Cold atoms

Lecture 6.

29th November, 2006

Preliminary plan/reality in the fall term

Lecture 1 …	Something about everything (see next slide) The textbook version of BEC in extended systems	Sep 22
Lecture 2 …	thermodynamics, grand canonical ensemble, extended gas: ODLRO, nature of the BE phase transition	Oct 4
Lecture 3 …	atomic clouds in the traps – independent bosons, what is BEC?, "thermodynamic limit", properties of OPDM	Oct 18
Lecture 4 …	atomic clouds in the traps – interactions, GP equation at zero temperature, variational prop., chem. potential	Nov 1
Lecture 5 …	Infinite systems: Bogolyubov theory	Nov 15
Lecture 6 …	BEC and symmetry breaking, coherent states	Nov 29
Lecture 7 …	Time dependent GP theory. Finite systems: BEC theory preserving the particle number	

The class before last:
Interacting atoms

L4: Scattering length, pseudopotential

Beyond the potential radius, say 3σ, the scattered wave propagates in free space

For small energies, the scattering is purely isotropic , the *s*-wave scattering. The outside wave is

$$\psi \propto \frac{\sin(kr + \delta_0)}{r}$$

For very small energies the radial part becomes just

$$r - a_s, \quad a_s \text{ ... the scattering length}$$

This may be extrapolated also into the interaction sphere

(we are not interested in the short range details)

Equivalent potential ("pseudopotential")

$$U(r) = g \cdot \delta(\boldsymbol{r})$$

$$g = \frac{4\pi a_s \hbar^2}{m}$$

The class before last:

Mean-field treatment of interacting atoms

L4: Many-body Hamiltonian and the Hartree approximation

$$\hat{H} = \sum_a \frac{1}{2m} p_a^2 + V(\boldsymbol{r}_a) + \frac{1}{2} \sum_{a \neq b} \sum U(\boldsymbol{r}_a - \boldsymbol{r}_b)$$

We start from the mean field approximation.

This is an educated way, similar to (almost identical with) the HARTREE APPROXIMATION we know for many electron systems.

Most of the interactions is indeed absorbed into the mean field and what remains are explicit quantum correlation corrections

$$\hat{H}_{\mathrm{GP}} = \sum_a \frac{1}{2m} p_a^2 + V(\boldsymbol{r}_a) + V_H(\boldsymbol{r}_a)$$

$$V_H(\boldsymbol{r}_a) = \int \mathrm{d}\boldsymbol{r}_b U(\boldsymbol{r}_a - \boldsymbol{r}_b) n(\boldsymbol{r}_b) = g \cdot n(\boldsymbol{r}_a)$$

$$n(\boldsymbol{r}) = \sum_\alpha n_\alpha \left| \varphi_\alpha(\boldsymbol{r}) \right|^2$$

$$\left(\frac{1}{2m} p^2 + V(\boldsymbol{r}) + V_H(\boldsymbol{r}) \right) \varphi_\alpha(\boldsymbol{r}) = E_\alpha \varphi_\alpha(\boldsymbol{r})$$

self-consistent system

L4: Gross-Pitaevskii equation at zero temperature

Consider a condensate. Then all occupied orbitals are the same and we have a single self-consistent equation for a single orbital

$$\left(\frac{1}{2m}p^2+V(\boldsymbol{r})+gN\left|\varphi_0(\boldsymbol{r})\right|^2\right)\varphi_0(\boldsymbol{r})=E_0\varphi_0(\boldsymbol{r})$$

Putting

$$\Psi(\boldsymbol{r})=\sqrt{N}\cdot\varphi_0(\boldsymbol{r})$$

we obtain a closed equation for the order parameter:

$$\left(\frac{1}{2m}p^2+V(\boldsymbol{r})+g\left|\Psi(\boldsymbol{r})\right|^2\right)\Psi(\boldsymbol{r})=\mu\Psi(\boldsymbol{r})$$

The lowest level coincides with the chemical potential

This is the celebrated Gross-Pitaevskii equation.

For a static condensate, the order parameter has ZERO PHASE. Then

$$\Psi(\boldsymbol{r})=\sqrt{N}\cdot\varphi_0(\boldsymbol{r})=\sqrt{n(\boldsymbol{r})}$$

$$\mathcal{N}[n]=N=\int \mathrm{d}^3\boldsymbol{r}\left|\Psi(\boldsymbol{r})\right|^2=\int \mathrm{d}^3\boldsymbol{r}\cdot n(\boldsymbol{r})=N$$

Gross-Pitaevskii equation – homogeneous gas

The GP equation simplifies

$$\left(-\frac{\hbar^2}{2m}\Delta + g\left|\Psi(\boldsymbol{r})\right|^2\right)\Psi(\boldsymbol{r}) = \mu\Psi(\boldsymbol{r})$$

For periodic boundary conditions in a box with $V = L_x \cdot L_y \cdot L_z$

$$\varphi_0(\boldsymbol{r}) = \frac{1}{\sqrt{V}}$$

$$\Psi(\boldsymbol{r}) = \sqrt{N}\cdot\varphi_0(\boldsymbol{r}) = \sqrt{\frac{N}{V}} = \sqrt{n}$$

$$g\left|\Psi(\boldsymbol{r})\right|^2\Psi(\boldsymbol{r}) = \mu\Psi(\boldsymbol{r}) \quad \text{... GP equation}$$

$$\mu = g\left|\Psi(\boldsymbol{r})\right|^2 = gn$$

$$\frac{E}{N} = \frac{1}{N}\int \mathrm{d}^3\boldsymbol{r}\left\{\cancel{\frac{\hbar^2}{2m}(\nabla\sqrt{n})^2} + \cancel{V(\boldsymbol{r})n} + \frac{1}{2}g\,n^2\right\} = \frac{1}{2}g\,n$$

Previous class:

Field theoretic reformulation

(second quantization)

L5: Field operator for spin-less bosons

Definition by commutation relations

$$\left[\psi(\boldsymbol{r}),\psi^{\dagger}(\boldsymbol{r}')\right]=\delta(\boldsymbol{r}-\boldsymbol{r}'),\quad \left[\psi(\boldsymbol{r}),\psi\ \ (\boldsymbol{r}')\right]=0,\quad \left[\psi^{\dagger}(\boldsymbol{r}),\psi^{\dagger}(\boldsymbol{r}')\right]=0$$

basis of single-particle states (κ complete set of quantum numbers)
decomposition of the field operator

$$\psi(\boldsymbol{r}) = \sum \varphi_{\kappa}(\boldsymbol{r})\ a_{\kappa},\quad a_{\kappa}=\text{"}\langle\kappa|\psi\rangle\text{"}=\int \mathrm{d}^3\ \varphi^{*}_{\kappa}(\boldsymbol{r})\psi(\boldsymbol{r})$$

$$\psi^{\dagger}(\boldsymbol{r}) = \sum \varphi^{*}_{\kappa}(\boldsymbol{r})\ a^{\dagger}_{\kappa}$$

commutation relations

$$\left[a_{\kappa},a^{\dagger}_{\lambda}\right]=\delta_{\kappa\lambda},\quad \left[a_{\kappa},a_{\lambda}\right]=0,\quad \left[a^{\dagger}_{\kappa},a^{\dagger}_{\lambda}\right]=0$$

Plane wave representation (BK normalization)

$$\psi(\boldsymbol{r}) = V^{-1/2}\sum \mathrm{e}^{\mathrm{i}\boldsymbol{kr}}\ a_{\boldsymbol{k}},\quad a_{\boldsymbol{k}}=V^{-1/2}\int \mathrm{d}^3\boldsymbol{r}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{kr}}\,\psi(\boldsymbol{r})$$

$$\psi^{\dagger}(\boldsymbol{r}) = V^{-1/2}\sum \mathrm{e}^{-\mathrm{i}\boldsymbol{kr}}\ a^{\dagger}_{\boldsymbol{k}}=V^{-1/2}\sum \mathrm{e}^{\mathrm{i}\boldsymbol{kr}}\ a^{\dagger}_{\boldsymbol{-k}}$$

$$\left[a_{\boldsymbol{k}},a^{\dagger}_{\boldsymbol{k}'}\right]=\delta_{\boldsymbol{kk}'},\quad \left[a_{\boldsymbol{k}},a_{\boldsymbol{k}'}\right]=0,\quad \left[a^{\dagger}_{\boldsymbol{k}},a^{\dagger}_{\boldsymbol{k}'}\right]=0$$

L5: Operators

Additive observable

$$X = \sum X_j \quad \rightarrow \quad X = \iint \mathrm{d}^3\boldsymbol{r}\,\mathrm{d}^3\boldsymbol{r}'\,\psi^\dagger(\boldsymbol{r})\langle\boldsymbol{r}|X|\boldsymbol{r}'\rangle\psi(\boldsymbol{r}')$$

General definition of the OPDM

$$\langle X\rangle = \left\langle \iint \mathrm{d}^3\boldsymbol{r}\,\mathrm{d}^3\boldsymbol{r}'\,\psi^\dagger(\boldsymbol{r})\langle\boldsymbol{r}|X|\boldsymbol{r}'\rangle\psi(\boldsymbol{r}')\right\rangle = \iint \mathrm{d}^3\boldsymbol{r}\,\mathrm{d}^3\boldsymbol{r}'\,\langle\boldsymbol{r}|X|\boldsymbol{r}'\rangle\underbrace{\langle\psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r}')\rangle}_{\langle\boldsymbol{r}'|\rho|\boldsymbol{r}\rangle}$$

$$\equiv \iint \mathrm{d}^3\boldsymbol{r}\,\mathrm{d}^3\boldsymbol{r}'\,\langle\boldsymbol{r}|X|\boldsymbol{r}'\rangle\langle\boldsymbol{r}'|\rho|\boldsymbol{r}\rangle = \mathrm{Tr}\,X\rho$$

Particle number

$$N = \sum 1_{\mathrm{OP},j} \quad \rightarrow \quad N = \int \mathrm{d}^3\boldsymbol{r}\,\psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})$$

$$N = \sum a_\kappa^\dagger a_\kappa$$

Hamiltonian

$$H = \sum_a \frac{1}{2m}p_a^2 + V(\boldsymbol{r}_a) + \frac{1}{2}\sum_{a\,\neq}\sum_{b} U(\boldsymbol{r}_a - \boldsymbol{r}_b)$$

$$= \int \mathrm{d}^3\boldsymbol{r}\,\psi^\dagger(\boldsymbol{r})\left(-\tfrac{\hbar^2}{2m}\Delta + V(\boldsymbol{r})\right)\psi(\boldsymbol{r}) + \tfrac{1}{2}\iint \mathrm{d}^3\boldsymbol{r}\,\mathrm{d}^3\boldsymbol{r}'\,\psi^\dagger(\boldsymbol{r})\psi^\dagger(\boldsymbol{r}')U(\boldsymbol{r}-\boldsymbol{r}')\psi(\boldsymbol{r}')\psi(\boldsymbol{r})$$

Previous class:

Bogolyubov method

L5: Basic idea

Bogolyubov method
is devised for boson quantum fluids with weak interactions – at T=0 now

no interaction	weak interaction
$g = 0$	$g \neq 0$
$N = N_{\mathrm{BE}} = \langle a_0^\dagger a_0 \rangle \gg 1$	$N = N_{\mathrm{BE}} + \sum_{\boldsymbol{k} \neq 0} \langle a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k}} \rangle \approx N_{\mathrm{BE}} \gg 1$

The condensate dominates.

Strange idea

$$N_0 = \langle a_0^\dagger a_0 \rangle \gg 1 \Rightarrow \langle a_0^\dagger a_0 \rangle \gg a_0^\dagger a_0 - a_0 a_0^\dagger \Rightarrow \text{like } c\text{-numbers}$$

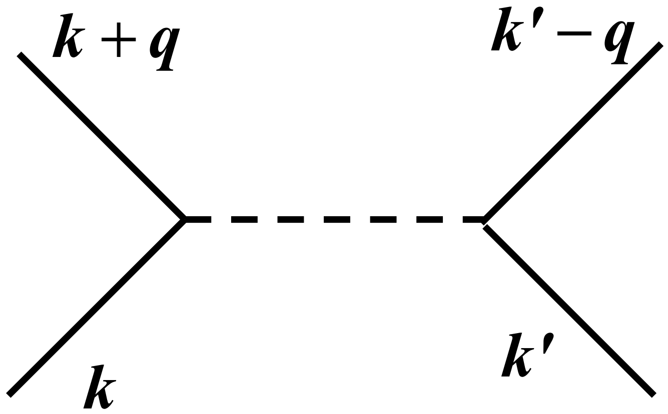
$$\boldsymbol{a}_0 \approx \sqrt{\boldsymbol{N}_0}, \quad \boldsymbol{a}_0^\dagger \approx \sqrt{\boldsymbol{N}_0}$$

$$N = N_0 + \sum_{\boldsymbol{k} \neq 0} a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k}} \quad \text{... mixture of } c\text{-numbers and } q\text{-numbers}$$

L5: Hamiltonian of the homogeneous gas

$$H = \sum \frac{\hbar^2}{2m} \boldsymbol{k}^2 a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k}} + \tfrac{1}{2} V^{-1} \sum_{\boldsymbol{k}\boldsymbol{k}'\boldsymbol{q}} U_{\boldsymbol{q}} a_{\boldsymbol{k}+\boldsymbol{q}}^\dagger , a_{\boldsymbol{k}'-\boldsymbol{q}}^\dagger a_{\boldsymbol{k}'} a_{\boldsymbol{k}} ,$$

$$U_{\boldsymbol{k}} = \int \mathrm{d}^3 \boldsymbol{r}\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\boldsymbol{r}}\, U(\boldsymbol{r})$$



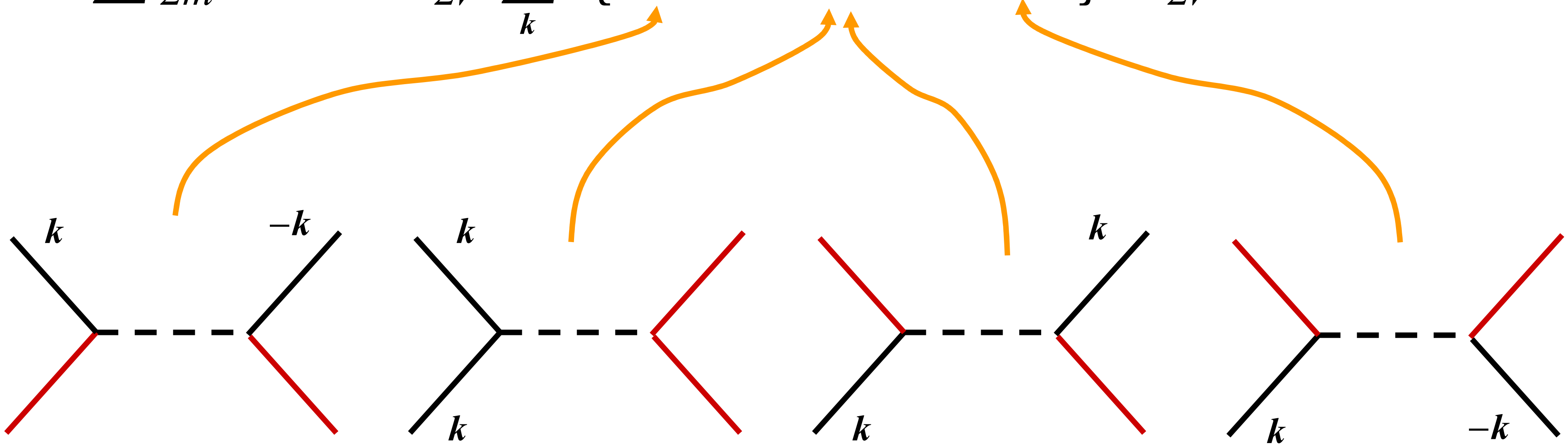
L5: Approximate Hamiltonian

Keep at most two particles out of the condensate

$$H = \sum \frac{\hbar^2}{2m}\boldsymbol{k}^2 a_k^\dagger a_k + \frac{1}{2}V^{-1}\sum_{kk'q} U_q a_{k+q}^\dagger a_{k'-q}^\dagger a_{k'} a_k$$

$$= \sum \frac{\hbar^2}{2m}\boldsymbol{k}^2 a_k^\dagger a_k + \frac{UN_0}{2V}\sum_k \left\{a_k^\dagger a_{-k}^\dagger + 4a_k^\dagger a_k + a_k a_{-k}\right\} + \frac{UN_0^2}{2V}$$

$$= \sum \frac{\hbar^2}{2m}\boldsymbol{k}^2 a_k^\dagger a_k + \frac{UN}{2V}\sum_k \left\{a_k^\dagger a_{-k}^\dagger + 2a_k^\dagger a_k + a_k a_{-k}\right\} + \frac{UN^2}{2V}$$



condensate particle

L5: Bogolyubov transformation

Last rearrangement

$$H = \frac{1}{2}\sum \underbrace{\left(\frac{\hbar^2}{2m}\boldsymbol{k}^2 + gn\right)}_{\text{mean field}}\left\{a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k}} + a_{-\boldsymbol{k}}^\dagger a_{-\boldsymbol{k}}\right\} + \frac{gn}{2}\sum_{\boldsymbol{k}} \underbrace{\left\{a_{\boldsymbol{k}}^\dagger a_{-\boldsymbol{k}}^\dagger + a_{\boldsymbol{k}} a_{-\boldsymbol{k}}\right\}}_{\text{anomalous}} + \frac{UN^2}{2V}$$

Conservation properties: momentum … YES, particle number … NO

<u>NEW FIELD OPERATORS</u> notice momentum conservation!!

$$\begin{array}{l|l} b_{\boldsymbol{k}} = u_{\boldsymbol{k}} a_{\boldsymbol{k}} + v_{\boldsymbol{k}} a_{-\boldsymbol{k}}^\dagger & a_{\boldsymbol{k}} = u_{\boldsymbol{k}} b_{\boldsymbol{k}} - v_{\boldsymbol{k}} b_{-\boldsymbol{k}}^\dagger \\ b_{-\boldsymbol{k}}^\dagger = v_{\boldsymbol{k}} a_{\boldsymbol{k}} + u_{\boldsymbol{k}} a_{-\boldsymbol{k}}^\dagger & a_{-\boldsymbol{k}}^\dagger = -v_{\boldsymbol{k}} b_{\boldsymbol{k}} + u_{\boldsymbol{k}} b_{-\boldsymbol{k}}^\dagger \end{array}$$

requirements

❶ New operators should satisfy the boson commutation rules

$$\left[b_{\boldsymbol{k}}, b_{\boldsymbol{k}'}^\dagger\right] = \delta_{\boldsymbol{k}\boldsymbol{k}'}, \quad \left[b_{\boldsymbol{k}}, b_{\boldsymbol{k}'}\right] = 0, \quad \left[b_{\boldsymbol{k}}^\dagger, b_{\boldsymbol{k}'}^\dagger\right] = 0$$

$$\textbf{iff} \quad u_{\boldsymbol{k}}^2 - v_{\boldsymbol{k}}^2 = 1$$

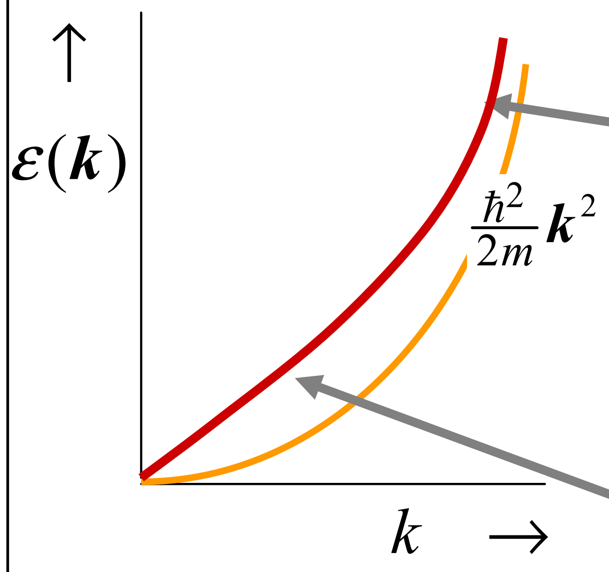
❷ When introduced into the Hamiltonian, the anomalous terms have to vanish

L5: Bogolyubov transformation – result

Without quoting the transformation matrix

$$H = \frac{1}{2}\sum \underbrace{\varepsilon(\boldsymbol{k})\left\{a_{\boldsymbol{k}}^{\dagger}a_{\boldsymbol{k}} + a_{-\boldsymbol{k}}^{\dagger}a_{-\boldsymbol{k}}\right\}}_{\text{independent quasiparticles}} + \frac{UN^2}{2V} + \text{higher order constant}$$

$$\varepsilon(\boldsymbol{k}) = \sqrt{\left(\frac{\hbar^2}{2m}\boldsymbol{k}^2 + gn\right)^2 - (gn)^2} = \sqrt{\frac{\hbar^2}{2m}\boldsymbol{k}^2}\sqrt{\frac{\hbar^2}{2m}\boldsymbol{k}^2 + 2gn}$$



asymptotically merge

high energy region

quasi-particles are nearly just particles

$$\omega(\boldsymbol{k}) = c \cdot k$$

$$c = \sqrt{\frac{gn}{m}}$$

sound region

quasi-particles are collective excitations

L5: More about the sound part of the dispersion law

- Entirely dependent on the interactions, both the magnitude of the velocity and the linear frequency range determined by g

$$\omega(\boldsymbol{k}) = c \cdot k$$
$$c = \sqrt{\frac{gn}{m}}$$

- Can be shown to really be a sound:

$$c = \text{"}\sqrt{\frac{\kappa}{\rho}}\text{"} = \sqrt{\frac{V \partial_{VV} E}{m \cdot n}}, \qquad E = \frac{UN^2}{2V} + \cdots$$

- Even a weakly interacting gas exhibits superfluidity; the ideal gas does not.
- The phonons are actually Goldstone modes corresponding to a broken symmetry
- The dispersion law has no **roton** region, contrary to the reality
- The dispersion law bends upwards $\Rightarrow$ quasi-particles are unstable, can decay

Particles and quasi-particles

At zero temperature, there are no quasi-particles, just the condensate.

Things are different with the true particles. Not all particles are in the condensate, but they are not thermally agitated in an incoherent way, they are a part of the fully coherent ground state

$$\left\langle a_k^\dagger a_k \right\rangle = \left\langle \left(-v_k b_k + u_k b_{-k}^\dagger\right)\left(u_k b_{-k} - v_k b_k^\dagger\right)\right\rangle = v_k^2 \neq 0$$

The total amount of the particles outside of the condensate is

$$\frac{N - N_0}{N} \approx \frac{8}{3\sqrt{\pi}} \underbrace{a_s^{3/2} n^{1/2}}_{\sqrt{a_s^3 n}}$$

the gas parameter
is
the expansion variable

Trying to understand the Bogolyubov method

L5: Action of the field operators in the Fock space

basis of single-particle states

$$\{|\kappa\rangle\} \quad \langle\kappa|\beta\rangle = \delta_{\kappa\beta} \quad |\psi\rangle = \sum |\kappa\rangle\langle\kappa|\psi\rangle, \quad \psi \text{ ... single particle state}$$

$$\langle \boldsymbol{r}|\kappa\rangle = \varphi_\kappa(\boldsymbol{r}) \qquad \langle \boldsymbol{r}|\psi\rangle = \sum \langle \boldsymbol{r}|\kappa\rangle\langle\kappa|\psi\rangle$$

FOCK SPACE space of many particle states

basis states … symmetrized products of single-particle states for bosons

specified by the set of **occupation numbers 0, 1, 2, 3, …**

$$\{\kappa_1, \kappa_2, \kappa_3, \ldots, \kappa_p, \ldots\}$$

$$\Psi_{\{n_\kappa\}} = |\, n_1, n_2, n_3, \ldots, n_p, \ldots \rangle \quad n\text{-particle state } n = \Sigma n_p$$

$$a_p^\dagger |\, n_1, n_2, n_3, \ldots, n_p, \ldots \rangle = \sqrt{n_p + 1}\, |\, n_1, n_2, n_3, \ldots, n_p + 1, \ldots \rangle$$

$$a_p |\, n_1, n_2, n_3, \ldots, n_p, \ldots \rangle = \sqrt{n_p}\, |\, n_1, n_2, n_3, \ldots, n_p - 1, \ldots \rangle$$

Average values of the field operators in the Fock states

$$a_p^\dagger \left| n_1, n_2, n_3, \ldots, n_p, \ldots \right\rangle = \sqrt{n_p + 1} \left| n_1, n_2, n_3, \ldots, n_p + 1, \ldots \right\rangle$$

$$a_p \left| n_1, n_2, n_3, \ldots, n_p, \ldots \right\rangle = \sqrt{n_p} \left| n_1, n_2, n_3, \ldots, n_p - 1, \ldots \right\rangle$$

$$\left\langle n_1, n_2, n_3, \ldots, n_p, \ldots \right| a_p \left| n_1, n_2, n_3, \ldots, n_p, \ldots \right\rangle =$$
$$\left\langle n_1, n_2, n_3, \ldots, n_p, \ldots \right| \sqrt{n_p} \left| n_1, n_2, n_3, \ldots, n_p - 1, \ldots \right\rangle = 0$$

Off-diagonal elements only!!!

L5: Hamiltonian conserves the particle number

Particle number conservation

$$[H, N] = 0$$

Equilibrium density operators and the ground state

$$P = P(H), \quad [N, P] = 0$$

Typical selection rule

$$\langle \psi(\boldsymbol{r}) \rangle = \mathrm{Tr}\, \psi(\boldsymbol{r}) P = 0$$

is a consequence of the gauge invariance of the 1st kind:

$$\mathrm{Tr}\, \psi P = \mathrm{Tr}\, \psi\, \mathrm{e}^{\mathrm{i}\varphi N} P\, \mathrm{e}^{-\mathrm{i}\varphi N} = \mathrm{Tr}\, \mathrm{e}^{-\mathrm{i}\varphi N} \psi\, \mathrm{e}^{\mathrm{i}\varphi N} P = \mathrm{e}^{-\mathrm{i}\varphi}\, \mathrm{Tr}\, \psi P$$

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This is at odds with the Bogolyubov idea

$$\boldsymbol{a}_0 \approx \sqrt{\boldsymbol{N}_0}, \quad \boldsymbol{a}_0^\dagger \approx \sqrt{\boldsymbol{N}_0}$$

which leads to

$$\langle \boldsymbol{a}_0 \rangle \approx \sqrt{\boldsymbol{N}_0}, \quad \langle \boldsymbol{a}_0^\dagger \rangle \approx \sqrt{\boldsymbol{N}_0}$$

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$$\mathrm{Tr}\,\psi P\ =\mathrm{Tr}\,\psi\,\mathrm{e}^{\mathrm{i}\varphi N}\,P\ \mathrm{e}^{-\mathrm{i}\varphi N}\ =\mathrm{Tr}\,\mathrm{e}^{-\mathrm{i}\varphi N}\ \psi\,\mathrm{e}^{\mathrm{i}\varphi N}\,P=\mathrm{e}^{-\mathrm{i}\varphi}\,\mathrm{Tr}\,\psi P$$

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Perhaps the ground state does not conserve the particle number??

First try:
Coherent ground state

Reformulation of the Bogolyubov requirements

Bogolyubov himself and his faithful followers, like LL IX, never speak of the many particle wave function. Looks like he wanted

$$a_0 \left|\Psi\right\rangle = \Lambda \left|\Psi\right\rangle, \quad \Lambda = \sqrt{N}\,\mathrm{e}^{\mathrm{i}\phi}\;,$$ so that

$$\left\langle a_0 \right\rangle = \Lambda$$

The ground state

This is in contradiction with the rule derived above, $\left\langle a_0 \right\rangle = 0$

The above equation is known and defines the ground state to be a **coherent state with the parameter** Λ

For a coherent state, there is no problem with the particle number conservation. It has a rather uncertain particle number, but a well defined phase:

$$\left|\Psi\right\rangle = \left|\Lambda\right\rangle = \mathrm{e}^{-|\Lambda|^2/2}\cdot \mathrm{e}^{\Lambda a_0^\dagger}\left|\mathrm{vac}\right\rangle$$ explicit form

$$\left\langle \Lambda \right| a_0 \left| \Lambda \right\rangle = \Lambda$$ Bogolyubov condition

$$\left\langle \Lambda \right| a_0^\dagger a_0 \left| \Lambda \right\rangle = |\Lambda|^2 \qquad \left\langle n_0 \right\rangle = N$$

$$\left\langle \Lambda \right| a_0^\dagger a_0 a_0^\dagger a_0 \left| \Lambda \right\rangle = |\Lambda|^4 + |\Lambda|^2 \qquad \left\langle n_0^2 \right\rangle = N^2 + N$$

$$\Delta N_0 = |\Lambda| \qquad \Delta n_0 = \sqrt{N} = \left\langle n_0 \right\rangle / \sqrt{N} \ll \left\langle n_0 \right\rangle$$

New vacuum and the shifted field operators

Does all that make sense? Try to work in the full Fock space F rather in its fixed N sub-space N_N This implies using the "grand Hamiltonian"

$$H - \mu N$$

Let us define the shifted field operator

$$b_0 = a_0 - \Lambda, \quad b_0^\dagger = a_0^\dagger - \Lambda^*$$

$$\left[b_0, b_0^\dagger \right] = 1, \quad b_0 \left| \Psi \right\rangle = 0 \quad \ldots \quad \text{new vacuum}$$

What next? Example: BE system without interactions – ideal Bose gas

$$(H - \mu N) \left| \Psi \right\rangle = \sum \left(\frac{\hbar^2}{2m} \boldsymbol{k}^2 - \mu \right) a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k}} \left| \Psi \right\rangle$$

$$= -\mu a_0^\dagger a_0 \left| \Psi \right\rangle = 0 \quad \text{for} \quad \mu = 0$$

Here, $\left| \Psi \right\rangle$ is a true eigenstate, μ coincides with the previous result for the particle number conserving state. **Two different, but macroscopically equivalent possibilities.**

General case: the approximate vacuum

$$H = \int \mathrm{d}^3\boldsymbol{r}\, \psi^\dagger(\boldsymbol{r})\left(-\tfrac{\hbar^2}{2m}\Delta + V(\boldsymbol{r})\right)\psi(\boldsymbol{r}) + \tfrac{1}{2}\iint \mathrm{d}^3\boldsymbol{r}\,\mathrm{d}^3\boldsymbol{r}'\, \psi^\dagger(\boldsymbol{r})\psi^\dagger(\boldsymbol{r}')U(\boldsymbol{r}-\boldsymbol{r}')\psi(\boldsymbol{r}')\psi(\boldsymbol{r})$$

$$g\delta(\boldsymbol{r}-\boldsymbol{r}')$$

Trial function … a coherent state

$$\psi(\boldsymbol{r})\left|\Psi\right\rangle = \Psi(\boldsymbol{r})\left|\Psi\right\rangle$$

We should minimize the average grand energy

$$\left\langle \Psi \right| H - \mu N \left| \Psi \right\rangle = \int \mathrm{d}^3\boldsymbol{r}\, \Psi^*(\boldsymbol{r})\left(-\tfrac{\hbar^2}{2m}\Delta + V(\boldsymbol{r}) - \mu\right)\Psi(\boldsymbol{r})$$

$$+\tfrac{1}{2}\iint \mathrm{d}^3\boldsymbol{r}\,\mathrm{d}^3\boldsymbol{r}'\, \Psi^*(\boldsymbol{r})\Psi(\boldsymbol{r})U(\boldsymbol{r}-\boldsymbol{r}')\Psi^*(\boldsymbol{r}')\Psi(\boldsymbol{r})$$

This is precisely the energy functional of the Hartree type we met already

and the Euler-Lagrange equation is the good old Gross-Pitaevski equation

$$\left(\frac{\hbar^2}{2m}\Delta + V(\boldsymbol{r}) + g\left|\Psi(\boldsymbol{r})\right|^2\right)\Psi(\boldsymbol{r}) = \mu\Psi(\boldsymbol{r})$$

with the normalization condition

$$N\,[n] = N = \int \mathrm{d}^3\,\boldsymbol{r}\left|\Psi(\boldsymbol{r})\right|^2$$

General case: the Bogolyubov transformation

Define $$\eta(\boldsymbol{r}) = \psi(\boldsymbol{r}) - \Psi(\boldsymbol{r}), \quad \eta^\dagger(\boldsymbol{r}) = \psi^\dagger(\boldsymbol{r}) - \Psi^*(\boldsymbol{r})$$

$$\left[\eta(\boldsymbol{r}), \eta^\dagger(\boldsymbol{r}')\right] = \delta(\boldsymbol{r} - \boldsymbol{r}'), \quad \eta(\boldsymbol{r})|\Psi\rangle = 0$$

If we keep only the terms not more than quadratic in the new operators, the approximate Hamiltonian becomes

$$H = \int \mathrm{d}^3\boldsymbol{r}\, \eta^\dagger(\boldsymbol{r})\left(-\tfrac{\hbar^2}{2m}\Delta + V(\boldsymbol{r}) - \mu\right)\eta(\boldsymbol{r})$$
$$+ \tfrac{g}{2}\int \mathrm{d}^3\boldsymbol{r}\; n_{\mathrm{BE}}(\boldsymbol{r})\left\{\eta^\dagger(\boldsymbol{r})\eta^\dagger(\boldsymbol{r}) + 4\eta^\dagger(\boldsymbol{r})\eta(\boldsymbol{r}) + \eta(\boldsymbol{r})\eta(\boldsymbol{r})\right\}$$

Now eliminate the anomalous terms by the Bogolyubov transformation

$$\boxed{\varphi(\boldsymbol{r}) = u(\boldsymbol{r})\eta(\boldsymbol{r}) + v(\boldsymbol{r})\eta^\dagger(\boldsymbol{r}) \qquad \varphi^\dagger(\boldsymbol{r}) = v^*(\boldsymbol{r})\eta(\boldsymbol{r}) + u^*(\boldsymbol{r})\eta^\dagger(\boldsymbol{r})}$$

$$\left[\varphi(\boldsymbol{r}), \varphi^\dagger(\boldsymbol{r}')\right] = \delta(\boldsymbol{r} - \boldsymbol{r}') \quad \textbf{iff} \quad |u(\boldsymbol{r})|^2 - |v(\boldsymbol{r})|^2 = 1$$

$$\eta(\boldsymbol{r}) = u^*(\boldsymbol{r})\varphi(\boldsymbol{r}) - v(\boldsymbol{r})\varphi^\dagger(\boldsymbol{r}) \qquad \eta^\dagger(\boldsymbol{r}) = -v^*(\boldsymbol{r})\varphi(\boldsymbol{r}) + u(\boldsymbol{r})\varphi^\dagger(\boldsymbol{r})$$

Second try:
Quasi-averages and broken symmetry

Zero temperature limit of the grand canonical ensemble

$$\mathcal{P} = Z\,\mathrm{e}^{-\beta(\mathcal{H}-\mu\mathcal{N})}$$

$$= Z\sum \mathrm{e}^{-\beta(E_{\alpha N}-\mu N)}$$

$$\to Z\sum \mathrm{e}^{-\beta(E_{0\tilde{N}}-\mu\tilde{N})}$$

Picks up the correct ground state energy,

all ground states are taken with equal statistical weight

SYSTEMS

- **NORMAL** — NON-DEGENERATE GROUND STATE
- **"ANOMALOUS"** — DEGENERATE GROUND STATE

Degenerate ground state

SYSTEMS
- NORMAL — NON-DEGENERATE GROUND STATE
- "ANOMALOUS" — DEGENERATE GROUND STATE

Characterized by a classical order parameter … **macroscopic quantity**
Typical cause: a symmetry degeneracy
Everything depends on the system characteristic parameters
Ginsburg – Landau phenomenological model

$$\mathcal{E}\left(\Psi\right) = a\Psi^2 + b\Psi^4$$

$a > 0$
stable equilibrium
non-degenerate

$a < 0$
metastable equilibrium
degenerate

Degenerate ground state

SYSTEMS
- NORMAL — NON-DEGENERATE GROUND STATE
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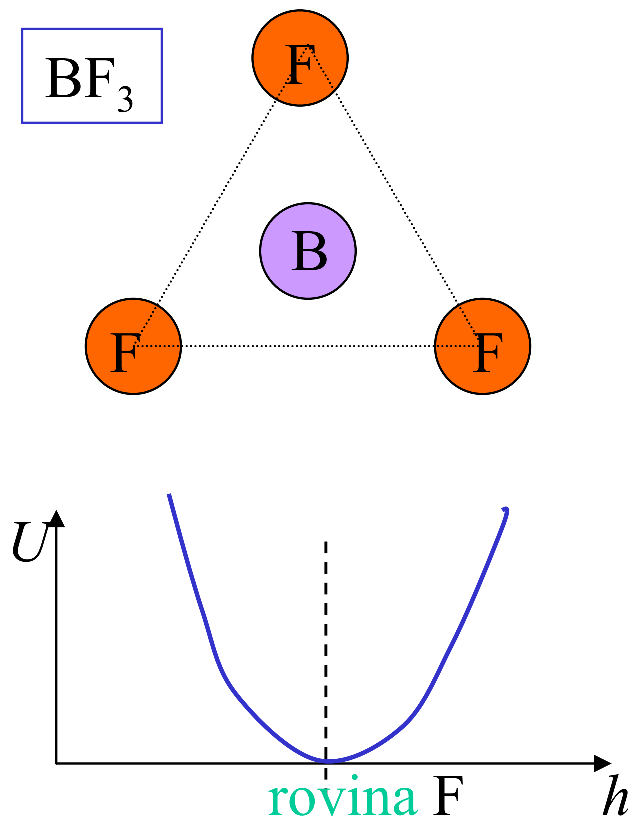
Everything depends on the system characteristic parameters
Ginsburg – Landau phenomenological model

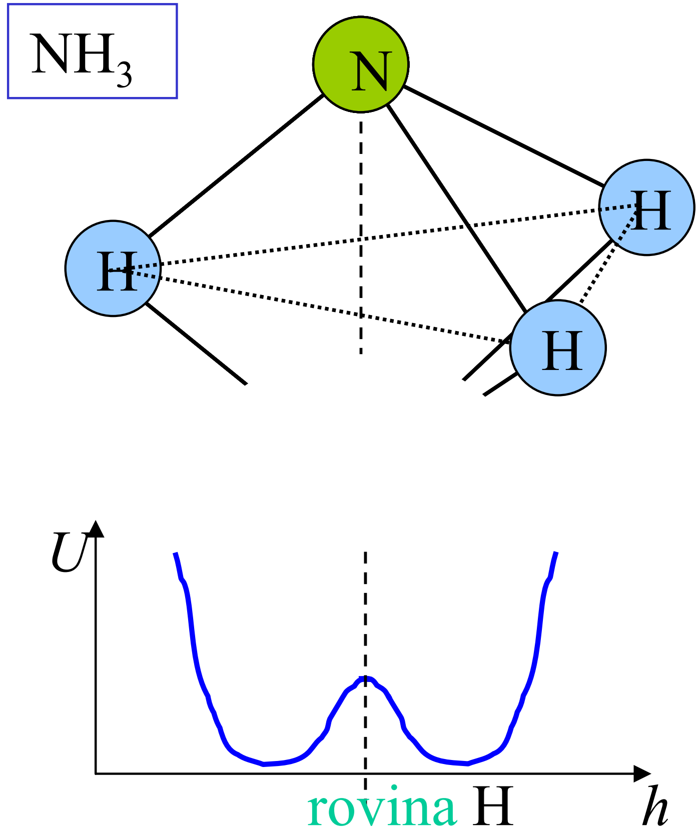
$$\mathcal{E}\left(\Psi\right) = a\Psi^2 + b\Psi^4$$

spontaneous symmetry breaking

$a > 0$	$a < 0$
stable equilibrium	metastable equilibrium
non-degenerate	degenerate

Rovnovážná struktura molekul AB_3





U adiabatická potenciální energie

The end

On the way to the mean-field Hamiltonian

ADDITIONAL NOTES

On the way to the mean-field Hamiltonian

① First, the following exact transformations are performed

$$\hat{H} = \overbrace{\sum_a \frac{1}{2m} p_a^2}^{\hat{W}} + \overbrace{\sum_a V(\boldsymbol{r}_a)}^{\hat{V}} + \overbrace{\frac{1}{2} \sum_{a \neq b} \sum U(\boldsymbol{r}_a - \boldsymbol{r}_b)}^{\hat{U}}$$

$$\hat{V} = \sum_a V(\boldsymbol{r}_a) = \int \mathrm{d}^3\boldsymbol{r}\, V(\boldsymbol{r}) \sum_a \delta(\boldsymbol{r} - \boldsymbol{r}_a) \equiv \int \mathrm{d}^3\boldsymbol{r}\, V(\boldsymbol{r}) \cdot \hat{n}(\boldsymbol{r})$$

particle density operator

$$\hat{U} = \frac{1}{2} \sum_{a \neq b} \sum U(\boldsymbol{r}_a - \boldsymbol{r}_b) = \frac{1}{2} \int \mathrm{d}^3\boldsymbol{r}\, \mathrm{d}^3\boldsymbol{r}'\, U(\boldsymbol{r} - \boldsymbol{r}') \sum_{a \neq b} \sum \delta(\boldsymbol{r} - \boldsymbol{r}_a) \delta(\boldsymbol{r}' - \boldsymbol{r}_b)$$

$$= \frac{1}{2} \int \mathrm{d}^3\mathbf{r}\, \mathrm{d}^3\mathbf{r}'\, U(\boldsymbol{r} - \boldsymbol{r}') \underbrace{\sum_a \delta(\boldsymbol{r} - \boldsymbol{r}_a)}_{\hat{n}(\boldsymbol{r})} \left\{ \underbrace{\sum_b \delta(\boldsymbol{r}' - \boldsymbol{r}_b)}_{\hat{n}(\boldsymbol{r}')} - \overbrace{\delta(\boldsymbol{r} - \boldsymbol{r}')}^{\text{TRICK!!}} \right\}$$

eliminates **SI** (self-interaction)

$$\hat{H} = \hat{W} + \int \mathrm{d}^3\boldsymbol{r}\, V(\boldsymbol{r}) \cdot \hat{n}(\boldsymbol{r}) + \frac{1}{2} \int \mathrm{d}^3\boldsymbol{r}\, \mathrm{d}^3\boldsymbol{r}'\, U(\boldsymbol{r} - \boldsymbol{r}') \hat{n}(\boldsymbol{r}) \{ \hat{n}(\boldsymbol{r}') - \delta(\boldsymbol{r} - \boldsymbol{r}') \}$$

On the way to the mean-field Hamiltonian

② Second, a specific many-body state is chosen, which defines the mean field:

$$\Psi \rightarrow n(\boldsymbol{r}) = \langle \hat{n}(\boldsymbol{r}) \rangle \equiv \langle \Psi | \hat{n}(\boldsymbol{r}) | \Psi \rangle$$

Then, the operator of the (quantum) density fluctuation is defined:

$$\hat{n}(\boldsymbol{r}) = n(\boldsymbol{r}) + \Delta\hat{n}(\boldsymbol{r})$$

$$\hat{n}(\boldsymbol{r})\hat{n}(\boldsymbol{r}') = \hat{n}(\boldsymbol{r})n(\boldsymbol{r}') + n(\boldsymbol{r})\hat{n}(\boldsymbol{r}') + \Delta\hat{n}(\boldsymbol{r})\Delta\hat{n}(\boldsymbol{r}') - n(\boldsymbol{r})n(\boldsymbol{r}')$$

The Hamiltonian, still exactly, becomes

$$\hat{H} = \hat{W} + \int \mathrm{d}^3\boldsymbol{r} \left\{ V(\boldsymbol{r}) + \int \mathrm{d}^3\boldsymbol{r}' U(\boldsymbol{r}-\boldsymbol{r}') n(\boldsymbol{r}') \right\} \cdot \hat{n}(\boldsymbol{r})$$

$$-\frac{1}{2}\int \mathrm{d}^3\boldsymbol{r}\, \mathrm{d}^3\boldsymbol{r}' U(\boldsymbol{r}-\boldsymbol{r}') n(\boldsymbol{r}) n(\boldsymbol{r}')$$

$$+\frac{1}{2}\int \mathrm{d}^3\boldsymbol{r}\, \mathrm{d}^3\boldsymbol{r}' U(\boldsymbol{r}-\boldsymbol{r}') \left\{ \Delta\hat{n}(\boldsymbol{r})\Delta\hat{n}(\boldsymbol{r}') - \hat{n}(\boldsymbol{r})\delta(\boldsymbol{r}-\boldsymbol{r}') \right\}$$

On the way to the mean-field Hamiltonian

③ In the last step, the third line containing exchange, correlation and the self-interaction correction is neglected. The mean-field Hamiltonian of the main lecture results:

$$V_H(\boldsymbol{r})$$

substitute back $\hat{n}(\boldsymbol{r}) = \sum_a \delta(\boldsymbol{r} - \boldsymbol{r}_a)$ and integrate

$$+\frac{1}{2}\int \mathrm{d}^3\boldsymbol{r}\,\mathrm{d}^3\boldsymbol{r}'\,U(\boldsymbol{r}-\boldsymbol{r}')\{\Delta\hat{n}(\boldsymbol{r})\Delta\hat{n}(\boldsymbol{r}') - \hat{n}(\boldsymbol{r})\delta(\boldsymbol{r}-\boldsymbol{r}')\}$$

REMARKS

BACK

• Second line … an additive constant compensation for double-counting of the Hartree interaction energy

• In the original (variational) Hartree approximation, the self-interaction is not left out, leading to non-orthogonal Hartree orbitals

Variational approach to the condensate ground state

ADDITIONAL NOTES

Variational estimate of the condensate properties

① VARIATIONAL PRINCIPLE OF QUANTUM MECHANICS

The ground state and energy are uniquely defined by

$$E = \langle \Psi | \hat{H} | \Psi \rangle \le \langle \Psi' | \hat{H} | \Psi' \rangle \quad \text{for all } |\Psi'\rangle \in \mathcal{H}_N^{\mathrm{S}}, \langle \Psi' | \Psi' \rangle = 1$$

In words, $|\Psi'\rangle$ is a normalized symmetrical wave function of *N* particles. The minimum condition in the variational form is

$$\delta \langle \Psi | \hat{H} | \Psi \rangle = 0 \quad \text{equivalent with the SR} \quad \hat{H} |\Psi\rangle = E |\Psi\rangle$$

② HARTREE VARIATIONAL ANSATZ FOR THE CONDENSATE WAVE F.

For our many-particle Hamiltonian,

$$\hat{H} = \sum_a \frac{1}{2m} p_a^2 + V(\boldsymbol{r}_a) + \frac{1}{2} \sum_{a \neq b} \sum U(\boldsymbol{r}_a - \boldsymbol{r}_b), \quad U(\boldsymbol{r}) = g \cdot \delta(\boldsymbol{r})$$

the true ground state is approximated by the condensate for non-interacting particles (Hartree Ansatz, here identical with the symmetrized Hartree-Fock)

$$\Psi\left(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_p, \ldots, \boldsymbol{r}_N\right) = \varphi_0\left(\boldsymbol{r}_1\right) \varphi_0\left(\boldsymbol{r}_2\right) \cdots \varphi_0\left(\boldsymbol{r}_p\right) \cdots \varphi_0\left(\boldsymbol{r}_N\right)$$

ADDITIONAL NOTES

Variational estimate of the condensate properties

Here, φ_0 is a normalized real spinless orbital. It is a functional variable to be found from the variational condition

$$\delta \mathcal{E}[\varphi_0] = \delta \langle \Psi[\varphi_0] | \hat{H} | \Psi[\varphi_0] \rangle = 0 \quad \text{with} \quad \langle \Psi[\varphi_0] | \Psi[\varphi_0] \rangle = 1 \Leftrightarrow \langle \varphi_0 | \varphi_0 \rangle = 1$$

Explicit calculation yields

$$\mathcal{E}[\varphi_0] = \frac{\hbar^2}{2m} N \int \mathrm{d}^3\boldsymbol{r} \left(\nabla \varphi_0(\boldsymbol{r})\right)^2 + N \int \mathrm{d}^3\boldsymbol{r}\, V(\boldsymbol{r}) \left(\varphi_0(\boldsymbol{r})\right)^2 + \frac{1}{2} N(N-1) g \int \mathrm{d}^3\boldsymbol{r} \left(\varphi_0(\boldsymbol{r})\right)^4$$

Variation of energy with the use of a Lagrange multiplier:

$$\delta \left\{ N^{-1} \mathcal{E}[\varphi_0] - \mu \langle \varphi_0 | \varphi_0 \rangle \right\}$$

$$\varphi_0 = \varphi_0(\boldsymbol{r}), \quad \delta\varphi_0 = \delta\varphi_0(\boldsymbol{r})$$

BY PARTS

$$= \frac{2\hbar^2}{2m} \int \mathrm{d}^3\boldsymbol{r}\ \delta\varphi_0 \cdot (-\Delta \varphi_0) + 2 \int \mathrm{d}^3\boldsymbol{r}\, \delta\varphi_0 \cdot \left(V(\boldsymbol{r}) - \mu\right) \varphi_0 + \frac{4}{2} (N-1) g \int \mathrm{d}^3\boldsymbol{r}\, \delta\varphi_0 \cdot \varphi_0^3$$

This results into the GP equation derived here in the variational way:

$$\left(\frac{1}{2m} p^2 + V(\boldsymbol{r}) + (N - 1) g \left| \varphi_0(\boldsymbol{r}) \right|^2 \right) \varphi_0(\boldsymbol{r}) = \mu \varphi_0(\boldsymbol{r})$$

eliminates self-interaction

BACK

Variational estimate of the condensate properties

② *ANNEX Interpretation of the Lagrange multiplier μ*

The idea is to identify it with the chemical potential. First, we modify the notation to express the particle number dependence

$$\mathcal{E}_N[\varphi] = N\left\{\langle\varphi|\frac{1}{2m}p^2|\varphi\rangle + \langle\varphi|V|\varphi\rangle + \frac{1}{2}(N-1)g\int \mathrm{d}^3\boldsymbol{r}\,\varphi^4\right\}$$

$$E_N = \mathcal{E}_N[\varphi_{0N}], \quad \left(\frac{1}{2m}p^2 + V(\boldsymbol{r}) + (N-1)g|\varphi_{0N}(\boldsymbol{r})|^2\right)\varphi_0(\boldsymbol{r}) = \mu_N\varphi_{0N}(\boldsymbol{r})$$

The first result is that μ is not the average energy per particle:

$$E_N/N = \mathcal{E}_N[\varphi_{0N}]/N = \langle\varphi_{0N}|\frac{1}{2m}p^2|\varphi_{0N}\rangle + \langle\varphi_{0N}|V|\varphi_{0N}\rangle + \frac{1}{2}(N-1)g\int \mathrm{d}^3\boldsymbol{r}\,\varphi_{0N}{}^4$$

from the GPE $$\mu_N = \langle\varphi_{0N}|\frac{1}{2m}p^2|\varphi_{0N}\rangle + \langle\varphi_{0N}|V|\varphi_{0N}\rangle + (N-1)g\int \mathrm{d}^3\boldsymbol{r}\,\varphi_{0N}{}^4$$

Variational estimate of the condensate properties

Compare now systems with *N* and *N* -1 particles:

$$E_N = \underbrace{\mathcal{E}_N[\varphi_{0N}] = \mathcal{E}_{N-1}[\varphi_{0N}] + \mu_N}_{} \geq \underbrace{\mathcal{E}_{N-1}[\varphi_{0,N-1}] + \mu_N}_{} = E_{N-1} + \mu_N$$

μ_N … energy to remove a particle without relaxation of the condensate

use of the variational principle for GPE

In the "thermodynamic"asymptotics of large *N*, the inequality tends to equality. This only makes sense, and can be proved, for *g* > 0.

BACK

Reminescent of the theorem in the HF theory of atoms.

Derivation:

$$\mathcal{E}_N[\varphi] = N\langle\varphi|\tfrac{1}{2m}p^2|\varphi\rangle + N\langle\varphi|V|\varphi\rangle + \tfrac{1}{2}N(N-1)g\int \mathrm{d}^3\boldsymbol{r}\,\varphi^4$$

$$\mathcal{E}_{N-1}[\varphi] = (N-1)\langle\varphi|\tfrac{1}{2m}p^2|\varphi\rangle + (N-1)\langle\varphi|V|\varphi\rangle + \tfrac{1}{2}(N-1)(N-2)g\int \mathrm{d}^3\boldsymbol{r}\,\varphi^4$$

$$\mathcal{E}_N - \mathcal{E}_{N-1} = \underbrace{\langle\varphi|\tfrac{1}{2m}p^2|\varphi\rangle + \langle\varphi|V|\varphi\rangle + \tfrac{1}{2}\big(N(N-1)-(N-1)(N-2)\big)g\int \mathrm{d}^3\boldsymbol{r}\,\varphi^4}_{\mu_N \text{ for } \varphi \mapsto \varphi_{0N}}$$

ADDITIONAL NOTES

Variational estimate of the condensate properties

③ SCALING ANSATZ FOR A SPHERICAL PARABOLIC TRAP

The potential energy has the form

$$V(\boldsymbol{r}) = \tfrac{1}{2} m\omega_0^2 \cdot r^2 = \tfrac{1}{2} m\omega_0^2 \left(x^2 + y^2 + z^2\right)$$

Without interactions, the GPE reduces to the SE for isotropic oscillator

$$\left(\frac{1}{2m} p^2 + \tfrac{1}{2} m\omega_0^2 \cdot r^2\right)\varphi_0(\boldsymbol{r}) = \tfrac{3}{2}\hbar\omega_0 \varphi_0(\boldsymbol{r})$$

The solution (for the ground state orbital) is

$$\varphi_{00}(\boldsymbol{r}) = A_0^3 \, \mathrm{e}^{-\frac{1}{2}\cdot\frac{r^2}{a_0^2}}, \quad a_0 = \sqrt{\frac{\hbar}{m\omega_0}}, \quad \hbar\omega_0 = \frac{\hbar^2}{m a_0^2} \quad A_0 = \left(a_0^2 \pi\right)^{-1/4}$$

We (have used and) will need two integrals:

$$I_1(\sigma) = \int_{-\infty}^{+\infty} \mathrm{d}u \, \mathrm{e}^{-\frac{u^2}{\sigma^2}} = \sigma\sqrt{\pi}, \quad I_2(\sigma) = \int_{-\infty}^{+\infty} \mathrm{d}u \, \mathrm{e}^{-\frac{u^2}{\sigma^2}} u^2 = \tfrac{1}{2}\sigma^3\sqrt{\pi}$$

ADDITIONAL NOTES

Variational estimate of the condensate properties

SCALING ANSATZ

The condensate orbital will be taken in the form

$$\varphi_0(\boldsymbol{r}) = A^3\,\mathrm{e}^{-\frac{1}{2}\cdot\frac{r^2}{b^2}}, \qquad A = \left(b^2\pi\right)^{-1/4}$$

It is just like the ground state orbital for the isotropic oscillator, but with a rescaled size. This is reminescent of the well-known scaling for the ground state of the helium atom.

Next, the total energy is calculated for this orbital

$$\mathcal{E}[\varphi_0] = \frac{\hbar^2}{2m} N \int \mathrm{d}^3\boldsymbol{r} \left(\nabla\varphi_0(\boldsymbol{r})\right)^2 + N \int \mathrm{d}^3\boldsymbol{r}\, V(\boldsymbol{r}) \left(\varphi_0(\boldsymbol{r})\right)^2 + \frac{1}{2} N(N-1) g \int \mathrm{d}^3\boldsymbol{r} \left(\varphi_0(\boldsymbol{r})\right)^4$$

$$= \frac{1}{2}\hbar\omega_0 N A^6 \left\{ \frac{a_0^2}{b^4} \int \mathrm{d}^3\boldsymbol{r}\,\mathrm{e}^{-\frac{r^2}{b^2}} r^2 + \frac{1}{a_0^2} \int \mathrm{d}^3\boldsymbol{r}\,\mathrm{e}^{-\frac{r^2}{b^2}} r^2 + (N-1) A^6 \frac{m a_0^2}{\hbar^2} g \int \mathrm{d}^3\boldsymbol{r}\,\mathrm{e}^{-\frac{2r^2}{b^2}} \right\}$$

ADDITIONAL NOTES

Variational estimate of the condensate properties

For an explicit evaluation, we (have used and) will use the identities:

$$\frac{\hbar^2}{m}=\hbar\omega_0 a_0^2,\ \ m\omega_0^2=\frac{\hbar\omega_0}{a_0^2},\ \ A^2=\frac{1}{I_1(b)}=\frac{1}{b\sqrt{\pi}},\ \ g=\frac{4\pi\hbar^2 a_s}{m}$$

$\mathcal{E}[\varphi_0]=$

The integrals, by the Fubini theorem, are a product of three:

$$=\hbar\omega_0 N\left\{\frac{3I_2(b)}{2b\sqrt{\pi}}\frac{\left(I_1(b)\right)^2}{\left(I_1(b)\right)^2}\left\{\frac{a_0^2}{b^4}+\frac{1}{a_0^2}\right\}+(N-1)\frac{1}{2b^3\pi^{3/2}}\frac{ma_0^2}{\hbar^2}\frac{4\pi\hbar^2 a_s}{m}\frac{\left(I_1\left(b/\sqrt{2}\right)\right)^3}{\left(I_1(b)\right)^3}\right\}$$

Finally,

$$\mathcal{E}[\varphi_0]=\hbar\omega_0 N\left\{\frac{3}{4}\left\{\frac{a_0^2}{b^2}+\frac{b^2}{a_0^2}\right\}+\frac{(N-1)}{\sqrt{2\pi}}\frac{a_s}{a_0}\cdot\frac{a_0^3}{b^3}\right\}\equiv\hbar\omega_0 N\cdot\tilde{E}(\tilde{\sigma})$$

dimension-less energy per particle

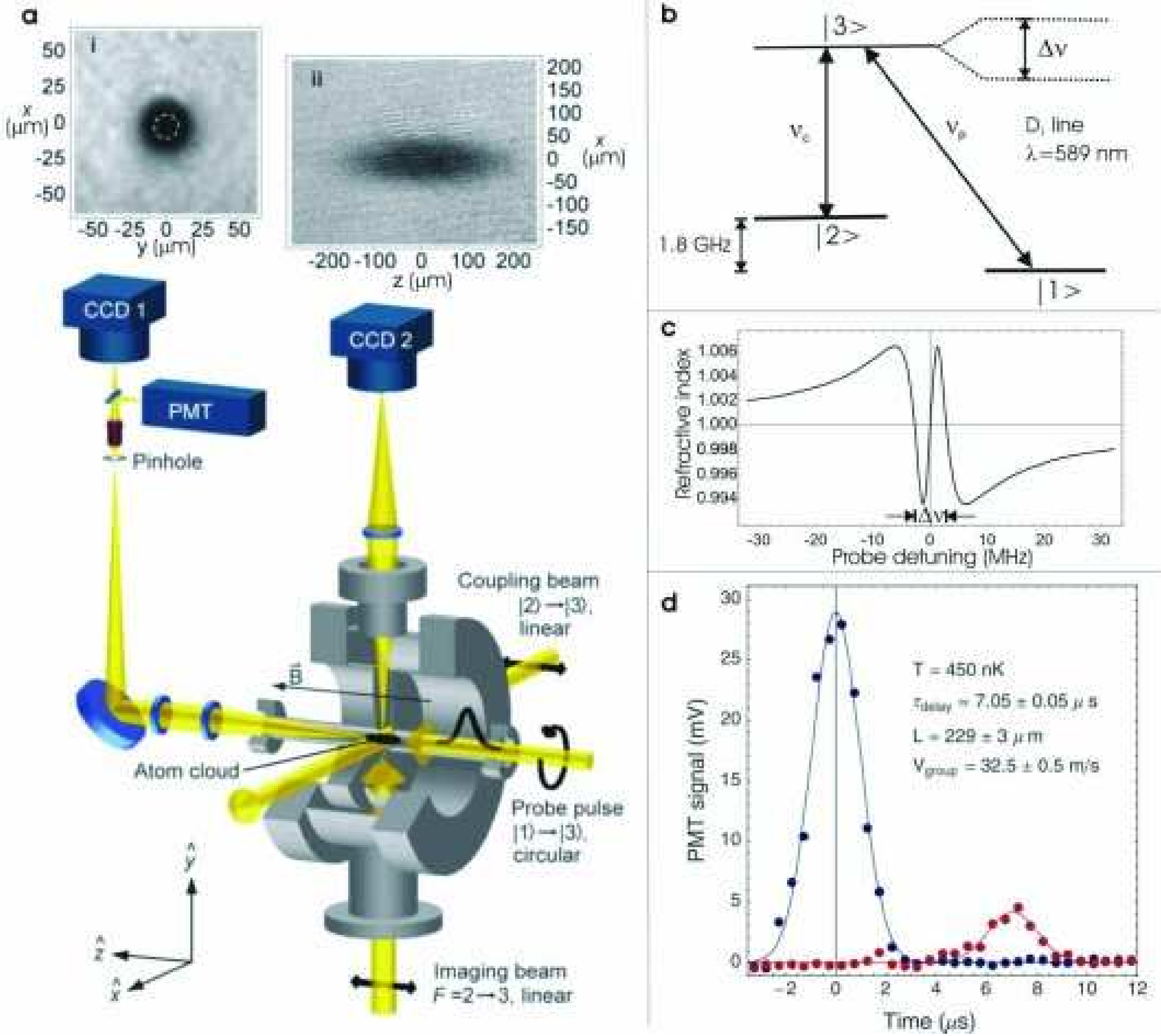
$$\tilde{E}(\tilde{\sigma})=\frac{3}{4}\left\{\frac{1}{\tilde{\sigma}^2}+\tilde{\sigma}^2\right\}+\eta\cdot\frac{1}{\tilde{\sigma}^3}\qquad\qquad \tilde{\sigma}=\frac{b}{a_0}$$

dimension-less orbital size

This expression is plotted in the figures in the main lecture.

BACK

ADDITIONAL NOTES


a
i
ii
x (μm)
y (μm)
z (μm)
CCD 1
CCD 2
PMT
Pinhole
Coupling beam
|2⟩→|3⟩, linear
Atom cloud
Probe pulse
|1⟩→|3⟩, circular
Imaging beam
F=2→3, linear
b
|3>
|2>
|1>
Δν
ν_c
ν_p
D1 line
λ=589 nm
1.8 GHz
c
Refractive index
Probe detuning (MHz)
d
PMT signal (mV)
Time (μs)
T = 450 nK
τ_{delay} = 7.05 ± 0.05 μs
L = 229 ± 3 μm
V_{group} = 32.5 ± 0.5 m/s