Příklad na dvoustavový systém
Nechť $\{X_n; n \in N_0\}$ je markovský řetězec s množinou stavů J = {0, 1} a maticí pravděpodobností přechodu 1. řádu $\mathbf{P}(n, n+1) = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$. Vektor počátečních pravděpodobností $\mathbf{p}(0) = (\gamma, 1-\gamma)$.

a) Pro n = 2 najděte pravděpodobnostní rozložení tohoto markovského řetězce.
b) Vypočtěte $P(X_1 \neq X_2)$

Řešení:
ad a) Hledáme vlastně simultánní pravděpodobnostní funkci $\pi(x_0, x_1, x_2)$ náhodného vektoru (X_0, X_1, X_2), kde náhodné veličiny X_0, X_1, X_2 mohou nabývat pouze hodnot 0 a 1. Výpočty uspořádáme do tabulky:

x_0, x_1, x_2	$\pi(x_0, x_1, x_2)$
0, 0, 0	$\gamma(1-\alpha)^2$
0, 0, 1	$\gamma(1-\alpha)\alpha$
0, 1, 0	$\gamma\alpha\beta$
1, 0, 0	$(1-\gamma)\beta(1-\alpha)$
0, 1, 1	$\gamma\alpha(1-\beta)$
1, 0, 1	$(1-\gamma)\beta\alpha$
1, 1, 0	$(1-\gamma)(1-\beta)\beta$
1, 1, 1	$(1-\gamma)(1-\beta)^2$

$$\pi(0,0,0) = P(X_0 = 0 \wedge X_1 = 0 \wedge X_2 = 0) = P(X_0 = 0)P(X_1 = 0 / X_0 = 0)P(X_2 = 0 / X_1 = 0 \wedge X_0 = 0) =$$
$$= P(X_0 = 0)P(X_1 = 0 / X_0 = 0)P(X_2 = 0 / X_1 = 0) = \gamma(1-\alpha)^2$$
$$\pi(0,0,1) = P(X_0 = 0 \wedge X_1 = 0 \wedge X_2 = 1) = P(X_0 = 0)P(X_1 = 0 / X_0 = 0)P(X_2 = 1 / X_1 = 0 \wedge X_0 = 0) =$$
$$= P(X_0 = 0)P(X_1 = 0 / X_0 = 0)P(X_2 = 1 / X_1 = 0) = \gamma(1-\alpha)\alpha$$
$$\pi(0,1,0) = P(X_0 = 0)P(X_1 = 1 / X_0 = 0)P(X_2 = 0 / X_1 = 1) = \gamma\alpha\beta$$
$$\pi(1,0,0) = P(X_0 = 1)P(X_1 = 0 / X_0 = 1)P(X_2 = 0 / X_1 = 0) = (1-\gamma)\beta(1-\alpha)$$
$$\pi(0,1,1) = P(X_0 = 0)P(X_1 = 1 / X_0 = 0)P(X_2 = 1 / X_1 = 1) = \gamma\alpha(1-\beta)$$
$$\pi(1,0,1) = P(X_0 = 1)P(X_1 = 0 / X_0 = 1)P(X_2 = 1 / X_1 = 0) = (1-\gamma)\beta\alpha$$
$$\pi(1,1,0) = P(X_0 = 1)P(X_1 = 1 / X_0 = 1)P(X_2 = 0 / X_1 = 1) = (1-\gamma)(1-\beta)\beta$$
$$\pi(1,1,1) = P(X_0 = 1)P(X_1 = 1 / X_0 = 1)P(X_2 = 1 / X_1 = 1) = (1-\gamma)(1-\beta)^2$$

ad b)
$$P(X_1 \neq X_2) = P(X_1 = 0 \wedge X_2 = 1) + P(X_1 = 1 \wedge X_2 = 0) =$$
$$= P(X_1 = 0)P(X_2 = 1 / X_1 = 0) + P(X_1 = 1)P(X_2 = 0 / X_1 = 1) = P(X_1 = 0)\alpha + P(X_1 = 1)\beta$$
Musíme vypočítat $P(X_1 = 0), P(X_1 = 1)$. Podle zákona evoluce dostaneme:
$$P(X_1 = 0) = P(X_0 = 0)P(X_1 = 0 / X_0 = 0) + P(X_0 = 1)P(X_1 = 0 / X_0 = 1) = \gamma(1-\alpha) + (1-\gamma)\beta$$
$$P(X_1 = 1) = P(X_0 = 0)P(X_1 = 1 / X_0 = 0) + P(X_0 = 1)P(X_1 = 1 / X_0 = 1) = \gamma\alpha + (1-\gamma)(1-\beta)$$
Po dosazení tedy máme:
$$P(X_1 \neq X_2) = P(X_1 = 0)\alpha + P(X_1 = 1)\beta = [\gamma(1-\alpha) + (1-\gamma)\beta]\alpha + [\gamma\alpha + (1-\gamma)(1-\beta)]\beta = \ldots$$
$$\ldots = \alpha(\gamma - \alpha\gamma + \beta) + \beta(1 - \gamma - \beta + \beta\gamma)$$