

Higher gauge theory — differential versus integral formulation

Florian Girelli¹

Perimeter Institute for Theoretical Physics, 35 King Street N, Waterloo, Ontario, N2J 2W9, Canada

Hendryk Pfeiffer²

Perimeter Institute for Theoretical Physics, 35 King Street N, Waterloo, Ontario, N2J 2W9, Canada

Emmanuel College, St Andrew's Street, Cambridge CB2 3AP, United Kingdom

DAMTP, Wilberforce Road, Cambridge CB3 0WA, United Kingdom

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Abstract

The term higher gauge theory refers to the generalization of gauge theory to a theory of connections at two levels, essentially given by 1- and 2-forms. So far, there have been two approaches to this subject. The differential picture uses non-Abelian 1- and 2-forms in order to generalize the connection 1-form of a conventional gauge theory to the next level. The integral picture makes use of curves and surfaces labeled with elements of non-Abelian groups and generalizes the formulation of gauge theory in terms of parallel transports. We recall how to circumvent the classic no-go theorems in order to define non-Abelian surface ordered products in the integral picture. We then derive the differential picture from the integral formulation under the assumption that the curve and surface labels depend smoothly on the position of the curves and surfaces. We show that some aspects of the no-go theorems are still present in the differential (but not in the integral) picture. This implies a substantial structural difference between non-perturbative and perturbative approaches to higher gauge theory. We finally demonstrate that higher gauge theory provides a geometrical explanation for the extended topological symmetry of BF -theory in both pictures.

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1 Introduction

Gauge theory can be formulated in two ways which we term the differential and the integral picture. As an illustration, recall, for example, Maxwell's equations which can be formulated either in terms of integral equations relating electric and magnetic fluxes through surfaces and currents through solenoids³ (integral picture) or alternatively in terms of the familiar differential equations (differential picture).

¹email: fgirelli@perimeterinstitute.ca

²email: hpfeiffer@perimeterinstitute.ca

³This is actually the form which corresponds to experimental setups and in which the laws of electrodynamics were originally discovered.

Similarly, any gauge theory can be formulated in two ways. Let M be some space-time manifold and the gauge group G be a Lie group with Lie algebra \mathfrak{g} . In the differential formulation of gauge theory, one considers a connection of some principal G -bundle $P \rightarrow M$. In local coordinates, i.e. using a local trivialization of the bundle, the connection is given by a \mathfrak{g} -valued connection 1-form A which transforms under changes of the coordinates as,

$$A \mapsto A' = g^{-1}Ag + g^{-1}dg, \quad (1.1)$$

where $g: U_1 \cap U_2 \rightarrow G$ denotes a transition function on the overlap of the coordinate patches $U_1, U_2 \subseteq M$.

The connection is usually taken to be the basic field of the theory, i.e. the variation in the action principle is with respect to A , and in a path integral quantum theory, one has to integrate over all possible connections. The Lagrangian and the action of the theory depend on the curvature,

$$F = dA + \frac{1}{2}[A, A], \quad (1.2)$$

which is a \mathfrak{g} -valued 2-form. It transforms under coordinate changes in a gauge covariant way,

$$F \mapsto F' = g^{-1}Fg. \quad (1.3)$$

As an alternative to this differential picture, there exists the integral formulation. In this formulation, one uses the group valued parallel transports,

$$U_\gamma = P \exp \left(\int_\gamma A \right) \in G, \quad (1.4)$$

along curves $\gamma: [0, 1] \rightarrow M, \tau \mapsto \gamma(\tau)$ as the basic variables. Locally, the parallel transport always exists, and it is uniquely determined as the solution of the first order matrix differential equation,

$$\frac{d}{dt}U_\gamma(t) = [A_\mu(\gamma(t))\dot{\gamma}^\mu(t)]U_\gamma(t), \quad (1.5)$$

where we have written,

$$U_\gamma(t) = P \exp \left(\int_0^t A_\mu(\gamma(\tau))\dot{\gamma}^\mu(\tau)d\tau \right), \quad (1.6)$$

for the parallel transport along γ from $\tau = 0$ to $\tau = t$. The initial condition is $U_\gamma(0) = 1 \in G$.

The curvature can then be calculated from the holonomy U_γ of a closed loop γ in the limit in which the loop shrinks to infinitesimal size. This integral picture of gauge theory closely resembles what is done in lattice gauge theory, but without the restriction that the curves have to live on some fixed lattice. Note that in the integral formulation, the parallel transports satisfy various relations.

Now let us illustrate the basic idea of higher gauge theory. Suppose first that the gauge group in conventional gauge theory is Abelian, say $G = U(1)$. Then the connection 1-form A is imaginary and the transition functions are of the form $g(x) = e^{\varphi(x)}$, where $\varphi: U \rightarrow i\mathbb{R}$ is a suitable imaginary valued function, and (1.1) becomes,

$$A \mapsto A' = A + d\varphi, \quad (1.7)$$

so that the curvature 2-form (1.2) is just the exterior derivative,

$$F = dA. \quad (1.8)$$

Abelian gauge theory with (1.7) and (1.8) now admits the following higher level generalization. Let A be some (imaginary valued) p -form which becomes the basic field of the theory. The Lagrangian and the action depend only on the curvature $(p+1)$ -form $F = dA$. The theory therefore enjoys a local gauge symmetry with the transformation law (1.7) for some $(p-1)$ -form φ . This is a consequence of the Poincaré lemma because locally any closed p -form $A - A'$ is of the form $d\varphi$. The Abelian theory at level p is therefore completely governed by the de Rham cohomology of M .

The Abelian theory at level $p = 2$ is known in the physics literature as Kalb–Ramond fields [1], and at generic level p as p -form electrodynamics [2]. Both refer to the differential picture of the theory. The corresponding integral picture makes use of p -dimensional surfaces labeled with elements of the Abelian group $U(1)$. If the p -surfaces were restricted to a fixed hyper-cubic lattice, we would have at level $p = 0$ the xy -model of statistical mechanics, at $p = 1$, $U(1)$ -lattice gauge theory, and at higher p the models of [3].

We refer to these higher level theories as *higher gauge theory*. Does there exist a *non-Abelian* higher gauge theory, at least at level $p = 2$? Many authors [4, 5, 6, 7, 8, 9, 10] have attempted to construct such models, but the necessity to find an underlying geometrical picture by suitably generalizing fibre bundles seems to impose serious constraints. To our knowledge, the only thoroughly studied model was the Freedman–Townsend model [11] which, however, has only an Abelian local symmetry and lacks a geometrical understanding of why it has to be Abelian.

What precisely are the geometrical conditions involved in higher gauge theory? The coboundary condition $d \circ d = 0$ of the Abelian case is, of course, no longer useful as the curvature F is in general no longer just dA , but rather given by (1.2). Assume we have some non-Abelian ‘connection’ 2-form and wish to define its ‘curvature’ 3-form. Geometrically, the 2-form would be associated with surfaces labeled by elements of some non-Abelian group H , and the curvature 3-form should be some group element associated with a closed surface, composed from several constituent surfaces that are all labeled with elements of H . Since the group H is non-Abelian and there is no canonical surface order available, we simply do not know how to compose the various non-Abelian labels.

Recall that the points of a curve have a natural order, and the definition of the parallel transport along a given curve indeed makes use of this order. For higher dimensional submanifolds, however, such a canonical order is not available. This lack of natural order has led Teitelboim [12] to the formulation of a no-go theorem ruling out the existence of non-Abelian gauge theories for extended objects. This applies essentially to any gauge theory whose connection is a non-Abelian p -form, $p \geq 2$.

With the introduction of 2-categories in mathematics, it recently became possible to sidestep this no-go theorem at $p = 2$. On the mathematical side, there is the construction of non-Abelian gerbes generalizing fibre bundles, see, for example [13]. It is expected that gerbes provide the desired generalization of fibre bundles. They are, for example, conjectured to play a role in theories on coincident 5-branes in string theory, see, for example [14]. In the present paper, we prefer a slightly different approach based on the definition of Lie 2-groups by Baez [15] which generalize ordinary Lie groups to higher level and which include the symmetries of gerbes at least in the case of strict categories. This will allow us to explicitly solve

the surface ordering problem, thereby providing a rigorous basis for Chepelev's conjectures [9], and to see in detail how Teitelboim's no-go theorem is avoided. Our results finally provide the geometrical background to most of the traditional approaches to non-Abelian 2-forms, at least as long as strict categories are sufficient, and explain geometrically why these models are so restricted. It comes as a surprise that BF -theory which is usually not mentioned in the context of non-Abelian 2-forms, does form a non-trivial example of higher gauge theory.

Let us now briefly outline our approach. Starting from the notion of Lie 2-groups, Baez has defined Lie 2-algebras and started to generalize the differential picture of gauge theory to a theory involving non-Abelian connection 1- and 2-forms [15]. Open questions in this approach are the precise form of the local gauge transformations and of the gauge invariant expressions which are required in order to define Lagrangians and actions in physics.

Also starting from the notion of Lie 2-groups, we have generalized the integral picture of gauge theory to a theory involving curves and surfaces labeled with elements of non-Abelian groups [16]. This formulation has the advantage that the theory of 2-categories dictates the form of the local gauge transformations and the expressions for the gauge invariant quantities. The no-go theorems can be avoided because the underlying 2-categorical structure leads to a non-trivial interplay of the curve and surface labels.

An important question is how the differential [15] and the integral approach [16] are related. In this article, we start from the integral picture of [16] and systematically derive the corresponding differential expressions by studying the non-Abelian curve and surface labels of the theory in the infinitesimal limit, assuming that the labels depend smoothly on the curves and surfaces. In the smooth case, we find an additional flatness condition at level 1 which has not yet appeared in the literature. It implies in particular that the non-Abelian part of the connection 2-form agrees with the curvature of the connection 1-form, that the curvature 2-form vanishes and that the curvature 3-form is Abelian.

We show that an interesting example of higher gauge theory is given by BF -theory with non-Abelian gauge group [17] in which the level-1 flatness is a key feature of the theory, in fact encoded in the field equations. The theory of 2-categories then provides the explanation for the extended local (topological) symmetry. Otherwise, the resulting conditions show that the classic no-go theorems reappear only in the differential picture in which they rule out the naive generalization of the Yang–Mills action. The algebraic structure of the integral picture, however, still allows us to have non-trivial central group elements that characterize the 2-curvature associated with closed labeled surfaces which we call 2-holonomies. If the centre of the gauge group is discrete such as, for example, for $SU(N)$, the differential picture would require these central elements to be trivial, but in the integral picture and in any non-perturbative quantum theory based on it, no such restriction applies. The integral picture is therefore more general than the differential one and is in some sense essentially non-perturbative.

Since the non-trivial central elements can be interpreted as the presence of singularities of codimension 2, in some sense the integral picture of higher gauge theory rather *predicts* the existence of topological defects in the differential formulation.

The present article is structured as follows. In Section 2, we recall the construction of higher gauge theory in the integral formulation as it was developed in [16]. Our presentation is self-contained. We emphasize the calculational aspects and try to hide as much as possible of the 2-category theory in our notation. In Section 3, we then derive the differential formulation of higher gauge theory starting from the integral picture and compare the result with [15]. We conclude in Section 4 with comments on interesting questions for further investigations.

2 The integral formulation

In this section, we review the integral picture of higher gauge theory following [16]. We call the higher level model a *2-gauge theory* whereas we refer to conventional gauge theory as a *1-gauge theory* in view of the hierarchy of models sketched in [16]. The theory is formulated at the integral level, i.e. it describes curves and surfaces which are labeled with data from some algebraic structure, supplementing the parallel transports of conventional 1-gauge theory by additional group elements which are used to label surfaces.

2.1 Lie 2-groups

The algebraic structure required to describe a 1-gauge theory is just some gauge group G , usually taken to be a Lie group. The geometric objects that are labeled with algebraic data, are the curves giving rise to the parallel transports of the theory. The group structure ensures that we can consistently compose (multiply) parallel transports and also reverse their direction (inversion).

The algebraic structure for a 2-gauge theory is a so-called 2-group [15, 18]. This is a pair G, H of groups with two maps⁴. The first map is a group homomorphism $t: H \rightarrow G$, i.e.

$$t(h_1 \cdot h_2) = t(h_1) \cdot t(h_2), \quad (2.1)$$

$$t(1) = 1, \quad (2.2)$$

for all $h_1, h_2 \in H$. The second map is an action of G on H by automorphisms. This is an operation $g \triangleright h$ taking values in H , which is a group action, i.e.

$$(g_1 \cdot g_2) \triangleright h = g_1 \triangleright (g_2 \triangleright h), \quad (2.3)$$

$$1 \triangleright h = h, \quad (2.4)$$

for all $g_1, g_2 \in G$ and $h \in H$, such that $h \mapsto g \triangleright h$ is a homomorphism for each $g \in G$, i.e.

$$g \triangleright (h_1 \cdot h_2) = (g \triangleright h_1) \cdot (g \triangleright h_2), \quad (2.5)$$

$$g \triangleright 1 = 1, \quad (2.6)$$

for all $h_1, h_2 \in H$. These maps are required to satisfy the following two compatibility conditions,

$$t(g \triangleright h) = g \cdot t(h) \cdot g^{-1}, \quad (2.7)$$

$$t(h) \triangleright h' = h \cdot h' \cdot h^{-1}. \quad (2.8)$$

for all $g \in G$, $h, h' \in H$. A Lie 2-group is a 2-group in which G and H are Lie groups and both maps t and \triangleright are smooth.

Plenty of examples of Lie 2-groups are known [15, 16]. Here we mention the following cases.

1. The *Euclidean* and *Poincaré* 2-groups. Here $H = \mathbb{R}^n$ is Euclidean or Minkowski space and $G = SO(n)$ or $SO(n-1, 1)$. The map t is trivial, i.e. $t(h) = 1$ for all $h \in H$, and \triangleright is the obvious action by rotation.

⁴We define here a *crossed module*, a structure from which we can construct a strict 2-group [15, 18].

2. More generally, one can choose H to be any vector space on which the Lie group G is represented. The map t is trivial in this case, and \triangleright is the action of G on its representation H . In this way, one defines, for example, the *adjoint* and the *co-adjoint 2-groups* in which $H = \mathfrak{g}$ or $H = \mathfrak{g}^*$ where \mathfrak{g} denotes the Lie algebra of G .
3. The *automorphism 2-groups*. Let H be any Lie group and G its group of automorphisms. The action $g \triangleright h$ is the application of the particular automorphism, and $t: H \rightarrow G$ associates with each element h the corresponding *inner automorphism* $h' \mapsto hh'h^{-1}$. This example is related to non-Abelian gerbes [15]. For example, if $H = SU(2)$, we have $G = SU(2)/\mathbb{Z}_2 \cong SO(3)$ where \mathbb{Z}_2 is the centre of $SU(2)$. In general, $\ker t \subseteq Z(H)$ is always contained in the centre of H .

2.2 2-gauge theory

In a 2-gauge theory, we have to label geometric objects at two levels. Curves are labeled by elements of G . Their composition and orientation reversal is defined as in conventional gauge theory.

In addition, surfaces are labeled with elements of H . For each surface⁵, we choose two reference points on the boundary (full dots in the diagram below, we are going to suppress them later on) and split the boundary into two curves with labels $g_1 \in G$ (*source*) and $g_2 \in G$ (*target*) as follows,

$$\bullet \begin{array}{c} \xrightarrow{g_1} \\ \Downarrow h \\ \xrightarrow{g_2} \end{array} \bullet \quad (2.9)$$

The label h of the surface is required to satisfy,

$$t(h) = g_2 \cdot g_1^{-1}, \quad (2.10)$$

i.e. $t(h)$ is the (inverse) holonomy along the boundary curve. This condition appears when we use the Lie crossed module in order to construct a Lie 2-group [15, 16, 18]. The first reference point is the base point of this holonomy and therefore plays a role in (2.10) whereas the second reference point does not enter this condition.

We can now compose surfaces in two different ways. Firstly, we can join them *horizontally* in one common reference point,

$$\begin{array}{c} \xrightarrow{g_1} \\ \Downarrow h \\ \xrightarrow{g_2} \end{array} \begin{array}{c} \xrightarrow{g'_1} \\ \Downarrow h' \\ \xrightarrow{g'_2} \end{array} = \begin{array}{c} \xrightarrow{g_1 \cdot g'_1} \\ \Downarrow \tilde{h} \\ \xrightarrow{g_2 \cdot g'_2} \end{array} \quad (2.11)$$

where the label of the composition is given by

$$\tilde{h} = h \cdot (g_1 \triangleright h'). \quad (2.12)$$

Note the asymmetry: the source of the first surface acts on the label of the second one⁶. As required, it follows that $t(\tilde{h}) = (g_2 g'_2)(g_1 g'_1)^{-1}$. Alternatively, we can glue the surfaces

⁵The elementary surfaces are chosen to have the topology of a disc.

⁶The pairs (h, g_1) of surface label and source curve label form the semi-direct product $H \rtimes G$ under horizontal composition.

vertically along a common curve,

$$\begin{array}{c} g_1 \\ \Downarrow h \\ g_2 \end{array} \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} \begin{array}{c} g_3 \\ \Downarrow h' \\ g_3 \end{array} = \begin{array}{c} g_1 \\ \Downarrow \tilde{h} \\ g_3 \end{array} \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} \quad (2.13)$$

where the composition is simply given by

$$\tilde{h} = h' \cdot h. \quad (2.14)$$

Observe that $t(\tilde{h}) = g_3 g_1^{-1}$ as expected.

The orientation of a surface can be reversed if it is labeled by the inverse element h^{-1} instead,

$$\begin{array}{c} g_1 \\ \Downarrow h \\ g_2 \end{array} \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} = \begin{array}{c} g_1 \\ \Uparrow h^{-1} \\ g_2 \end{array} \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} \quad (2.15)$$

Both source and target curve of some surface can be reversed,

$$\begin{array}{c} g_1 \\ \Downarrow h \\ g_2 \end{array} \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} = \begin{array}{c} g_1^{-1} \\ \Downarrow \tilde{h} \\ g_2^{-1} \end{array} \begin{array}{c} \xleftarrow{\quad} \\ \xleftarrow{\quad} \\ \xleftarrow{\quad} \end{array} \quad (2.16)$$

if the surface label is replaced by $\tilde{h} = g_1^{-1} \triangleright h^{-1}$. Observe that $t(\tilde{h}) = g_2^{-1} (g_1^{-1})^{-1}$ as required.

An important operation is known as *whiskering*. By attaching whiskers to a surface h , for example attaching whiskers g_1 and g_1'' to some surface h' with source g_1' and target g_2' ,

$$\xrightarrow{g_1} \begin{array}{c} g_1' \\ \Downarrow h' \\ g_2' \end{array} \xrightarrow{g_1''} = \begin{array}{c} g_1 g_1' g_1'' \\ \Downarrow \tilde{h}' \\ g_1 g_2' g_1'' \end{array} \quad (2.17)$$

we can construct a surface with source $g_1 g_1' g_1''$ and target $g_1 g_2' g_1''$, carrying the label

$$\tilde{h}' = g_1 \triangleright h'. \quad (2.18)$$

The attachment of the left whisker can be understood as a special case of the horizontal composition (2.11) in which $g_1 = g_2$ and $h = 1$ so that the left surface collapses to a line. A similar argument is available for the right whisker. The asymmetry in the expression (2.18) originates from the asymmetry of the horizontal composition (2.12).

Whiskering allows us to change the reference points of a surface. For example, starting from a surface h with reference points x and y , i.e. source $g_1 \cdot g_2$ and target g_3 ,

$$\begin{array}{ccc} & x & \\ g_1^{-1} \nearrow & & \searrow g_3 \\ z & \xrightarrow{g_2} & y \end{array} \begin{array}{c} \Downarrow h \\ \Downarrow h \end{array} = \begin{array}{ccc} & x & \\ g_1^{-1} \nearrow & & \searrow g_3 \\ z & \xrightarrow{g_2} & y \end{array} \begin{array}{c} \Uparrow \tilde{h} \\ \Uparrow \tilde{h} \end{array} \quad (2.19)$$

we can whisker from the left by g_1^{-1} and obtain the surface $\tilde{h} = g_1^{-1} \triangleright h$ with reference points z and y , i.e. source g_2 and target $g_1^{-1} \cdot g_3$.

Given any collection of curves and surfaces, a *configuration* of 2-gauge theory is an assignment of elements of G to the curves and of elements of H to the surfaces so that the following conditions hold. Compositions of curves are labeled by the product of elements in G , curves of opposite orientation are labeled by the inverse group element. For each surface labeled by $h \in H$, we have $t(h) = g_2 \cdot g_1^{-1}$ where g_1 and g_2 are the source and target curve, respectively. Finally, compositions of surfaces, and surfaces whose reference points have been changed, are labeled as described above in this section. The configurations thus defined can be viewed as the classical configurations of 2-gauge theory or, in a path integral quantum theory, these are the configurations over which we sum in the path integral. The path integral was given in detail in [16].

2.3 Local 2-gauge transformations

The 2-gauge theory defined in the preceding section enjoys an extended local gauge symmetry which we call a *local 2-gauge symmetry*.

First recall the conventional local 1-gauge symmetry in a formulation of gauge theory in the language of parallel transports. A local gauge transformation is given by a *generating function* assigning group elements $\eta_x, \eta_y \in G$ to the points. For each curve γ from point x to point y with label $g_\gamma \in G$, the transformed parallel transport is then calculated by,

$$\tilde{g}_\gamma = \eta_x^{-1} g_\gamma \eta_y, \quad (2.20)$$

which we visualize by the following diagram,

$$\begin{array}{ccc} x & \xrightarrow{g_\gamma} & y \\ \eta_x \downarrow & & \downarrow \eta_y \\ x & \xrightarrow{\tilde{g}_\gamma} & y \end{array} \quad (2.21)$$

We say that the diagram *commutes*, i.e. it does not matter which way round we go from one corner to another. If we view all four labeled curves g_γ , \tilde{g}_γ , η_x and η_y as a gauge connection, then this connection is *flat*, i.e. the parallel transport is path independent.

In 2-gauge theory, the local gauge transformation (2.20) is weakened by extending the generating function to the next level. The *2-generating function* not only assigns group elements $\eta_x, \eta_y \in G$ to the points, but there is the additional freedom of choosing elements $\eta_\gamma \in H$ for the curves⁷. Diagram (2.21) is generalized to

$$\begin{array}{ccc} x & \xrightarrow{g_\gamma} & y \\ \eta_x \downarrow & \eta_\gamma \swarrow & \downarrow \eta_y \\ x & \xrightarrow{\tilde{g}_\gamma} & y \end{array} \quad (2.22)$$

⁷These curve labels are in general path dependent.

where we require $t(\eta_\gamma) = (\eta_x \tilde{g}_\gamma)(g_\gamma \eta_y)^{-1}$. The full diagram involving g_γ , \tilde{g}_γ , η_x , η_y and η_γ can therefore be viewed as a configuration of 2-gauge theory in which the surface labeled with η_γ has the source $g_\gamma \cdot \eta_y$ and the target $\eta_x \cdot \tilde{g}_\gamma$. We can thus calculate the gauge transformed parallel transport by,

$$\tilde{g}_\gamma = \eta_x^{-1} t(\eta_\gamma) g_\gamma \eta_y, \quad (2.23)$$

which generalizes the conventional local gauge transformation (2.20).

This is the prescription of how to transform the curve labels. In 2-gauge theory, we have to specify in addition how to transform the surface labels. Therefore we write down the surface analogue of the diagram (2.21) and require that for each surface labeled $h \in H$ with source and target curves γ, γ' labeled by g_γ and $g_{\gamma'}$, the following ‘tin can’ diagram

2-commutes. This means that it does not matter which way round we compose the labeled surfaces, *i.e.* the configuration of 2-gauge theory shown in diagram (2.24) is 2-flat. The top of this ‘tin can’ is the old configuration and the bottom the new one with curve labels $\tilde{g}_\gamma, \tilde{g}_{\gamma'}$ and surface label \tilde{h} . The transformed surface label is thus given by,

$$\tilde{h} = \eta_x^{-1} \triangleright (\eta_{\gamma'} h \eta_\gamma^{-1}). \quad (2.25)$$

We can summarize this paragraph as follows. The local 2-gauge transformations are given by 2-generating functions which assign elements of G to the points and elements of H to the curves. The transformed curve and surface labels are then determined by (2.23) and (2.25). Although, at first sight, these transformation rules look quite artificial, they follow immediately from the underlying 2-categorical structure [16].

2.4 Pure 2-gauge and 2-flatness

In conventional 1-gauge theory, we say that a configuration is *pure gauge* if it is gauge equivalent to the trivial connection in which all curves are assigned the group unit. A configuration is therefore pure gauge if there exists a generating function associating group elements $\eta_j \in G$ with all points so that the parallel transports are given by

$$g_{12} = \eta_1^{-1} \eta_2, \quad (2.26)$$

for any curve from 1 to 2, *c.f.* (2.20). Observe that any configuration which is pure gauge, is also *flat*, *i.e.* its parallel transports are path independent.

In complete analogy, we say that a configuration of 2-gauge theory is *pure 2-gauge* if it is 2-gauge equivalent to the trivial configuration in which all curves are labeled by the group unit of G and all surfaces by the group unit of H . A configuration with curve labels $g_\gamma, g_{\gamma'} \in G$ and surface labels $h \in H$ is therefore pure 2-gauge if there exists a 2-generating function that assigns elements $\eta_x, \eta_y \in G$ to the points and $\eta_\gamma, \eta_{\gamma'} \in H$ to the curves such that for any curve γ from x to y ,

$$g_\gamma = \eta_x^{-1} t(\eta_\gamma) \eta_y, \quad (2.27)$$

and for any surface with source curve g_γ and target curve $g_{\gamma'}$,

$$h = \eta_x^{-1} \triangleright (\eta_{\gamma'} \eta_\gamma^{-1}), \quad (2.28)$$

c.f. (2.23) and (2.25).

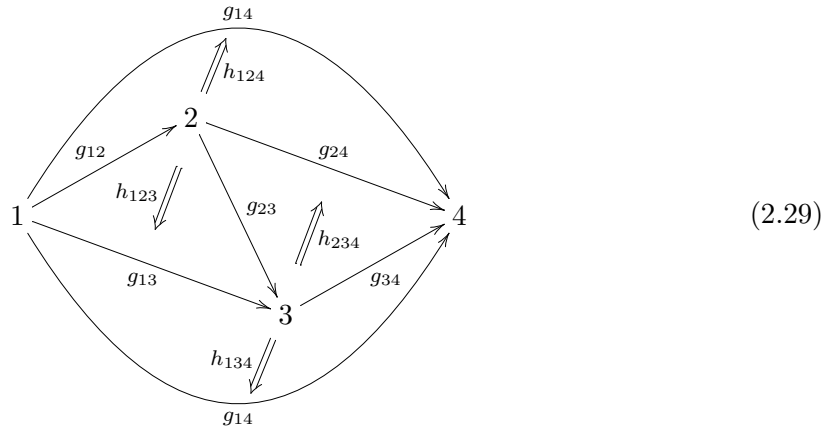
A configuration of 2-gauge theory is called *2-flat* if the surface label on any surface of topology S^2 which is the boundary of a 3-ball, is just the group unit $1 \in H$. As a consequence, in 2-flat configurations, the surface label of any disc shaped surface depends only on the boundary (source and target) curve labels in G . Note also that being pure 2-gauge implies 2-flatness.

2.5 Composition of labeled surfaces

In this section, we describe how the language of 2-gauge theory can be used in order to define compositions of labeled surfaces. We will make use of this surface composition in Section 2.6 in order to construct gauge invariant quantities that are associated with closed surfaces, and in Section 3 in order to derive the differential formulation.

In Section 2.2, we have introduced a number of operations by which we can modify and combine labeled surfaces: vertical and horizontal composition, two types of orientation reversal and the change of reference point by whiskering. These rules can be employed in order to calculate the composition of elementary surfaces to arbitrarily large ones.

We illustrate this procedure for the boundary surface of a tetrahedron,



We have numbered the vertices by 1, 2, 3, 4. The edges (j, k) , $j < k$, are labeled by group elements $g_{jk} \in G$ and the triangles (j, k, ℓ) , $j < k < \ell$, by elements $h_{jkl} \in H$. We have oriented the triangles (j, k, ℓ) so that they have the source $g_{jk} \cdot g_{kl}$ and the target g_{jl} , i.e. $t(h_{jkl}) = g_{jl}(g_{jk}g_{kl})^{-1}$.

We choose reference points, here 1 and 4, and cut the tetrahedron surface along the edge (14). This *base edge* forms both the source and the target curve of the surface. Imagine that a curve starting from the source sweeps out the entire surface until it reaches the target. This determines the ordering of the vertical composition of the constituent surfaces. We just have to make sure that all surfaces are composable, *i.e.* they have the suitable reference points and the correct orientation in order to compose them vertically by (2.13).

Consider the diagram (2.29). We first move the curve from g_{14} to $g_{12}g_{24}$ via h_{124}^{-1} . At this stage we cannot compose the result with the triangle (123) because source and target would not match, but we can use the orientation reversed triangle (234), whiskered from the left by g_{12} . This moves our curve to $g_{12}g_{23}g_{34}$ using the label $g_{12} \triangleright h_{234}^{-1}$ of the whiskered and reversed surface. In the next step, we can use the triangle (123), whiskered from the right by g_{34} which does not change the label h_{123} . Finally, we move our curve from $g_{13}g_{34}$ to g_{14} along h_{134} .

The label associated to the boundary surface of the tetrahedron is therefore the vertical composition, *c.f.* (2.13),

$$\tilde{h} = h_{134}h_{123}(g_{12} \triangleright h_{234}^{-1})h_{124}^{-1}. \quad (2.30)$$

This is a useful notation for the automorphism 2-group in which typically both G and H are non-Abelian. In the case of the Euclidean and the Poincaré 2-groups, it is preferable to write the group structure of H additively, *i.e.*

$$\tilde{h} = h_{134} + h_{123} - g_{12} \triangleright h_{234} - h_{124}. \quad (2.31)$$

The following geometrical picture illustrates the surface composition. Imagine the surface labels h_{jkl} are interpreted in a local coordinate system associated with their first reference point j , the common starting point of their source and target curves. If we vertically compose surfaces that are based at the same reference point, *i.e.* whose labels are given in the same coordinate system, the composition is just the group product in H , *c.f.* (2.14). If the reference points and therefore the coordinate systems are different, however, then we have to parallel transport before we can compare and multiply their surface labels. In the example (2.30), this is relevant for the surface h_{234} , the only surface that is not based at point 1 but rather at 2. We have to whisker h_{234}^{-1} from the left by g_{12} in order to obtain a surface $g_{12} \triangleright h_{234}^{-1}$ with reference point 1.

For a closed surface of topology S^2 , *i.e.* of genus zero, source and target curve agree so that $t(\tilde{h}) = 1$. Recall that $\ker t \subseteq Z(H)$ is always contained in the centre $Z(H)$ of H and therefore Abelian. We call the labels $h \in \ker t$ of closed surfaces the *2-holonomies* of the theory.

2.6 Gauge invariant expressions

For all the assignments of algebraic data to geometric objects, we should understand how they depend on the choices made. Consider, for example, the holonomy along a closed loop (*Wilson loop*) in conventional gauge theory. It still depends on the base point of the loop. It does so, however, in a well-understood way. Changing the base point leads to the conjugation of the holonomy with the parallel transport from the old to the new base point. Any group character applied to the holonomy yields an invariant. Observe that the independence of the base point and the invariance under local gauge transformations are both implemented by the same operation, namely by calculating the character. Due to its gauge invariance, the character can then serve as the Lagrangian or as the action of a physical theory.

An analogous result can be shown for the integral picture of 2-gauge theory [16]. Consider a closed surface of topology S^2 , for example the surface of the tetrahedron (2.29). We have to chose a *base edge* at which we start and finish the surface composition. In our tetrahedron example this was the edge (14). When we change the base edge, holding its two end points fixed, then the 2-holonomy h' of such a closed surface (*Wilson surface*) is conjugated,

$$h' \mapsto hh'h^{-1}, \quad (2.32)$$

by the label $h \in H$ associated with the surface enclosed between the old and the new base edge. If we change the reference point of a closed surface (the starting point of its base edge) by whiskering with $g \in G$, then the 2-holonomy is acted upon by the corresponding parallel transport,

$$h' \mapsto g \triangleright h'. \quad (2.33)$$

We have seen that the 2-holonomies, *i.e.* the labels h' associated with closed surfaces, are contained in $\ker t$. The functions $s: \ker t \rightarrow \mathbb{R}$ that are independent of the base edge and of the reference points, *i.e.* that satisfy for all $g \in G$, $h \in H$ and $h' \in \ker t$,

$$\begin{aligned} s(hh'h^{-1}) &= s(h'), \\ s(g \triangleright h') &= s(h'), \end{aligned} \quad (2.34)$$

are called *2-actions*. We have shown in [16] that these are precisely the functions of the 2-holonomy that are invariant under the local 2-gauge transformations (2.23) and (2.25), hence the name. They form the generalization of the Wilson action to 2-gauge theory.

For the Euclidean and Poincaré 2-groups, the 2-actions are the maps $s: H \rightarrow \mathbb{R}$ that are constant on the orbits of G on $H = \mathbb{R}^n$, *i.e.* they are functions of the invariant Euclidean or Minkowski norm, $s(v) = f(\eta(v, v))$ where $f: \mathbb{R} \rightarrow \mathbb{R}$ is any function, $v \in \mathbb{R}^n$, and η denotes the Euclidean or Minkowski scalar product. For the automorphism 2-group, any map $s: Z(H) \rightarrow \mathbb{R}$ gives rise to an acceptable 2-action.

Even though there exists no canonical surface ordering, we have shown, using ideas from the theory of 2-categories, that the interplay of curves and surfaces not only circumvents the no-go theorems, but also provides us with an unambiguous and gauge covariant composition of labeled surfaces.

Whereas in conventional gauge theory, the gauge invariant expressions are associated with closed loops, we have seen that in 2-gauge theory, we can form 2-gauge invariant expressions for closed surfaces. Is there also a 2-gauge invariant expression associated with loops, *i.e.* a direct generalization of the Wilson loop to 2-gauge theory?

Recall that in a 1-gauge theory, we would just calculate the (real part of a) character,

$$\chi(g_2^{-1}g_1), \quad (2.35)$$

of the holonomy $g_2^{-1}g_1 \in G$ in order to obtain a locally 1-gauge invariant expression. If we take into account a possibly non-trivial transport of curves along surfaces, then we cannot directly compare the two curve labels g_1 and g_2 , but rather have to surface transport one curve onto the other,

$$\begin{array}{ccc} \begin{array}{c} \xrightarrow{g_1} \\ \Downarrow h \\ \xrightarrow{g_2} \end{array} & \longrightarrow & \begin{array}{c} \xrightarrow{t(h)g_1} \\ \xrightarrow{g_2} \end{array} \end{array} \quad (2.36)$$

Rather than the usual holonomy $g_2^{-1}g_1$, we should therefore consider the expression,

$$\mathcal{F} = g_2^{-1}t(h)g_1, \quad (2.37)$$

which takes the surface transport into account. Due to condition (2.10), however, this expression always gives the group unit of G , $\mathcal{F} = 1$.

There is therefore no loop based gauge invariant expression in 2-gauge theory which would generalize the Wilson loop of 1-gauge theory.

3 The differential formulation

In this section, we derive the differential formulation of 2-gauge theory which corresponds to the integral picture of the previous section. We therefore study the integral formulation of 2-gauge theory on squares, cubes and hypercubes, assume that the labels depend smoothly on the positions of the curves and surfaces and consider the limit in which these shrink to infinitesimal size.

The theory we derive uses the same connection 1- and 2-forms as Baez [15] which have also been found independently by Hofman [10], but with a flatness condition at level 1 which has not yet appeared in the literature. As a bonus, we can also derive the local gauge transformations.

3.1 Lie 2-algebras

Just as the differential picture of conventional gauge theory involves the Lie algebra \mathfrak{g} of the gauge group G , we need here the appropriate generalized notion of a ‘Lie algebra’ associated with the gauge 2-group.

A *Lie 2-algebra* consists of two Lie algebras \mathfrak{g} and \mathfrak{h} with two maps⁸. The first map, $\tau: \mathfrak{h} \rightarrow \mathfrak{g}$, is a homomorphism of Lie algebras, i.e. a linear map that satisfies,

$$\tau([Y_1, Y_2]) = [\tau(Y_1), \tau(Y_2)], \quad (3.1)$$

for all $Y_1, Y_2 \in \mathfrak{h}$. The second map is an action of \mathfrak{g} on \mathfrak{h} by derivations, i.e. an operation $X \triangleright Y$ for $X \in \mathfrak{g}$, $Y \in \mathfrak{h}$, taking values in \mathfrak{h} , such that it is an action, i.e.

$$[X_1, X_2] \triangleright Y = X_1 \triangleright (X_2 \triangleright Y) - X_2 \triangleright (X_1 \triangleright Y), \quad (3.2)$$

for all $X_1, X_2 \in \mathfrak{g}$ and $Y \in \mathfrak{h}$, and such that for any $X \in \mathfrak{g}$, the map $Y \mapsto X \triangleright Y$ is a derivation on \mathfrak{h} , i.e.

$$X \triangleright [Y_1, Y_2] = [X \triangleright Y_1, Y_2] + [Y_1, X \triangleright Y_2], \quad (3.3)$$

for all $Y_1, Y_2 \in \mathfrak{h}$. These maps are required to satisfy the following two compatibility conditions,

$$\tau(X \triangleright Y) = [X, \tau(Y)], \quad (3.4)$$

$$\tau(Y) \triangleright Y' = [Y, Y'], \quad (3.5)$$

for all $X \in \mathfrak{g}$, $Y, Y' \in \mathfrak{h}$.

⁸We describe here a differential crossed module, a structure from which we can construct a strict Lie 2-algebra [15, 19].

Given some Lie 2-group in terms of the Lie groups G , H and the maps t and \triangleright (Section 2.1), one can construct its Lie 2-algebra as follows [19]. The Lie algebras \mathfrak{g} and \mathfrak{h} are the Lie algebras of the Lie groups G and H . The map $\tau: \mathfrak{h} \rightarrow \mathfrak{g}$ is the derivative $\tau = dt$ of the map $t: H \rightarrow G$. Finally, let the map,

$$\alpha: G \rightarrow \text{Aut } H, \quad \alpha(g)[h] := g \triangleright h, \quad (3.6)$$

associate an automorphism $\alpha(g)$ of H with each element $g \in G$. Then the derivative of α ,

$$d\alpha: \mathfrak{g} \rightarrow \text{Der } \mathfrak{h}, \quad X \mapsto d\alpha(X), \quad (3.7)$$

associates with each element $X \in \mathfrak{g}$ a derivation $d\alpha(X)$ of \mathfrak{h} . The operation \triangleright in the definition of the Lie 2-algebra is chosen to be $X \triangleright Y := d\alpha(X)[Y]$.

Consider first the Lie 2-algebra of the Euclidean and Poincaré 2-groups. In this case, $\mathfrak{g} = \mathfrak{so}(n)$ or $\mathfrak{g} = \mathfrak{so}(n-1, 1)$, and $\mathfrak{h} = \mathbb{R}^n$. The action of \mathfrak{g} on \mathfrak{h} is in the defining representation of \mathfrak{g} , and the map $\tau: \mathfrak{h} \rightarrow \mathfrak{g}$ is the null map.

For the automorphism 2-group of $H = SU(2)$, we have $G = SU(2)/\mathbb{Z}_2$. In this case, both Lie algebras agree, $\mathfrak{g} = \mathfrak{h}$, and we have $t(Y) = Y$ for all $Y \in \mathfrak{h}$. Finally, the action of \mathfrak{g} on $\mathfrak{h} = \mathfrak{g}$ is the adjoint action, $X \triangleright Y = [X, Y]$.

Let us conclude this section with a remark on the category theory underlying the construction of Lie 2-algebras. When we construct a Lie 2-algebra from the differential crossed module, there is the condition (analogous to (2.10)),

$$\tau(Y) = X_2 - X_1, \quad (3.8)$$

for each 2-cell,

$$\begin{array}{c} X_1 \\ \curvearrowright \\ \Downarrow Y \\ \curvearrowleft \\ X_2 \end{array} \quad (3.9)$$

$X_i \in \mathfrak{g}$, $Y \in \mathfrak{h}$, of the 2-category which is defined by the Lie 2-algebra. In the gauge theory language, this would correspond to an infinitesimally small surface. The condition (3.8) is in fact already present in the 2-vector spaces of [19].

3.2 Notation

For the discussion of the differential picture of higher gauge theory, we restrict ourselves to trivial bundles and present the theory in the language of the \mathfrak{g} - and \mathfrak{h} -valued connection 1- and 2-forms. As we will see in the following section, the basic fields of the differential picture are a \mathfrak{g} -valued connection 1-form A and an \mathfrak{h} -valued connection 2-form B .

We denote by d_A the exterior covariant derivative for the connection A which acts on \mathfrak{g} -valued p -forms φ by,

$$d_A(\varphi) = d\varphi + [A, \varphi]. \quad (3.10)$$

Here the bracket of a 1-form A with a p -form φ , both taking values in \mathfrak{g} , is defined by,

$$[A, \varphi] := A^a \wedge \varphi^b [T_a, T_b], \quad (3.11)$$

where we have chosen a basis (T_a) of \mathfrak{g} and written $A = A^a T_a$, $\varphi = \varphi^b T_b$ with coefficient forms A^a and φ^b . Summation over repeated indices is understood. Similarly, we define the action of d_A on \mathfrak{h} -valued p -forms ψ , using the action of \mathfrak{g} on \mathfrak{h} via the operation \triangleright ,

$$d_A(\psi) = d\psi + A \triangleright \psi, \quad (3.12)$$

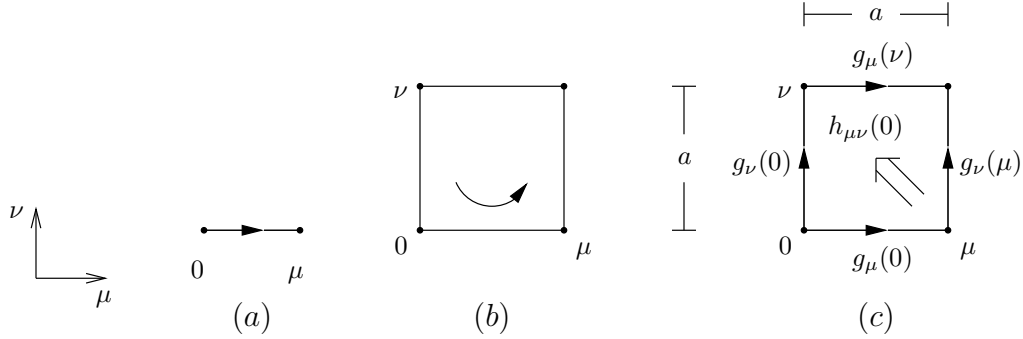


Figure 1: Our conventions for the integral formulation of 2-gauge theory on the squares, in particular (c) the generalized holonomy (3.22).

where the \triangleright of a \mathfrak{g} -valued 1-form with an \mathfrak{h} -valued p -form is defined by,

$$A \triangleright \psi := A^a \wedge \psi^b (T_a \triangleright T'_b), \quad (3.13)$$

where (T_a) denotes a basis of \mathfrak{g} and (T'_b) a basis of \mathfrak{h} . We calculate for \mathfrak{g} -valued p -forms φ ,

$$(d_A \circ d_A)(\varphi) = [F, \varphi], \quad (3.14)$$

and for \mathfrak{h} -valued p -forms ψ ,

$$(d_A \circ d_A)(\psi) = F \triangleright \psi. \quad (3.15)$$

3.3 Configuration variables and their curvature

We shall first identify the basic fields and their associated curvature. We then compute their respective transformations under the local 2-gauge transformations.

We assume that all labels depend smoothly on the curves and surfaces. The key idea for the derivation of the differential picture is to write down the integral formulation on squares and cubes and then to study the limit in which these shrink to infinitesimal size. We write the labels $g \in G$, $h \in H$ as exponentiated curve and surface integrals over approximately constant differential forms,

$$\begin{aligned} g_\mu(0) &= e^{\int_\gamma A} \sim e^{aA_\mu}, \\ h_{\mu\nu}(0) &= e^{\int_S B} \sim e^{a^2 B_{\mu\nu}}. \end{aligned} \quad (3.16)$$

Here γ denotes a curve of length a from $x = 0$ to $x = \mu$ and S a square of area a^2 in the $(\mu\nu)$ -plane. We abbreviate the coordinates by $x = \mu := ae_\mu$ where e_μ is a vector of unit length. All A_μ , etc. without argument are at $x = 0$.

The basic fields in the differential picture are the \mathfrak{g} -valued connection 1-form $A = A_\mu dx^\mu$ and the \mathfrak{h} -valued connection 2-form $B = \frac{1}{2} B_{\mu\nu} dx^\mu \wedge dx^\nu$. Note that $h_{\mu\nu} = h_{\nu\mu}^{-1}$.

We will make use of the usual Taylor expansion,

$$g_\mu(\alpha) \sim e^{aA_\mu + a^2 \partial_\alpha A_\mu}, \quad (3.17)$$

$$h_{\mu\nu}(\alpha) \sim e^{a^2 B_{\mu\nu} + a^3 \partial_\alpha B_{\mu\nu}}. \quad (3.18)$$

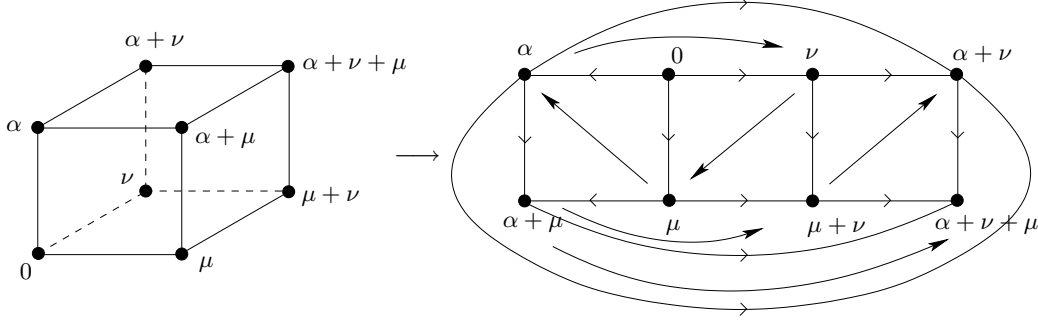


Figure 2: A flattened cube in order to read off the 2-holonomy (3.26).

When we have a product of Lie group elements, the Baker–Hausdorff formula allows us to get the corresponding operation at the Lie algebra level,

$$e^x e^y = e^{x+y+\frac{1}{2}[x,y]+\dots}. \quad (3.19)$$

The action $d\alpha$ of \mathfrak{g} on \mathfrak{h} is the infinitesimal version of the action of G over H ,

$$g_\beta(0) \triangleright h_{\mu\nu}(\beta) \sim e^{a^2 B_{\mu\nu} + a^3 \partial_\beta B_{\mu\nu} + a^3 d\alpha(A_\beta)(B_{\mu\nu})}. \quad (3.20)$$

The map $\tau: \mathfrak{h} \rightarrow \mathfrak{g}$ is the infinitesimal version of the map $t: H \rightarrow G$,

$$t(h_{\mu\nu}) \sim e^{a^2 \tau(B_{\mu\nu})}. \quad (3.21)$$

As mentioned earlier, they satisfy the compatibility conditions (3.5) and (3.4). The approximations (3.16–3.20) together with (3.4, 3.5) are all we need in order to derive the differential picture.

So far we have identified as the basic fields the generalized connection (A, B) in agreement with [15]. Let us now calculate a curvature 2-form, using the holonomy around an infinitesimal square, and a curvature 3-form, using the 2-holonomy around an infinitesimal cube.

In 1-gauge theory, the curvature 2-form is given by an infinitesimal Wilson loop. In the context of 2-gauge theory, it depends also on the B -field because of (2.37). The expression \mathcal{F} of (2.37) reads for the square of Figure 1(c),

$$e^{a^2 \tilde{F}_{\mu\nu}} \sim \mathcal{F}_{\mu\nu} = g_\mu^{-1}(\nu) g_\nu(0)^{-1} t(h_{\mu\nu}(0)) g_\mu(0) g_\nu(\mu). \quad (3.22)$$

Using the approximations (3.16–3.19) and dropping all terms of order a^3 in the exponent, we obtain the curvature 2-form $\tilde{F}_{\mu\nu} = \frac{1}{2} \tilde{F}_{\mu\nu} dx^\mu \wedge dx^\nu$ as follows,

$$\begin{aligned} \tilde{F}_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] + \tau(B_{\mu\nu}) \\ &= F_{\mu\nu} + \tau(B_{\mu\nu}), \end{aligned} \quad (3.23)$$

where F denotes the conventional curvature of A . This agrees with Baez' expression [15] (modulo a sign which is matter of convention),

$$\tilde{F} = dA + \frac{1}{2}[A, A] + \tau(B) = F + \tau(B). \quad (3.24)$$

We have therefore found a geometrical interpretation for the generalized curvature \tilde{F} from the surface transport of the curve (2.36).

It is important here to remember that in the definition of a strict Lie 2-group, we have $\mathcal{F} = 1 \in H$ in (2.37) and therefore at the differential level,

$$\tau(B) = -F. \quad (3.25)$$

This condition is the alter ego of the equations (2.10) and (3.8) and has some drastic consequences: it means that the curvature 2-form \tilde{F} is always zero.

Let us now compute the curvature 3-form $G = \frac{1}{6}G_{\alpha\mu\nu}dx^\alpha \wedge dx^\mu \wedge dx^\nu$ associated with A and B . It is the differential counterpart of the 2-holonomy around a cube (Figure 2).

To calculate it, we use the same technique as for the tetrahedron in (2.29, 2.30). We obtain,

$$\begin{aligned} e^{a^3 G_{\alpha\mu\nu}} &\sim \mathcal{G}_{\alpha\mu\nu} \\ &= [g_\alpha(0) \triangleright h_{\mu\nu}(\alpha)] h_{\mu\alpha}(0) [g_\mu(0) \triangleright h_{\nu\alpha}(\mu)] h_{\nu\mu}(0) [g_\nu(0) \triangleright h_{\alpha\mu}(\nu)] h_{\alpha\nu}(0). \end{aligned} \quad (3.26)$$

Let us use once again the the approximations (3.16–3.19) and drop all terms of order a^4 in the exponent, so that we get,

$$G_{\alpha\mu\nu} = \partial_\alpha B_{\mu\nu} + d\alpha(A_\alpha)(B_{\mu\nu}) + \partial_\nu B_{\alpha\mu} + d\alpha(A_\nu)(B_{\alpha\mu}) + \partial_\mu B_{\nu\alpha} + d\alpha(A_\mu)(B_{\nu\alpha}), \quad (3.27)$$

and using the simplified notation,

$$G = dB + A \triangleright B = d_A(B). \quad (3.28)$$

This coincides with Baez' definition of the curvature 3-form [15].

3.4 'Differential' gauge transformations

In order to derive the differential form of the local 2-gauge transformations (2.23) and (2.25), we draw the analogous diagrams for a square and a cube, respectively, *c.f.* Figure 1 and 2. Here the old configuration corresponds to the bottom of the diagram, the new one to the top. We parameterize 'differential' gauge transformations by the height ε of these diagrams, *i.e.* the 2-generating function,

$$\eta_\alpha(0) \sim e^{\varepsilon X}, \quad (3.29)$$

$$\eta_{\mu\alpha}(0) \sim e^{\varepsilon a Y_\mu}, \quad (3.30)$$

is parameterized by a \mathfrak{g} -valued function X and by an \mathfrak{h} -valued 1-form $Y = Y_\mu dx^\mu$. Similarly to Section 3.3, we use the Taylor expansion,

$$\eta_\alpha(\mu) \sim e^{\varepsilon(X + a\partial_\mu X)}, \quad (3.31)$$

$$\eta_{\mu\alpha}(\nu) \sim e^{\varepsilon(aY_\mu + a^2\partial_\nu Y_\mu)}, \quad (3.32)$$

the convention $\eta_{\mu\alpha} = \eta_{\alpha\mu}^{-1}$, the derivative $\tau = dt$,

$$t(\eta_{\mu\alpha}(0)) \sim e^{a\varepsilon\tau(Y_\mu)}, \quad (3.33)$$

and the group actions,

$$g_\nu(0) \triangleright \eta_{\mu\alpha}(\nu) \sim e^{\varepsilon(aY_\mu + a^2\partial_\nu Y_\mu + a^2 d\alpha(A_\nu)(Y_\mu))}, \quad (3.34)$$

$$\eta_\alpha(0) \triangleright h_{\mu\nu}(0) \sim e^{a^2(B_{\mu\nu} + \varepsilon d\alpha(X)(B_{\mu\nu}))}. \quad (3.35)$$

The gauge transformation for the connection 1-form A is read off from the square of Figure 1 and the formula (2.20),

$$e^{aA_\mu + a\varepsilon\delta A_\mu} \sim \tilde{g}_\mu(\alpha) = \eta_\alpha^{-1}(0)t(\eta_{\mu\alpha}(0))g_\mu(0)\eta_\alpha(\mu). \quad (3.36)$$

Using the approximations (3.16–3.19) and dropping terms of order ε^2 and a^2 in the exponent, we get the gauge transformation,

$$A_\mu \mapsto A_\mu + \varepsilon\delta A_\mu, \quad \delta A_\mu = \partial_\mu X + [A_\mu, X] + \tau(Y_\mu), \quad (3.37)$$

that is

$$\delta A = d_A(X) + \tau(Y). \quad (3.38)$$

The 2-gauge transformations of the B -field can be deduced from the flattened cube (Figure 2) whose height in α -direction is ε for the gauge transformation,

$$\begin{aligned} e^{a^2 B_{\mu\nu} + a^2 \varepsilon \delta B_{\mu\nu}} &\sim \tilde{h}_{\mu\nu}(\beta) \\ &= \eta_\alpha(0)^{-1} \triangleright \{ \eta_{\nu\alpha}(0)[g_\nu(0) \triangleright \eta_{\mu\alpha}(\nu)] h_{\mu\nu}(0)[g_\mu(0) \triangleright \eta_{\nu\alpha}^{-1}(\mu)] \eta_{\mu\alpha}^{-1}(0) \}. \end{aligned} \quad (3.39)$$

The infinitesimal transformation is calculated as usual with the help of (3.16–3.19), dropping terms of order ε^2 and a^3 in the exponent, and we get,

$$\begin{aligned} B_{\mu\nu} &\mapsto B_{\mu\nu} + \varepsilon\delta B_{\mu\nu}, \\ \delta B_{\mu\nu} &= -\partial_\mu Y_\nu + \partial_\nu Y_\mu - d\alpha(A_\mu)(Y_\nu) + d\alpha(A_\nu)(Y_\mu) - d\alpha(X)(B_{\mu\nu}). \end{aligned} \quad (3.40)$$

So by using the shorthand notation, we have

$$\delta B = -dY - A \triangleright Y - X \triangleright B = -d_A(Y) - X \triangleright B. \quad (3.41)$$

From the gauge transformations (3.37) and (3.40) for A and B , we can deduce the transformation of the curvature 2-form,

$$\tilde{F} \mapsto \tilde{F} + \varepsilon\delta\tilde{F}, \quad \delta\tilde{F} = [\tilde{F}, X], \quad (3.42)$$

up to terms of order ε^2 . This means that \tilde{F} transforms covariantly even without the assumption $\tilde{F} = 0$, and on the other hand that the transformation preserves the condition $\tilde{F} = 0$.

For the curvature 3-form G , we obtain after a rather lengthy calculation,

$$G \mapsto G + \varepsilon\delta G, \quad \delta G = -\tilde{F} \triangleright Y - X \triangleright G, \quad (3.43)$$

up to terms of order ε^2 . This transformation shows that G transforms covariantly if and only if $\tilde{F} = 0$. However, if we are considering the case of a strict Lie 2-algebra, then this flatness condition is naturally present, and G transforms covariantly, moreover it sees only the level 1 of the generating function,

$$\delta G = -X \triangleright G. \quad (3.44)$$

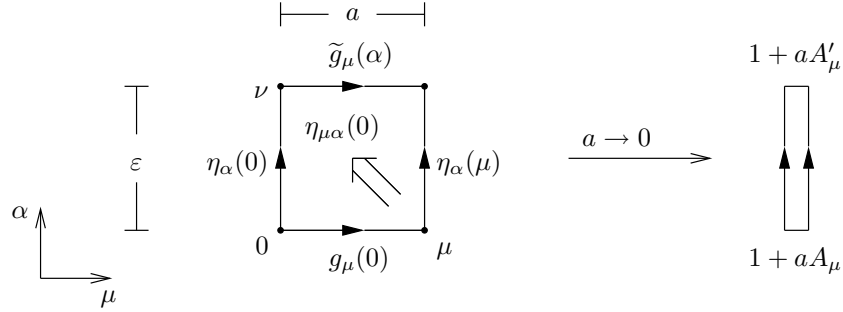


Figure 3: The local gauge transformation of the edge labels in the integral picture, *c.f.* (3.50). The bottom layer is the old configuration, the top layer the new one. In order to pass to the differential picture, we shrink the rectangle to infinitesimal width, $a \rightarrow 0$, but keep its height ε fixed.

3.5 ‘Large’ gauge transformations

In the previous section, we have derived the ‘differential’ gauge transformations in the differential picture. What can we conclude from their existence?

Recall first the role of ‘large’ and ‘differential’ gauge transformations in conventional non-Abelian gauge theory with gauge group G . A ‘large’ gauge transformation is a bundle automorphism of the principal bundle $P \rightarrow M$. In a local trivialization on $U \subseteq M$, it is given by a G -valued generating function $g: U \rightarrow G$. The connection 1-form and the curvature 2-form transform as,

$$A \mapsto g^{-1}Ag + g^{-1}dg, \quad (3.45)$$

$$F \mapsto g^{-1}Fg. \quad (3.46)$$

It is often convenient to consider only the tangents to the above transformations which means to parameterize g in terms of the Lie algebra,

$$g = e^{\varepsilon X}, \quad (3.47)$$

where $X: U \rightarrow \mathfrak{g}$ is a Lie algebra valued function. If G is compact and connected, then the exponential map is surjective, see, for example [20], and any generating function is of this form. If G is non-compact or not connected, this is in general no longer true. Usually, one considers the parameterization (3.47) only for small ε and finds,

$$A \mapsto A + \varepsilon(dX + [A, X]) = A + \varepsilon d_A(X), \quad (3.48)$$

$$F \mapsto F + \varepsilon[F, X], \quad (3.49)$$

dropping terms of order ε^2 . Since we know that we can always integrate these ‘differential’ gauge transformations, we can recover the ‘large’ transformations as long as the exponential map is surjective.

Let us now try to derive the ‘large’ counterparts of the ‘differential’ 2-gauge transformations of Section 3.4. Therefore, we again consider the integral formulation on squares and cubes, but this time we keep the height ε fixed and consider only the limit $a \rightarrow 0$, see Figure 3.

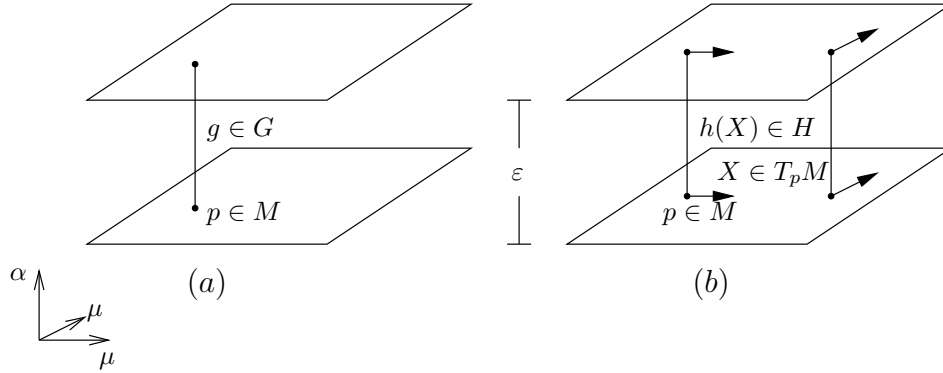


Figure 4: (a) After shrinking the square of Figure 3 to infinitesimal width $a \rightarrow 0$, its vertical edges carry a group label $g(p) = \eta_\alpha(p) \in G$ at each point $p \in M$. (b) The vertical surface label $\eta_{\mu\alpha}(p)$ would ideally yield an H -valued 1-form $h_\mu(p)dx^\mu = \eta_{\mu\alpha}(p)dx^\mu$ which associates with each vector $X \in T_p M$ an element $h(X) \in H$. The linear structure of $T_p M$ here imposes a serious constraint as we explain in the text.

First we restrict ourselves to the case in which the curve labels of the gauge generating transformation are trivial, i.e. the surface in Figure 3 has the trivial label $\eta_{\mu\alpha}(0) = 1 \in H$. Rather than (3.36), we now write $e^{aA_\mu} \sim g_\mu(0)$ and $e^{aA'_\mu} \sim \tilde{g}_\mu(\alpha)$ and obtain,

$$(1 + aA'_\mu) \sim e^{aA'_\mu} \sim \eta_\alpha(0)^{-1} e^{aA_\mu} \eta_\alpha(\mu) \sim \eta_\alpha(0)^{-1} (1 + aA_\mu) (\eta_\alpha(0) + a\partial_\mu \eta_\alpha(0)). \quad (3.50)$$

Dropping terms of order a^2 , this gives the familiar transformation rule (3.45) for the G -valued function $g(p) = \eta_\alpha(p)$, $p \in U$. The index α was just used in order to denote the vertical ‘gauge’ direction in our figure.

All the data required in order to describe the gauge transformation are associated with vertical curves or surfaces which link the bottom with the top layer of Figure 3. If the surface label is trivial, i.e. if the gauge generating transformation assigns the group unit $1 \in H$ to each curve, we have to deal only with vertical lines labeled by elements,

$$\eta_\alpha(p) = g(p) \in G, \quad (3.51)$$

at each point $p \in U$, Figure 4(a). Indeed, the gauge generating function,

$$g: U \rightarrow G, \quad (3.52)$$

can be visualized by a bunch of such vertical lines between the old configuration (bottom) and the new one (top). It poses no problem that the labels are in the group G . The transformation is just a change of coordinates.

Let us now consider the case in which the curve labels of the 2-generating function are non-trivial, i.e. the square in Figure 3 has a non-trivial label $\eta_{\mu\alpha}(0) \in H$. The index α just indicates that the surface is vertical, but the index μ has indeed a geometrical meaning. In order to derive the ‘differential’ gauge transformations we have expanded (3.30) in terms of both a and ε in order to obtain a 1-form $Y_\mu dx^\mu$ as the differential expression.

If we expand only in terms of a , but keep ε fixed, we run into the following geometrical obstruction as depicted in Figure 4(b). By expansion in terms of a , we wish to obtain a

1-form, say $h_\mu dx^\mu$, from the vertical surface. For any tangent vector $X \in T_p M$ at some point $p \in U$, we should therefore be able to evaluate $h(X) = h_\mu X^\mu$. On the other hand, from the surface label, $\eta_{\mu\alpha}(0) \in H$, there remains for each choice of μ a group element in H . We are tempted to write,

$$\eta_{\mu\alpha}(p) = h_\mu(p) \in H. \quad (3.53)$$

Since $T_p M$ is a linear space, however, this is possible only if h_μ gives rise to a *linear* map,

$$h: TM \rightarrow H. \quad (3.54)$$

This condition is stronger than that in (3.51) and (3.52).

The construction of the ‘large’ gauge transformations in the differential picture is therefore possible only if $H = \mathbb{R}^n$ for some n . In this case, the action \triangleright of G on H is actually a representation of the Lie group G and induces a representation \triangleright of the Lie algebra \mathfrak{g} on H .

We parameterize the ‘large’ gauge transformations by $h_\mu(p) \in H$, $p \in M$, i.e.

$$\eta_{\mu\alpha}(p) = a h_\mu(p), \quad (3.55)$$

identifying H with its Lie algebra and indicating that h_μ is already a quantity of order a . Rather than (3.36), we obtain for the transformation of the connection 1-form,

$$A \mapsto A' = g^{-1} A g + g^{-1} dg + \tau(h). \quad (3.56)$$

In order to derive the ‘large’ gauge transformations for the connection 2-form $B_{\mu\nu}$, we replace (3.39) by,

$$e^{a^2 B'_{\mu\nu}} \sim \eta_\alpha^{-1}(0) \triangleright \left[\eta_{\nu\alpha}(0) (e^{a A_\nu} \triangleright \eta_{\mu\alpha}(\nu)) e^{a^2 B_{\mu\nu}} (e^{a A_\mu} \triangleright \eta_{\nu\alpha}^{-1}(\mu)) \eta_{\mu\alpha}^{-1}(0) \right]. \quad (3.57)$$

Expanding everything up to order a^2 , using $\eta_\alpha(0) = g(0)$, (3.55), and $h_\mu(\nu) = h_\mu(0) + a \partial_\nu h_\mu(0)$, we obtain the transformation,

$$B \mapsto B' = g^{-1} \triangleright B - d_A(h). \quad (3.58)$$

From the ‘large’ gauge transformations (3.56) and (3.58) one can recover the ‘differential’ transformations (3.38) and (3.41) using the parameterizations (3.47) and (3.55).

We conclude that we have a full extended local gauge symmetry only if $H \cong \mathbb{R}^n$. In general, it is not possible to integrate the ‘differential’ gauge transformations and to obtain proper (‘large’) transformations. The ‘large’ transformations are given by (3.56), (3.58) and,

$$G \mapsto G' = g^{-1} \triangleright G. \quad (3.59)$$

3.6 Pure 2-gauge and 2-flatness

For the case $H \cong \mathbb{R}^n$, we can now express the condition of being pure 2-gauge (Section 2.4) in the differential language. A generalized connection (A, B) is pure gauge if there exists (locally) a G -valued function $g: U \rightarrow G$ and an H -valued 1-form $h: T^*M|_U \rightarrow H$ such that,

$$A = g^{-1} dg + \tau(h), \quad (3.60)$$

$$B = -dh. \quad (3.61)$$

It is straightforward to show that these configurations are also 2-flat.

3.7 Flatness at level 1

The level-1 flatness condition $\tilde{F} = 0$, *c.f.* (3.25), has the following effect on the curvature 3-form. The definition $G = d_A(B)$ implies $\tau(G) = d_A(\tau(B)) = -d_A(F) = 0$ by the Bianchi identity for the conventional curvature F of A . This implies that G takes values in $\ker \tau \trianglelefteq \mathfrak{h}$ which is an Abelian ideal of \mathfrak{h} . This is the differential counterpart of the result [16] that the 2-holonomy of an S^2 surface in the integral picture takes values in the Abelian normal subgroup $\ker t \trianglelefteq H$ (Section 2.6).

We can always decompose the \mathfrak{h} -valued connection 2-form $B = z \oplus B'$ where $z \in \ker \tau$ and $\mathfrak{h} = \ker \tau \oplus \mathfrak{h}'$ is split into a direct sum of vector spaces. The non-Abelian part of B is therefore contained in $B' \in \mathfrak{h}'$ and related to the conventional curvature F of A by $F = -\tau(B')$ due to (3.25). The only contribution to B unrelated to the curvature of A is contained in some Abelian sub-algebra of \mathfrak{h} .

3.8 Higher Bianchi identities

As a generalization of the Bianchi identity $d_A(F) = 0$ of conventional 1-gauge theory, we have in 2-gauge theory,

$$d_A(\tilde{F}) = d_A(F) + \tau(d_A(B)) = \tau(G) = 0, \quad (3.62)$$

where we have used the condition (3.25) only in the last step, and

$$d_A(G) = F \triangleright B = -\tau(B) \triangleright B = -[B, B] = 0, \quad (3.63)$$

where we have used (3.25) in the second step. They can be derived from the integral picture by drawing the square and cube from the definition of the curvature 2- and 3-form and by parallel transporting the entire diagram in an independent direction.

3.9 Examples

BF-theory. Consider first the special example in which we use the adjoint 2-group of some Lie group G , *i.e.* $H = \mathfrak{g}$ is the Lie algebra, G acts on $H = \mathfrak{g}$ by the adjoint action, and the map $t: H \rightarrow G$ is $t(h) = 1$ for all $h \in H$. The corresponding Lie 2-algebra is given by the Lie algebras $\mathfrak{g} = \mathfrak{h}$, the adjoint action of \mathfrak{g} on \mathfrak{h} and the null map $\tau = dt = 0$.

In this case, the differential forms A , B , F , \tilde{F} and G are all \mathfrak{g} -valued. In four dimensions, one can therefore consider the Lagrangian of *BF*-theory [17],

$$\mathcal{L} = \text{tr}_{\mathfrak{g}}(B \wedge F). \quad (3.64)$$

The local 2-gauge transformations are generated by a \mathfrak{g} -valued function X and a \mathfrak{g} -valued 1-form Y ,

$$\delta A = d_A(X), \quad (3.65)$$

$$\delta B = d_A(Y) - [X, B], \quad (3.66)$$

$$\delta F = -[X, F], \quad (3.67)$$

$$\delta G = -[X, G]. \quad (3.68)$$

They encompass both the ordinary local gauge symmetry (generated by X) and the extended, so-called topological, local symmetry which is a special feature of *BF*-theory (generated by

Y). Both are unified in the local 2-gauge symmetry. We have therefore discovered the actual geometrical reason for the topological symmetry of BF -theory. Notice that the level-1 flatness condition (3.25) reads in this case $F = 0$ which is actually one of the field equations of BF -theory.

Notice that in the case of BF -theory, the Abelian ideal is $\ker t = \mathfrak{g}$ which is an Abelian group using the addition of elements of \mathfrak{g} , even though \mathfrak{g} as a Lie algebra can be non-Abelian if the gauge group G is non-Abelian.

Yang–Mills theory. Let us now try to construct a higher level analogue of the Yang–Mills action. In conventional gauge theory, the Yang–Mills Lagrangian reads,

$$\mathcal{L} = \text{tr}_{\mathfrak{g}}(F \wedge *F), \quad (3.69)$$

where $\text{tr}_{\mathfrak{g}}$ denotes the Cartan–Killing form of \mathfrak{g} . A candidate for a Lagrangian density in higher gauge theory is therefore given by the expression,

$$\mathcal{L} = \text{tr}_{\mathfrak{h}}(G \wedge *G). \quad (3.70)$$

We could have tried $\text{tr}_{\mathfrak{g}}(\tilde{F} \wedge *\tilde{F})$ which, however, vanishes because of (3.25) in the case of strict Lie 2-groups. We have seen that the curvature 3-form G is always Abelian.

If we choose the Euclidean or Poincaré 2-group, we have $\mathfrak{g} = \mathfrak{so}(n)$ or $\mathfrak{so}(n-1, 1)$, $\mathfrak{h} = \mathbb{R}^n$ and $\tau: \mathfrak{h} \rightarrow \mathfrak{g}$ the null map. This implies in particular that $\tilde{F} = F$, and the condition (3.25) furthermore states that the connection A is flat. In addition, we have a connection 2-form B taking values in $\mathfrak{h} = \mathbb{R}^n$ with a curvature 3-form $G = d_A(B)$. Locally, the flatness of A implies that it is pure gauge, *i.e.* gauge equivalent to $A = 0$, so that locally G is just the exterior derivative, $G = dB$. The Yang–Mills Lagrangian (3.70) therefore agrees locally with that of Abelian 2-form electrodynamics.

A similar result can be shown for all 2-groups in which $H = V$ is a vector space on which G is represented. The connection 1-form A is locally pure gauge and the Yang–Mills Lagrangian (3.70) reduces to that of Abelian 2-form electrodynamics.

Consider finally the automorphism 2-group of $H = SU(2)$, *i.e.* $\mathfrak{g} = \mathfrak{h} = \mathfrak{su}(2)$, τ is the identity map, and \mathfrak{g} acts on $\mathfrak{h} = \mathfrak{g}$ by the adjoint action. In this case, the condition (3.25) implies that $B = -F$ is just (minus) the ordinary curvature 2-form of A . The Yang–Mills Lagrangian (3.70) therefore vanishes because of the conventional Bianchi identity $G = d_A(B) = -d_A(F) = 0$.

With the known Lie 2-groups alone, it is therefore not possible to find a non-trivial generalization of the Yang–Mills action. This is in outright contrast to the integral picture for which we have shown [16] that non-trivial generalizations exist. This result points towards a genuine discrepancy between perturbative and non-perturbative formulations of higher gauge theory on which we comment in the conclusion.

4 Conclusion and Outlook

In this article, we have reviewed both the integral and the differential picture of higher gauge theory. One main result is the appearance of the condition (3.25) at the differential level as soon as the curve and surface labels depend smoothly on the positions of the curves and surfaces.

Another main result is that we are able to construct ‘large’ (as opposed to ‘differential’) 2-gauge transformations in the differential picture only in the case in which $H \cong \mathbb{R}^n$ as an Abelian group. This seriously restricts the applications of the differential formulation and prevents us from obtaining an interesting level-2 generalization of Yang–Mills theory. BF -theory, however, forms an interesting example of a 2-gauge theory. The local 2-gauge transformations unify the two types of local symmetries of BF -theory and thereby provide a structural explanation for the existence of the topological symmetry of BF -theory.

We have chosen the language of 2-groups [15] in order to study higher gauge theory. The categorical structure of 2-groups leads directly to the integral picture [16] and as a consequence to the differential formulation as derived in the present article. Alternatively it would be possible to use the language of gerbes and to start with a differential formulation of higher gauge theory. One can then ask under which conditions it is possible to integrate the connection 1-form along curves and the connection 2-form along surfaces in a consistent way. The result of the present article suggests the conjecture that (3.25) is the required integrability condition.

How serious are the restrictions we have found, in particular the Abelianness of the curvature 3-form?

First note that all the Lie 2-groups and Lie 2-algebras used in the present article are *strict*. They form only the simplest examples of these structures which can be constructed in a general 2-categorical framework, but there exist the more general notions of *weak* and *coherent* 2-groups and their Lie 2-algebras. For 2-groups, see, for example [21, 22] and for Lie 2-algebras [19]. One can hope that the differential picture becomes less restrictive once we generalize from strict Lie 2-groups and 2-algebras to weak ones. Since the origin of the level-1 flatness is the very basic condition (2.10), a fully successful weakening should therefore allow for a non-Abelian kernel of the map $t: H \rightarrow G$.

We also observe that the non-trivial 2-holonomies are ruled out in the differential picture only if one requires the connection 1- and 2-forms to be both continuous and well defined everywhere in space-time. In particular, in the integral picture with the automorphism 2-group of $SU(2)$, we can have \mathbb{Z}_2 -valued 2-holonomies associated with surfaces of topology S^2 . If we assume that a smooth deformation of the surface changes the 2-holonomy only smoothly, then the non-trivial \mathbb{Z}_2 -element indicates that the S^2 is not smoothly contractible. This can be interpreted as an indication that there are singularities of codimension 2 in the theory which are actually *predicted* by the algebraic structure. Soliton-like solutions of some classical field equations come to mind. In fact, the integral picture for the inner automorphism group of $SU(3)$ is related to the symmetries of centre vortices in QCD as sketched in [16].

The difference of the differential and the integral picture is much deeper, though. As an illustration, we refer to a result in the context of the path integral quantization of conventional gauge theory. For simplicity, assume that we work in the Euclidean setting (i.e. with ‘imaginary’ time) on some Riemannian manifold M .

The obvious naive choice is to consider the set \mathcal{A} of all smooth connections A on M which form an affine space, and then to divide out the action of the gauge transformations. This step, however, destroys the linear structure so that the standard techniques fail to construct a useful path integral measure on the quotient \mathcal{A}/\mathcal{G} . This failure to implement the gauge symmetry correctly can be seen as a main reason why perturbative QCD does not predict confinement as observed in Nature.

The sophisticated approach, see, for example [23], is to consider the collection of all graphs embedded in M , to study gauge theory in the integral picture on these graphs, i.e. all

‘connections’ that are given by group labels attached to the edges of the graph, and finally to make use of a refinement relation on the class of all graphs which facilitates the construction of a projective continuum limit for the set of connections. Not only does this set of generalized connections form a compact Hausdorff space, it is also possible to fully divide out the gauge symmetry. This set of generalized connections modulo gauge transformations is a huge space that includes not only smooth or continuous connections, but rather mainly distributional ones. In order to appreciate the physical significance of this space of generalized connections it is useful to recall the most basic example of a field theory which admits a rigorous Euclidean path integral quantization, the free relativistic scalar field. Its path integral measure [24] is supported mainly on non-continuous scalar fields. In fact, the subsets of continuous and smooth fields form sets of measure zero!

This is a strong indication that a restriction to smooth fields does not yield an adequate description of the corresponding quantum theory and that we should take the integral formulation seriously. In a proper continuum limit, constructed from a suitable refinement of the integral formulation of higher gauge theory, the above-mentioned codimension-2 singularities will not only be allowed, they may actually be abundant in the path integral.

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