_j(x_{j+1} - x_j)}{\Delta t} - \frac{p_j^2}{2m} - V(x_j) + \frac{i\tilde{\theta}_j(\theta_{j+1} - \theta_j)}{\Delta t} + \tilde{\theta}_j W(x_j) \frac{(\theta_{j+1} + \theta_j)}{2} \right) \right] \\ &\equiv \int \mathcal{D}x(t) \mathcal{D}\bar{\Psi}(t) \mathcal{D}\Psi(t) \exp \left[ \frac{i}{\hbar} \int_0^T dt \left( \frac{1}{2} m \dot{x}^2(t) - V(x(t)) + \bar{\Psi}^\dagger(t) [i\partial_t + W(x(t))] \Psi(t) \right) \right] \end{aligned} \quad (15.36)$$

A careful derivation of this formula and a step by step comparison with the matrix representation in the spin degrees of freedom can be found in sections 1 to 3 of the paper “Fermionic coordinates and supersymmetry in quantum mechanics”, Nuclear Physics **B196** (1982) 509, by P. Salomonson and J.W. van Holten. For further details see the lectures by L. Faddeev in “Methods in field theory”, Les Houches 1975, ed. R. Balian and J. Zinn-Justin.

It is now also straightforward to derive the path integral for the Dirac Hamiltonian of the previous section. Using as a basis the plane waves constructed there, the Hamiltonian becomes a decoupled sum (in a finite volume) for each  $\vec{k}$  of four fermions, described by  $b_1^\dagger(\vec{k})$ ,  $b_2^\dagger(\vec{k})$ ,  $d_1^\dagger(\vec{k})$  and  $d_2^\dagger(\vec{k})$ . Each of which can be described by its own  $\theta$ . Associating the annihilation operators with their respective  $\bar{\theta}$ , and performing the Fourier transformation back to coordinate space, it is left as an exercise to show that the path integral for fermions is given by

$$\int \mathcal{D}\bar{\Psi}(x) \mathcal{D}\Psi(x) \exp \left[ \frac{i}{\hbar} \int d_4x \left( \bar{\Psi}(x) (i\gamma^\mu \partial_\mu - m) \Psi(x) - \bar{\mathcal{J}}(x) \Psi(x) - \bar{\Psi}(x) \mathcal{J}(x) \right) \right] \quad (15.37)$$

Since the fields  $\Psi(x)$  and  $\bar{\Psi}(x)$  are Grassmann variables, also the sources  $\mathcal{J}(x)$  and  $\bar{\mathcal{J}}(x)$  are Grassmann variables. Their order in the above equation is therefore important, when used in further calculations. As promised, it is as simple as for scalar field theories to calculate the dependence of this path integral on the sources. Since Grassmann variables form a complex linear space we can perform all calculations as in the scalar case, provided we keep track of the order of the Grassmann odd variables. In particular we can make the replacement

$$\begin{aligned} \Psi(x) &\rightarrow \Psi(x) + \int d_4x' G_F(x - x') \mathcal{J}(x') \quad , \\ \bar{\Psi}(x) &\rightarrow \bar{\Psi}(x) + \int d_4x' \bar{\mathcal{J}}(x') G_F(x' - x) \quad , \end{aligned} \quad (15.38)$$



where  $G_F(x)$  is the Green's function defined in eq. (14.9). The integration measure, as for commuting variables, is invariant under a shift by a constant Grassmann variable, such that we obtain (as for the scalar case we normalise the path integral to 1 for vanishing sources)

$$\begin{aligned} \langle 0|U_{\mathcal{J}\bar{\mathcal{J}}}(T)|0\rangle &= \langle 0|U_{\mathcal{J}=\bar{\mathcal{J}}=0}(T)|0\rangle \exp\left(-i\int d_4x d_4y \bar{\mathcal{J}}(x)G_F(x-y)\mathcal{J}(y)\right) \\ &\equiv \langle 0|U_{\mathcal{J}=\bar{\mathcal{J}}=0}(T)|0\rangle Z_2(\mathcal{J},\bar{\mathcal{J}}) \quad . \end{aligned} \quad (15.39)$$

Again, this result holds to arbitrary order in the sources, and agrees with what can be derived to second order within Hamiltonian perturbation theory (see problem 24).

Interactions are taken into account by adding higher order terms to the Lagrangian, where the order of the fermion fields is important. For example, the Lagrangian for a fermionic and a scalar field is given by

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_2 - V(\bar{\Psi}, \Psi, \sigma) - J(x)\sigma(x) - \bar{\mathcal{J}}(x)\Psi(x) - \bar{\Psi}(x)\mathcal{J}(x) \quad , \\ \mathcal{L}_2 &= \bar{\Psi}(x)(i\gamma^\mu\partial_\mu - m)\Psi(x) + \tfrac{1}{2}\partial_\mu\sigma(x)\partial^\mu\sigma(x) - \tfrac{1}{2}M^2\sigma^2(x) \quad . \end{aligned} \quad (15.40)$$

We find as in eq. (8.6)

$$\begin{aligned} Z(\mathcal{J}, \bar{\mathcal{J}}, J, g_n) &= \exp\left(-i\int d_4x V\left(\frac{-i\delta}{\delta\mathcal{J}(x)}, \frac{i\delta}{\delta\bar{\mathcal{J}}(x)}, \frac{i\delta}{\delta J(x)}\right)\right) Z_2(\mathcal{J}, \bar{\mathcal{J}}, J) \quad , \\ Z_2(\mathcal{J}, \bar{\mathcal{J}}, J) &= \exp\left[-i\int d_4x d_4y \left(\tfrac{1}{2}J(x)G(x-y)J(y) + \bar{\mathcal{J}}(x)G_F(x-y)\mathcal{J}(y)\right)\right] \quad . \end{aligned} \quad (15.41)$$

Note the minus sign for the derivative with respect to  $\mathcal{J}(x)$ , which is because the source stands behind the field component  $\bar{\Psi}(x)$ . Derivatives of Grassmann variables are simply defined as one would intuitively expect

$$\frac{d}{d\theta} 1 = 0 \quad , \quad \frac{d}{d\theta} \theta = 1 \quad , \quad \frac{d}{d\theta} \theta' = 0 \quad , \quad (15.42)$$

together with a generalised Leibnitz rule for functions  $f$  and  $g$  that are either even or odd Grassmann variables, a property denoted by the *sign* or *grading*  $s(=\pm 1)$ ,

$$\frac{d}{d\theta} (fg) = s_f f \frac{d}{d\theta} g + \left(\frac{d}{d\theta} f\right) g \quad . \quad (15.43)$$

By declaring the derivative to be a linear function on the Grassmann algebra, it can be uniquely extended to this algebra from the above set of rules. Note that these rules imply that the Grassmann integral over a total Grassmann derivative vanishes. Comparing with eq. (15.21) we note that apparently Grassmann integration and differentiation is one and the same thing. Both project on the coefficient in front of the Grassmann variable. The vanishing of the integral over a total derivative and of the derivative of an integral is in that perspective trivial. More importantly, to make sense of eq. (15.41), one easily shows the following identity to hold

$$\frac{d}{d\theta} \exp(\theta x) = x \quad . \quad (15.44)$$

An example of an interaction between the fermions and a scalar field  $\sigma$  is given by the so-called Yukawa interaction

$$V(\bar{\Psi}(x), \Psi(x), \sigma(x)) = g\bar{\Psi}(x)\Psi(x)\sigma(x) \quad . \quad (15.45)$$

We can also consider the interaction of the fermions with the electromagnetic field, whose quantisation will be undertaken in the next section. For this we can take the minimal coupling in eq. (14.12) (see also eq. (14.10)), such that

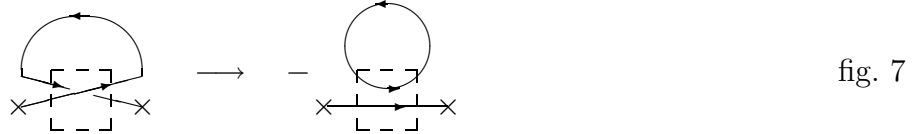
$$V(\bar{\Psi}(x), \Psi(x), A_\mu(x)) = -e\bar{\Psi}(x)\gamma^\mu\Psi(x)A_\mu(x) \quad , \quad (15.46)$$

which will play a dominating role in describing Quantum Electrodynamics (QED).

As before,  $\log Z(\bar{\mathcal{J}}, \mathcal{J}, J)$  is the sum over connected diagrams. Diagrams that involve fermions necessarily have as many lines ending in a source  $\mathcal{J}$  as in a source  $\bar{\mathcal{J}}$ . This is because the Lagrangian is Grassmann even, a requirement that can be related to the Lorentz invariance. It does not in general require the Lagrangian to be bilinear in  $\bar{\Psi}$  and  $\Psi$ . In section 19 and problem 30 we will discuss the four-Fermi interaction,  $\bar{\Psi}(x)\gamma^\mu\Psi(x)\bar{\Psi}(x)\gamma_\mu\Psi(x)$ , which is clearly Lorentz invariant and Grassmann even. However, for many of the theories we discuss, the Lagrangian is bilinear in the fermionic fields, because higher order terms will generally not be renormalisable (except in one space and one time dimension). If no higher order fermionic interactions occur, a fermionic line either forms a loop or it goes from a source  $\mathcal{J}$  to a source  $\bar{\mathcal{J}}$ . As changing the order of fermionic fields and sources gives a sign change, this has consequences for the Feynman diagrams too. However, it is cumbersome to determine the overall sign of a diagram. Fortunately, all we need is the relative sign of the various diagrams that contribute to the Green's function with a fixed number of sources, since the overall sign drops out in our computations of cross sections and decay rates. If one diagram can be obtained from the other by crossing two fermion lines, this gives a relative minus sign, as for



It also implies that each loop formed by a fermion line carries a minus sign. Intuitively this follows, as is indicated in the following figure by the dashed box, from the identity above



More accurately a fermion loop that connects vertices  $x_k$  for  $k = 1$  to  $n$  is associated to

$$\prod_{k=1}^n \bar{\Psi}(x_k)\Psi(x_k) \rightarrow \prod_{k=1}^n \left( \frac{-i\delta}{\delta\mathcal{J}(x_k)} \frac{i\delta}{\delta\bar{\mathcal{J}}(x_k)} \right) \rightarrow - \sum_{\{k\}} \text{Tr} \left[ iG_F(x_{k(1)} - x_{k(2)}) iG_F(x_{k(2)} - x_{k(3)}) \cdots iG_F(x_{k(n)} - x_{k(1)}) \right] \quad , \quad (15.47)$$

where  $\{k\}$  stands for the various orders in which the vertices are connected. For each vertex we have only indicated the fermionic part  $\bar{\Psi}(x_k)\Psi(x_k)$ . For the examples of eqs. (15.45) and (15.46), the scalar or vector field contributions are not indicated, as they are not relevant for the fermion loop. The trace is with respect to the spinor indices which are not displayed explicitly to keep the notation simple. We used that

$$\Psi(x_{k(j)})\bar{\Psi}(x_{k(j+1)}) \rightarrow \frac{i\delta}{\delta\bar{\mathcal{J}}(x_{k(j)})} \frac{-i\delta}{\delta\mathcal{J}(x_{k(j+1)})} \rightarrow iG_F(x_{k(j)} - x_{k(j+1)}) \quad , \quad (15.48)$$

where an extra minus sign arises since in  $\log Z_2(\mathcal{J}, \bar{\mathcal{J}})$  (see eq. (15.39))  $\bar{\mathcal{J}}$  comes first, and has to be anticommutated with  $\delta/\delta\mathcal{J}$ , before this derivative can be taken. The overall minus sign comes from the term that closes the loop

$$\bar{\Psi}(x_{k(1)})A\Psi(x_{k(n)}) = -A\Psi(x_{k(n)})\bar{\Psi}(x_{k(1)}) \quad , \quad (15.49)$$

where  $A$  is Grassmann even. We contrast this, as an example, with a scalar loop for the field  $\varphi$  that arises in the theory discussed in section 11, which is described by the Lagrangian  $\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi)^2 - \frac{1}{2}m^2\varphi^2 + \frac{1}{2}(\partial_\mu\sigma)^2 - \frac{1}{2}M^2\sigma^2 - \frac{1}{2}\varphi^2\sigma - J\sigma - j\varphi$ . One finds  $(k(n+1) \equiv k(1))$

$$\prod_{k=1}^n \varphi(x_k)\varphi(x_k) \rightarrow \prod_{k=1}^n \left( \frac{i\delta}{\delta J(x_k)} \frac{i\delta}{\delta J(x_k)} \right) \rightarrow \sum_{\{k\}} \prod_{j=1}^n iG(x_{k(j)} - x_{k(j+1)}) \quad , \quad (15.50)$$

which completes the demonstration of the extra minus sign for fermion loops. Note that the factors of  $i$ , associated to the derivatives with respect to the sources, are absorbed in the vertices for the Feynman rules in the table on page 35, which is why the propagator in that table equals  $-i$  times the Green's function. This is also the Feynman rule for the fermion propagator. But the extra minus sign in the derivative with respect to the fermionic source  $\mathcal{J}$  (see eq. (15.41)) is *not* absorbed in the vertex in order to guarantee that vertex factors are assigned as in the scalar theory. That minus sign is, however, absorbed in eq. (15.48) due to the anticommuting nature of the fermionic variables, which was in the first place the reason for the extra minus sign in eq. (15.41) to appear. Only the overall minus sign required in closing a fermion loop remains as an extra factor.

Before we convert these Feynman rules to the ones involved in computing the scattering matrix, cross sections and decay rates (see pg. 42 and eq. (9.25)), we need to determine the wavefunction factors to be inserted for the external lines that correspond to the in- and outgoing fermion lines. For this we express the creation and annihilation operators in terms of the fermionic fields (at  $t = 0$ ), such that their insertion in the operator formulation can be converted in the path integral, as in the scalar case, to derivatives with respect to the sources. Using eq. (13.17) and the orthogonality relations of eq. (13.14) one finds (cmp. eq. (7.26))

$$b_\alpha(\vec{k}) = \frac{u^{(\alpha)}(\vec{k})^\dagger}{\sqrt{2k_0(\vec{k})}} \int \frac{d_3\vec{x}}{\sqrt{(2\pi)^3}} e^{-i\vec{k}\cdot\vec{x}} \hat{\Psi}(\vec{x}) \quad , \quad d_\alpha^\dagger(\vec{k}) = \frac{v^{(\alpha)}(\vec{k})^\dagger}{\sqrt{2k_0(\vec{k})}} \int \frac{d_3\vec{x}}{\sqrt{(2\pi)^3}} e^{i\vec{k}\cdot\vec{x}} \hat{\Psi}(\vec{x}) \quad (15.51)$$

and through conjugation we get

$$b_\alpha^\dagger(\vec{k}) = \int \frac{d_3\vec{x}}{\sqrt{(2\pi)^3}} e^{i\vec{k}\cdot\vec{x}} \hat{\Psi}^\dagger(\vec{x}) \frac{u^{(\alpha)}(\vec{k})}{\sqrt{2k_0(\vec{k})}} \quad , \quad d_\alpha(\vec{k}) = \int \frac{d_3\vec{x}}{\sqrt{(2\pi)^3}} e^{-i\vec{k}\cdot\vec{x}} \hat{\Psi}^\dagger(\vec{x}) \frac{v^{(\alpha)}(\vec{k})}{\sqrt{2k_0(\vec{k})}} \quad . \quad (15.52)$$

In the Hamiltonian formulation the scattering matrix is given by

$$\begin{aligned} &_{\text{out}} \langle (\vec{p}_1, \alpha_1), (\vec{p}_2, \alpha_2), \dots, (\vec{p}_\ell, \alpha_\ell) | (\vec{k}_1, \beta_1), (\vec{k}_2, \beta_2), \dots, (\vec{k}_n, \beta_n) \rangle_{\text{in}} = \\ & \quad < 0 | c_{\alpha_1}(\vec{p}_1) c_{\alpha_2}(\vec{p}_2) \dots c_{\alpha_\ell}(\vec{p}_\ell) U(T_{\text{out}}, T_{\text{in}}) c_{\beta_1}^\dagger(\vec{k}_1) c_{\beta_2}^\dagger(\vec{k}_2) \dots c_{\beta_n}^\dagger(\vec{k}_n) | 0 \rangle , \end{aligned} \quad (15.53)$$

where in a short-hand notation we separate particles from antiparticles by the helicity index

$$c_1(\vec{k}) \equiv b_1(\vec{k}) \quad , \quad c_2(\vec{k}) \equiv b_2(\vec{k}) \quad , \quad c_3(\vec{k}) \equiv d_1(\vec{k}) \quad , \quad c_4(\vec{k}) \equiv d_2(\vec{k}) \quad . \quad (15.54)$$

Like in eq. (9.3), the insertion of a field operator at the appropriate time is in the path integral represented by a derivative with respect to the source

$$\begin{aligned}
\hat{b}_\alpha^\dagger(\vec{k}) &= \frac{[\gamma^0 u^{(\alpha)}(\vec{k})]_a}{\sqrt{2k_0(\vec{k})(2\pi)^3}} \int d_3\vec{x} e^{i\vec{k}\cdot\vec{x}} \frac{-i\delta}{\delta\mathcal{J}_a(\vec{x}, t = T_{\text{in}})} = \frac{[\gamma^0 u^{(\alpha)}(\vec{k})]_a}{\sqrt{2k_0(\vec{k})}} \frac{-i\delta}{\delta\tilde{\mathcal{J}}_a(\vec{k}, t = T_{\text{in}})}, \\
\hat{d}_\alpha^\dagger(\vec{k}) &= \frac{v_a^{(\alpha)}(\vec{k})^*}{\sqrt{2k_0(\vec{k})(2\pi)^3}} \int d_3\vec{x} e^{i\vec{k}\cdot\vec{x}} \frac{i\delta}{\delta\tilde{\mathcal{J}}_a(\vec{x}, t = T_{\text{in}})} = \frac{v_a^{(\alpha)}(\vec{k})^*}{\sqrt{2k_0(\vec{k})}} \frac{i\delta}{\delta\tilde{\tilde{\mathcal{J}}}_a(\vec{k}, t = T_{\text{in}})}, \\
\hat{b}_\alpha(\vec{k}) &= \frac{u_a^{(\alpha)}(\vec{k})^*}{\sqrt{2k_0(\vec{k})(2\pi)^3}} \int d_3\vec{x} e^{-i\vec{k}\cdot\vec{x}} \frac{i\delta}{\delta\tilde{\mathcal{J}}_a(\vec{x}, t = T_{\text{out}})} = \frac{u_a^{(\alpha)}(\vec{k})^*}{\sqrt{2k_0(\vec{k})}} \frac{i\delta}{\delta\tilde{\tilde{\mathcal{J}}}_a(-\vec{k}, t = T_{\text{out}})}, \\
\hat{d}_\alpha(\vec{k}) &= \frac{[\gamma^0 v^{(\alpha)}(\vec{k})]_a}{\sqrt{2k_0(\vec{k})(2\pi)^3}} \int d_3\vec{x} e^{-i\vec{k}\cdot\vec{x}} \frac{-i\delta}{\delta\mathcal{J}_a(\vec{x}, t = T_{\text{out}})} = \frac{[\gamma^0 v^{(\alpha)}(\vec{k})]_a}{\sqrt{2k_0(\vec{k})}} \frac{-i\delta}{\delta\tilde{\mathcal{J}}_a(-\vec{k}, t = T_{\text{out}})},
\end{aligned} \tag{15.55}$$

such that

$$\begin{aligned} \text{out} &< (\vec{p}_1, \alpha_1), (\vec{p}_2, \alpha_2), \dots, (\vec{p}_\ell, \alpha_\ell) | (\vec{k}_1, \beta_1), (\vec{k}_2, \beta_2), \dots, (\vec{k}_n, \beta_n) >_{\text{in}} \\ &= \prod_{i=1}^{\ell} \hat{c}_{\alpha_i}(\vec{p}_i) \prod_{j=1}^n \hat{c}_{\beta_j}^\dagger(\vec{k}_j) \exp(G_{\mathcal{J}, \vec{\mathcal{J}}})|_{\mathcal{J}=\vec{\mathcal{J}}=0} \quad . \end{aligned} \quad (15.56)$$

The  $\hat{c}_\alpha(\vec{k})$  are of course defined in terms of  $\hat{b}_\alpha(\vec{k})$  and  $\hat{d}_\alpha(\vec{k})$  as in eq. (15.54). We continue as in section 9 by first fixing the wavefunction and mass renormalisations in terms of the connected two-point function (we leave it as an exercise to show that in the absence of interactions  $\text{out} \langle (\vec{p}, \alpha) | (\vec{k}, \beta) \rangle_{\text{in}} = e^{-ip_0(\vec{p})T} \delta_{\vec{k}, \vec{p}} \delta_{\alpha, \beta}$ ).

$$\begin{aligned}
G_c^{(2)}(\mathcal{J}, \bar{\mathcal{J}}) &= \text{diagram 1} + \text{diagram 2} + \text{diagram 3} + \dots \\
&\equiv \text{diagram 4} \quad , \quad (15.57)
\end{aligned}$$

where the self-energy is now a  $4 \times 4$  matrix given by  $(-i \times)$  the amputated  $1PI$  2-point function

$$i\Sigma_{ab}(p) \equiv \text{diagram with a circle containing diagonal lines, with an incoming line from the left labeled } b \text{ and an outgoing line to the right labeled } a. \quad (15.58)$$

(The 1PI diagram *equals*  $\Sigma_{ab}(p)$ , when evaluated with the Feynman rules for the reduced matrix elements of table 4 (pg. 69) by *dropping* the wavefunction factors.) The convention for these sort of diagrams is that momentum flows in the direction of the arrow, which points to the *first* spinor index (here  $a$ ). With these definitions the two-point function becomes

$$G_c^{(2)}(\mathcal{J}, \bar{\mathcal{J}}) = -i \int d_4p \, \tilde{\tilde{\mathcal{J}}}_a(-p) \left[ \frac{1}{\not{p} - m - \Sigma(p) + i\varepsilon} \right]_{ab} \tilde{\tilde{\mathcal{J}}}_b(p) \quad , \quad (15.59)$$

with between square brackets the inverse of the  $4 \times 4$  matrix  $(p_\mu \gamma^\mu - m - \Sigma(p) + i\varepsilon)_{ab}$ , which is the full propagator in the momentum representation for the conventions on pg. 42. As long as we don't break the Lorentz invariance, the full propagator near the poles is of the form of the free propagator with a wavefunction renormalisation factor  $Z_F$  and a renormalised mass  $\tilde{m}$ , such that on the mass-shell one has (cmp. eq. (9.12))

$$G_c^{(2)}(\mathcal{J}, \bar{\mathcal{J}}) = -iZ_F \int d_4p \, \tilde{\mathcal{J}}(-p) \left[ \frac{1}{\not{p} - \tilde{m} + i\varepsilon} \right] \tilde{\mathcal{J}}(p) \quad . \quad (15.60)$$

Performing the wavefunction renormalisation, we have to modify eqs. (15.55) and (15.56) accordingly (cmp. eqs. (9.13) and (9.14)).

$$\begin{aligned}\hat{b}_+^\alpha(\vec{k}) &= \frac{[\gamma^0 u^{(\alpha)}(\vec{k})]_a}{\sqrt{2Z_F\sqrt{\vec{k}^2 + \tilde{m}^2}}} \frac{-i\delta}{\delta\tilde{\mathcal{J}}_a(\vec{k}, t = T_{\text{in}})} \quad , \quad \hat{b}_-^\alpha(\vec{k}) = \frac{u_a^{(\alpha)}(\vec{k})^*}{\sqrt{2Z_F\sqrt{\vec{k}^2 + \tilde{m}^2}}} \frac{i\delta}{\delta\tilde{\mathcal{J}}_a(-\vec{k}, t = T_{\text{out}})} \quad , \\ \hat{d}_+^\alpha(\vec{k}) &= \frac{v_a^{(\alpha)}(\vec{k})^*}{\sqrt{2Z_F\sqrt{\vec{k}^2 + \tilde{m}^2}}} \frac{i\delta}{\delta\tilde{\mathcal{J}}_a(\vec{k}, t = T_{\text{in}})} \quad , \quad \hat{d}_-^\alpha(\vec{k}) = \frac{[\gamma^0 v^{(\alpha)}(\vec{k})]_a}{\sqrt{2Z_F\sqrt{\vec{k}^2 + \tilde{m}^2}}} \frac{-i\delta}{\delta\tilde{\mathcal{J}}_a(-\vec{k}, t = T_{\text{out}})} \quad ,\end{aligned}\tag{15.61}$$

and, with the  $\hat{c}_\pm^\alpha(\vec{k})$  defined as in eq. (15.54),

$$\begin{aligned}\text{out} < (\vec{p}_1, \alpha_1), (\vec{p}_2, \alpha_2), \dots, (\vec{p}_\ell, \alpha_\ell) | (\vec{k}_1, \beta_1), (\vec{k}_2, \beta_2), \dots, (\vec{k}_n, \beta_n) >_{\text{in}} \\ = \prod_{i=1}^{\ell} \hat{c}_-^{\alpha_i}(\vec{p}_i) \prod_{j=1}^n \hat{c}_+^{\beta_j}(\vec{k}_j) \exp(G_{\mathcal{J}, \bar{\mathcal{J}}})|_{\mathcal{J}=\bar{\mathcal{J}}=0} \quad .\end{aligned}\tag{15.62}$$

To compute the wavefunction factors for the external lines we express the  $n$ -point function in terms of amputated  $n$ -point functions, as in eq. (9.15), with the difference that there has to be an even number of external lines, since the number of  $\mathcal{J}$  and  $\bar{\mathcal{J}}$  sources has to be equal (we ignore for the moment any other bosonic fields that might be present, including those in the external lines will be obvious). The amputated  $n$ -point function will now carry the spinor index of each of the external lines and one has

$$\begin{aligned}G_c^{(2n)}(\mathcal{J}, \bar{\mathcal{J}}) &\equiv \int \prod_{j=1}^n d_4 p_j d_4 k_j G_c^{(\text{amp})}(p_1, p_2, \dots, p_n; k_1, k_2, \dots, k_n)_{a_1, a_2, \dots, a_n}^{b_1, b_2, \dots, b_n} \times \\ &\prod_{j=1}^n \left\{ \tilde{\mathcal{J}}_{c_j}(p_j) \left[ \frac{-i}{-\not{p}_j - m - \Sigma(-p_j) + i\varepsilon} \right]_{c_j b_j} \left[ \frac{-i}{\not{k}_j - m - \Sigma(k_j) + i\varepsilon} \right]_{a_j d_j} \tilde{\mathcal{J}}_{d_j}(k_j) \right\} \quad .\end{aligned}\tag{15.63}$$

Note that we have assumed one particular ordering for the sources. Relative signs of the diagrams are determined by the rules that were described above (it is not difficult to convince oneself that with respect to the fermion lines, any diagram can be generated from a given one by permuting fermion lines). As in eqs. (9.20) and (9.21) we can compute the action of  $\hat{c}_\pm^\alpha(\vec{k})$ , from which the wavefunction factors will be obtained. Like in eq. (9.19), computing the action of  $\hat{b}_-^\alpha(\vec{p})$  and  $\hat{d}_+^\alpha(\vec{p})$  we can restrict our attention to

$$\begin{aligned}A(\dots)^\dots &\equiv \int d_4 p \tilde{\mathcal{J}}_a(p) \left[ \frac{-i}{-\not{p} - m - \Sigma(-p) + i\varepsilon} \right]_{ab} G_c^{(\text{amp})}(p, \dots)^{b\dots} = \\ &\int \frac{d_4 p}{\sqrt{2\pi}} \int dt \tilde{\mathcal{J}}_a(\vec{p}, t) \left[ \frac{-ie^{ip_0 t}}{-\not{p} - m - \Sigma(-p) + i\varepsilon} \right]_{ab} G_c^{(\text{amp})}(p, \dots)^{b\dots} \quad .\end{aligned}\tag{15.64}$$

If we define as usual  $p_0(\vec{p}) = \sqrt{\vec{p}^2 + \tilde{m}^2}$  and for convenience of notation change  $p$  to  $-p$  in eq. (15.64), we find

$$\begin{aligned}\hat{b}_-^\alpha(\vec{p}) A(\dots)^\dots &= \frac{u_a^{(\alpha)}(\vec{p})^*}{\sqrt{4\pi p_0(\vec{p}) Z_F}} \int d p_0 \left[ \frac{e^{-ip_0 T_{\text{out}}}}{\not{p} - m - \Sigma(p) + i\varepsilon} \right]_{ab} G_c^{(\text{amp})}(-p, \dots)^{b\dots} = \\ &-i\sqrt{(2\pi)^4 Z_F} u_a^{(\alpha)}(\vec{p})^* \frac{(\not{p} + \tilde{m})_{ab}}{2p_0(\vec{p})} \frac{G_c^{(\text{amp})}(-p, \dots)^{b\dots}}{\sqrt{2p_0(\vec{p})(2\pi)^3}} e^{-ip_0(\vec{p}) T_{\text{out}}} \quad ,\end{aligned}\tag{15.65}$$

which is obtained by deforming the  $p_0$  integration contour to the lower half-plane (since  $T_{\text{out}} \rightarrow \infty$ ), and computing the contribution from the pole at  $p_0 = p_0(\vec{p})$ , taking into account eq. (15.60). Its residue is proportional to the matrix  $Z_F(\not{p} + \tilde{m})/2p_0(\vec{p})$ , which is most easily found using  $1/(\not{p} - \tilde{m} + i\varepsilon) = (\not{p} + \tilde{m})/(p^2 - \tilde{m}^2 + i\varepsilon)$ . We can now use the fact that the spinor  $u^{(\alpha)}(\vec{p})$  satisfies the equations of motion (see eq. (13.2)), such that

$$u_a^{(\alpha)}(\vec{p})^* \frac{(\not{p} + \tilde{m})_{ab}}{2p_0(\vec{p})} = \left( \frac{(\not{p} + \tilde{m})^\dagger}{2p_0(\vec{p})} u^{(\alpha)}(\vec{p}) \right)_b^* = \left( \frac{(\gamma^0 \not{p} \gamma^0 + \tilde{m})}{2p_0(\vec{p})} u^{(\alpha)}(\vec{p}) \right)_b^* = \overline{u^{(\alpha)}(\vec{p})}_b. \quad (15.66)$$

Consequently, eq.(15.65) becomes

$$\hat{b}_-^\alpha(\vec{p}) A(\cdots)_{\cdots} = -i \overline{u^{(\alpha)}(\vec{p})}_b \sqrt{(2\pi)^4 Z_F} \frac{G_c^{(\text{amp})}(-p, \cdots)_{\cdots}^{b\cdots}}{\sqrt{2p_0(\vec{p})(2\pi)^3}} e^{-ip_0(\vec{p})T_{\text{out}}} \quad (15.67)$$

The wavefunction factor for an *outgoing* electron is therefore given by  $\overline{u^{(\alpha)}(\vec{p})}_b \sqrt{Z_F}$ . The convention is that the momentum flows out of the diagram, along the arrow (see the table below), this is why the amputated  $n$ -point function has  $-p$  as its argument, like for the scalar case, where we defined for the amputated  $n$ -point function all momenta to flow into the diagram. This means that in the reduced matrix element  $\mathcal{M}$  the outgoing electron momenta occur precisely as indicated in eq. (9.18).

By similar arguments we obtain from eq. (15.64)

$$\begin{aligned} \hat{d}_+^\alpha(\vec{p}) A(\cdots)_{\cdots} &= \frac{v_a^{(\alpha)}(\vec{p})^*}{\sqrt{4\pi p_0(\vec{p}) Z_F}} \int dp_0 \left[ \frac{e^{ip_0 T_{\text{in}}}}{-\not{p} - m - \Sigma(-p) + i\varepsilon} \right]_{ab} G_c^{(\text{amp})}(p, \cdots)_{\cdots}^{b\cdots} = \\ &= -i \sqrt{(2\pi)^4 Z_F} v_a^{(\alpha)}(\vec{p})^* \frac{(\not{p} + \tilde{m})_{ab}}{2p_0(\vec{p})} \frac{G_c^{(\text{amp})}(p, \cdots)_{\cdots}^{b\cdots}}{\sqrt{2p_0(\vec{p})(2\pi)^3}} e^{ip_0(\vec{p})T_{\text{in}}} = \\ &= i \overline{v^{(\alpha)}(\vec{p})}_b \sqrt{(2\pi)^4 Z_F} \frac{G_c^{(\text{amp})}(p, \cdots)_{\cdots}^{b\cdots}}{\sqrt{2p_0(\vec{p})(2\pi)^3}} e^{ip_0(\vec{p})T_{\text{in}}} \quad , \end{aligned} \quad (15.68)$$

such that the wavefunction factor for an *incoming* antiparticle (positron) is  $-\overline{v^{(\alpha)}(\vec{p})}_b \sqrt{Z_F}$ . In this case the momentum flows *against* the arrow of the fermion line, but does flow into the diagram as is required in the convention of the reduced matrix element.

To compute the action of  $\hat{b}_+^\alpha(\vec{p})$  and  $\hat{d}_-^\alpha(\vec{p})$ , we restrict our attention to

$$\begin{aligned} B(\cdots)_{\cdots} &\equiv \int d_4p G_c^{(\text{amp})}(p, \cdots)_{b\cdots} \left[ \frac{-i}{\not{p} - m - \Sigma(p) + i\varepsilon} \right]_{ba} \tilde{\mathcal{J}}_a(p) = \\ &= \int \frac{d_4p}{\sqrt{2\pi}} \int dt G_c^{(\text{amp})}(p, \cdots)_{b\cdots} \left[ \frac{-ie^{ip_0 t}}{\not{p} - m - \Sigma(p) + i\varepsilon} \right]_{ba} \tilde{\mathcal{J}}_a(\vec{p}, t) \quad . \end{aligned} \quad (15.69)$$

One finds

$$\begin{aligned} \hat{b}_+^\alpha(\vec{p}) B(\cdots)_{\cdots} &= \left\{ \int dp_0 G_c^{(\text{amp})}(p, \cdots)_{b\cdots} \left[ \frac{e^{ip_0 T_{\text{in}}}}{\not{p} - m - \Sigma(p) + i\varepsilon} \right]_{ba} \right\} \frac{[\gamma^0 u^{(\alpha)}(\vec{p})]_a}{\sqrt{4\pi p_0(\vec{p}) Z_F}} = \\ &= -i \sqrt{(2\pi)^4 Z_F} \frac{G_c^{(\text{amp})}(p, \cdots)_{b\cdots}}{\sqrt{2p_0(\vec{p})(2\pi)^3}} \frac{(\not{p} + \tilde{m})_{ba}}{2p_0(\vec{p})} [\gamma^0 u^{(\alpha)}(\vec{p})]_a e^{ip_0(\vec{p})T_{\text{in}}} = \\ &= -i \sqrt{(2\pi)^4 Z_F} \frac{G_c^{(\text{amp})}(p, \cdots)_{b\cdots}}{\sqrt{2p_0(\vec{p})(2\pi)^3}} u_b^{(\alpha)}(\vec{p}) e^{ip_0(\vec{p})T_{\text{in}}} \quad , \end{aligned} \quad (15.70)$$

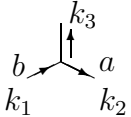
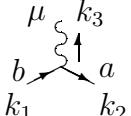
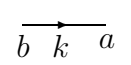
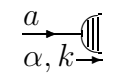
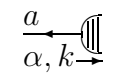
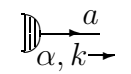
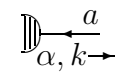
and (again for convenience changing  $p$  to  $-p$ )

$$\begin{aligned}
\hat{d}_-^\alpha(\vec{p})B(\cdots)_{b\cdots} &= \left\{ \int dp_0 G_c^{(\text{amp})}(-p, \cdots)_{b\cdots} \left[ \frac{e^{-ip_0 T_{\text{out}}}}{-\not{p} - m - \Sigma(-p) + i\varepsilon} \right]_{ba} \right\} \frac{[\gamma^0 v^{(\alpha)}(\vec{p})]_a}{\sqrt{4\pi p_0(\vec{p})Z_F}} = \\
&= -i\sqrt{(2\pi)^4 Z_F} \frac{G_c^{(\text{amp})}(-p, \cdots)_{b\cdots}}{\sqrt{2p_0(\vec{p})(2\pi)^3}} \frac{(-\not{p} + \tilde{m})_{ba}}{2p_0(\vec{p})} [\gamma^0 v^{(\alpha)}(\vec{p})]_a e^{-ip_0(\vec{p})T_{\text{in}}} = \\
&= i\sqrt{(2\pi)^4 Z_F} \frac{G_c^{(\text{amp})}(-p, \cdots)_{b\cdots}}{\sqrt{2p_0(\vec{p})(2\pi)^3}} v_b^{(\alpha)}(\vec{p}) e^{-ip_0(\vec{p})T_{\text{out}}} .
\end{aligned} \tag{15.71}$$

In both cases there is an extra minus sign from pulling  $\delta/\delta\mathcal{J}$  through  $\bar{\mathcal{J}}$  in eq. (15.63). The wavefunction factor for an *incoming* electron is hence  $u_b^{(\alpha)}(\vec{p})\sqrt{Z_F}$  with the momentum flowing along the fermionic arrow, whereas the wavefunction factor of an *outcoming* antiparticle (or positron) is given by  $-v_b^{(\alpha)}(\vec{p})\sqrt{Z_F}$ , where the momentum flows opposite to the fermionic arrow. The minus signs in front of some of the wavefunction factors are irrelevant (they can be absorbed in the overall sign ambiguity).

In the table below we summarise the Feynman rules that correspond to the fermionic pieces in computing the reduced matrix elements. We have chosen the convention that the incoming momenta flow in, and the outgoing momenta flow out of the diagram. This guarantees that eqs. (9.18), (10.12) and (11.11) (resp. the scattering matrix, cross section and decay rate), remain valid in the presence of fermions. Consequently, all fermion momenta in the table flow from left to right. For conventions where momenta always flow in the direction of the fermion arrow the four momentum for wavefunction factors associated to in- and outgoing antiparticles (positrons) should be reversed. Signs from fermion loops and exchanges of external lines will *not* be implicit in diagrams, as only *relative* signs are known.

table 4

 $\equiv g\delta_{ab} \text{ and } k_1 = k_2 + k_3$	Yukawa vertex
 $\equiv -e\gamma_{ab}^\mu \text{ and } k_1 = k_2 + k_3$	photon vertex
 $\equiv \left[ \frac{1}{\not{k} - m + i\varepsilon} \right]_{ab}$	fermion propagator
 $\equiv \sqrt{Z_F} u_a^{(\alpha)}(\vec{k}) \text{ , } k_0 = \sqrt{\vec{k}^2 + \tilde{m}^2}$	incoming electron
 $\equiv \sqrt{Z_F} \overline{v^{(\alpha)}(\vec{k})}_a \text{ , } k_0 = \sqrt{\vec{k}^2 + \tilde{m}^2}$	incoming positron
 $\equiv \sqrt{Z_F} \overline{u^{(\alpha)}(\vec{k})}_a \text{ , } k_0 = \sqrt{\vec{k}^2 + \tilde{m}^2}$	outgoing electron
 $\equiv \sqrt{Z_F} v_a^{(\alpha)}(\vec{k}) \text{ , } k_0 = \sqrt{\vec{k}^2 + \tilde{m}^2}$	outgoing positron
$-1 \times i \int \frac{d_4 k}{(2\pi)^4}$	for each <i>fermion</i> loop
-1	interchange of <i>fermion</i> lines

## 16 Feynman rules for vector fields

As before, the quantisation for vector fields starts by expanding the field in plane waves and identifying the Fourier coefficients with creation and annihilation operators.

$$A_\mu(x) = \int \frac{d_3\vec{k}}{\sqrt{2k_0(\vec{k})(2\pi)^3}} \sum_\lambda \left( a_\lambda(\vec{k}) \varepsilon_\mu^{(\lambda)}(\vec{k}) e^{-ikx} + a_\lambda^\dagger(\vec{k}) \varepsilon_\mu^{(\lambda)*}(\vec{k}) e^{ikx} \right) \quad , \quad (16.1)$$

where  $\varepsilon_\mu^{(\lambda)}(\vec{k}) e^{-ikx}$  are independent plane wave solutions of the equations of motion. The index  $\lambda$  enumerates the various solutions for fixed momentum. They will be identified with the spin components or helicity eigenstates of the vector.

We will first discuss the simpler case of a massive vector field, expected to describe a massive spin one particle. In problem 12 we already saw that its Lagrangian is given by

$$\mathcal{L}_A = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu - A_\mu(x) J^\mu(x) \quad , \quad (16.2)$$

and that the free equations of motion (i.e.  $J^\mu = 0$ ) are equivalent to

$$\partial_\mu A^\mu(x) = 0 \quad \text{and} \quad (\partial_\mu \partial^\mu + m^2) A_\nu(x) = 0 \quad . \quad (16.3)$$

As usual, this implies the on-shell condition  $k_0^2 = \vec{k}^2 + m^2$ , but also

$$k^\mu \varepsilon_\mu^{(\lambda)}(\vec{k}) = 0 \quad . \quad (16.4)$$

It removes one of the four degrees of freedom of a four-vector. Three independent components remain, exactly what would be required for a particle with spin one. We may for example choose

$$\varepsilon_\mu^{(\lambda)}(\vec{0}) = \delta_\mu^\lambda \quad (\lambda = 1, 2, 3) \quad . \quad (16.5)$$

in the restframe of the particle, which is extended to an arbitrary frame by applying a Lorentz boost. They satisfy the Lorentz invariant normalisation

$$g^{\mu\nu} \varepsilon_\mu^{(\lambda)}(\vec{k})^* \varepsilon_\nu^{(\lambda')}(\vec{k}) = -\delta^{\lambda\lambda'} \quad . \quad (16.6)$$

The minus sign is just a consequence of the fact that in our conventions  $g_{ij} = -\delta_{ij}$ . The spin wave functions also satisfy a completeness relation which is given by

$$\Lambda_{\mu\nu}(\vec{k}) \equiv \sum_\lambda \varepsilon_\mu^{(\lambda)}(\vec{k}) \varepsilon_\nu^{(\lambda)*}(\vec{k}) = - \left( g_{\mu\nu} - \frac{k_\mu k_\nu}{m^2} \right) \quad , \quad (16.7)$$

most easily proven in the restframe. Since the spin wave functions are by construction Lorentz vectors, the above expression forms a Lorentz tensor and its *on-shell* extension to an arbitrary frame is therefore unique. It is easily seen to project arbitrary vectors  $w_\mu$  on to vectors that satisfy  $k^\mu w_\mu = 0$ . The propagator for the massive spin one field was already computed in problem 12

$$\mu \overset{k}{\text{~~~~~}} \nu = \frac{- \left( g_{\mu\nu} - \frac{k_\mu k_\nu}{m^2} \right)}{k^2 - m^2 + i\varepsilon} = \frac{\Lambda_{\mu\nu}(\vec{k})}{k^2 - m^2 + i\varepsilon} \quad . \quad (16.8)$$

Especially for the massless spin one field (i.e. the photon field) to be discussed below, it is advantageous to decompose the spin with respect to the direction of the particle's motion,



which are called helicity eigenstates. We have helicities 0 and  $\pm 1$ , described by the spin wave functions

$$\varepsilon_\mu^{(0)}(k) = \left( \frac{|\vec{k}|}{m}, \frac{k_0}{m|\vec{k}|} \vec{k} \right) \quad , \quad \varepsilon_\mu^{(\pm)}(k) = \frac{1}{\sqrt{2}} \left( \bar{\varepsilon}_\mu^{(1)}(k) \pm i \bar{\varepsilon}_\mu^{(2)}(k) \right) \quad , \quad (16.9)$$

where  $k_\mu$ ,  $\varepsilon_\mu^{(0)}(k)$ ,  $\bar{\varepsilon}_\mu^{(1)}(k)$  and  $\bar{\varepsilon}_\mu^{(2)}(k)$  form a complete set of real orthogonal four-vectors. These new spin wave functions satisfy the same properties as in eqs. (16.6) and (16.7) and are also defined off-shell, where they satisfy

$$\Lambda_{\mu\nu}(k) \equiv \sum_\lambda \varepsilon_\mu^{(\lambda)}(k) \varepsilon_\nu^{(\lambda)}(k)^* = - \left( g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \quad , \quad (16.10)$$

Sums over  $\lambda$  will of course run over the set  $\{0, +, -\}$  in this case. We leave it as an exercise to verify that rotations over an angle  $\alpha$  around the axis pointing in the direction of  $\vec{k}$  leaves  $\varepsilon_\mu^{(0)}(k)$  invariant and transforms  $\varepsilon_\mu^{(\pm)}(k)$  to  $e^{\pm i\alpha} \varepsilon_\mu^{(\pm)}(k)$ .

The Hamiltonian for the massive spin one particles is given by

$$H = \int d_3\vec{k} \frac{1}{2} k_0(\vec{k}) \sum_\lambda \left( a_\lambda^\dagger(\vec{k}) a_\lambda(\vec{k}) + a_\lambda(\vec{k}) a_\lambda^\dagger(\vec{k}) \right) \quad . \quad (16.11)$$

which can be expressed in terms of three scalar fields  $\varphi_\lambda$ ,  $\lambda = 1, 2, 3$ , as

$$H = \int d_3\vec{k} \left( \frac{1}{2} |\tilde{\pi}_\lambda(\vec{k})|^2 + \frac{1}{2} (\vec{k}^2 + m^2) |\tilde{\varphi}_\lambda(\vec{k})|^2 \right) = \int d_3\vec{x} \left( \frac{1}{2} \pi_\lambda^2(\vec{x}) + \frac{1}{2} (\partial_i \varphi_\lambda(\vec{x}))^2 + \frac{1}{2} m^2 \varphi_\lambda^2(\vec{x}) \right) \quad . \quad (16.12)$$

where we defined, as in eqs.(2.6) and (2.7),

$$\tilde{\varphi}_\lambda(\vec{k}) = \left( a_\lambda(\vec{k}) + a_\lambda^\dagger(-\vec{k}) \right) / \sqrt{2k_0(\vec{k})} \quad , \quad \tilde{\pi}_\lambda(\vec{k}) = \frac{i}{2} \sqrt{2k_0(\vec{k})} \left( a_\lambda^\dagger(\vec{k}) - a_\lambda(-\vec{k}) \right) \quad . \quad (16.13)$$

The corresponding Lagrangian would be given by

$$\mathcal{L}_\varphi = \sum_\lambda \left( \frac{1}{2} (\partial_\mu \varphi_\lambda(x))^2 - \frac{1}{2} m^2 \varphi_\lambda^2(x) - \varphi_\lambda(x) J^{(\lambda)}(x) \right) \quad , \quad (16.14)$$

where we have added a source for each scalar field. We introduce also a field  $\sigma$  for the component of the vector field along  $k_\mu$ . Writing

$$\tilde{A}_\mu(k) \equiv \frac{1}{m} \tilde{\sigma}(k) k_\mu + \sum_\lambda \tilde{\varphi}_\lambda(k) \varepsilon_\mu^{(\lambda)}(k) \quad , \quad \tilde{J}_\mu(k) \equiv \frac{1}{m} \tilde{j}(k) k_\mu + \sum_\lambda \tilde{J}^{(\lambda)}(k) \varepsilon_\mu^{(\lambda)}(k) \quad , \quad (16.15)$$

a simple calculation shows that  $\mathcal{L}_A \equiv \mathcal{L}_\varphi - \frac{1}{2} (\partial_\mu \sigma(x))^2 - \sigma(x) j(x)$ . We see that  $\sigma$  decouples from the other components and behaves like a scalar particle with the *wrong* sign for the kinetic term. This would lead to serious inconsistencies, which are circumvented if we take  $\partial_\mu J^\mu(x) = 0$ , such that we can put  $\sigma \equiv 0$ .

It is important to realise that it is the Lorentz invariance that requires us to describe a spin one particle by a four-vector. From the point of view of the scalar degrees of freedom,  $\varphi_\lambda$ , this invariance seems to be lost when we introduce interactions in the Lagrangian of eq. (16.14). Nevertheless, if we treat  $\lambda$  as a three-vector index (in some *internal* space) and demand the interactions to be O(3) invariant with respect to this index (i.e. invariance under rigid rotations *and* reflections in the internal space), then we claim that the resulting

interactions do respect the Lorentz invariance. The reason is simple, because the O(3) invariance requires that the  $\lambda$  index is always pairwise contracted. Eqs. (16.10) and (16.15) guarantee that such a pair, written in terms of the vector field  $A_\mu(x)$ , is a Lorentz scalar as far as it concerns the dependence on  $A$ , which is sufficient if the Lagrangian is Lorentz invariant when treating  $\lambda$  as a dummy label. To eliminate the  $\sigma$  field we have to enforce  $\partial_\mu A^\mu(x) = 0$ , not only on-shell, but also off-shell. This can be achieved by adding to eq. (16.2) a term  $-\lambda(x)\partial_\mu A^\mu(x)$  (cmp. problem 8). Using

$$\int \mathcal{D}A_\mu(x) \mathcal{D}\lambda(x) \exp \left[ -i \int d_4x \lambda(x) \partial_\mu A^\mu(x) \right] = \int \mathcal{D}\tilde{A}_\mu(k) \prod_k \delta(k_\mu \tilde{A}^\mu(k)) \quad , \quad (16.16)$$

we see that the so-called Lagrange multiplier field  $\lambda(x)$  plays the role of removing the unwanted degree of freedom. Since we modified the theory off-shell, the propagator in eq. (16.8) has to be changed also, by replacing  $\Lambda(\vec{k})$  by its off-shell value  $\Lambda(k)$  (eq. (16.10)). If none of the vertices or sources couple to the  $\sigma$  field, we might just as well replace it by  $-g_{\mu\nu}$ . There are a number of other ways to eliminate the  $\sigma$  degree of freedom, see e.g. section 3-2-3 in Itzykson and Zuber. We will come back to massive vector particles in section 19.

For the computation of the scattering matrix we express the creation and annihilation operators in terms of the vector fields at  $t = 0$ , using the relations

$$\begin{aligned} a_\lambda(\vec{k}) &= -\sqrt{2k_0(\vec{k})} \varepsilon_\mu^{(\lambda)}(\vec{k})^* \int \frac{d_3\vec{x}}{\sqrt{(2\pi)^3}} \hat{A}^\mu(\vec{x}) e^{-i\vec{k}\cdot\vec{x}} \quad , \\ a_\lambda^\dagger(\vec{k}) &= -\sqrt{2k_0(\vec{k})} \varepsilon_\mu^{(\lambda)}(\vec{k}) \int \frac{d_3\vec{x}}{\sqrt{(2\pi)^3}} \hat{A}^\mu(\vec{x}) e^{i\vec{k}\cdot\vec{x}} \quad . \end{aligned} \quad (16.17)$$

which in the path integral turn into

$$\hat{a}_\lambda^\dagger(\vec{k}) = \sqrt{2k_0(\vec{k})} \varepsilon_\mu^{(\lambda)}(\vec{k}) \frac{-i\delta}{\delta \tilde{J}_\mu(\vec{k}, t = T_{\text{in}})} \quad , \quad \hat{a}_\lambda(\vec{k}) = \sqrt{2k_0(\vec{k})} \varepsilon_\mu^{(\lambda)}(-\vec{k}) \frac{-i\delta}{\delta \tilde{J}_\mu(-\vec{k}, t = T_{\text{out}})} \quad , \quad (16.18)$$

which is *identical* to eq. (9.3), when re-expressed in terms of  $J^{(\lambda)}$ . Like for scalar and fermion fields there will be a mass and wavefunction (denoted by  $Z_A$ ) renormalisation, determined through the self-energy of the vector field, which is now a Lorentz tensor of rank two. It is proportional to  $\Lambda_{\mu\nu}(k)$  (to guarantee that the scalar field  $\sigma$  introduced above decouples from the other fields; alternatively it can be seen as a consequence of the O(3) invariance with respect to the index  $\lambda$ ). We can consequently define

$$\Sigma_{\mu\nu}(p) \equiv \Lambda_{\mu\nu}(p) \Sigma_A(p) \quad . \quad (16.19)$$

The  $n$ -point Green's functions can now be written in terms of amputated Green's functions that carry four-vector indices for each external spin one line,

$$G_c^{(n)}(J_\mu) \equiv \int \left\{ \prod_{j=1}^n d_4p_j \tilde{J}^{\mu_j}(p_j) \frac{-i\Lambda_{\mu_j\nu_j}(p_j)}{p_j^2 - m^2 - \Sigma_A(p_j) + i\varepsilon} \right\} G_c^{\text{amp}}(p_1, p_2, \dots, p_n)^{\nu_1, \nu_2, \dots, \nu_n} \quad . \quad (16.20)$$

Using the fact that on-shell  $\Lambda_\mu^\nu(\vec{k}) \varepsilon_\nu^{(\lambda)}(\vec{k}) = -\varepsilon_\mu^{(\lambda)}(\vec{k})$ , we find for the *incoming* spin one line a wavefunction factor  $\sqrt{Z_A} \varepsilon_\mu^{(\lambda)}(\vec{k})$  and for the *outgoing* line a factor  $\sqrt{Z_A} \varepsilon_\mu^{(\lambda)}(\vec{k})^*$ .

For a massless spin one field (the photon) we would expect the helicity zero component of the vector field to be absent. First we have to redefine, however, what we would mean

with the zero helicity component, because eq. (16.9) is singular in the limit of zero mass. We take as our definition

$$\varepsilon_\mu^{(0)}(k) = n_\mu(k) \equiv \frac{1}{2}\sqrt{2} \left( 1, -\frac{\vec{k}}{|\vec{k}|} \right) \quad , \quad \varepsilon_\mu^{(\pm)}(k) = (0, \vec{s}_\pm(\vec{k})) \quad , \quad (16.21)$$

with  $\vec{k} \cdot \vec{s}_\pm(\vec{k}) = 0$ . These still form with  $k_\mu$  four independent four-vectors and  $\varepsilon_\mu^{(\pm)}(\vec{k})$  are still transverse polarisations. However, it is not longer true that  $k^\mu \varepsilon_\mu^{(0)}(\vec{k})$  will vanish on-shell (i.e. at  $k^2 = 0$ ). This is easily seen to imply that *on-shell*  $a^{(0)}(\vec{k}) = 0$ . In other words, on-shell there is no longitudinal component for the photon. The extra degree of freedom is removed by the gauge invariance of the massless vector field, as was discussed at the end of section 4 and in problem 9. To perform the quantisation of the theory, one can go about as in the massive case. Due to the presence of the four-vector  $n(k)$  it will be much more cumbersome to demonstrate the Lorentz invariance. In section 20 it will be shown how in principle in any gauge the path integral can be defined and that the result is independent of the chosen gauge. One could then choose a gauge that allows us to show the equivalence between the path integral and the canonical quantisation. However, it is the great advantage of the path integral formulation that calculations can be performed in a gauge in which the Lorentz invariance is manifest. The gauge most suitable for that purpose is of course the Lorentz gauge  $\partial_\mu A^\mu(x) = 0$ , see eqs. (4.21) and (4.22). The propagator is read off from eq. (4.26)

$$\mu \text{---} \text{wavy line} \text{---} \nu = \frac{-\left(g_{\mu\nu} - (1 - \frac{1}{\alpha}) \frac{k_\mu k_\nu}{k^2 + i\varepsilon}\right)}{k^2 + i\varepsilon} \equiv \frac{\Lambda_{\mu\nu}^{(\alpha)}(k)}{k^2 + i\varepsilon} \quad , \quad (16.22)$$

where  $\alpha$  is an arbitrary parameter, on which physical observables like cross sections and decay rates should not depend. It is in general *not* true anymore that the self-energy is proportional to  $\Lambda_{\mu\nu}^{(\alpha)}(k)$ , but the gauge invariance does guarantee that  $\Sigma_{\mu\nu}(k)\tilde{j}^\nu(k)$  is independent of  $\alpha$  for any conserved current, i.e.  $k_\mu \tilde{j}^\mu(k) = 0$ . It can be shown that this in general implies

$$\Sigma_{\mu\nu}(k) = \Lambda_{\mu\nu}^{(\alpha')} (k) \Sigma(k) \quad , \quad (16.23)$$

for some, possibly infinite,  $\alpha'$ . Apart from a wavefunction renormalisation ( $Z_A$ ), the gauge fixing parameter  $\alpha$  will in principle have to be renormalised too. One still has for any value of  $\alpha$  that  $\Lambda_{\mu\nu}^{(\alpha)}(k) \varepsilon_\nu^{(\pm)}(\vec{k}) = -\varepsilon_\mu^{(\pm)}(\vec{k})$ . The wavefunction factors for external photon lines are therefore identical to the ones for the massive case, except that now only two helicity states can appear. It is an important consequence of gauge invariance that unphysical degrees of freedom decouple in the physical amplitudes. It also implies that the self-energy vanishes on shell (see problem 39), such that it will *not* give rise to a renormalisation of the mass. The photon remains massless. That the gauge invariance must be crucial here is clear, as a massive photon would have one extra degree of freedom. We will come back to this point in section 19. In the table below we summarise the Feynman rules.

table 5

$\mu \text{---} \text{wavy line} \text{---} \nu$	$\equiv \frac{-\left(g_{\mu\nu} - (1 - \frac{1}{\alpha}) \frac{k_\mu k_\nu}{k^2 + i\varepsilon}\right)}{k^2 + i\varepsilon}$	photon propagator (Lorentz gauge)
$\begin{array}{c} \mu \\ \text{wavy line} \\ \lambda, k \rightarrow \end{array}$	$\equiv \sqrt{Z_A} \varepsilon_\mu^{(\lambda)}(\vec{k})$	incoming photon
$\begin{array}{c} \text{wavy line} \\ \lambda, k \rightarrow \\ \mu \end{array}$	$\equiv \sqrt{Z_A} \varepsilon_\mu^{(\lambda)}(\vec{k})^*$	outgoing photon

# 17 Quantum Electrodynamics - QED

QED is the field theory that describes the interaction between the photon and the charged fermions. In the Lorentz gauge (see eqs. (4.21) and (4.22)) the Lagrangian is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) - \frac{1}{2}\alpha(\partial_\mu A^\mu(x))^2 + \sum_f \bar{\Psi}_f(x)(i\gamma^\mu D_\mu - m_f)\Psi_f(x) \quad . \quad (17.1)$$

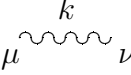
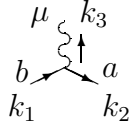
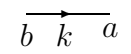
Here  $f$  is the so-called flavour index, which distinguishes the various types of fermions (electrons, protons, etc.). The covariant derivative  $D_\mu$  is given as before (see eqs. (3.35) and (14.11)) by

$$D_\mu \Psi_f(x) = (\partial_\mu + iq_f A_\mu(x))\Psi_f(x) \quad . \quad (17.2)$$

For electrons we have  $q_f = -e$  and for protons  $q_f = e$ . For  $\alpha = 0$  the Lagrangian is invariant under gauge transformations

$$A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu \Lambda(x) \quad , \quad \Psi_f(x) \rightarrow \exp(-iq_f \Lambda(x))\Psi_f(x) \quad . \quad (17.3)$$

The Feynman rules are collected in the table below.

table 6		
	$\equiv \frac{-\left(g_{\mu\nu} - (1 - \frac{1}{\alpha}) \frac{k_\mu k_\nu}{k^2 + i\varepsilon}\right)}{k^2 + i\varepsilon}$	photon propagator (Lorentz gauge)
	$\equiv q\gamma_{ab}^\mu \text{ and } k_1 = k_2 + k_3$	photon vertex (fermion charge is $q$ )
	$\equiv \left[\frac{1}{\not{k} - m + i\varepsilon}\right]_{ab}$	fermion propagator

Before calculating cross sections we wish to discuss in more detail the helicity of the fermions and its relation to charge conjugation  $C$ . The latter relates, say electrons to positrons, or in general particles to anti-particles, which is an important symmetry of the theory. It, as well as parity ( $P$ ) and time reversal ( $T$ ) symmetry can be separately broken, but the combination CPT is to be unbroken to allow for a local, relativistic invariant field theory. The spin components of the solutions in eq. (13.8) were based on a decomposition along the  $z$ -axis in the restframe. Helicity, as for the photon, is defined by decomposing the spin in the direction of motion,  $\vec{k}$ . It is hence defined in terms of the eigenvalues of the operator

$$\hat{k} \cdot \vec{J} \equiv \frac{k_i}{4|\vec{k}|} \varepsilon_{ijk} \sigma_{jk} = \begin{pmatrix} \frac{1}{2}\hat{k} \cdot \vec{\sigma} & \oslash \\ \oslash & \frac{1}{2}\hat{k} \cdot \vec{\sigma} \end{pmatrix} \quad . \quad (17.4)$$

( $\vec{J}$  is the spin part of the angular momentum operator, the equivalent of  $\frac{1}{2}\vec{\sigma}$  for a two-component spinor.) This holds both in the Dirac and Weyl representations. It is easy to verify that  $[\hat{k} \cdot \vec{J}, \not{k}] = 0$ , e.g. by making use of the fact that in the Dirac representation

$$\not{k} = \begin{pmatrix} k_0 1_2 & \vec{k} \cdot \vec{\sigma} \\ -\vec{k} \cdot \vec{\sigma} & -k_0 1_2 \end{pmatrix} \quad . \quad (17.5)$$

This implies that we can choose  $u_0^{(\alpha)}$  and  $v_0^{(\alpha)}$  to be eigenstates of the helicity operator  $\hat{k} \cdot \vec{J}$  (consequently they become functions of  $\hat{k}$ )

$$\hat{k} \cdot \vec{J} \tilde{\Psi}(\vec{k}) = \pm \frac{1}{2} \tilde{\Psi}(\vec{k}) \quad . \quad (17.6)$$

Instead of the label  $\alpha$  we can use  $\pm$  to indicate the helicity and we have

$$\begin{aligned} u^\pm(\vec{k}) &= \frac{(\not{k} + m)}{\sqrt{m + |k_0|}} u_0^\pm(\hat{k}) \quad , \quad \hat{k} \cdot \vec{J} u_0^\pm(\hat{k}) = \pm \frac{1}{2} u_0^\pm(\hat{k}) \quad , \\ v^\pm(\vec{k}) &= \frac{(-\not{k} + m)}{\sqrt{m + |k_0|}} v_0^\pm(\hat{k}) \quad , \quad \hat{k} \cdot \vec{J} v_0^\pm(\hat{k}) = \mp \frac{1}{2} v_0^\pm(\hat{k}) \quad . \end{aligned} \quad (17.7)$$

Note the flip of helicity for the positron wave functions. For  $\vec{k} = (0, 0, k)$  these eigenstates coincide with the decomposition in eq. (13.8). It is clear that we can define

$$\begin{aligned} u_0^\pm(\hat{k}) &= \begin{pmatrix} \varphi_\pm(\hat{k}) \\ 0 \\ 0 \end{pmatrix} \quad , \quad \hat{k} \cdot \vec{\sigma} \varphi_\pm(\hat{k}) = \pm \varphi_\pm(\hat{k}) \quad , \\ v_0^\pm(\hat{k}) &= \begin{pmatrix} 0 \\ 0 \\ \chi_\pm(\hat{k}) \end{pmatrix} \quad , \quad \hat{k} \cdot \vec{\sigma} \chi_\pm(\hat{k}) = \mp \chi_\pm(\hat{k}) \quad , \end{aligned} \quad (17.8)$$

with  $\varphi_\pm$  and  $\chi_\pm$  each an orthonormal set of two-component spinors. They can be related to each other by

$$\chi_\pm(\hat{k}) \equiv -i\sigma_2 \varphi_\pm^*(\hat{k}) \quad . \quad (17.9)$$

Indeed, when we use that

$$\sigma_2 \sigma_i \sigma_2 = -\sigma_i^* \quad , \quad i = 1, 2, 3 \quad , \quad (17.10)$$

which expresses the fact that SU(2) is so-called pseudo real, we find

$$\hat{k} \cdot \vec{\sigma} (-i\sigma_2 \varphi_\pm^*(\hat{k})) = (-i\hat{k} \cdot \vec{\sigma}^* \sigma_2 \varphi_\pm(\hat{k}))^* = (i\sigma_2 \hat{k} \cdot \vec{\sigma} \varphi_\pm(\hat{k}))^* = \mp (-i\sigma_2 \varphi_\pm^*(\hat{k})) \quad . \quad (17.11)$$

As eq. (17.9) relates the components of the electron wave function to those of the positron wave function, it is the basis of the charge conjugation symmetry, which relates the solutions of the Dirac equation to solutions of the complex conjugate Dirac equation (see eq. (12.31)), which indeed interchanges positive and negative energy solutions, i.e. particles and antiparticles. To formulate this symmetry in the four-component spinor language one introduces the charge conjugation matrix (in the Dirac representation)

$$C \equiv -i\gamma^0 \gamma^2 = \begin{pmatrix} \oslash & -i\sigma_2 \\ -i\sigma_2 & \oslash \end{pmatrix} \quad , \quad (17.12)$$

which satisfies

$$C^{-1} = C^\dagger = -C \quad , \quad C\gamma_\mu C^{-1} = -\gamma_\mu^t \quad . \quad (17.13)$$

This can be proven from the explicit form of the Dirac matrices. The equivalent of eq. (17.10) is given by

$$\gamma^2 \gamma_\mu \gamma_2 = -\gamma_\mu^* \quad . \quad (17.14)$$

It is now easy to verify that

$$v_{\pm}(\vec{k}) = C\bar{u}_{\pm}^t(\vec{k}) \quad , \quad u_{\pm}(\vec{k}) = C\bar{v}_{\pm}^t(\vec{k}) \quad . \quad (17.15)$$

We just need to prove one of these identities, because charge conjugation is an involution, i.e. applying it twice gives the identity

$$C\overline{(C\bar{\Psi}^t)}^t = C\gamma_0^t C^* \gamma_0^\dagger \Psi = \Psi \quad . \quad (17.16)$$

We find

$$\begin{aligned} C\bar{u}_{\pm}^t(\vec{k}) &= C\gamma_0^t u_{\pm}^*(\vec{k}) = i\gamma^2 u_{\pm}^*(\vec{k}) = i\gamma^2 \frac{(\not{k}^* + m)}{\sqrt{m + |k_0|}} \begin{pmatrix} \varphi_{\pm}^*(\hat{k}) \\ 0 \\ 0 \end{pmatrix} = i \frac{(-\not{k} + m)}{\sqrt{m + |k_0|}} \gamma^2 \begin{pmatrix} \varphi_{\pm}^*(\hat{k}) \\ 0 \\ 0 \end{pmatrix} \\ &= \frac{(-\not{k} + m)}{\sqrt{m + |k_0|}} \begin{pmatrix} \oslash & i\sigma_2 \\ -i\sigma_2 & \oslash \end{pmatrix} \begin{pmatrix} \varphi_{\pm}^*(\hat{k}) \\ 0 \\ 0 \end{pmatrix} = \frac{(-\not{k} + m)}{\sqrt{m + |k_0|}} \begin{pmatrix} 0 \\ 0 \\ \chi_{\pm}(\hat{k}) \end{pmatrix} = v_{\pm}(\vec{k}) \quad . \end{aligned} \quad (17.17)$$

Under charge conjugation the charge that appears in the covariant derivative in eq. (17.2) should change sign too. To show this we multiply the complex conjugate of the Dirac equation with  $-i\gamma^2$

$$\begin{aligned} -i\gamma^2 [(-i\gamma_{\mu}(\partial^{\mu} - ieA^{\mu}) + m) \Psi]^* &= (i\gamma^2 \gamma_{\mu}^* \gamma^2 (\partial^{\mu} + ieA^{\mu}) + m) (-i\gamma^2 \Psi^*) \\ &= (-i\gamma^{\mu}(\partial_{\mu} + ieA_{\mu}) + m) (C\bar{\Psi}^t) \quad . \end{aligned} \quad (17.18)$$

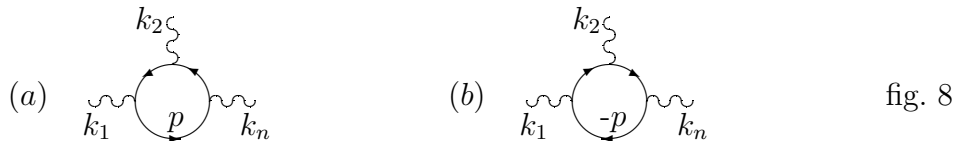
That charge conjugation is really a symmetry of the quantum theory, is most convincingly demonstrated by the fact that the Dirac Lagrangian is invariant under charge conjugation. Using  $\overline{C\bar{\Psi}^t} = \Psi^t \gamma_0^* C^{\dagger} \gamma_0 = -\Psi^t C^{-1}$ , the anti-commuting properties of the fermi fields and partial integration, we find

$$\begin{aligned} \int d_4x \overline{C\bar{\Psi}^t} (i\gamma^{\mu}(\partial_{\mu} + ieA_{\mu}) - m) C\bar{\Psi}^t &= \int d_4x -\Psi^t C^{-1} (i\gamma^{\mu}(\partial_{\mu} + ieA_{\mu}) - m) C\bar{\Psi}^t = \\ \int d_4x -\Psi^t (-i\gamma_{\mu}^t(\partial_{\mu} + ieA_{\mu}) - m) \bar{\Psi}^t &= \int d_4x \bar{\Psi} (i\gamma^{\mu}(\partial_{\mu} - ieA_{\mu}) - m) \Psi \quad . \end{aligned} \quad (17.19)$$

In particular we see that the electromagnetic current generated by the fermi fields transforms as required for the interchange of particles and anti-particles, under which the charge changes sign

$$j_{\mu} = \bar{\Psi} \gamma_{\mu} \Psi \xrightarrow{C} -\bar{\Psi} \gamma_{\mu} \Psi \quad . \quad (17.20)$$

An important consequence of the charge conjugation symmetry is Furry's theorem, which states that a fermionic loop with an odd number of vertices will not contribute to the amplitude. Consider a fermion loop as in figure (8a) below



for which the Feynman rules lead to the expression (note that spinor index contractions run against the arrow of the fermion line and  $\sum_i k_i = 0$ )

$$q^n \text{Tr} \left( \frac{1}{\not{p} - m + i\varepsilon} \gamma^{\mu_1} \frac{1}{\not{p} + \not{k}_1 - m + i\varepsilon} \gamma^{\mu_2} \frac{1}{\not{p} + \not{k}_1 + \not{k}_2 - m + i\varepsilon} \gamma^{\mu_3} \dots \gamma^{\mu_n} \right) \quad . \quad (17.21)$$

Using the fact that for any matrix  $A$  we have  $\text{Tr}(A) = \text{Tr}(A^t) = \text{Tr}(CA^tC^{-1})$ , we convert eq. (17.21) to the expression

$$(-q)^n \text{Tr} \left( \gamma^{\mu_n} \dots \gamma^{\mu_3} \frac{1}{-\not{p} - \not{k}_1 - \not{k}_2 - m + i\varepsilon} \gamma^{\mu_2} \frac{1}{-\not{p} - \not{k}_1 - m + i\varepsilon} \gamma^{\mu_1} \frac{1}{-\not{p} - m + i\varepsilon} \right), \quad (17.22)$$

which is exactly  $(-1)^n$  times the result of the Feynman diagram that is obtained by inverting the orientation of the fermion line (i.e. the vertices are connected in the reversed order) as indicated in figure (b). As both diagrams will occur, their contributions will cancel whenever  $n$  is odd. It confirms the intuition that particles and anti-particles contribute equally, except for their opposite charge factors  $(\pm q)^n$ .

We will now calculate the cross section for electron-electron scattering (the so-called Møller cross section). In lowest non-trivial order there are only two diagrams that contribute, as indicated in the figure.

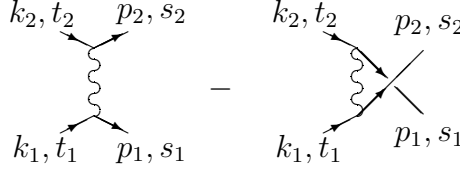


fig. 9

The labels  $t_i$  and  $s_i$  indicate the helicities of the incoming and outgoing electrons. The scattering matrix (ignoring the time dependent phase factor) for this process is given by

$$\begin{aligned} \text{out} < (\vec{p}_1, s_1), (\vec{p}_2, s_2) | (\vec{k}_1, t_1), (\vec{k}_2, t_2) >_{\text{in}} = \\ & -i(2\pi)^4 \frac{\delta_4(p_1 + p_2 - k_1 - k_2) \mathcal{M}(\{(-p_1, s_1), (-p_2, s_2)\}, \{(k_1, t_1), (k_2, t_2)\})}{\sqrt{2p_0^{(1)}(\vec{p}_1)(2\pi)^3 2p_0^{(2)}(\vec{p}_2)(2\pi)^3 2k_0^{(1)}(\vec{k}_1)(2\pi)^3 2k_0^{(2)}(\vec{k}_2)(2\pi)^3}} = \\ & \frac{i\delta_4(p_1 + p_2 - k_1 - k_2)}{(4\pi)^2 \sqrt{p_0^{(1)} p_0^{(2)} k_0^{(1)} k_0^{(2)}}} \left\{ \frac{\bar{u}_{s_1}(p_1) e\gamma^\mu u_{t_1}(k_1) g_{\mu\nu} \bar{u}_{s_2}(p_2) e\gamma^\nu u_{t_2}(k_2)}{(k_1 - p_1)^2 + i\varepsilon} \right. \\ & \quad \left. - \frac{\bar{u}_{s_1}(p_1) e\gamma^\mu u_{t_2}(k_2) g_{\mu\nu} \bar{u}_{s_2}(p_2) e\gamma^\nu u_{t_1}(k_1)}{(k_1 - p_2)^2 + i\varepsilon} \right\} . \quad (17.23) \end{aligned}$$

The relative minus-sign is of course a consequence of the so-called Fermi-Dirac statistics, which implements the Pauli principle. We got rid of the gauge dependent part of the photon propagator (see table 6) by using the fact that the currents generated by  $\bar{u}_s(p)\gamma^\mu u_t(k)$  are conserved, such that

$$\bar{u}_s(p)\gamma^\mu u_t(k)(p_\mu - k_\mu) = \bar{u}_s(p)[(\not{p} - m) - (\not{k} - m)]u_t(k) = 0, \quad (17.24)$$

because on-shell  $(\not{k} - m)u(k) = 0$  (and hence also  $\bar{u}(k)(\not{k} - m) = 0$ ). Indirectly, through current conservation, this is related to gauge invariance. It guarantees that the longitudinal component of the photon does not contribute to the scattering matrix, which is thus seen not to depend on the gauge fixing parameter  $\alpha$ .

The differential cross section for *unpolarised* electron-electron scattering is given by (see eq. (10.12), from now on we will drop the distinction between  $\vec{k}_i$  and  $k_i$ )

$$\begin{aligned} d\sigma = \sum_{s_1, s_2} \frac{d_3 \vec{p}_1}{2p_0(\vec{p}_1)(2\pi)^3} \frac{d_3 \vec{p}_2}{2p_0(\vec{p}_2)(2\pi)^3} \frac{(2\pi)^4 \delta_4(p_1 + p_2 - k_1 - k_2)}{4\sqrt{(k_1 \cdot k_2)^2 - m^4}} \times \\ \frac{1}{4} \sum_{t_1, t_2} |\mathcal{M}(\{(-p_1, s_1), (-p_2, s_2)\}, \{(k_1, t_1), (k_2, t_2)\})|^2, \quad (17.25) \end{aligned}$$

where  $\frac{1}{4} \sum_{t_1, t_2}$  stands for averaging over the polarisations of the incoming electrons. For the total cross section we should multiply with a factor  $\frac{1}{2}$  to avoid double counting the identical outgoing electrons, or restrict the scattering angle  $\theta$  to the interval  $\theta \in [0, \pi/2]$ , when integrating over the outgoing momenta. The latter convention will be followed here. In the center of mass system the scattered particles move back to back in a direction which is only determined modulo  $\pi$ , which is why  $\theta \in [0, \pi/2]$ , with  $\theta$  measured from the incoming particle direction (also defined modulo  $\pi$ ).

To calculate  $|\mathcal{M}|^2$  we use

$$\begin{aligned} \sum_{s,t} \bar{u}_s(p) \gamma_\mu u_t(k) (\bar{u}_s(p) \gamma_\nu u_t(k))^* &= \sum_{s,t} \bar{u}_s(p) \gamma_\mu u_t(k) \bar{u}_t(k) \gamma_0 \gamma_\nu^\dagger \gamma_0 u_s(p) = \\ \text{Tr} \left( \gamma_\mu \sum_t u_t(k) \otimes \bar{u}_t(k) \gamma_\nu \sum_s u_s(p) \otimes \bar{u}_s(p) \right) &= \text{Tr} \left( \gamma_\mu (\not{k} + m) \gamma_\nu (\not{p} + m) \right), \end{aligned} \quad (17.26)$$

which can be represented graphically as follows

$$\sum_{s,t} \begin{array}{c} k, t \quad p, s \\ \mu \quad \nu \\ \mathcal{M} \quad \mathcal{M}^* \end{array} = \sum_t \begin{array}{c} k, t \quad p \\ \mu \quad \nu \end{array} = \sum_t \begin{array}{c} k, t \quad p \\ \mu \quad \nu \end{array} = \begin{array}{c} k \\ \mu \quad \nu \end{array} \quad \text{fig. 10}$$

Hence, we add a Feynman rule for the so-called *cut* fermion propagator

$$\begin{array}{c} k \\ \text{---} \text{---} \text{---} \\ b \quad a \end{array} = \text{sign}(k_0) (\not{k} + m)_{ab} \quad . \quad (17.27)$$

For anti-particles  $k_0 < 0$  (see eq. (13.18) for the extra minus sign). These results can be generalised to other fields too, by noting that our conventions have been such that the propagators can be written as

$$\frac{\sum_\beta \phi_\beta(\vec{k}) \otimes \bar{\phi}_\beta(\vec{k})^*}{k^2 - m^2 + i\varepsilon} \quad , \quad (17.28)$$

where  $\phi_\beta(\vec{k})$  are the wave functions for the incoming lines and  $\bar{\phi}_\beta(\vec{k})^*$  for the outgoing lines, with  $\beta$  labelling the internal degrees of freedom (cmp. eq. (16.8)).

We can now use this result to compute  $|\mathcal{M}|^2$

$$\begin{aligned} \sum_{s_1, s_2, t_1, t_2} |\mathcal{M}|^2 &= \left( \begin{array}{c} k_2 \\ p_2 \\ p_1 \\ k_1 \end{array} \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \end{array} - \begin{array}{c} k_2 \\ p_2 \\ p_1 \\ k_1 \end{array} \begin{array}{c} \text{---} \text{---} \end{array} \right) + (p_1 \longleftrightarrow p_2) \\ &= e^4 \left\{ \frac{\text{Tr}[\gamma^\mu (\not{k}_1 + m) \gamma^\nu (\not{p}_1 + m)] \text{Tr}[\gamma_\mu (\not{k}_2 + m) \gamma_\nu (\not{p}_2 + m)]}{((k_1 - p_1)^2 + i\varepsilon) ((k_2 - p_2)^2 + i\varepsilon)} \right. \\ &\quad \left. - \frac{\text{Tr}[\gamma^\mu (\not{k}_1 + m) \gamma^\nu (\not{p}_2 + m) \gamma_\mu (\not{k}_2 + m) \gamma_\nu (\not{p}_1 + m)]}{((k_1 - p_1)^2 + i\varepsilon) ((k_1 - p_2)^2 + i\varepsilon)} \right\} \\ &\quad + (p_1 \longleftrightarrow p_2) \quad . \end{aligned} \quad (17.29)$$

To compute the traces over the gamma matrices we use the following identities (problem 21)

$$\begin{aligned} \text{Tr}(\gamma_\mu \gamma_\nu) &= 4g_{\mu\nu} \quad , \quad \text{Tr}(\gamma_\mu \gamma_\nu \gamma_\alpha \gamma_\beta) = 4(g_{\mu\nu} g_{\alpha\beta} + g_{\mu\beta} g_{\alpha\nu} - g_{\mu\alpha} g_{\nu\beta}) \quad , \\ \sum_\mu \gamma_\mu \gamma_\alpha \gamma_\beta \gamma^\mu &= 4g_{\alpha\beta} \quad , \quad \sum_\mu \gamma_\mu \gamma_\nu \gamma_\alpha \gamma_\beta \gamma^\mu = -2\gamma_\beta \gamma_\alpha \gamma_\nu \quad , \end{aligned} \quad (17.30)$$



and the fact that the trace over an odd number of gamma matrices vanishes. This implies

$$\text{Tr}(\gamma_\mu(\not{k}+m)\gamma_\nu(\not{p}+m)) = \text{Tr}(\gamma_\mu\not{k}\gamma_\nu\not{p}) + 4m^2 g_{\mu\nu} = 4(m^2 - k \cdot p)g_{\mu\nu} + 4k_\mu p_\nu + 4k_\nu p_\mu \quad , \quad (17.31)$$

and

$$\sum_\mu \gamma_\mu(\not{k}+m)\gamma_\nu(\not{p}+m)\gamma^\mu = -2\not{p}\gamma_\nu\not{k} - 2m^2\gamma_\nu + 4mk_\nu + 4mp_\nu \quad . \quad (17.32)$$

Together with momentum conservation ( $p_1 + p_2 = k_1 + k_2$ ) and the on-shell conditions ( $p_1^2 = p_2^2 = k_1^2 = k_2^2 = m^2$ ), which imply identities like  $p_1 \cdot p_2 = k_1 \cdot k_2$ , we find

$$\begin{aligned} \sum_{s_1, s_2, t_1, t_2} |\mathcal{M}|^2 &= e^4 \left\{ \frac{16 \left( g^{\mu\nu}(m^2 - k_1 \cdot p_1) + k_1^\mu p_1^\nu + k_1^\nu p_1^\mu \right) \left( g_{\mu\nu}(m^2 - k_2 \cdot p_2) + k_2^\mu p_2^\nu + k_2^\nu p_2^\mu \right)}{\left( (k_1 - p_1)^2 + i\varepsilon \right) \left( (k_2 - p_2)^2 + i\varepsilon \right)} \right. \\ &\quad \left. - \kappa \frac{\text{Tr} \left[ \left( 2\not{p}_2 \gamma^\nu \not{k}_1 + 2m^2 \gamma^\nu - 4mk_1^\nu - 4mp_2^\nu \right) (\not{k}_2 + m) \gamma_\nu (\not{p}_1 + m) \right]}{\left( (k_1 - p_1)^2 + i\varepsilon \right) \left( (k_1 - p_2)^2 + i\varepsilon \right)} \right\} + (p_1 \leftrightarrow p_2) \\ &= 32e^4 \left\{ \frac{(k_1 \cdot k_2)^2 + (k_1 \cdot p_2)^2 + 2m^2(k_1 \cdot p_2 - k_1 \cdot k_2)}{\left( (k_1 - p_1)^2 + i\varepsilon \right)^2} \right. \\ &\quad \left. - \kappa \frac{(k_1 \cdot k_2)^2 - 2m^2 k_1 \cdot k_2}{\left( (k_1 - p_1)^2 + i\varepsilon \right) \left( (k_1 - p_2)^2 + i\varepsilon \right)} \right\} + (p_1 \leftrightarrow p_2) \quad . \quad (17.33) \end{aligned}$$

The parameter  $\kappa$  determines the relative sign of the “crossed” diagrams in eq. (17.29), which arise from multiplying the direct electron-electron scattering diagram with the complex conjugate of the one where the outgoing fermion lines were crossed. For Fermi-Dirac statistics  $\kappa \equiv -1$ . By keeping track of the dependence on  $\kappa$  one sees how scattering experiments can be used to verify the anticommuting nature of the electrons.

We finally perform some kinematics and express the differential cross section in terms of the scattering angle  $\theta$ . We define in the center of mass frame

$$k_1^0 = k_2^0 = p_1^0 = p_2^0 \equiv E \quad , \quad \vec{p}_1 = -\vec{p}_2 \equiv \vec{p} \quad , \quad \vec{k}_1 = -\vec{k}_2 \equiv \vec{k} \quad \text{with} \quad |\vec{p}| = |\vec{k}| \quad . \quad (17.34)$$

Defining  $\theta$  to be the angle between  $\vec{k}$  and  $\vec{p}$ , i.e.  $\vec{p} \cdot \vec{k} = \vec{k}^2 \cos \theta$ , we have

$$\begin{aligned} k_1 \cdot k_2 &= E^2 + \vec{k}^2 = 2E^2 - m^2 \quad , \quad k_1 \cdot p_1 = E^2 - \vec{k}^2 \cos \theta = E^2(1 - \cos \theta) + m^2 \cos \theta \quad , \\ (p_1 - k_1)^2 &= -4\vec{k}^2 \sin^2(\tfrac{1}{2}\theta) \quad , \quad k_1 \cdot p_2 = E^2 + \vec{k}^2 \cos \theta = E^2(1 + \cos \theta) - m^2 \cos \theta \quad , \\ (p_2 - k_1)^2 &= -4\vec{k}^2 \cos^2(\tfrac{1}{2}\theta) \quad , \quad (k_2 \cdot k_1)^2 - m^4 = 4E^2(E^2 - m^2) = 4\vec{k}^2 E^2 \quad . \quad (17.35) \end{aligned}$$

Finally we use that identity ( $\Omega$  is the solid angle,  $d\Omega = \sin \theta d\theta d\phi$ )

$$\int d_3\vec{p}_1 d_3\vec{p}_2 \delta_4(p_1 + p_2 - k_1 - k_2) = \int p^2 dp d\Omega \delta \left( 2\sqrt{\vec{p}^2 + m^2} - 2\sqrt{\vec{k}^2 + m^2} \right) = \int d\Omega \tfrac{1}{2} E |\vec{k}| \quad . \quad (17.36)$$

Collecting all terms we find for eq. (17.25) the result

$$\frac{d\sigma}{d\Omega} = \frac{1}{(2E(2\pi)^3)^2} \cdot \frac{(2\pi)^4 \tfrac{1}{2} E |\vec{k}|}{4\sqrt{4\vec{k}^2 E^2}} \cdot \tfrac{1}{4} \sum_{s_1, s_2, t_1, t_2} |\mathcal{M}|^2 = \frac{1}{2^{10} \pi^2 E^2} \sum_{s_1, s_2, t_1, t_2} |\mathcal{M}|^2 \quad , \quad (17.37)$$

with  $\sum_{s_1, s_2, t_1, t_2} |\mathcal{M}|^2$  given by

$$\begin{aligned}
& 32e^4 \left\{ \frac{(2E^2 - m^2)^2 + [E^2(1 + \cos \theta) - m^2 \cos \theta]^2 + 2m^2[E^2(1 + \cos \theta) - m^2 \cos \theta + m^2 - 2E^2]}{16(\vec{k}^2)^2 \sin^4(\frac{1}{2}\theta)} + \right. \\
& \quad \left. \frac{(2E^2 - m^2)^2 + [E^2(1 - \cos \theta) + m^2 \cos \theta]^2 + 2m^2[E^2(1 - \cos \theta) + m^2 \cos \theta + m^2 - 2E^2]}{16(\vec{k}^2)^2 \cos^4(\frac{1}{2}\theta)} - \kappa \frac{2(2E^2 - m^2)(2E^2 - 3m^2)}{16(\vec{k}^2)^2 \sin^2(\frac{1}{2}\theta) \cos^2(\frac{1}{2}\theta)} \right\} \\
&= \frac{32e^4}{(\vec{k}^2)^2} \left\{ \frac{[(2E^2 - m^2)^2 + E^4 + (E^2 - m^2)^2 \cos^2 \theta + 2m^2(m^2 - E^2)](\cos^4(\frac{1}{2}\theta) + \sin^4(\frac{1}{2}\theta))}{\sin^4 \theta} + \right. \\
& \quad \left. \frac{\cos \theta [2E^2(E^2 - m^2) + 2m^2(E^2 - m^2)](\cos^4(\frac{1}{2}\theta) - \sin^4(\frac{1}{2}\theta))}{\sin^4 \theta} - \kappa \frac{2(2E^2 - m^2)(2E^2 - 3m^2)}{4 \sin^2 \theta} \right\} \\
&= \frac{16e^4}{(E^2 - m^2)^2} \left\{ (E^2 - m^2)^2 + \frac{4(2E^2 - m^2)^2}{\sin^4 \theta} - \kappa \frac{4(E^2 - m^2)^2 - 3(2E^2 - m^2)^2}{4 \sin^2 \theta} \right\} , \quad (17.38)
\end{aligned}$$

yielding ( $\alpha_e = \frac{e^2}{4\pi} = \frac{e^2}{4\pi\hbar c} \approx \frac{1}{137}$  is the fine-structure constant)

$$\frac{d\sigma}{d\Omega} = \frac{\alpha_e^2 (2E^2 - m^2)^2}{4E^2 (E^2 - m^2)^2} \left\{ \frac{4}{\sin^4 \theta} + \frac{3\kappa}{\sin^2 \theta} + \frac{(E^2 - m^2)^2}{(2E^2 - m^2)^2} \left( 1 - \frac{4\kappa}{\sin^2 \theta} \right) \right\} . \quad (17.39)$$

This cross section is invariant under  $\theta \rightarrow \pi - \theta$ , such that we cannot tell the two outgoing electrons apart, as it should be. For electron-electron scattering we have to put  $\kappa = -1$  and we see that one can easily distinguish experimentally if electrons behave according to the Fermi-Dirac statistics.

In problem 29 electron-positron scattering is studied with in the final state an electron and a positron (Bhabha scattering) or a muon and an antimuon. Both for  $e^-e^- \rightarrow e^-e^-$  and  $e^-e^+ \rightarrow e^-e^+$  one cannot take  $\theta$  too close to zero (or  $\pi$  for  $e^-e^-$ ). Apart from the fact that the detector would be in the way of the beam, it is fundamentally impossible to distinguish the scattered particles at  $\theta = 0$  (and  $\theta = \pi$  for  $e^-e^-$ ) from those in the beam. The divergence of the differential cross section was therefore to be expected. For  $e^-e^+ \rightarrow \mu^-\mu^+$  this divergence is absent and one can define the total cross section by integrating over all angles. For  $E \gg m_e$  and  $E \gg m_\mu$  one finds (see problem 29)  $\sigma = \frac{1}{3}\pi\alpha_e^2\hbar^2c^2/E^2 = 21.7\text{nb}/E^2(\text{GeV})$ .

We now discuss electron-photon scattering, also known as Compton scattering. The resulting cross section is called the Klein-Nishina formula. There are again two diagrams that contribute in lowest non-trivial order to the scattering matrix.

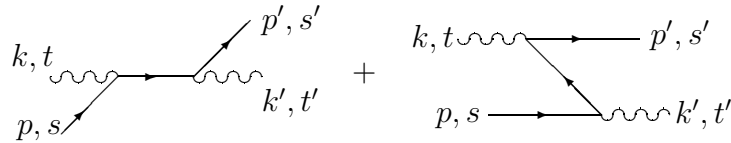


fig. 11

The cross section is now given by

$$\begin{aligned}
d\sigma = \sum_{s', t'} \frac{d_3 \vec{p}'}{2p'_0(\vec{p}')(2\pi)^3} \frac{d_3 \vec{k}'}{2k'_0(\vec{k}')(2\pi)^3} \frac{(2\pi)^4 \delta_4(p + p' - k - k')}{4|p \cdot k|} \times \\
\frac{1}{4} \sum_{s, t} |\mathcal{M}(\{(-p', s'), (-k', t')\}, \{(p, s), (k, t)\})|^2 , \quad (17.40)
\end{aligned}$$

where, as for electron-electron scattering, we will discuss unpolarised cross sections. This requires averaging over the polarisations of the incoming particles (at the end we will mention the dependence on the photon polarisations). Note that the photon has also two helicity eigenstates, together with the electron  $\sum_{s,t}$  contains four terms. The reduced matrix element for the two diagrams is given by

$$\mathcal{M} = \frac{\bar{u}_{s'}(p')e\gamma^\mu\varepsilon_\mu^{(t')}(k')^*(\not{p} + \not{k} + m)e\gamma^\nu\varepsilon_\nu^{(t)}(k)u_s(p)}{(p+k)^2 - m^2 + i\varepsilon} + ((k, t) \longleftrightarrow (-k', t')) \quad . \quad (17.41)$$

We leave it as an exercise to verify that the cut photon propagator, for the choice of polarisations discussed in eq. (16.21), is given by ( $k^2 \equiv 0$ )

$$\mu \text{---}\text{wavy line}\text{---}\nu \equiv \sum_{t=\pm} \varepsilon_\mu^{(t)}(k)\varepsilon_\nu^{(t)}(k)^* = -\left(g_{\mu\nu} - \frac{k_\mu n_\nu(k) + k_\nu n_\mu(k)}{k \cdot n(k)}\right) \quad . \quad (17.42)$$

Like for electron-electron scattering, we can compute  $|\mathcal{M}|^2$  graphically by

$$\begin{aligned} \sum_{s_1, s_2, t_1, t_2} |\mathcal{M}|^2 &= \left( \text{Diagram 1} + \text{Diagram 2} \right) + (k' \longleftrightarrow -k) \\ &= e^4 \left\{ \frac{\text{Tr}[\gamma^\mu(\not{p} + \not{k} + m)\gamma^\nu(\not{p} + m)\gamma^{\mu'}(\not{p} + \not{k} + m)\gamma^{\nu'}(\not{p}' + m)]}{((p+k)^2 - m^2 + i\varepsilon)^2} \right. \\ &\quad \left. + \frac{\text{Tr}[\gamma^\mu(\not{p} + \not{k} + m)\gamma^\nu(\not{p} + m)\gamma^{\mu'}(\not{p} - \not{k}' + m)\gamma^{\nu'}(\not{p}' + m)]}{((p+k)^2 - m^2 + i\varepsilon)((p-k')^2 - m^2 + i\varepsilon)} \right\} \times \\ &\quad \sum_{t'=\pm} \varepsilon_\mu^{(t')}(k')\varepsilon_{\mu'}^{(t')}(k')^* \sum_{t=\pm} \varepsilon_\nu^{(t)}(k)\varepsilon_\nu^{(t)}(k)^* + (k' \longleftrightarrow -k) \quad . \quad (17.43) \end{aligned}$$

The gauge invariance, i.e. conservation of the fermionic current, is again instrumental in decoupling the longitudinal component of the photon field. In this case the argument is somewhat more subtle. Consider for example the term from the cut photon propagators that contains  $k_\nu$ . It gives rise to the combination (using that  $p^2 = m^2$ )

$$\begin{aligned} (\not{p} + \not{k} + m)\gamma^\nu k_\nu(\not{p} + m) &= (\not{p} + \not{k} + m)\{(\not{p} + \not{k} - m) - (\not{p} - m)\}(\not{p} + m) \\ &= ((p+k)^2 - m^2)(\not{p} + m) \quad . \quad (17.44) \end{aligned}$$

This means that one of the photon vertices is removed. There remain two diagrams, each with one fermion loop and with an odd number of vertices. Furry's theorem tells us that these two diagrams add to zero. We may therefore just as well replace the cut photon propagator by  $-g_{\mu\nu}$ . Using this we find

$$\begin{aligned} \sum_{s_1, s_2, t_1, t_2} |\mathcal{M}|^2 &= e^4 \left\{ \frac{\text{Tr}[\gamma^\mu(\not{p} + \not{k} + m)\gamma^\nu(\not{p} + m)\gamma_\nu(\not{p} + \not{k} + m)\gamma_\mu(\not{p}' + m)]}{((p+k)^2 - m^2 + i\varepsilon)^2} + \right. \\ &\quad \left. \frac{\text{Tr}[\gamma^\mu(\not{p} + \not{k} + m)\gamma^\nu(\not{p} + m)\gamma_\mu(\not{p} - \not{k}' + m)\gamma_\nu(\not{p}' + m)]}{((p+k)^2 - m^2 + i\varepsilon)((p-k')^2 - m^2 + i\varepsilon)} \right\} + (k' \leftrightarrow -k) \quad . \quad (17.45) \end{aligned}$$

Taking the incoming electron at rest ( $p = (m, \vec{0})$ ), following similar steps as for electron-electron scattering, one will arrive at the result

$$\frac{d\sigma}{d\Omega} = \frac{\alpha_e^2}{2m^2} \left(\frac{k'_0}{k_0}\right)^2 \left(\frac{k_0}{k'_0} + \frac{k'_0}{k_0} - \sin^2 \theta\right) \quad , \quad (17.46)$$

where  $\theta$  is the angle of the scattered photon with the direction of the incident photon. From energy and momentum conservation one finds that

$$k'_0 = \frac{k_0}{1 + (k_0/m)(1 - \cos \theta)} \quad . \quad (17.47)$$

For a detailed derivation we refer to section 5-2-1 of Itzykson and Zuber and to section 86 of Berestetskii, e.a. (see section 1 for the reference).

In Itzykson and Zuber, as for most other textbooks, the result is derived by choosing the photon polarisation such that  $\varepsilon(k) \cdot p = 0$  (keeping  $\varepsilon(k) \cdot k = 0$ ). With this choice it is even possible to determine the polarised cross section (the polarisation of the electron is assumed not to be observed)

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{pol}} = \frac{\alpha_e^2}{4m^2} \left(\frac{k'_0}{k_0}\right)^2 \left(\frac{k_0}{k'_0} + \frac{k'_0}{k_0} + 4(\varepsilon' \cdot \varepsilon)^2 - 2\right) \quad , \quad (17.48)$$

where  $\varepsilon$  and  $\varepsilon'$  are the polarisations of resp. the incident and scattered photon. When  $k'_0 \ll m$  one obtains the well-known Thomson formula

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{pol}} = \frac{\alpha_e^2}{m^2} (\varepsilon' \cdot \varepsilon)^2 \quad . \quad (17.49)$$

The unpolarised cross section in this limit is obtained by summing over the scattered and averaging over the incident polarisations

$$\frac{d\sigma}{d\Omega} = \frac{\alpha_e^2}{2m^2} (1 + \cos^2 \theta) \quad \text{and} \quad \sigma = \frac{8\pi\alpha_e^2}{3m^2} \quad . \quad (17.50)$$

## 18 Non-Abelian gauge theories

Quantum Electrodynamics is an example of a U(1) gauge theory. U(1) is the group of the unimodular complex numbers and determines the transformation of the charged fields

$$\Psi(x) \rightarrow \exp(-iq\Lambda(x))\Psi(x) \equiv g(x)\Psi(x) \quad . \quad (18.1)$$

It forms a group, which means that for any two elements  $g, h \in \text{U}(1)$ , the product is also in U(1). Furthermore, any element has an inverse  $g^{-1}$ , which satisfies  $gg^{-1} = g^{-1}g = 1$ . The unit 1 satisfies  $g1 = 1g = g$ , for any  $g \in \text{U}(1)$ . U(1) is called an Abelian group because its product is commutative. For every  $g, h \in \text{U}(1)$ ,  $gh = hg$ .

It is now tempting to generalise this to other, in general non-commutative groups, which are called non-Abelian groups. It was the way how Yang and Mills discovered SU(2) gauge theories in 1954. Like for U(1) gauge theories they made the SU(2) transformation into a local one, where at every point the field can be transformed independently. (It should be noted that they were originally after describing the isospin symmetry, that relates protons to neutrons, which form a so-called isospin doublet.)

The simplest non-Abelian gauge group, for which no longer  $gh = hg$ , is SU(2). This group is well-know from the description of spin one-half particles. It has a two dimensional (spinor) representation, which can also be seen as a representation of the rotation group SO(3). As a local gauge theory it does no longer act on the spinor indices, but on indices related to some internal space, giving rise to so-called internal symmetries. The way the

gauge group  $G$  acts on the fields  $\Psi$  is described by a representation of the group  $G$ . A representation defines a mapping  $\rho$  from  $G$  to the space of linear mappings  $\text{Map}(V)$ , of the linear vector space  $V$  into itself.

$$\rho : G \rightarrow \text{Map}(V), \quad \rho(g) : V \rightarrow V \quad . \quad (18.2)$$

Mostly,  $V$  will be either  $\mathbb{R}^n$  or  $\mathbb{C}^n$ , in which case  $\rho(g)$  is resp. a real or a complex  $n \times n$  matrix. For  $\rho$  to be a representation, it has to preserve the group structure of  $G$

$$\rho(g)\rho(h) = \rho(gh) \quad , \quad \rho(1) = \text{id}_V \quad . \quad (18.3)$$

We will generally restrict the gauge symmetries to Lie-groups for which one can write any group element as an exponential of a Lie-algebra element

$$g \equiv \exp(X) \quad , \quad X \in L_G \quad . \quad (18.4)$$

This Lie-algebra has an associative, but non-commutative, bilinear product

$$(X, Y) \in L_G \times L_G \rightarrow [X, Y] \in L_G \quad . \quad (18.5)$$

The Campbell-Baker-Hausdorff formula expresses that the logarithm of  $\exp(X)\exp(Y)$  is an element of the Lie algebra, i.e. the product of two exponentials is again an exponential.

$$F(X, Y) \equiv \log(\exp(X)\exp(Y)) = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] + \frac{1}{12}[Y, [Y, X]] + \dots \in L_G \quad . \quad (18.6)$$

This formula will be of great help in finding a simple criterion for  $\rho$  to be a representation, satisfying eq. (18.3). Apart from the group structure of  $\text{Map}(V)$ , it also has a Lie-algebra structure (the commutator of two  $n \times n$  matrices is again a  $n \times n$  matrix). The representations of the group can be easily restricted to the Lie-algebra.

$$\rho : L_G \rightarrow \text{Map}(V) \quad , \quad (18.7)$$

in a way that preserves the Lie-algebra structure

$$\rho([X, Y]) = [\rho(X), \rho(Y)] = \rho(X)\rho(Y) - \rho(Y)\rho(X) \quad (18.8)$$

It is more or less by construction that we require

$$\rho(\exp(X)) = \exp(\rho(X)) \quad , \quad (18.9)$$

where on the left-hand side  $\rho$  is the group representation and on the right-hand side it is the Lie-algebra representation. Without causing too much confusion we can use the same symbol for the two objects. As a Lie-algebra forms a linear vector space we can define a basis on  $L_G$ .

$$Z = \sum_{a=1}^n z_a T^a \in L_G \quad , \quad z_a \in \mathbb{R} \text{ (or } \mathbb{C}) \quad , \quad T^a \in L_G. \quad (18.10)$$

In here  $n$  is the dimension of the Lie-algebra (and the Lie-group if, as we will assume throughout, the exponential is locally an invertible mapping). The commutator, also called Lie-product, is completely determined by the structure constants  $f_{abc}$

$$[T^a, T^b] = \sum_c f_{abc} T^c \quad . \quad (18.11)$$

Using the Jacobi identity

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \quad , \quad (18.12)$$

applied to  $X = T^a$ ,  $Y = T^b$  and  $Z = T^c$ , we find (from now on sums over repeated group indices are implicit)

$$f_{bcd}f_{ade} + f_{cad}f_{bde} + f_{abd}f_{cde} = 0 \quad . \quad (18.13)$$

This precisely coincides with the commutation relations of the so-called adjoint representation

$$(T_{\text{ad}}^a)_{bc} \equiv \rho_{\text{ad}}(T^a)_{bc} = -f_{abc} \quad . \quad (18.14)$$

Indeed, one easily verifies that

$$[\rho_{\text{ad}}(T^a), \rho_{\text{ad}}(T^b)] = f_{abc}\rho_{\text{ad}}(T^c) \quad . \quad (18.15)$$

In general, since a representation preserves the commutation relations, it also preserves the structure constants in terms of  $\rho(T^a) \equiv T_\rho^a$ , which forms a basis for the linear representation space which is contained in  $V$ . They are called the generators of the representation. With the help of eq. (18.6) we easily verify that  $\rho$  is a representation if and only if

$$[T_\rho^a, T_\rho^b] = f_{abc}T_\rho^c \quad . \quad (18.16)$$

This is because under the action of  $\rho$  one simply replaces  $T^a$  by  $T_\rho^a$

$$\rho(\exp(x_a T^a)) = \rho(\exp(X)) = \exp(\rho(X)) = \exp(x_a T_\rho^a) \quad . \quad (18.17)$$

Similarly, the Campbell-Baker-Hausdorff formula, when expressed with respect to the Lie-algebra basis  $\{T^a\}$

$$\exp(x_a T^a) \exp(y_b T^b) = \exp\left(\{x_a + y_a + \tfrac{1}{2}x_b y_c f_{abc} + \tfrac{1}{12}(x_d x_b y_c + y_d y_b x_c) f_{bce} f_{dea} + \dots\} T^a\right) . \quad (18.18)$$

directly determines the multiplication of the representation of group elements by replacing  $T^a$  by  $T_\rho^a$ , provided eq. (18.16) is satisfied. Note that the structure constants are antisymmetric with respect to the first two indices. They are also invariant under cyclic permutations of the indices. This follows from the cyclic property of the trace

$$f_{abd} \text{Tr}(T_\rho^d T_\rho^c) = \text{Tr}([T_\rho^a, T_\rho^b] T_\rho^c) = \text{Tr}(T_\rho^a [T_\rho^b, T_\rho^c]) = f_{bcd} \text{Tr}(T_\rho^a T_\rho^d) \quad , \quad (18.19)$$

and from the fact that for compact groups the generators can be normalised such that

$$\text{Tr}(T_{\text{fnd}}^a T_{\text{fnd}}^b) = -\tfrac{1}{2} \delta_{ab} \quad , \quad (18.20)$$

where  $T_{\text{fnd}}^a$  are the generators of the so-called fundamental or defining representation of the group  $G$ . This matrix representation is usually identified with the group (or algebra) itself, which till now was seen more as an abstract entity. The simplest example is  $\text{SU}(2)$ , the set of complex unitary  $2 \times 2$  matrices with unit determinant. Its fundamental representation coincides with the spinor or spin one-half representation. The structure constants and the generators of the fundamental and adjoint representations were considered in sect. 12 (see eq. (12.9))

$$T_{\text{fnd}}^a = -\tfrac{i}{2} \sigma^a \quad , \quad f_{abc} = \varepsilon_{abc} \quad , \quad \rho_{\text{ad}}(T^a) = -L^a \quad . \quad (18.21)$$

Because the Campbell-Baker-Hausdorff formula plays such a crucial role in the theory and in the practical implementation of group representations, we will now provide a more abstract derivation of eq. (18.6) to all orders. The proof simply states how in the Taylor expansion products of Lie-algebra elements are regrouped in multiple commutators. A crucial ingredient for deriving the Campbell-Baker-Hausdorff is the so-called *derivation*  $\mathcal{D}$ , that maps a product of Lie-algebra elements into a multiple commutator.

$$\mathcal{D}X = X \quad , \quad \mathcal{D}X_{i_1}X_{i_2} \cdots X_{i_s} \equiv [X_{i_1}, [X_{i_2}, \cdots [X_{i_{s-1}}, X_{i_s}] \cdots]] \quad , \quad s > 1 \quad . \quad (18.22)$$

We also define for these products the adjoint map,  $\text{ad}$ , introduced in eq. (12.12)

$$\text{ad}X_{i_1}X_{i_2} \cdots X_{i_s} \equiv \text{ad}X_{i_1}\text{ad}X_{i_2} \cdots \text{ad}X_{i_s} \quad , \quad (18.23)$$

which is easily seen to satisfy

$$\text{ad}([X, Y]) = [\text{ad}X, \text{ad}Y] \quad . \quad (18.24)$$

It is more or less by definition that for any two products  $u$  and  $v$  of Lie-algebra elements

$$\mathcal{D}(uv) = \text{ad}u\mathcal{D}v \quad . \quad (18.25)$$

For two Lie-algebra elements  $X$  and  $Y$ , it can easily be shown that

$$\mathcal{D}[X, Y] = \mathcal{D}(XY) - \mathcal{D}(YX) = \text{ad}X\mathcal{D}Y - \text{ad}Y\mathcal{D}X = [X, \mathcal{D}Y] + [\mathcal{D}X, Y] \quad (18.26)$$

and this allows us to prove by induction that a monomial  $Q$  (a polynomial of which all terms are of the same order) of degree  $m$  in terms of Lie-algebra elements  $X_i$ ,  $i = 1, 2, \dots, s$  is an element of the Lie-algebra (i.e. can be written as multiple commutators, called a Lie-monomial) if and only if  $\mathcal{D}Q = mQ$ . If this equation is satisfied it is clear from the definition of a derivation that  $Q$  is a Lie-monomial. So it is sufficient to prove that the equation is satisfied for  $Q$  a Lie-monomial. In that case  $Q$  is a sum of terms, each of which can be written as  $\text{ad}(X_{i_1})Q^{(1)}$  with  $Q^{(1)}$  a Lie-monomial of degree  $m-1$ . Using eq. (18.26) therefore yields  $\mathcal{D}\text{ad}(X_{i_1})Q^{(1)} = \text{ad}(X_{i_1})\mathcal{D}Q^{(1)} + \text{ad}(X_{i_1})Q^{(1)}$ . Induction in  $m$  gives the required result.

Now it is trivial to regroup the terms in the Taylor expansion of eq. (18.6) in multiple commutators. From the fact that any group element can be written as the exponent of a Lie-algebra element, we know that  $F(X, Y) \in L_G$  (at the worst one needs to restrict  $X$  and  $Y$  to sufficiently small neighbourhoods of the origin in  $L_G$ ). Consequently, each term in the Taylor expansion of  $F(X, Y)$  of order  $m$  (denoted by  $F_m(X, Y)$ , being a monomial in  $X$  and  $Y$ ) has to be an element of the Lie-algebra and satisfies

$$F(X, Y) \equiv \sum_m F_m(X, Y) \quad , \quad F_m(X, Y) = \frac{1}{m} \mathcal{D}F_m(X, Y) \quad . \quad (18.27)$$

It is not difficult to work out the Taylor expansion for  $F(X, Y)$

$$F(X, Y) \equiv \log(\exp(X)\exp(Y)) = \log\left(1 + \sum_{i+j>0} \frac{X^i Y^j}{i!j!}\right) = \sum_k \frac{(-1)^{k-1}}{k} \left(\sum_{i+j>0} \frac{X^i Y^j}{i!j!}\right)^k \quad , \quad (18.28)$$

from which we easily obtain the explicit expression for the Campbell-Baker-Hausdorff formula in terms of multiple commutators,

$$F(X, Y) = \sum_m \sum_{\{k, \sum_{j=1}^k p_j + q_j = m, p_j + q_j > 0\}} \frac{(-1)^{k-1}}{km} \frac{\mathcal{D}(X^{p_1} Y^{q_1} X^{p_2} Y^{q_2} \cdots X^{p_k} Y^{q_k})}{p_1! q_1! p_2! q_2! \cdots p_k! q_k!} \quad . \quad (18.29)$$

We leave it to the industrious student to verify that

$$\begin{aligned}
F_1(X, Y) &= X + Y, \quad F_2(X, Y) = \tfrac{1}{2}[X, Y], \quad F_3(X, Y) = \tfrac{1}{12} \{(\text{ad} X)^2 Y + (\text{ad} Y)^2 X\}, \\
F_4(X, Y) &= -\tfrac{1}{24} \text{ad} X \text{ad} Y \text{ad} X(Y), \\
F_5(X, Y) &= -\tfrac{1}{720} \{(\text{ad} X)^4 Y + (\text{ad} Y)^4 X\} + \tfrac{1}{360} \{\text{ad} X(\text{ad} Y)^3 X + \text{ad} Y(\text{ad} X)^3 Y\} \\
&\quad - \tfrac{1}{120} \{\text{ad} X \text{ad} Y(\text{ad} X)^2 Y + \text{ad} Y \text{ad} X(\text{ad} Y)^3 X\}. \quad (18.30)
\end{aligned}$$

After this intermezzo we return to the issue of constructing non-Abelian gauge theories. The simplest way is by generalising first the covariant derivative.  $U(1)$  gauge transformations act on a complex field as in eq. (18.1), and the covariant derivative is designed such that

$$D_\mu \Psi(x) \rightarrow g(x) D_\mu \Psi(x) \quad . \quad (18.31)$$

Since the gauge field transforms as in eq. (17.3), this is easily seen to imply that the covariant derivative is defined as in eq. (17.2) (these formula are of course also valid for complex scalar fields, compare eq. (3.36)). For a non-Abelian gauge theory we consider first a field  $\Psi$  that transforms as an irreducible representation (i.e. only the origin is left invariant under the action of *all* gauge group elements)

$$\Psi \rightarrow {}^g \Psi \equiv \rho(g) \Psi \quad . \quad (18.32)$$

In the following, as in the literature, we shall no longer make a distinction between  $g$  and  $\rho(g)$ . It will always be clear from the context what is intended. The vector potential should now be an element of the Lie-algebra  $L_G$  (more precisely a representation thereof)

$$A_\mu = A_\mu^a T^a \quad . \quad (18.33)$$

For  $U(1)$ , which is one-dimensional, we need to define  $T^1 \equiv i$ . The Lie-algebra of the group consisting of the unimodular complex numbers is the set of imaginary numbers,  $L_{U(1)} = i\mathbb{R}$ . Note that as an exception this generator is normalised different from eq. (18.20), so as not to introduce unconventional normalisations elsewhere. The real valued vector potential  $A_\mu$  will now be denoted by  $A_\mu^1$  and we see that under a gauge transformation

$$A_\mu \rightarrow {}^g A_\mu = g A_\mu g^{-1} - q^{-1} (\partial_\mu g) g^{-1} = g A_\mu g^{-1} + q^{-1} g \partial_\mu (g^{-1}) \quad . \quad (18.34)$$

This is the form that generalises directly to the non-Abelian gauge groups with the covariant derivative defined by

$$D_\mu \Psi = (\partial_\mu + q A_\mu) \Psi \quad , \quad (18.35)$$

where  $A_\mu \equiv A_\mu^a T_\rho^a$  is a matrix acting on the fields  $\Psi$ . We leave it as an exercise to verify that under a gauge transformation, eq. (18.31) remains valid for the non-Abelian case.

It is now a trivial matter to construct a Lagrangian that is invariant under local gauge transformation. For a scalar field  $\Psi$  one has

$$\mathcal{L}_\Psi = (D_\mu \Psi)^\dagger D^\mu \Psi - m^2 \Psi^\dagger \Psi \quad , \quad (18.36)$$

whereas if  $\Psi$  is a Dirac field, carrying both spinor (representation of the Lorentz group) and group indices, one has

$$\mathcal{L}_\Psi = \bar{\Psi} (i \gamma^\mu D_\mu - m) \Psi \quad , \quad \bar{\Psi} = \Psi^\dagger \gamma^0 \quad , \quad (18.37)$$



where  $\Psi^\dagger$  is the hermitian conjugate both with respect to the spinor and the group (representation) indices.

The part of the Lagrangian that describes the self interactions of the vector field  $A_\mu$  has to be invariant under local gauge transformations too. In that respect U(1) or Abelian gauge theories are special, since the homogeneous part of the transformation of the vector potential is trivial,  $gA_\mu g^{-1} = A_\mu$ . For non-Abelian gauge transformations this is no longer true. For U(1) one easily verifies that

$$D_\mu D_\nu \Psi - D_\nu D_\mu \Psi \equiv [D_\mu, D_\nu] \Psi = iqF_{\mu\nu}^1 \Psi \quad , \quad (18.38)$$

where  $F_{\mu\nu}^1 = \partial_\mu A_\nu^1 - \partial_\nu A_\mu^1$  is the electromagnetic field strength, compare eq. (3.27). Because the covariant derivative transforms in a simple way under gauge transformations, this formula can be directly generalised to non-Abelian gauge theories

$$F_{\mu\nu} \equiv q^{-1} [D_\mu, D_\nu] \xrightarrow{g} g F_{\mu\nu} g^{-1} \quad . \quad (18.39)$$

For U(1), where  $g$  is a number, this means that the field strength is gauge invariant, as was noted before. For non-Abelian gauge theories the field strength itself is not gauge invariant. Nevertheless, it is simple to construct a gauge invariant action for the gauge field

$$\mathcal{L}_A = \frac{1}{2} \text{Tr} (F_{\mu\nu} F^{\mu\nu}) \equiv -\frac{1}{4} F_{\mu\nu}^a F_a^{\mu\nu} \quad , \quad (18.40)$$

where  $F_{\mu\nu}^a$  are the components of the field strength with respect to the Lie-algebra basis,

$$F_{\mu\nu} \equiv F_{\mu\nu}^a T^a = (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + q f_{abc} A_\mu^b A_\nu^c) T^a \quad . \quad (18.41)$$

We see from  $\mathcal{L}_A$  and  $\mathcal{L}_\Psi$  that  $q$  plays the role of an expansion parameter. For  $q = 0$  we have  $n = \dim(G)$  non-interacting photon fields. They couple with strength  $q$  to the scalar or Dirac fields. For non-Abelian gauge theories, in addition the vector field couples to itself. These *self-interactions* guarantee that there is invariance under the gauge group  $G$ , which is much bigger than U(1) <sup>$n$</sup> , which is the symmetry that seems implied by the  $q = 0$  limit. The non-Abelian gauge invariance fixes the “charges” of the fields with respect to each of these U(1) gauge factors. Without the non-Abelian gauge symmetry there would have been  $n$  independent “charges”.

The Lagrangian  $\mathcal{L}_A$  is the one that was discovered in 1954 by C.N. Yang and R.L. Mills. The Euler-Lagrange equations for the Lagrangian  $\mathcal{L}_A$  are called the Yang-Mills equations. One easily shows that

$$\partial_\mu F_a^{\mu\nu} + q f_{abc} A_\mu^b F_c^{\mu\nu} = 0 \quad \text{or} \quad [D_\mu, F^{\mu\nu}] \equiv \partial_\mu F^{\mu\nu} + q[A_\mu, F^{\mu\nu}] = 0 \quad . \quad (18.42)$$

For the coupling to fermions we read off from eq. (18.37) what the current for the Yang-Mills field is

$$\mathcal{L}_\Psi = \bar{\Psi}(i\gamma^\mu D_\mu - m)\Psi = \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi + iqA_\mu^a \bar{\Psi}\gamma^\mu T^a \Psi \quad . \quad (18.43)$$

The current is therefore given by

$$J_\mu^a \equiv -iq\bar{\Psi}\gamma_\mu T^a \Psi \quad . \quad (18.44)$$

The coupled Yang-Mills equations read

$$\partial_\mu F_a^{\mu\nu} + q f_{abc} A_\mu^b F_c^{\mu\nu} = J_\mu^a \quad . \quad (18.45)$$

In problem 31 it will be shown that the current is not gauge invariant, unlike for Abelian gauge symmetries. Closely related is the fact that it is no longer true that the current is conserved, i.e.  $\partial^\mu J_\mu^a \neq 0$ . Instead, it will be shown in problem 31 that  $\partial^\mu J_\mu^a + q f_{abc} A_b^\mu J_\mu^c = 0$ .

## 19 The Higgs mechanism

We have seen in problem 30 that the four-Fermi interaction in good approximation can be written in terms of the exchange of a heavy vector particle. In lowest order we have resp. the diagrams in fig. 12a and fig. 12b



fig. 12

The first diagram comes from a four fermion interaction term that can be written in terms of the product of two currents  $J_\mu J^\mu$ , where  $J_\mu = \bar{\Psi}\gamma_\mu\Psi$ . Here each fermion line typically carries its own flavour index, which was suppressed for simplicity. Fig. 12b can be seen to effectively correspond to

$$-\tilde{J}^\mu(-k) \left( \frac{g_{\mu\nu} - k_\mu k_\nu / M^2}{k^2 - M^2 + i\varepsilon} \right) \tilde{J}^\nu(k) \quad (19.1)$$

At values of the exchanged momentum  $k^2 \ll M^2$ , one will not see a difference between these two processes, provided the coupling constant for the four-Fermi interactions (fig. 12a) is chosen suitably (see problem 30). This is because for small  $k^2$ , the propagator can be replaced by  $g_{\mu\nu}/M^2$ , which indeed converts eq. (19.1) to  $J^\mu J_\mu/M^2$ . It shows that the four-Fermi coupling constant is proportional to  $M^{-2}$ , such that its weakness is explained by the heavy mass of the vector particle that mediates the interactions. Examples of four-Fermi interactions occur in the theory of  $\beta$ -decay, for example the decay of a neutron into a proton, an electron and an anti-neutrino. In that case the current also contains a  $\gamma^5$  (problem 40).

It turns out that the four-Fermi theory cannot be renormalised. Its quantum corrections give rise to an infinite number of divergent terms, that can not be reabsorbed in a redefinition of a Lagrangian with a finite number of interactions. With the interaction resolved at higher energies by the exchange of a massive vector particle, the situation is considerably better. But it becomes crucial for the currents in question to be conserved, such that the  $k_\mu k_\nu$  part in the propagator has no effect. It would give rise to violations of unitarity in the scattering matrix at high energies (the  $\sigma$  field defined in eq. (16.15) has the wrong sign for its kinetic part). To enforce current conservation, we typically use gauge invariance. But gauge invariance would protect the vector particle from having a mass. The big puzzle therefore was how to describe a massive vector particle that is nevertheless associated to the vector potential of a gauge field.

The answer can be found in the theory of superconductivity, which prevents magnetic field lines to penetrate in a superconducting sample. If there is, however, penetration in the form of a quantised flux tube, the magnetic field decays exponentially outside the flux tube. This would indicate a mass term for the electromagnetic field within the superconductor. The Landau-Ginzburg theory that gives an effective description of this phenomenon (the microscopic description being given by the BCS theory of Cooper pairs) precisely coincides with scalar quantum electrodynamics.

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\varphi)^* D^\mu\varphi - \kappa\varphi^*\varphi - \frac{1}{4}\lambda(\varphi^*\varphi)^2 \quad (19.2)$$

In the Landau-Ginzburg theory  $\varphi$  describes the Cooper pairs. It is also called the order parameter of the BCS theory. In usual scalar quantum electrodynamics we would put  $\kappa = m^2$ , where  $m$  is the mass of the charged scalar field. But in the Landau-Ginzburg theory of

superconductivity, it happens to be the case that  $\kappa$  is *negative*. In that case the potential  $V(\varphi)$  for the scalar field has the shape of a Mexican hat.

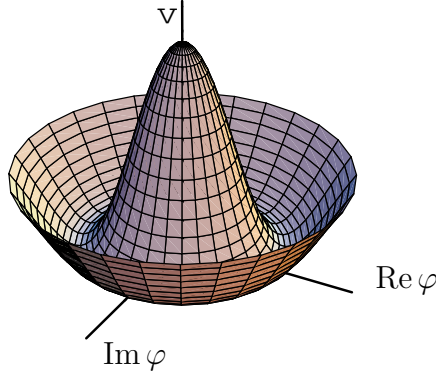


fig. 13

The minimum of the potential is no longer at  $\varphi = 0$ , but at  $\varphi^* \varphi = -2\kappa/\lambda$ , and is *independent* of the phase of  $\varphi$ . To find the physical excitations of this theory we have to expand around the minimum. With a global phase rotation we can chose the point to expand around to be real,

$$\varphi_0 = \sqrt{-2\kappa/\lambda} \quad . \quad (19.3)$$

But this immediately implies that the terms quadratic in the gauge field give rise to a mass term for the photon field

$$|D_\mu \varphi_0|^2 = e^2 \varphi_0^2 A_\mu A^\mu \equiv \frac{1}{2} M^2 A_\mu A^\mu \quad , \quad M = 2e\sqrt{-\kappa/\lambda} \quad . \quad (19.4)$$

Furthermore, from the degeneracy of the minimum of the potential it follows that the fluctuation along that minimum (the phase in  $\varphi = \varphi_0 \exp(i\chi)$ ) has no mass (this is related to the famous Goldstone theorem, which states that if choosing a minimum of the potential would break the symmetry, called *spontaneous* symmetry breaking, there is always a massless particle). However, this phase  $\chi$  is precisely related to the gauge invariance, and can be rotated away by a gauge transformation. On the one hand  $\chi$  corresponds to a massless excitation, on the other hand it is the unphysical longitudinal component of the gauge field. But the photon became massive, and has to develop an additional physical polarisation, which is precisely the longitudinal component. In a prosaic way one states that the massless excitation (called a *would-be* Goldstone boson) was “eaten” by the longitudinal component of the photon, which in the process got a mass (“got fat”).

This means we have four massive degrees of freedom, three for the massive vector particle and one for the absolute value of the complex scalar field (its mass is determined by the quadratic part of the potential in the radial direction at  $\varphi = \varphi_0$ ). This is exactly the same number as for ordinary scalar electrodynamics where  $\kappa > 0$ , because in that case the massless photon has only two degrees of freedom, whereas the complex scalar field represents two massive real scalar fields. It looks, however, that there is a discontinuity in the description of these degrees of freedom, when approaching  $\kappa = 0$ . But the interpretation of the phase of the complex field as a longitudinal component of the vector field is simply a matter of choosing a particular gauge. To count the number of degrees of freedom we implicitly made two different gauge choices

$$\begin{array}{lll} \kappa > 0 : & \partial_\mu A^\mu = 0, & \text{Lorentz gauge} \quad , \\ \kappa < 0 : & \text{Im } \varphi = 0, & \text{Unitary gauge} \quad . \end{array} \quad (19.5)$$

There is a gauge, called the 't Hooft gauge, that interpolates between these two gauges

$$\mathcal{F} \equiv \partial_\mu A^\mu - 2ie\xi\varphi_0 \text{Im } \varphi = 0, \quad \text{'t Hooft gauge} \quad . \quad (19.6)$$

Rather than adding to the Lagrangian the gauge fixing term  $\mathcal{L}_{\text{gf}} = -\frac{1}{2}\alpha(\partial_\mu A^\mu)^2$ , one adds  $\mathcal{L}_{\text{gf}} = -\frac{1}{2}\alpha\mathcal{F}^2$ . At  $\xi = 0$  this corresponds to the Lorentz gauge, at  $\xi = \infty$  to the unitary gauge. For the choice 't Hooft made ( $\xi = 1/\alpha$ ) the terms that mix  $(\varphi - \varphi_0)$  and  $A_\mu$  at quadratic order disappear and one easily reads off the masses. Gauge fixing will be discussed in the next section, where it will be shown how extra unphysical degrees of freedom appear in the path integral, so as to cancel the unphysical components of the gauge and scalar fields. The scalar field, whose interactions give the gauge field a mass, is called the Higgs field. Problems 34 and 35 discuss the Higgs mechanism in detail for the Georgi-Glashow model, which is a non-Abelian gauge theory with gauge group  $\text{SO}(3)$ , coupled to an  $\text{SO}(3)$ -vector of scalar fields  $\varphi^a$ .

## 20 Gauge fixing and ghosts

The quantisation of gauge theories in the path integral formalism requires more discussion, since the gauge condition (like the Lorentz gauge  $\partial_\mu A^\mu = 0$ ) seems to remove only one degree of freedom of the two that are eliminated in the Hamiltonian formulation (see section 16). From a simple example it is easily demonstrated what the effect of gauge fixing on a (path) integral is. For this we take  $f(\vec{x})$  to be a function on  $\mathbb{R}^3$ , which is invariant under rotations around the origin, such that it is a function  $f(r)$  of the radius  $r = |\vec{x}|$  only. The symmetry group is hence  $\text{SO}(3)$  and we can attempt to compute the integral  $\int d_3\vec{x} f(\vec{x})$  by introducing a “gauge” fixing condition like  $x_2 = x_3 = 0$ . But it is clear that

$$\int d_3\vec{x} f(\vec{x}) \neq \int d_3\vec{x} \delta(x_2)\delta(x_3)f(\vec{x}) = \int dx_1 f(x_1) \quad (20.1)$$

We know very well that we need a Jacobian factor for the radial integral

$$\int d_3\vec{x} f(\vec{x}) = 4\pi \int_0^\infty r^2 dr f(r) \quad . \quad (20.2)$$

This Jacobian, arising in the change of variable to the invariant radial coordinates and the angular coordinates, can be properly incorporated following the method introduced by Faddeev and Popov. The starting point is a straightforward generalisation of the identity  $\int dx |f'(x)|\delta(f(x)) = 1$ , assuming the equation  $f(x) = 0$  to have precisely one solution (in a sense the right-hand side of the equation counts the number of zeros). It reads

$$1 = \int \mathcal{D}g |\det(M(^gA))| \delta(\mathcal{F}(^gA)) \quad , \quad (20.3)$$

where  $\mathcal{F}(A) \in L_G$  is the gauge fixing function (with the gauge condition  $\mathcal{F}(A) = 0$ , e.g.  $\mathcal{F}(A) = \partial_\mu A^\mu = 0$ ). The gauge transformation  $g$  of the gauge field  $A$  is indicated by  $^gA$ , see eq. (18.34). Furthermore,  $M(A) : L_G \rightarrow L_G$  plays the role of the Jacobian,

$$M(^gA) = \frac{\partial \mathcal{F}(^gA)}{\partial g} \equiv \frac{\partial \mathcal{F}(e^{Xg}A)}{\partial X} \quad \text{at } X = 0 \quad . \quad (20.4)$$

Equivalently, with respect to the Lie-algebra basis, where  $\mathcal{F}(A) \equiv \mathcal{F}_a(A)T^a$ , one has

$$M_{ab}(A) = \frac{d\mathcal{F}_a(\exp(tT^b)A)}{dt} \quad \text{at } t = 0 \quad . \quad (20.5)$$

To relate this to the previous equation, one makes use of the fact that

$$h({}^g A) = (hg)A \quad , \quad (20.6)$$

which states that two successive gauge transformations,  $g$  and  $h$ , give the same result as a single gauge transformation with  $hg$ .

As an example we consider the Lorentz gauge, with  $\mathcal{F}(A) = \partial_\mu A^\mu$ , for which

$$\mathcal{F}(\exp(X)A) - \mathcal{F}(A) = -q^{-1}\partial_\mu D_{\text{ad}}^\mu(A)(X) + \mathcal{O}(X^2) \quad , \quad (20.7)$$

where  $D_{\text{ad}}^\mu(A)$  is the covariant derivative in the adjoint representation

$$D_{\text{ad}}^\mu(A)(X) \equiv \partial^\mu X + q[A^\mu, X] \quad . \quad (20.8)$$

With respect to the Lie-algebra basis this gives

$$qM_{ab}(A) = -\delta_{ab}\partial_\mu\partial^\mu + qf_{abc}(\partial^\mu A_\mu^c + A_\mu^c\partial^\mu) \quad . \quad (20.9)$$

For an Abelian gauge theory the structure constants  $f_{abc}$  vanish and  $M(A)$  becomes independent of the gauge field. This means that  $\det(M(A))$  can be absorbed in an overall normalisation of the path-integral. For non-Abelian gauge theories this is no longer possible. Before describing how the  $A$  dependence of  $\det(M(A))$  is incorporated, it is important to note that we assumed the gauge condition  $\mathcal{F}({}^g A) = 0$  to have precisely one solution, which can be arranged with the help of eq. (20.6) to occur at  $g = 1$ , in which case  $A$  is said to satisfy the gauge condition. This is in general not correct, as was discovered by Gribov. Even in our simple problem on  $\mathbb{R}^3$ , the gauge condition  $x_2 = x_3 = 0$  does not uniquely specify the gauge, because we can go from  $(r, 0, 0)$  to  $(-r, 0, 0)$  through a rotation over 180 degrees. We have to introduce a further restriction to get the identity

$$\int d_3\vec{x} f(\vec{x}) = 4\pi \int d_3\vec{x} x_1^2 \delta(x_2)\delta(x_3)\theta(x_1)f(\vec{x}) \quad , \quad (20.10)$$

where  $\theta(x) = 0$  for  $x < 0$  and  $\theta(x) = 1$  for  $x \geq 0$ . In perturbation theory only the gauge fields near the origin in field space are relevant, and gauge conditions are chosen so as to avoid this problem in a small neighbourhood of the origin. The Lorentz gauge is such a gauge condition, and the gauge fixing or Gribov ambiguity is not an issue for computing quantities in perturbation theory in  $q$ .

We still need to define what we mean with  $\mathcal{D}g$  in eq. (20.3). It stands for the integration measure  $\prod_x dg(x)$ , with  $dg(x)$  for every  $x$  defined as the so-called Haar-measure on the group. It is best described in the example of  $\text{SU}(2)$ , which as a space is isomorphic with  $S^3$ . When  $S^3$  is embedded in  $\mathbb{R}^4$  as a unit sphere,  $n_\mu^2 = 1$ , it is not too difficult to see that  $g = n_4 + i\sigma_k n_k$  gives an element of  $\text{SU}(2)$ , whereas  $\exp(i\chi s_k \sigma_k) = \cos(\chi) + i\sin(\chi)s_k \sigma_k$ , with  $s_k^2 = 1$ , shows that any element of  $\text{SU}(2)$  can be written in terms of  $n_\mu$ . The Haar-measure coincides with the standard integration measure on  $S^3$ ,  $\int d_4 n \delta(n_\mu^2 - 1)$ . The Haar-measure is in general invariant under the change of variables  $g \rightarrow hg$  and  $g \rightarrow gh$ , for  $h$  some fixed group element. We can insert eq. (20.3) in the path integral to obtain

$$Z = \int \mathcal{D}A_\mu \mathcal{D}g \det(M({}^g A)) \delta(\mathcal{F}({}^g A)) \exp(iS(A)) \quad . \quad (20.11)$$

We now use that the action  $S(A)$  is invariant under gauge transformations. We leave it as an exercise to verify that likewise  $\mathcal{D}A_\mu$  is invariant under the change of variables  $A \rightarrow {}^g A$ , which trivially implies that

$$Z = \int \mathcal{D}A_\mu \mathcal{D}g \det(M(A)) \delta(\mathcal{F}(A)) \exp(iS(A)) \quad . \quad (20.12)$$

The dependence of the integrand on  $g$  has disappeared, and the integration over  $g$  gives an overall (infinite) normalisation factor, which is irrelevant. We next note that  $Z$  has to be independent of the gauge fixing function  $\mathcal{F}$ , in particular  $\mathcal{F}(A) - Y$  is just as good for the gauge fixing (provided of course we show that  $\mathcal{F}(^g A) = Y$  has a solution). This modification does not affect the so-called Faddeev-Popov operator  $M(A)$  and we find

$$Z = \int \mathcal{D}A_\mu \det(M(A)) \delta(\mathcal{F}(A) - Y) \exp(iS(A)) \quad , \quad (20.13)$$

independent of  $Y$ . Suitably normalising  $\mathcal{D}Y$  we can define

$$\int \mathcal{D}Y \exp\left(-\frac{\alpha i}{2} \int d_4x Y_a^2(x)\right) = 1 \quad , \quad (20.14)$$

which combined with the previous equation gives

$$\begin{aligned} Z &= \int \mathcal{D}Y \mathcal{D}A_\mu \det(M(A)) \delta(\mathcal{F}(A) - Y) \exp\left(i \int d_4x \mathcal{L}(A) - \frac{\alpha}{2} Y_a^2(x)\right) \\ &= \int \mathcal{D}A_\mu \det(M(A)) \exp\left(i \int d_4x \mathcal{L}(A) - \frac{\alpha}{2} \mathcal{F}_a^2(A)\right) . \end{aligned} \quad (20.15)$$

For U(1) gauge theories with  $\mathcal{F}(A) = \partial_\mu A^\mu$  this precisely reproduces the action of eq. (4.22) in the Lorentz gauge, and in that case  $\det(M(A))$  is a constant.

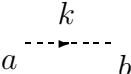
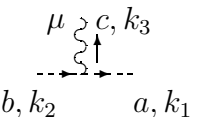
For non-Abelian gauge theories we are left with the task of computing  $\det(M(A))$  for each  $A$ , which is no longer constant. But here the path integral over Grassmann variables comes to the rescue. In problem 25 we have seen that

$$\int \mathcal{D}\bar{\eta} \mathcal{D}\eta \exp\left(i \int d_4x \bar{\eta}^a(x) q M_{ab}(A) \eta^b(x)\right) = \det(M(A)) \quad , \quad (20.16)$$

up to an overall normalisation. This implies that the path integral can also be written in the Lorentz gauge as

$$Z = \int \mathcal{D}A_\mu \mathcal{D}\bar{\eta} \mathcal{D}\eta \exp\left\{i \int d_4x \mathcal{L}(A) - \frac{\alpha}{2} (\partial_\mu A_a^\mu(x))^2 + \partial^\mu \bar{\eta}^a(x) [\partial_\mu \eta^a(x) + q f_{abc} A_\mu^b(x) \eta^c(x)]\right\}. \quad (20.17)$$

Since  $\bar{\eta}$  and  $\eta$  are auxiliary fields, they should never appear as external lines. They are therefore called ghosts. Ghosts can only appear in loops and because of the fermionic nature of the ghost variables, every such loop gives a minus sign. The Feynman rules for the Lorentz gauge are given in the following table.

table 7	
	<i>no external ghost lines</i>
 $\equiv (qM)^{-1}(A=0)_{ab} = \frac{\delta_{ab}}{k^2 + i\epsilon}$	ghost propagator (Lorentz gauge)
 $\equiv -iq f_{abc} k_1^\mu$	ghost vertex (Lorentz gauge)
$-1 \times i \int \frac{d_4k}{(2\pi)^4}$	loop factor

Because one can easily derive that for a complex scalar field (up to an overall constant)

$$\int \mathcal{D}\varphi^* \mathcal{D}\varphi \exp \left( i \int d_4x \varphi_a^*(x) q M^{ab}(A) \varphi_b(x) \right) = \frac{1}{\det(M(A))} \quad , \quad (20.18)$$

we can view a ghost as the elimination of a complex degree of freedom. It is in this way that in the path integral the two unphysical degrees of freedom of a Lorentz vector are eliminated. For QED both the ghost and the unphysical degrees of freedom have no interactions, and can not appear as external lines either, which is why in QED the introduction of ghosts was never necessary for a consistent description of the theory. For non-Abelian gauge theories, because of the interaction of the ghost with the gauge field, ghosts can no longer be ignored. To have the ghosts eliminate the unphysical degrees of freedom, one should have the “masses” (poles) of the ghosts to coincide with the “masses” of the unphysical degrees of freedom. Furthermore the couplings of the ghost and unphysical fields to the physical fields should be related. This is verified explicitly for the Georgi-Glashow model in problem 35. In general it is guaranteed by the existence of an extra symmetry, discovered by Becchi, Rouet and Stora, called the BRS symmetry  $s$ , which for example acts on the gauge field as follows

$$sA^\mu = D_{\text{ad}}^\mu \eta \quad . \quad (20.19)$$

This is precisely an infinitesimal gauge transformation. For more details see Itzykson and Zuber, section 12.4-1.

## 21 The Standard Model

The standard model describes the electromagnetic and weak interactions, unified in the so-called electro-weak theory of Glashow-Weinberg-Salam with the gauge group  $U(1) \times SU(2)$  and the strong interactions, known as Quantum Chromodynamics (QCD) with gauge group  $SU(3)$ . Theory and experiment, where tested, agree very well up to about 100 GeV, the energies reached by presentday accelerators. Now the top quark has been found, at a mass of 174 GeV, only the Higgs particle remains to be detected. Its mass should be smaller than 1000 GeV (i.e. 1 TeV=Terra electronvolt) according to presentday theoretical insight. Gravitation has been left out so far. Its natural scale in energy where quantum effects would become important is the Planck energy,  $E_{pl} = \sqrt{\hbar c^5/G} \approx 10^{19}$  GeV. It is very well possible that a number of the fundamental parameters in the standard model will be determined, either directly or indirectly, by gravitational interactions. The standard model should then be considered as an effective field theory. The theory for which the standard model describes its effective low-energy behaviour is called a unified theory. An intermediate stage, which does not yet include gravity is a so-called Grand Unified Theory (GUT). The simplest version unifies the electro-weak and strong interactions using a gauge group  $SU(5)$  (which has  $U(1) \times SU(2) \times SU(3)$  as a subgroup), thereby reducing the number of free parameters considerably. These GUT's predict proton decay, albeit at the tremendously low rate of one decay in every  $10^{30-31}$  years. Nevertheless, a swimming pool of  $(10\text{ m})^3$  contains enough protons to verify that the proton decay is slower than can be comfortably accommodated by GUT's. Candidates that unify the standard model with gravity in the form of string theories and supergravity have been unable to provide predictions that either rule them out experimentally or provide evidence in favour of these theories. Much is therefore still to be discovered, in particular because theoretical insight of the last ten years has shown that a Higgs field is most likely not fundamental, although it is not yet ruled out that it will show its structure

only at Planck energies. If that is the case the mass of the Higgs should, however, not be much bigger than 100 GeV.

The standard model consists of gauge fields  $B_\mu$  (for U(1)),  $W_\mu^a$  (for SU(2)) and  $A_\mu^a$  (for SU(3), where  $a$  runs from 1 to 8, to be discussed later). These gauge fields have interactions with a Higgs field  $\phi \in \mathcal{C}^2$ , which transforms under SU(2) as a spin one-half representation (i.e. the fundamental representation) with a coupling constant  $g$ . Under U(1) this Higgs field transforms with a coupling constant  $-\frac{1}{2}g'$ , whereas it is neutral under SU(3). These couplings are represented in the covariant derivative

$$D_\mu \phi = \partial_\mu \phi - \frac{i}{2}g' B_\mu \phi - \frac{ig}{2} \sum_{a=1}^3 W_\mu^a \sigma_a \phi \quad . \quad (21.1)$$

The potential for the Higgs field causes spontaneous breaking of part of the symmetries

$$V(\phi) = \frac{\lambda}{4}(\phi^\dagger \phi - F^2)^2 = \kappa \phi^\dagger \phi + \frac{\lambda}{4}(\phi^\dagger \phi)^2 + \text{const.} \quad , \quad (21.2)$$

where  $\kappa \equiv -\frac{1}{2}\lambda F^2$ . In this case the minimum of the potential, also called the vacuum, is degenerate on a three dimensional sphere, specified by  $\phi^\dagger \phi = F^2$  ( $\phi \in \mathcal{C}^2 \sim \mathbb{R}^4$ ), which would give rise to three massless scalar particles according to the Goldstone theorem, but all three will be “eaten” by longitudinal components of the gauge fields to which the Higgs field couples. We note that there are four gauge field components,  $B_\mu$  and  $W_\mu^a$  for  $a = 1, 2$  and 3. Indeed one combination among these four will not have something to “eat” and will therefore stay massless. It plays the role of the photon field as we got to know it in QED. To see this write

$$\phi = \begin{pmatrix} 0 \\ F \end{pmatrix} + \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \equiv \phi_0 + \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \quad , \quad (21.3)$$

such that

$$(D_\mu \phi_0)^\dagger D^\mu \phi_0 = \frac{g^2 F^2}{4} [(W_\mu^1)^2 + (W_\mu^2)^2] + \frac{F^2}{4} [g' B_\mu - g W_\mu^3]^2 \quad . \quad (21.4)$$

Apparently, the vector fields  $W_\mu^{1,2}$  will have a mass  $M_W = \frac{1}{2}\sqrt{2}gF$ , whereas the *linear combination*

$$Z_\mu = \frac{g' B_\mu - g W_\mu^3}{\sqrt{g^2 + g'^2}} = \sin \theta_W B_\mu - \cos \theta_W W_\mu^3 \quad , \quad \tan \theta_W = \frac{g'}{g} \quad , \quad (21.5)$$

receives a mass  $m_Z = \frac{1}{2}\sqrt{2}F(g^2 + g'^2)^{\frac{1}{2}} = M_W / \cos \theta_W$ . The linear combination perpendicular to  $Z_\mu$ ,

$$A_\mu^{\text{em}} = \cos \theta_W B_\mu + \sin \theta_W W_\mu^3 \quad , \quad (21.6)$$

remains massless. This gauge field defines a U(1) subgroup of SU(2)×U(1) that leaves  $\phi_0$  invariant. This U(1) subgroup is a combination of the U(1) subgroup of SU(2) generated by  $\exp(i\chi\sigma_3)$  and the phase rotations  $\exp(i\chi)$  associated with the explicit U(1) group with  $B_\mu$  as its gauge field. It is trivial to verify that the product of these group elements,  $\exp(i\chi)\exp(i\chi\sigma_3)$  indeed leaves  $\phi_0$  invariant. The gauge symmetry associated to this so-called diagonal U(1) subgroup therefore remains *unbroken* and corresponds to electromagnetism.

The Higgs field has three massless components  $\text{Re } \varphi_1$ ,  $\text{Im } \varphi_1$  and  $\text{Im } \varphi_2$ , all eaten by the vector particles  $W$  and  $Z$ , and one massive component  $\eta \equiv \text{Re } \varphi_2$  with a mass

$$m_\eta = \sqrt{-2\kappa} = \sqrt{\lambda}F \quad . \quad (21.7)$$



It is this component that is called the Higgs field. It does *not* couple to  $A_\mu^{\text{em}}$ , because like  $\varphi_0$ , also  $\eta$  is not affected by the transformation  $\exp(i\chi)\exp(i\chi\sigma_3)$ . Alternatively, this can be seen from the covariant derivative

$$D_\mu \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \left\{ \partial_\mu - \frac{ig}{2}(\sigma_1 W_\mu^1 + \sigma_2 W_\mu^2) + \frac{ig}{2\cos\theta_W}(\cos^2\theta_W\sigma_3 - \sin^2\theta_W)Z_\mu - \frac{ig}{2}\sin\theta_W(\sigma_3 + 1)A_\mu^{\text{em}} \right\} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} . \quad (21.8)$$

Using the fact that

$$\sigma_3 + 1 = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \quad (21.9)$$

it is clear that  $\varphi_2$  has no electric charge, whereas  $\varphi_1$  has a charge  $q = -g\sin\theta_W$ . As these are would-be Goldstone bosons, “eaten” by the vector fields, it will turn out that the combinations  $W_\mu^\pm = \frac{1}{\sqrt{2}}(W_\mu^1 \mp iW_\mu^2)$  are charged with an electric charge of  $\pm e$ , where

$$e = -g\sin\theta_W . \quad (21.10)$$

As a consequence, the two coupling constants  $g$  and  $g'$  are determined by the electric charge  $e$  and the so-called weak mixing angle  $\theta_W$ , also called the Weinberg angle. From experiment it follows that  $\sin^2\theta_W \approx 0.23$ . The  $Z$  vector field will remain neutral under the electromagnetic interactions. To verify the charge assignment to the vector fields, we have to find the coupling of the various fields to  $A_\mu^{\text{em}}$ . For this it is sufficient to consider the following part of the Lagrangian

$$\mathcal{L}_{W,B} = -\frac{1}{4} \left( \partial_\mu W_\nu^a - \partial_\nu W_\mu^a + g\varepsilon_{abc}W_\mu^b W_\nu^c \right)^2 - \frac{1}{4} (\partial_\mu B_\nu - \partial_\nu B_\mu)^2 . \quad (21.11)$$

with the obvious short-hand notations like  $(F_{\mu\nu}^a)^2 = F_{\mu\nu}^a F_a^{\mu\nu}$ . After some algebra the above equation can be rewritten as

$$\begin{aligned} \mathcal{L}_{W,B} = & -\frac{1}{4} \left( F_{\mu\nu}^{\text{em}} + ie(W_\mu^+ W_\nu^- - W_\nu^+ W_\mu^-) \right)^2 \\ & -\frac{1}{4} \left( \partial_\mu Z_\nu - \partial_\nu Z_\mu + ig\cos\theta_W(W_\mu^+ W_\nu^- - W_\nu^+ W_\mu^-) \right)^2 \\ & -\frac{1}{2} |D_\mu^{\text{em}} W_\nu^- - D_\nu^{\text{em}} W_\mu^- - ig\cos\theta_W(Z_\mu W_\nu^- - Z_\nu W_\mu^-)|^2 , \end{aligned} \quad (21.12)$$

where we have defined

$$D_\mu^{\text{em}} = \partial_\mu - ieA_\mu^{\text{em}} , \quad F_{\mu\nu}^{\text{em}} = \partial_\mu A_\nu^{\text{em}} - \partial_\nu A_\mu^{\text{em}} . \quad (21.13)$$

We immediately read-off that our charge assignments for  $Z$  and  $W^\pm$  have been correct. Note that the vector field  $W_\mu^\pm$  has an extra magnetic moment, because of its coupling to  $F_{\mu\nu}^{\text{em}}$

$$\mathcal{L}_{\text{magn.mom.}} = -ieF_{\mu\nu}^{\text{em}} W_\mu^+ W_\nu^- , \quad (21.14)$$

which is a direct consequence of the spin of the vector field (the magnetic moment for the Dirac field is discussed in problem 32).

We now introduce the fermions in the standard model. They are arranged according to *families*. The first family with the smallest masses consists of the electron, the neutrino, the up and the down quark. Essential in the standard model is that invariance under parity is broken explicitly by the weak interactions (as has been observed in the beta decay of

Cobalt-60, see problem 40). This is achieved by coupling the left- and right-handed helicity eigenstates of the fermions differently to the gauge fields. It should be stressed that the standard model does not explain why parity is violated; it was put in “by hand”. For each fermion we define

$$\Psi^L = \frac{1}{2}(1 + \gamma_5)\Psi \quad , \quad \Psi^R = \frac{1}{2}(1 - \gamma_5)\Psi \quad . \quad (21.15)$$

The Dirac Lagrangian in terms of these helicity eigenstates can be written as

$$\mathcal{L}_\Psi = \bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi = \bar{\Psi}^R(i\gamma^\mu\partial_\mu)\Psi^R + \bar{\Psi}^L(i\gamma^\mu\partial_\mu)\Psi^L - m(\bar{\Psi}^R\Psi^L + \bar{\Psi}^L\Psi^R) \quad , \quad (21.16)$$

such that different transformation rules for  $\Psi^{R,L}$  enforce  $m = 0$ , i.e. the absence of an explicit mass term. The beauty of the Higgs mechanism is that it also provides a mass for the fermions. This is achieved by coupling the scalar field  $\phi$  to the fermions, using a Yukawa coupling

$$\mathcal{L}_{\Psi,\phi} = -y(\bar{\Psi}^R\phi^\dagger\Psi^L + \bar{\Psi}^L\phi\Psi^R) \quad , \quad (21.17)$$

where  $y$  is the Yukawa coupling constant. It also immediately fixes the representation to which  $\Psi^{R,L}$  should belong. Since the Lagrangian has to be invariant with respect to the gauge symmetries, and since the scalar field is in the fundamental representation of  $SU(2)$ , we require that  $\Psi^L$  is also in the fundamental representation, i.e. it is a doublet. On the other hand  $\Psi^R$  is taken to be invariant under  $SU(2)$  (also called the singlet representation). The couplings of the fermions to the gauge field  $B_\mu$  have to be chosen such that the Lagrangian is neutral. This coupling is parametrised by the so-called hypercharge  $Y$ , in units of  $-\frac{1}{2}g'$ .

$$Y_H \equiv Y(\phi) = 1 \quad , \quad Y_R = Y_L - 1 \quad , \quad Y_{R,L} \equiv Y(\Psi^{R,L}) \quad . \quad (21.18)$$

The mass of the fermions is now read-off from eq. (21.17) by replacing  $\phi$  with its so-called vacuum expectation value  $\phi_0$

$$\mathcal{L}_{\Psi,\phi} = -yF(\bar{\Psi}^R\Psi_2^L + \bar{\Psi}_2^L\Psi^R) \quad , \quad (21.19)$$

where the index on  $\Psi^L$  indicates the so-called isospin index, the spinor index of the two dimensional fundamental representation for the internal  $SU(2)$  symmetry group. We also see that  $\Psi_1^L$  remains massless and this is exactly the neutrino. The electron is identified with the pair  $(\Psi^R, \Psi_2^L)$  and has a mass  $m_e = yF$ . We want the neutrino to have no electric charge and this fixes the hypercharge of  $\Psi^L$ . It is most easily determined from the covariant derivative, acting on the left-handed fermion, defined as in eqs. (21.1) and (21.8), since both are in the same representation (electron and neutrino are also neutral with respect to the strong interactions. The situation for the quarks will be discussed below).

$$\begin{aligned} D_\mu\Psi^L &= \left\{ \partial_\mu - \frac{ig}{2}(\sigma_1 W_\mu^1 + \sigma_2 W_\mu^2) + \frac{ig}{2\cos\theta_W}(\cos^2\theta_W\sigma_3 - Y_L\sin^2\theta_W)Z_\mu \right. \\ &\quad \left. - \frac{ig\sin\theta_W}{2}(\sigma_3 + Y_L)A_\mu^{\text{em}} \right\} \Psi^L \quad . \end{aligned} \quad (21.20)$$

To make  $\Psi_1^L$  decouple from the electromagnetic field we require

$$Y_L = -1 \quad , \quad Y_R = -2 \quad . \quad (21.21)$$

This also allows us to find the electric charge of  $\Psi_2^L$  to be  $g\sin\theta_W = -e$ , which as it should be, coincides with the electron charge. The right-handed component should of course have

the same electric charge. In that case the covariant derivative is given by

$$\begin{aligned} D_\mu \Psi^R &= (\partial_\mu - \frac{ig'}{2} Y_R B_\mu) \Psi^R = (\partial_\mu - \frac{ig'}{2} Y_R [\sin \theta_W Z_\mu + \cos \theta_W A_\mu^{\text{em}}]) \Psi^R \\ &= (\partial_\mu - ie A_\mu^{\text{em}} - ie \tan \theta_W Z_\mu) \Psi^R \quad , \end{aligned} \quad (21.22)$$

with the expected coupling to the electromagnetic field. Note that we can summarise our assignments of the electric charge by introducing the charge operator in terms of the hypercharge and the so-called isospin operator  $I_3$

$$Q^{\text{em}} = (\frac{1}{2} Y + I_3) e \quad . \quad (21.23)$$

On a doublet ( $\Psi^L$  and  $\varphi$ ) one has  $I_3 = \frac{1}{2} \sigma_3$ , whereas  $I_3 = 0$  on a singlet ( $\Psi^R$ ).

We now discuss quarks. Also there the weak interactions act differently on the left- and right-handed components. The left-handed up and down quarks are combined in a doublet representation for SU(2). If we denote the quark fields by  $q$ , we assign  $q_1^L$  to the left-handed component of the up-quark (also denoted by  $u^L$ ) and  $q_2^L$  to the left-handed component of the down-quark ( $d^L$ ). This doublet gets a hypercharge  $Y(q^L) = \frac{1}{3}$ , from which we read off the electric charges

$$Q^{\text{em}} q^L = Q^{\text{em}} \begin{pmatrix} u^L \\ d^L \end{pmatrix} = \begin{pmatrix} \frac{2e}{3} u^L \\ -\frac{e}{3} d^L \end{pmatrix} \quad . \quad (21.24)$$

The right-handed components of both the up and down quarks are singlets under SU(2) and their hypercharges are chosen to insure that they have the same electric charge as for their left-handed partners

$$Y(u^R) = \frac{4}{3} \quad \text{and} \quad Y(d^R) = -\frac{2}{3} \quad . \quad (21.25)$$

The quarks transform non-trivially under SU(3), the gauge group of the strong interactions. They form complex vectors in the three dimensional defining or fundamental representation of SU(3). The generators for SU(3) are given by

$$\begin{aligned} T^1 &= -\frac{i}{2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T^2 = -\frac{i}{2} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T^3 = -\frac{i}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ T^4 &= -\frac{i}{2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad T^5 = -\frac{i}{2} \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad T^6 = -\frac{i}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\ T^7 &= -\frac{i}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad T^8 = -\frac{i}{2\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \quad , \end{aligned} \quad (21.26)$$

normalised in accordance with eq. (18.20). (In terms of the so-called Gell-Mann matrices one has  $T^a = -i\lambda_a/2$ .) We leave it as an exercise to determine the structure constants. Note that the Lie-algebra for the group SU(N) is given by traceless and antihermitian ( $X^\dagger = -X$ ) complex  $N \times N$  matrices. The dimension of this Lie-algebra is easily seen to be  $N^2 - 1$ . Note that  $\det(\exp(X)) = \exp(\text{Tr}(X))$ , such that  $\exp(X)$  has determinant one. Also,  $\exp(X)^{-1} = \exp(-X) = \exp(X)^\dagger = \exp(X)^\dagger$  guarantees that  $\exp(X)$  is a unitary matrix.

The *fractional* electric charge of the quarks is not observable (otherwise we would have had a different unit for electric charge). The reason is that quarks are conjectured to always form bound states that are neutral under SU(3). This can be achieved by either taking

three quarks in an antisymmetric combination to form a  $SU(3)$  singlet, or by combining a quark and an antiquark. In the first case one has a baryon, of which the proton ( $uud$ ) and the neutron ( $udd$ ) are examples. The quark-antiquark bound state is called a meson, of which the pions are examples (e.g.  $\pi^+ = u\bar{d}$  and  $\pi^- = \bar{u}d$ ). The bar over the symbol of a particle of course denotes the antiparticle. Rather prosaically one associates to the three  $SU(3)$  components of the quark field the property colour. Choosing the three basic colours red, blue and green makes a bound state of three quarks in an antisymmetric wave function, where hence all colours are different, into a colourless composite. Similarly, combining a quark and an antiquark gives a colourless combination. It is not too difficult to show that a bound state of quarks and antiquarks is a singlet under  $SU(3)$  if and only if the net colour is white. It is now also easily verified that with the particular fractional electric charges assigned to the quarks, a colourless combination always has an electric charge that is an integer multiple of the electron charge. For this note that both quarks have modulo  $e$  an electric charge equal to  $-\frac{1}{3}e$ , whereas both antiquarks have modulo  $e$  a charge of  $\frac{1}{3}e$ . Three quarks bound together therefore have zero charge modulo  $e$ . The same holds for a quark-antiquark bound state.

That the strong interactions really are strong, follows from the fact that a quark and antiquark can not be separated without creating a quark-antiquark pair from the vacuum, to make sure that the separated components remain neutral under  $SU(3)$ . This is achieved by combining the quark (antiquark) of the pair created with the antiquark (quark) we try to separate. The mechanism that prevents free quarks to appear is called confinement, which still lacks a solid theoretical understanding. Because the coupling constant is strong, a perturbative expansion is no longer applicable. That nevertheless the theory of the strong interactions is believed to be the correct theory to describe the forces amongst the quarks (and therefore indirectly the nuclear forces) follows from the remarkable property that at high energies the effective coupling constant is small; at infinite energy even zero. This is called asymptotic freedom and will only briefly be discussed in the next section. For a more detailed discussion we refer to Itzykson and Zuber. In the table below we list the gauge particles of the standard model.

**Gauge particles** table 8

	name	charge	spin	mass	force
$\gamma$	photon	0	1	0	electromagnetism
A	gluon	0	1	0	strong force
$W^\pm$	W-particle	$\pm e$	1	80 GeV	weak force
Z	Z-particle	0	1	91 GeV	weak force

The strong interactions do not break parity invariance, i.e. the eight gluons  $A_\mu^a$  couple to the left-handed and right-handed components of the quark fields in the same way. However, the so-called Cabibbo mixing with two other families of quarks (the strange and charm quark on the one hand and the bottom and top quark on the other hand) gives in a very subtle way rise to violation of  $CP$ , that is the combination of charge conjugation and parity (equivalent to time reversal  $T$ , since  $CPT$  is conserved). The electron and neutrino, called leptons, in the first family are replaced by the muon and its neutrino for the second family and by the tau and associated neutrino for the third family. The experiments described in the

introduction (see problem 37) have shown that there are not more than three of these families with massless neutrinos. In the standard model there is room to add a right-handed partner for the neutrino field, which couples to none of the gauge fields. With a suitably chosen Yukawa coupling the neutrino can be given an arbitrarily small mass. It is experimentally very hard to measure the mass of the neutrino; only upper bounds have been established. The table below lists the properties of all the fermions observed in the standard model (the top quark was only discovered in 1994 at Fermilab). For much more on the standard model see in particular the book by J.C. Taylor mentioned in the introduction.

**Fermion families** table 9

	name	charge	spin	mass
d	down quark	$-e/3$	$1/2$	10 MeV
u	up quark	$2e/3$	$1/2$	5 MeV
e	electron	$-e$	$1/2$	511 keV
$\nu_e$	neutrino	0	$1/2$	$0(<10 \text{ eV})$
s	strange quark	$-e/3$	$1/2$	250 MeV
c	charm quark	$2e/3$	$1/2$	1,5 GeV
$\mu$	muon	$-e$	$1/2$	106 MeV
$\nu_\mu$	muon-neutrino	0	$1/2$	$0(<0,5 \text{ MeV})$
b	bottom quark	$-e/3$	$1/2$	4,8 GeV
t	top quark	$2e/3$	$1/2$	174 GeV
$\tau$	tau	$-e$	$1/2$	1,8 GeV
$\nu_\tau$	tau-neutrino	0	$1/2$	$0(<164 \text{ MeV})$

## 22 Loop corrections and renormalisation

Up to now, we have only considered the lowest order calculations of cross sections, for which it is sufficient to consider tree-level diagrams that do not contain any loops. Loop integrals typically give rise to infinities, which can be regularised by considering for example a cutoff in momentum space, as was discussed in sect. 7. Another possibility of regularising the theory is by discretising space-time, amounting to a lattice formulation, see eq. (7.5). In both of these cases there exists a maximal energy (equivalent to a minimal distance). The parameters, like the coupling constants, masses and field renormalisation constants will depend on this cutoff parameter, generically denoted by an energy  $\Lambda$  or a distance  $a = 1/\Lambda$ . How to give a physical definition of the mass in terms of the full propagator and why field renormalisation is necessary was discussed in sect. 9. For the renormalisation of the coupling constant it is best to define the physical coupling constant in terms of a particular scattering process, as that is what can be measured in experiment. Alternatively, as these are strongly related, the physical couplings can be defined in terms of an amputated  $1PI$   $n$ -point function, with prescribed momenta assigned to the external lines, all proportional to an energy scale called  $\mu \ll \Lambda$ . As an example consider the self-interacting scalar field, with a four-point coupling  $\lambda$

(see for example eq. (21.2)). We define the physical four-point coupling constant in terms of the  $1PI$  four-point function with the momenta on the amputated lines set to some particular value proportional to  $\mu$  (the precise choice is not important for the present discussion). It is clear that this gives a function  $\lambda_{\text{reg}}(\lambda, \mu, \Lambda)$ . The dependence on other coupling constants and the mass parameters is left implicit.

The theory is considered renormalisable if we can remove the cut-off by adjusting  $\lambda$  (also called the bare coupling constant) in such a way that at a fixed value  $\mu = \mu_0$  the renormalised coupling  $\lambda_R$  stays finite and takes on a prescribed (i.e. measured) value. It is then obvious that the renormalised coupling constant  $\lambda_R(\mu) \equiv \lambda_{\text{reg}}(\lambda(\Lambda), \mu, \Lambda)$  is a function of  $\mu$ , coinciding at  $\mu_0$  with the prescribed value  $\lambda_R$ . Since the physical coupling constant is computed in terms of the full  $1PI$  four-point function, the dependence on the energy scale is caused by quantum corrections. Since the vacuum in field theories is not really empty, as was discussed in the context of the Casimir effect in sect. 2, the computation is not much different from calculating effective interactions in a polarised medium. In this case the polarisation is due to the virtual particles that describe the quantum fluctuations (zero-point fluctuations) of the vacuum, and is therefore also called the vacuum polarisation. The energy dependent couplings are called running couplings. It should be emphasised that the running of the couplings is a manifestation of an anomaly (called the conformal anomaly), which is the breaking of a symmetry of the Lagrangian by the quantum corrections. In the absence of a mass, the scalar field theory with a  $\varphi^4$  interaction is at the classical level invariant under scale transformations,  $\varphi(x) \rightarrow \kappa\varphi(x/\kappa)$ , where  $\kappa$  is the scale parameter. It is obvious that the regularised couplings are not invariant under such a rescaling, because of the presence of a cutoff. What is not obvious is that, for the simple field theories we have been considering in four dimensions, the scale independence can not be recovered by removing the cutoff (i.e. taking  $\Lambda \rightarrow \infty$ ).

By adjusting the bare coupling constants of the theory as a function of the cutoff  $\Lambda$ , to ensure that all regularised couplings stay finite when the cutoff is moved to infinity, the calculations can be arranged such that nowhere explicit infinities occur. When we say that the contributions of the loops diverge, we mean that without adjusting the bare coupling constant, their contributions are infinite in the limit  $\Lambda \rightarrow \infty$ . A theory is called renormalisable if only a finite number of bare coupling constants needs to be adjusted to have all  $1PI$   $n$ -point functions finite. This can be shown to be equivalent to all  $1PI$   $n$ -point functions to be completely determined as a function of a finite number of renormalised couplings, called relevant couplings. It is only in such an instance that quantum field theory has predictive power. Renormalisability is therefore a necessary requirement for the theory to be insensitive to what happens at very high energies with a maximal amount of predictive power. The standard model falls in this class of theories.

Theoretical studies of the last five years or so have shown that the self-coupling of the Higgs field will most likely vanish if we really take  $\Lambda \rightarrow \infty$ , albeit in a logarithmic way. Loosely speaking the running of this coupling is such that the renormalised coupling increases with increasing energy. The only way it can be avoided that the renormalised coupling will become infinite at some finite energy, is to either take the renormalised coupling equal to zero or to keep the cutoff finite. It depends on the parameters of the model, in particular the Higgs mass, how large the cutoff should be. If the Higgs mass is relatively light, this can be at the Planck scale and has little consequence for the theory. If, however, the Higgs mass turns out to be in the order of 1 TeV, the cutoff has to be roughly smaller than 10 TeV.

As we can measure the self-coupling of the Higgs field and related quantities to be non-trivial (which is of course crucial for the spontaneous symmetry breaking and giving a mass

to the  $W$  and  $Z$  particles), the scalar sector of the standard model depends in a rather subtle way on what happens at higher energies. This sensitivity to high energies is, however, much and much weaker than in non-renormalisable theories like for the four-fermi interactions. It is outside the scope of these lectures to describe the computations necessary to make the above more precise. In the following we give a sample calculation that will provide the technical ingredients to perform such calculations and to illustrate some of these issues in a simple setting. Also, problems 2(!), 38 and 39 illustrate further ingredients that are pertinent to renormalising field theories.

Let us end this discussion by noting that a running coupling can of course either increase or decrease at increasing energy. The Higgs self-coupling and the electromagnetic coupling constant  $e$  are examples of couplings that increase at high energies. For the electric charge  $e$ , this increase is very tiny and the cutoff can be chosen much bigger than the Planck energy. The analogy with a polarised medium is that the virtual particles in the field of a charged particle will screen its charge at large distances. When we probe the charged particle at ever smaller distances the effective charge becomes less screened and increases. Due to the self-interactions of a non-Abelian gauge field, its charges show the effect of anti-screening. Here the effective charge becomes bigger at larger distances. For the strong interactions this is one way to understand confinement. The energy of a single quark within a spherical shell would increase without bound with increasing radius. A free quark would carry an infinite energy. To the contrary, at decreasing separations, the effective charge becomes weaker and weaker and the quarks start to behave as free particles. This is the asymptotic freedom mentioned in the previous section.

We will now consider to one-loop order the self-energy for the scalar field  $\varphi$  with a mass  $m$ , coupled to two flavours of fermions with masses  $m_1$  and  $m_2$ , coupled through Yukawa couplings described by the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{1}{2}m^2\varphi^2 - \frac{1}{6}\lambda\varphi^3 + \sum_i \bar{\Psi}_{(i)}(i\gamma^\mu\partial_\mu - m_i)\Psi_{(i)} - g\varphi\left(\bar{\Psi}_{(1)}\Psi_{(2)} + \bar{\Psi}_{(2)}\Psi_{(1)}\right) \quad . \quad (22.1)$$

The self-energy for the scalar field in one-loop order splits in two contributions  $\Sigma_1$  and  $\Sigma_2$  from a fermion and a scalar loop (in this order  $Z \equiv 1$ ).

$$\Sigma_\varphi(q) \equiv \Sigma_1 + \Sigma_2 = - \frac{\text{Diagram 1}}{q} + \frac{\text{Diagram 2}}{q} \quad (22.2)$$

The numerical expressions for  $\Sigma_1$  and  $\Sigma_2$  are given by

$$\begin{aligned}\Sigma_1 &= -\frac{ig^2}{(2\pi)^4} \int d_4k \frac{\text{Tr}((\not{k} + m_1)(\not{k} + \not{q} + m_2))}{(k^2 - m_1^2 + i\varepsilon)((k+q)^2 - m_2^2 + i\varepsilon)} \quad , \\ \Sigma_2 &= \frac{i\lambda^2}{2(2\pi)^4} \int d_4k \frac{1}{(k^2 - m^2 + i\varepsilon)((k+q)^2 - m^2 + i\varepsilon)} \quad .\end{aligned}\tag{22.3}$$

Using eq. (11.1), requires us to employ the Feynman rules of table 3 to obtain these expressions. (Alternatively the Feynman rules of table 2 can be used, provided the self-energy is defined through eq. (9.9).) These integrals are obviously divergent. Introducing a momentum cutoff  $\Lambda$  we find  $\Sigma_1 \sim \Lambda^2$  and  $\Sigma_2 \sim \log \Lambda$  to lowest non-trivial order in  $1/\Lambda$ . One says that  $\Sigma_1$  is quadratically and  $\Sigma_2$  logarithmically divergent. To simplify the integrands we discuss the Feynman trick

$$\frac{1}{ab} = \int_0^1 dx \frac{1}{(ax + b(1-x))^2} \quad , \quad (22.4)$$

which we can apply to the computation of  $\Sigma_1$  by substituting  $a = (k + q)^2 - m_2^2 + i\varepsilon$  and  $b = k^2 - m_1^2 + i\varepsilon$ . For  $\Sigma_2$  we have the same assignment, with in addition  $m_1 = m_2 = m$ . It is useful to also have the generalisation of the Feynman trick for an arbitrary product of scalar propagators.

$$\left( \prod_{i=1}^k a_i^{n_i} \right)^{-1} = \frac{\Gamma(\sum_i n_i)}{\prod_{i=1}^k \Gamma(n_i)} \prod_{i=1}^k \left( \int_0^1 x_i^{n_i-1} dx_i \right) \delta \left( \sum_{i=1}^k x_i - 1 \right) \left( \sum_{i=1}^k a_i x_i \right)^{-\sum_i n_i}, \quad (22.5)$$

which is proven by induction. In here  $\Gamma(z)$  is the gamma function, which satisfies  $\Gamma(z+1) = z\Gamma(z)$ ,  $\Gamma(n+1) = n!$  and  $\Gamma(\frac{1}{2}) = \sqrt{\pi}$  (see problem 2b). Consequently we find

$$\begin{aligned} \Sigma_1 &= -\frac{4ig^2}{(2\pi)^4} \int d_4k \int_0^1 dx \frac{k^2 + k \cdot q + m_1 m_2}{\left( (k + (1-x)q)^2 + x(1-x)q^2 - xm_1^2 - (1-x)m_2^2 + i\varepsilon \right)^2}, \\ \Sigma_2 &= \frac{i\lambda^2}{2(2\pi)^4} \int d_4k \int_0^1 dx \frac{1}{\left( (k + (1-x)q)^2 + x(1-x)q^2 - m^2 + i\varepsilon \right)^2}. \end{aligned} \quad (22.6)$$

We will show how to regularise these two integrals in two different ways. First we use dimensional regularisation introduced by 't Hooft and Veltman (see problem 2) and then discuss Pauli-Villars regularisation. In dimensional regularisation the loop integrations are replaced by integrals in  $n$ , instead of 4, dimensions. The momentum integrations are always of the form

$$I_{n,\alpha,\beta}(M) \equiv \int d_n k \frac{(k^2)^\alpha}{(k^2 - M^2 + i\varepsilon)^\beta}. \quad (22.7)$$

We can evaluate this integral by performing the so-called Wick rotation, where we replace the integral over  $\text{Re}k_0$  by an integration over  $\text{Im}k_0$ . The integral over the two quarter circles indicated in the figure, will vanish as the radius tends to infinity. As there are no poles inside the contour of integration, we find

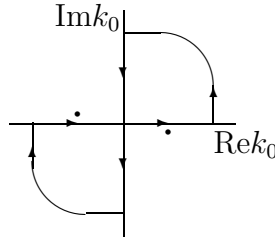


fig. 14

$$I_{n,\alpha,\beta}(M) = i(-1)^{\alpha-\beta} \int d_n k \frac{(k^2)^\alpha}{(k^2 + M^2 - i\varepsilon)^\beta}, \quad k^2 = k_0^2 + \vec{k}^2. \quad (22.8)$$

We note that the integrand is a purely radial integral and as the surface area of a  $n$ -dimensional sphere is analytically known ( $S_n = 2\pi^{n/2}/\Gamma(n/2)$ , e.g.  $S_2 = 2\pi$ ,  $S_3 = 4\pi$ ,  $S_4 = 2\pi^2, \dots$ ), we obtain

$$\begin{aligned} I_{n,\alpha,\beta}(M) &= i(-1)^{\alpha-\beta} \frac{2\pi^{n/2}}{\Gamma(n/2)} \int r^{n-1} dr \frac{r^{2\alpha}}{(r^2 + M^2 - i\varepsilon)^\beta} \\ &= \frac{i(-1)^{\alpha-\beta} \pi^{n/2} \Gamma(\alpha + n/2) \Gamma(\beta - \alpha - n/2)}{(M^2 - i\varepsilon)^{\beta-\alpha-n/2} \Gamma(\beta) \Gamma(n/2)}. \end{aligned} \quad (22.9)$$

We used the integral representation of the beta function

$$B(n, k) \equiv \frac{\Gamma(n)\Gamma(k)}{\Gamma(n+k)} = \int_0^\infty dz \frac{z^{n-1}}{(z+1)^{k+n}}. \quad (22.10)$$



Shifting the integration variable  $k \rightarrow k - (1-x)q$  we find

$$\begin{aligned}\Sigma_1 &= -\frac{4ig^2}{(2\pi)^4} \int d_n k \int_0^1 dx \frac{k^2 - (1-2x)k \cdot q + m_1 m_2 + x(x-1)q^2}{(k^2 - \hat{M}_{x,q}^2 + i\varepsilon)^2} , \\ \Sigma_2 &= \frac{i\lambda^2}{2(2\pi)^4} \int d_n k \int_0^1 dx \frac{1}{(k^2 - M_{x,q}^2 + i\varepsilon)^2} ,\end{aligned}\quad (22.11)$$

where

$$\hat{M}_{x,q}^2 = xm_1^2 + (1-x)m_2^2 - x(1-x)q^2 \quad \text{and} \quad M_{x,q}^2 = m^2 - x(1-x)q^2 , \quad (22.12)$$

which allows us to express  $\Sigma_i$  in terms of the integrals  $I_{n,\alpha,\beta}(m)$

$$\begin{aligned}\Sigma_1 &= -\frac{4ig^2}{(2\pi)^4} \int_0^1 dx \left\{ I_{n,1,2}(\hat{M}_{x,q}) + (m_1 m_2 - x(1-x)q^2) I_{n,0,2}(\hat{M}_{x,q}) \right\} , \\ \Sigma_2 &= \frac{i\lambda^2}{2(2\pi)^4} \int_0^1 dx I_{n,0,2}(M_{x,q}) .\end{aligned}\quad (22.13)$$

Substituting the expressions for  $I_{n,\alpha,\beta}(M)$  from eq. (22.9), we find

$$\begin{aligned}\Sigma_1 &= \frac{4g^2\pi^{n/2}}{(2\pi)^4} \int_0^1 dx \left\{ \Gamma(2-n/2) \frac{(m_1 m_2 - x(1-x)q^2)}{(\hat{M}_{x,q}^2 - i\varepsilon)^{2-n/2} \Gamma(2)} - \frac{\Gamma(1+n/2)\Gamma(1-n/2)}{(\hat{M}_{x,q}^2 - i\varepsilon)^{1-n/2} \Gamma(2)\Gamma(n/2)} \right\} , \\ \Sigma_2 &= -\frac{\lambda^2\pi^{n/2}}{2(2\pi)^4} \int_0^1 dx \frac{\Gamma(2-n/2)}{(M_{x,q}^2 - i\varepsilon)^{2-n/2} \Gamma(2)} .\end{aligned}\quad (22.14)$$

This can be further simplified using

$$\frac{\Gamma(1+n/2)\Gamma(1-n/2)}{\Gamma(2-n/2)\Gamma(n/2)} = \frac{n}{2-n} , \quad (22.15)$$

such that

$$\begin{aligned}\Sigma_1 &= \frac{4g^2\pi^{n/2}\Gamma(2-n/2)}{(2\pi)^4} \int_0^1 dx \left\{ \frac{(m_1 m_2 - x(1-x)q^2)}{(\hat{M}_{x,q}^2 - i\varepsilon)^{2-n/2}} - \frac{n/(2-n)}{(\hat{M}_{x,q}^2 - i\varepsilon)^{1-n/2}} \right\} , \\ \Sigma_2 &= -\frac{\lambda^2\pi^{n/2}\Gamma(2-n/2)}{2(2\pi)^4} \int_0^1 dx \frac{1}{(M_{x,q}^2 - i\varepsilon)^{2-n/2}} .\end{aligned}\quad (22.16)$$

The divergent part is now fully contained in  $\Gamma(2-n/2)$ , because

$$\Gamma(2-n/2) = \frac{\Gamma(1+\frac{1}{2}(4-n))}{\frac{1}{2}(4-n)} = \frac{2}{4-n} - \gamma + \mathcal{O}(4-n) , \quad (22.17)$$

where  $\gamma = 0.57721 \dots$  is the Euler constant. We expand  $\Sigma_i$  around  $n = 4$

$$\begin{aligned}\Sigma_1 &= \frac{g^2}{(2\pi)^2} \left( \frac{2}{4-n} - \gamma \right) \pi^{(n-4)/2} \int_0^1 dx \left( \frac{m_1 m_2 - x(1-x)q^2 + (2+\frac{1}{2}(4-n))\hat{M}_{x,q}^2}{(\hat{M}_{x,q}^2 - i\varepsilon)^{(4-n)/2}} \right) \\ &= \frac{g^2}{(2\pi)^2} \left( \frac{2}{4-n} - \gamma \right) \int_0^1 dx \left\{ (m_1 m_2 - 3x(1-x)q^2 + 2xm_1^2 + 2(1-x)m_2^2) \times \right. \\ &\quad \left. \left[ 1 - \frac{1}{2}(4-n) \log(\pi[\hat{M}_{x,q}^2 - i\varepsilon]) \right] + \frac{1}{2}(4-n)\hat{M}_{x,q}^2 \right\} \equiv \frac{\Sigma_1^{(-1)}}{4-n} + \Sigma_1^{(0)} + \mathcal{O}(4-n) , \\ \Sigma_2 &= \frac{\lambda^2}{8(2\pi)^2} \left\{ \int_0^1 dx \left( \log(\pi[M_{x,q}^2 - i\varepsilon]) + \gamma \right) - \frac{2}{4-n} \right\} \equiv \frac{\Sigma_2^{(-1)}}{4-n} + \Sigma_2^{(0)} + \mathcal{O}(4-n) .\end{aligned}\quad (22.18)$$

Note that  $q^2 = q_0^2 - \bar{q}^2$  and that the coupling constant  $\lambda$  for the scalar three-point coupling has the dimension of mass. We have split the result for  $\Sigma_i$  in a pole term with residue  $\Sigma_i^{(-1)}$  and a finite part  $\Sigma_i^{(0)}$  for  $n \rightarrow 4$ .

$$\begin{aligned}\Sigma_1^{(0)} &= \frac{g^2}{4\pi^2} \int_0^1 dx \left( \hat{M}_{x,q}^2 - \left\{ \log(\pi[\hat{M}_{x,q}^2 - i\varepsilon]) + \gamma \right\} \times \right. \\ &\quad \left. (m_1 m_2 + 3x(x-1)q^2 + 2xm_1^2 + 2(1-x)m_2^2) \right) , \\ \Sigma_2^{(0)} &= \frac{\lambda^2}{8(2\pi)^2} \int_0^1 dx \left( \log(\pi[M_{x,q}^2 - i\varepsilon]) + \gamma \right) , \\ \Sigma_1^{(-1)} &= \frac{g^2}{2\pi^2} (m_1 m_2 + m_1^2 + m_2^2 - \tfrac{1}{2}q^2) , \quad \Sigma_2^{(-1)} = -\frac{\lambda^2}{(4\pi)^2} .\end{aligned}\tag{22.19}$$

We now note that the pole terms are of the same form as the tree-level expressions obtained from the following extra term in the Lagrangian

$$\Delta\mathcal{L} = \tfrac{1}{2}a(\partial_\mu\varphi)^2 - \tfrac{1}{2}b\varphi^2 .\tag{22.20}$$

This means that we can choose  $a$  and  $b$  so as to precisely cancel the pole terms. To lowest order we therefore have

$$\Sigma_\varphi(\mathcal{L} + \Delta\mathcal{L}) = \Sigma_\varphi(\mathcal{L}) + b - aq^2 = \Sigma_1^{(0)} + \Sigma_2^{(0)} + \mathcal{O}(n-4) ,\tag{22.21}$$

from which we can solve for  $a$  and  $b$  in terms of  $\Sigma_i^{(-1)}$

$$a = -\frac{g^2}{(2\pi)^2} \frac{1}{4-n} , \quad b = \left( \frac{\lambda^2}{(4\pi)^2} - \frac{g^2}{2\pi^2} (m_1 m_2 + m_1^2 + m_2^2) \right) \frac{1}{4-n} .\tag{22.22}$$

Note that as long as we stay away from  $n = 4$  everything is well defined, including  $a$  and  $b$ . The limit  $n \rightarrow 4$  is to be taken after we have expressed everything in terms of the renormalised coupling constants and masses. We have taken here a slightly different approach for renormalising the theory. Rather than computing at  $n \neq 4$  physical processes to fix the renormalised couplings, we have started with renormalised couplings and determined how they have to depend on the bare couplings so as to cancel any infinities that might arise as  $n \rightarrow 4$ . It is clear that these two procedures are equivalent. For the physical interpretation the first procedure (due to Wilson) is more transparent, in a loop expansion the second procedure is more natural. To find the bare mass and the field renormalisation (for the bare  $\lambda$  coupling we should have considered the  $1PI$  three-point function with three  $\varphi$  external lines) we write to one-loop order

$$\mathcal{L}_B = \mathcal{L} + \Delta\mathcal{L} \equiv \tfrac{1}{2}(\partial_\mu\varphi_B)^2 - \tfrac{1}{2}m_B^2\varphi_B^2 , \quad m_B^2 \equiv \frac{m^2 + b}{1 + a} , \quad \varphi_B \equiv \sqrt{Z_\varphi}\varphi = \sqrt{1 + a}\varphi .\tag{22.23}$$

Often it can be determined by power-counting (of momenta) which diagrams give rise to divergencies for  $n \rightarrow 4$  or  $\Lambda \rightarrow \infty$ . The infinities correspond to local counter terms (i.e with a finite number of space-time derivatives) in the Lagrangian. For the theory in eq. (22.1), power counting easily shows that the  $\varphi$  four-point function is logarithmically divergent at one-loop order, see the Feynman diagram below.

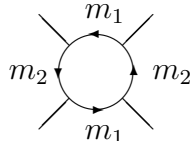


fig. 15

We therefore need to introduce an independent parameter for the  $\varphi$  four-point coupling, so as to adjust its bare coupling to depend in the proper way on the cutoff, to ensure that we can remove it. It can be shown that after adding to the Lagrangian in eq. (22.1) the term  $-\lambda_4\varphi^4/4!$ , the theory becomes renormalisable to all orders in the loop expansion. The relevant parameters are  $m$ ,  $m_1$ ,  $m_2$ ,  $g$ ,  $\lambda$  and  $\lambda_4$ .

As we have seen in section 11,  $\Sigma_\varphi(q)$  should have a non-vanishing imaginary part if the scalar particle is unstable. It is clear that the scalar particle itself can not decay in two scalar particles, but when  $m_1 + m_2 < m$  it could decay in two fermions. Indeed, it is not difficult to show that on the mass shell ( $q^2 = m^2$ )  $\Sigma_2^{(0)}$  is real

$$\Sigma_2^{(0)}(q^2 = m^2) = \frac{\lambda^2}{8(2\pi)^2} \int_0^1 dx \log[(x - \tfrac{1}{2})^2 + \tfrac{3}{4}] - \gamma - \log(m^2\pi) \in \mathbb{R} \quad . \quad (22.24)$$

We will show that  $\text{Im}\Sigma_1^{(0)}(q^2 = m^2) \neq 0$  if and only if  $q^2 > (m_1 + m_2)^2$ , called the threshold for decay. The only way  $\Sigma_1^{(0)}$  can develop an imaginary part is when the argument of the logarithm in eq. (22.19) becomes negative. The threshold is therefore determined by

$$\min\{\hat{M}_{x,q}^2 | x \in [0, 1]\} = \min\{xm_1^2 + (1-x)m_2^2 - x(1-x)q^2 | x \in [0, 1]\} < 0 \quad . \quad (22.25)$$

Let us first consider the simplest case of equal fermion masses,  $m_1 = m_2$ . In that case

$$\min\{\hat{M}_{x,q}^2 | x \in [0, 1]\} = m_1^2 - \tfrac{1}{4}q^2 \quad (m_1 = m_2) \quad , \quad (22.26)$$

and the threshold is determined by  $q^2 > 4m_1^2 = (m_1 + m_2)^2$ . For the general case of unequal fermion masses, the minimum is obtained for  $x = \tfrac{1}{2}(1 + (m_2^2 - m_1^2)/q^2)$ . After some algebra we find

$$\min\{\hat{M}_{x,q}^2 | x \in \mathbb{R}\} = \frac{q^2}{4} \left(1 - \frac{(m_1 - m_2)^2}{q^2}\right) \left(\frac{(m_1 + m_2)^2}{q^2} - 1\right) \quad , \quad (22.27)$$

which is indeed negative for  $q^2 > (m_1 + m_2)^2$ . The value of  $x$  where this minimum is attained does not lie in the interval  $[0, 1]$  if  $|(m_1^2 - m_2^2)/q^2| > 1$ , which can be used to rule out the other region,  $q^2 < (m_1 - m_2)^2$ , where eq. (22.27) is negative. This therefore proves that the kinematically determined threshold coincides with the threshold for  $\text{Im}\Sigma(q) \neq 0$ , as was assumed in section 11.

A major advantage of dimensional regularisation is that it preserves the Lorentz and gauge invariances. Furthermore, it is a local regulator. The lattice regularisation also can be arranged to preserve the gauge invariance, but locality and Lorentz invariance are only valid at distances much bigger than the lattice spacing  $a$ . A momentum cutoff breaks both the Lorentz and gauge invariance. Pauli-Villars regularisation is aimed at having a regulator that preserves the Lorentz invariance. We will describe it for the Lagrangian of eq. (22.1), using again the computation of  $\Sigma_\varphi(q)$  to one-loop order as an illustration. For each of the original fields  $\varphi$  and  $\Psi_{(i)}$  one adds extra (ghost) fields, with either the same ( $e_\ell > 0$ ) or opposite ( $e_\ell < 0$ ) statistics. This means that a loop of these ghost fields gets an additional factor  $e_\ell$ . Furthermore, the mass of these ghost fields is shifted over  $M_\ell$  with respect to the original (“parent”) field (alternatively, if the original field is a boson, one can shift  $m^2$  over  $M_\ell^2$ , see problem 39. With the present prescription we can treat bosons and fermions on the same footing). By defining  $e_0 = 1$  and  $M_0 = 0$ , the index  $\ell = 0$  describes the original fields of the model. We define furthermore  $M_\ell \equiv b_\ell\Lambda$  with  $\Lambda \gg m, m_1, m_2$ . To regularise  $\Sigma_\varphi$  by Pauli-Villars’ method, we choose

$$e_\ell = (1, -1, 2, -2) \quad \text{and} \quad b_\ell = (0, 4, 3, 1) \quad , \quad (22.28)$$

in other words the scalar and two fermion fields each have three ghost fields associated to them but with non-standard weights for the loops. We could stick to standard weights, such that these ghost fields can be described in terms of either grassmann or bosonic variables by taking  $|e_\ell|$  fields, having either the same ( $e_\ell > 0$ ) or reversed ( $e_\ell < 0$ ) statistics with respect to the original (“parent”) field. It is straightforward to give the self-energy including the contribution of the ghost fields

$$\Sigma_\varphi^{\text{PV}}(q) = \sum_{\ell=0}^3 e_\ell \left\{ \Sigma_1(m_1 + b_\ell \Lambda, m_2 + b_\ell \Lambda; q) + \Sigma_2(m + b_\ell \Lambda; q) \right\} , \quad (22.29)$$

in an obvious notation. The weights are chosen such that the momentum integrals can all be performed. Nevertheless, the masses of the ghost particles, all proportional to  $\Lambda$ , now play the role of a momentum cutoff, as the Lagrangian will at that energy scale no longer describe a physical theory. It is still convenient to evaluate the integrals in  $n$ , rather than in 4 dimensions. We will see that eq. (22.28) guarantees that the terms proportional to  $(4-n)^{-1}$  exactly cancel. Indeed, using eq. (22.19)

$$\begin{aligned} \sum_{\ell=0}^3 e_\ell \left( \Sigma_1^{(-1)}(m_1 + b_\ell \Lambda, m_2 + b_\ell \Lambda; q) + \Sigma_2^{(-1)}(m + b_\ell \Lambda; q) \right) = \\ \sum_{\ell=0}^3 e_\ell \left( \frac{g^2}{2\pi^2} \left( (m_1 + b_\ell \Lambda)(m_2 + b_\ell \Lambda) + (m_1 + b_\ell \Lambda)^2 + (m_2 + b_\ell \Lambda)^2 \right) - \frac{g^2 q^2}{(2\pi)^2} - \frac{\lambda^2}{(4\pi)^2} \right) = \\ \frac{g^2(m_1 m_2 + m_1^2 + m_2^2 - \frac{1}{2} q^2) - \frac{1}{8} \lambda^2}{2\pi^2} \sum_{\ell} e_\ell + \frac{3(m_1 + m_2)g^2}{2\pi^2} \Lambda \sum_{\ell} e_\ell b_\ell + \frac{3g^2}{2\pi^2} \Lambda^2 \sum_{\ell} e_\ell b_\ell^2 = 0 . \end{aligned} \quad (22.30)$$

The finite result that remains (replacing  $\Sigma^{(-1)}$  in the equation above by  $\Sigma^{(0)}$ ) is nevertheless still dependent on  $\Lambda$ . To keep the following computation transparent we take  $m_1 = m_2$

$$\begin{aligned} \Sigma_\varphi^{\text{PV}}(q) &= \frac{g^2}{(2\pi)^2} \int_0^1 dx \sum_{\ell} e_\ell \left\{ 1 - 3\gamma - 3 \log \left( \pi [(m_1 + b_\ell \Lambda)^2 - x(1-x)q^2] \right) \right\} [(m_1 + b_\ell \Lambda)^2 - x(1-x)q^2] \\ &\quad + \frac{\lambda^2}{8(2\pi)^2} \int_0^1 dx \sum_{\ell} e_\ell \left\{ \log \left( \pi [(m + b_\ell \Lambda)^2 - x(1-x)q^2] \right) + \gamma \right\} \\ &= -\frac{3g^2}{(2\pi)^2} \int_0^1 dx \sum_{\ell} e_\ell \log \left( (m_1 + b_\ell \Lambda)^2 - x(1-x)q^2 \right) [(m_1 + b_\ell \Lambda)^2 - x(1-x)q^2] \\ &\quad + \frac{\lambda^2}{8(2\pi)^2} \int_0^1 dx \sum_{\ell} e_\ell \log \left( (m + b_\ell \Lambda)^2 - x(1-x)q^2 \right) \\ &= \frac{\lambda^2}{8(2\pi)^2} \int_0^1 dx \log \left( m^2 - x(1-x)q^2 \right) - \frac{3g^2}{(2\pi)^2} \int_0^1 dx \log \left( m_1^2 - x(1-x)q^2 \right) [m_1^2 - x(1-x)q^2] \\ &\quad + a_1 \Lambda^2 + a_2 \Lambda + a_3 \log \Lambda + a_4 q^2 \log \Lambda + a_5 + a_6 q^2 + \mathcal{O}(1/\Lambda) . \end{aligned} \quad (22.31)$$

The precise values of the coefficients  $a_i$  are not very important, but can be calculated explicitly with some effort. All  $\Lambda$  dependent terms can be absorbed in a redefinition of  $Z_\varphi$  and the mass of the scalar field, such that

$$\begin{aligned} \Sigma_\varphi^{\text{PV}}(q) &= -\frac{3g^2}{(2\pi)^2} \int_0^1 dx \log \left( m_1^2 - x(1-x)q^2 \right) [m_1^2 - x(1-x)q^2] \\ &\quad + \frac{\lambda^2}{8(2\pi)^2} \log \left( m^2 - x(1-x)q^2 \right) + a_5 + a_6 q^2 . \end{aligned} \quad (22.32)$$

Note that in dimensional regularisation (DR) we found the result  $\Sigma_1^{(0)} + \Sigma_2^{(0)}$ , or

$$\begin{aligned} \Sigma_\varphi^{\text{DR}}(q) = & -\frac{3g^2}{(2\pi)^2} \int_0^1 dx \left\{ \log \left[ \pi \left( m_1^2 - x(1-x)q^2 \right) \right] + \gamma - \frac{1}{3} \right\} \left[ m_1^2 - x(1-x)q^2 \right] \\ & + \frac{\lambda^2}{8(2\pi)^2} \int_0^1 dx \left\{ \log \left[ \pi \left( m^2 - x(1-x)q^2 \right) \right] + \gamma \right\} . \end{aligned} \quad (22.33)$$

However, the difference  $\Sigma_\varphi^{\text{PV}}(q) - \Sigma_\varphi^{\text{DR}}(q) = a_5 - b_1 + (a_6 - b_2)q^2$ , where

$$b_1 = \frac{\lambda^2(\gamma + \log \pi)}{8(2\pi)^2} - \frac{g^2 m_1^2(3\gamma + 3\log \pi - 1)}{(2\pi)^2} , \quad b_2 = \frac{g^2 m_1^2(3\gamma + 3\log \pi - 1)}{6(2\pi)^2} , \quad (22.34)$$

can be absorbed in a finite redefinition of the mass and of  $Z_\varphi$ . If we define the renormalised coupling in terms of some physical scattering process, such an ambiguity of course cannot arise. In that case there is a unique relation between the bare and renormalised parameters. This relation, however, depends on the regularisation used.

We will now discuss, without a detailed derivation, the renormalisation of gauge theories to one-loop order in dimensional regularisation. The bare Lagrangian is given by

$$\mathcal{L} = -\frac{1}{4}(\partial^\mu A_B^\nu - \partial^\nu A_B^\mu)^2 - \frac{1}{2}\alpha_B(\partial_\mu A_B^\mu)^2 + \bar{\Psi}_B(i\gamma^\mu \partial_\mu - m_B)\Psi_B + e_B A_B^\mu \bar{\Psi}_B \gamma_\mu \Psi_B . \quad (22.35)$$

In  $n$  dimensions we still want the action to be dimensionless ( $\hbar = 1$ ), which implies that  $\mathcal{L}/\mu^n$  is dimensionless. From this we derive the dimensions of the fields and the parameters in  $n$  dimensions,

$$[A_B^\mu] = \mu^{\frac{1}{2}n-1} , \quad [\alpha_B] = 1 , \quad [\Psi_B] = [\bar{\Psi}_B] = \mu^{\frac{1}{2}n-\frac{1}{2}} , \quad [m_B] = \mu , \quad [e_B] = \mu^{2-\frac{1}{2}n} . \quad (22.36)$$

If we define (as is customary)  $\varepsilon \equiv 4 - n$ , one finds (for details see Itzykson and Zuber, e.g. sections 7-1 and 8-4. They use slightly different notations.)

$$\begin{aligned} A_B^\mu &\equiv \mu^{-\frac{1}{2}\varepsilon} \sqrt{Z_A} A^\mu = \mu^{-\frac{1}{2}\varepsilon} \left( 1 - \frac{e^2}{6\pi^2} \cdot \frac{1}{\varepsilon} + \dots \right)^{\frac{1}{2}} A^\mu , \quad m_B \equiv Z_m m = \left( 1 - \frac{3e^2}{8\pi^2} \cdot \frac{1}{\varepsilon} + \dots \right) m , \\ \Psi_B &\equiv \mu^{-\frac{1}{2}\varepsilon} \sqrt{Z_\Psi} \Psi = \mu^{-\frac{1}{2}\varepsilon} \left( 1 - \frac{e^2}{8\pi^2} \cdot \frac{1}{\varepsilon} + \dots \right)^{\frac{1}{2}} \Psi , \quad \alpha_B \equiv Z_\alpha \alpha = \left( 1 + \frac{e^2}{6\pi^2} \cdot \frac{1}{\varepsilon} + \dots \right) \alpha , \\ e_B &\equiv \mu^{\frac{1}{2}\varepsilon} Z_e e = \mu^{\frac{1}{2}\varepsilon} \left( 1 + \frac{e^2}{12\pi^2} \cdot \frac{1}{\varepsilon} + \dots \right) e . \end{aligned} \quad (22.37)$$

We note that to one-loop order  $A^\mu \equiv e_B A_B^\mu$  and  $\alpha_B/e_B^2$  are finite for  $n \rightarrow 4$ . This is not an accident, but the consequence of a so-called Ward identity, which as a consequence of the gauge symmetry (through the BRS invariance mentioned at the end of sect. 20) relates different  $Z$  factors,

$$Z_e^2 = Z_\alpha = 1/Z_A . \quad (22.38)$$

It is therefore sometimes much more convenient to use

$$\mathcal{L} = -\frac{1}{4e_B^2}(\partial^\mu A^\nu - \partial^\nu A^\mu)^2 - \frac{\alpha_B}{2e_B^2}(\partial_\mu A^\mu)^2 + \bar{\Psi}_B(i\gamma^\mu D_\mu - m_B)\Psi_B , \quad D^\mu \equiv \partial^\mu - iA^\mu . \quad (22.39)$$

To all orders in the loop expansion the field  $A^\mu$  and the gauge fixing parameter  $\alpha/e^2$  remain free of renormalisations. The same holds for non-Abelian gauge theories. By absorbing the

charge  $q$  (called coupling constant  $g \equiv q$  henceforth) in the gauge field, the Lagrangian can be expressed as

$$\mathcal{L} = \frac{1}{2g_B^2} \text{Tr}(F_{\mu\nu}^2) + \frac{\alpha_B}{g_B^2} \text{Tr}(\partial_\mu A^\mu)^2 + \bar{\Psi}_B(i\gamma^\mu D_\mu - m_B)\Psi_B \quad , \quad (22.40)$$

where the gauge field and the gauge fixing parameter receive no renormalisations, in other words they are already the renormalised field and gauge fixing parameter. The field strength  $F_{\mu\nu}$  and covariant derivative  $D_\mu$  are now given by (compare eqs. (18.35) and (18.41))

$$D_\mu = \partial_\mu + A_\mu \quad , \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \quad . \quad (22.41)$$

If the gauge group is  $\text{SU}(N)$  and there are  $n_f$  flavours of fermions the renormalisation of the coupling constant is given by (see Itzykson and Zuber sect. 12.3-4)

$$g_B = \mu^{\frac{1}{2}\varepsilon} Z_g g = \mu^{\frac{1}{2}\varepsilon} \left( 1 - \frac{(11N - 2n_f)g^2}{48\pi^2} \cdot \frac{1}{\varepsilon} + \dots \right) g \quad . \quad (22.42)$$

As long as the number of fermion flavours is small enough, we see that the one loop corrections to the bare coupling constant differs in sign from the equivalent expression for the Abelian case. It is the self-interactions of the non-Abelian gauge fields that are responsible for the asymptotic freedom of its running coupling constant. The running of the coupling is expressed in terms of the so-called beta-function

$$\beta(g) \equiv \mu \frac{\partial g(g_B(\varepsilon), \mu, \varepsilon)}{\partial \mu} \quad , \quad (22.43)$$

where the derivative is taken at fixed  $\varepsilon$  and  $g_B$  ( $g \equiv g_R$ ). For non-Abelian gauge theories one finds ( $\mu_0$  is an integration constant)

$$\beta(g) = -\frac{(11N - 2n_f)g^3}{48\pi^2} + \mathcal{O}(g^5) \quad , \quad \frac{1}{g^2(\mu)} = \frac{(11N - 2n_f)}{24\pi^2} \log(\mu/\mu_0) + \mathcal{O}(g^2(\mu)) \quad , \quad (22.44)$$

whereas for QED (coupled to  $n_f$  flavours of fermions)

$$\beta(e) = \frac{n_f e^3}{12\pi^2} + \mathcal{O}(e^5) \quad , \quad \frac{1}{e^2(\mu)} = -\frac{n_f}{6\pi^2} \log(\mu/\mu_0) + \mathcal{O}(e^2(\mu)) \quad . \quad (22.45)$$

For other regularisations the computation of the running coupling constant is similar, except that  $\varepsilon$  is replaced roughly by  $1/\log(\Lambda)$ . To the order displayed, the beta-functions do not depend on the regularisation scheme.

It is perhaps appropriate to end these lecture notes with as classic an experimental test of renormalisation effects in field theory as the one for the Casimir energy in section 2. It concerns the Lamb shift, measured in 1947, which is the very small energy splitting of the  $2S_{\frac{1}{2}}$  and  $2P_{\frac{1}{2}}$  orbitals in hydrogen atoms, receiving a contribution from vacuum polarisation effects (for a discussion of the other contributions see section 7-3-2 of Itzykson and Zuber). In problem 39 it will be shown that to one-loop order the photon vacuum polarisation is given by (compare eqs. (16.22) and (16.23). In the Landau gauge,  $\alpha \rightarrow \infty$ , we can drop the  $\Lambda^{(\alpha)}$  factors)

$$\Sigma_{\mu\nu}(q) = -\Lambda_{\mu\beta}^{(\alpha)}(q)(q^2 g^{\beta\gamma} - q^\beta q^\gamma) \omega(q^2) \Lambda_{\gamma\nu}^{(\alpha)}(q) \quad . \quad (22.46)$$

From the results of problem 39, where  $\omega$  is computed with Pauli-Villars regularisation, it can be deduced that ( $m$  is the electron mass)

$$\omega(q^2) = a \log(\Lambda/m) + b + cq^2 \quad \text{for} \quad q^2 \rightarrow 0, \Lambda \rightarrow \infty \quad . \quad (22.47)$$

The precise values of the coefficients  $a$  and  $b$  are not very important, as the combination  $a \log(\Lambda/m) + b$  can be absorbed in the field renormalisation (this means that  $a$  can be read off from  $Z_A$  given above). In section 7-1-1 of Itzykson and Zuber it is shown that  $c = e^2/(60\pi^2 m^2)$ . In the static limit, as is relevant for the hydrogen atom,  $q^2 = -\vec{q}^2$  and the photon exchange can be accurately described by the Coulomb potential

$$-\frac{e^2}{4\pi r} = -e^2 \int d_3\vec{q} \frac{e^{i\vec{q}\cdot\vec{r}}}{\vec{q}^2} \quad , \quad (22.48)$$

which due to the vacuum polarisation is replaced by

$$-e^2 \int d_3\vec{q} \frac{e^{i\vec{q}\cdot\vec{r}}}{\vec{q}^2(1 + \omega(-\vec{q}^2))} = -e^2 \int d_3\vec{q} \left( \frac{1}{\vec{q}^2} + c + \dots \right) e^{i\vec{q}\cdot\vec{r}} = -\frac{e^2}{4\pi r} - \frac{e^4}{60m^2\pi^2} \delta_3(\vec{r}) \quad . \quad (22.49)$$

The extra delta-function interaction, that arises from the vacuum fluctuations, only affects the wave functions that do not vanish in the origin. Consequently, only the energy of the S orbitals will be shifted by this correction

$$\Delta E(nS_{\frac{1}{2}}) = -\frac{4m\alpha_e^5}{15\pi n^3} \quad , \quad \alpha_e = \frac{e^2}{4\pi} \quad , \quad (22.50)$$

where  $n$  is the radial quantum number and  $\alpha_e$  is the fine-structure constant.