COMPUTING THE STEADY STATE OF LINEAR QUADRATIC OPTIMIZATION MODELS WITH RATIONAL EXPECTATIONS

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Abstract. In this paper we present a simple algorithm for computing the steady state of Linear-Quadratic control model with rational expectations. The method uses Sims' approach for solving rational expectations models first, and then solves the steady state through an iterative scheme.

1. Introduction

In recent work, Amman, Kendrick and Achath (1995), Amman (1996), we presented a procedure that introduces RE in a linear-quadratic (LQ) control framework based on the Blanchard and Kahn method. Due to the limitations of the Blanchard and Kahn approach and the fact that we had to rely on the diagonalization of the transition matrix, this work could only deal with a limited set of models. Recently, Sims (1996) proposed a different method for solving linear models with RE allowing for a broader range of models. This method is not based on the Jordan canonical form, but uses the more widely available *QZ* form that is based on generalized eigenvalues and which is numerically stable. In this paper we will follow the paper of Sims and incorporate his approach for solving the steady state of the linear-quadratic optimization model with rational expectations.

2. Problem statement and solution

Following Kendrick (1981), the standard single-agent stochastic linear-quadratic optimization problem is written as:

Find the steady state solution of the control vector u_∞ and the corresponding state vector x_∞ that minimizes the welfare loss function

$$J = \sum_{t=0}^{\infty} \beta^t L_t(x_t, u_t)\} \tag{1}$$

with

$$L_t = \begin{aligned} &\frac{1}{2}(x_t - \bar{x})' W (x_t - \bar{x}) + \\ &\frac{1}{2}(u_t - \bar{u})' R (u_t - \bar{u}) + (x_t - \bar{x})' F (u_t - \bar{u}) \end{aligned} \tag{2}$$

subject to the model

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$$x_{t+1} = Ax_t + Bu_t + Cz \tag{3}$$

The vector $x_t \in \Re^n$ is the state of the economy at time t and the vector $u_t \in \Re^m$ contains the policy instruments. The initial state of the economy x_0 is known, $\bar{x}$ and $\bar{u}$ are target values and β is some discount factor. W, R and F are penalty matrices of conformable size.

The above model is straightforward to solve for finite time and there are a number of packages available for computing its solution. However, a serious drawback for economics is that equation (2) does not allow for rational expectations and the model involves a finite horizon. One way of allowing RE to enter the model is to augment equation (2) in the following fashion

$$x_{t+1} = Ax_t + Bu_t + Cz + \sum_{j=1}^{k} D_j E_t x_{t+j} + \epsilon_t \tag{2a}$$

where the matrix D_j is a parameter matrix, $E_t x_{t+1}$ is the expected state for time $t+j$ at time t, k the maximum lead in the expectations formation and ϵ_t is a white noise vector, see Amman et al. (1995). In order to compute the admissible set of instruments we have to eliminate the rational expectations from the model. In a previous paper Amman and Kendrick (1997), we used Sims' approach to solve the rational expectations in the model. Sims (1996) proposes a method based on *generalized eigenvalues*, see Moler and Stewart (1973) or Coleman and Van Loan (1988). In order to apply Sims' method we first put equation (2) in the form[1]

$$\Gamma_0 \tilde{x}_{t+1} = \Gamma_1 \tilde{x}_t + \Gamma_2 u_t + \Gamma_3 z + \Gamma_4 \epsilon_t \tag{4}$$

where

$$\Gamma_{0,t} = \begin{bmatrix} I - D_1 & -D_2 & \dots & -D_{k-1} & -D_k \\ I & 0 & \dots & 0 & 0 \\ 0 & I & \dots & 0 & 0 \\ \vdots & & \ddots & 0 & 0 \\ 0 & \dots & & I & 0 \end{bmatrix}$$

$$\Gamma_1 = \begin{bmatrix} A & 0 & \dots & 0 \\ 0 & I & \dots & 0 \\ 0 & 0 & \ddots & 0 \\ \vdots & \ddots & & I \\ 0 & \dots & & 0 \end{bmatrix} \quad \Gamma_2 = \begin{bmatrix} B \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \Gamma_3 = \begin{bmatrix} C \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \Gamma_4 = \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

[1] Note that in contrast to Sims (1996) the variable z contains exogenous variables and not random variables. Hence, the matrix Π in Sims' paper is set to zero.

and the augmented state vector

$$\tilde{x}_t = \begin{bmatrix} x_t \\ Ex_{t+1} \\ Ex_{t+2} \\ \vdots \\ Ex_{t+k-1} \end{bmatrix} \tag{5}$$

Taking the generalized eigenvalues of equation (3) allows us to decompose the system matrices Γ_0 and Γ_1 in the following manner

$$\Lambda = Q\Gamma_0 Z$$
$$\Omega = Q\Gamma_1 Z$$

with $Z'Z = I$ and $Q'Q = I$. The matrices Λ and Ω are upper triangular matrices and the generalized eigenvalues are $\forall i\ \omega_{i,i}/\lambda_{i,i}$. If we use the transformation $w_t = Z'\tilde{x}_t$ we can write equation (3) as

$$\Lambda w_{t+1} = \Omega_1 w_t + Q\Gamma_2 u_t + Q\Gamma_3 z + Q\Gamma_4 \epsilon_t \tag{6}$$

Given the triangular structure of Λ and Ω we can partition (4) as follows

$$\begin{aligned} \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ 0 & \Lambda_{22} \end{bmatrix} \begin{bmatrix} w_{1,t+1} \\ w_{2,t+1} \end{bmatrix} &= \\ \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ 0 & \Omega_{22} \end{bmatrix} \begin{bmatrix} w_{1,t} \\ w_{2,t} \end{bmatrix} &+ \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \Gamma_2 u_t + \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \Gamma_3 z + \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \Gamma_4 \epsilon_t \end{aligned} \tag{7}$$

where the unstable eigenvalues are in lower right corner, i.e. in the matrices Λ_{22} and Ω_{22}. By forward propagation and taking expectations, it is possible to derive $w_{2,t}$ as a function of future instruments and exogenous variables, Sims (1996, page 5)

$$\gamma_t = w_{2,t} = -\sum_{j=1}^{\infty} M^{j-1}\Omega_{22}^{-1}Q_2(\Gamma_2 u_{t+j-1} + \Gamma_3 z) \tag{8}$$

with

$$M = \Omega_{22}^{-1}\Lambda_{22}$$

In the steady state, however, equation (7) can be rewritten as

$$\gamma = w_2 = -\sum_{j=1}^{\infty} M^{j-1}\Omega_{22}^{-1}Q_2(\Gamma_2 u_\infty + \Gamma_3 z)$$

where u_∞ is the optimal steady state solution of the control vector, which can be reduced to

$$\gamma = w_2 = -(I - M)^{-1}\Omega_{22}^{-1}Q_2(\Gamma_2 u_\infty + \Gamma_3 z) \tag{9}$$

Reinserting equation (9) into equation (7) and taking expectations gives us

$$\tilde{\Lambda} w_{t+1} = \tilde{\Omega} w_t + \tilde{\Gamma}_2 u_t + \tilde{\Gamma}_3 z + \tilde{\Gamma}_4 \epsilon_t + \tilde{\gamma} \tag{10}$$

with

$$\tilde{\Lambda} = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ 0 & I \end{bmatrix} \quad \tilde{\Omega} = \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ 0 & 0 \end{bmatrix} \quad \tilde{\Gamma}_2 = \begin{bmatrix} Q_1 \\ 0 \end{bmatrix} \Gamma_2$$

$$\tilde{\Gamma_3} = \begin{bmatrix} Q_1 \\ 0 \end{bmatrix} \Gamma_3 \quad \tilde{\Gamma_4} = \begin{bmatrix} Q_1 \\ 0 \end{bmatrix} \Gamma_4 \quad \tilde{\gamma} = \begin{bmatrix} 0 \\ \gamma \end{bmatrix}$$

Knowing that $\tilde{x}_t = Z w_t$ we can write equation (7) as

$$\tilde{x}_{t+1} = \tilde{A} \tilde{x}_t + \tilde{B} u_t + \tilde{C} \tilde{z} + \tilde{\epsilon}_t \tag{11}$$

with

$$\tilde{A} = Z \tilde{\Lambda}^{-1} \tilde{\Omega} Z' \quad \tilde{B} = Z \tilde{\Lambda}^{-1} \tilde{\Gamma}_2 \quad \tilde{C} = \begin{bmatrix} Z \tilde{\Lambda}^{-1} \tilde{\Gamma}_3 & Z \tilde{\Lambda}^{-1} \end{bmatrix} \tag{12}$$

and

$$\tilde{z} = \begin{bmatrix} z \\ \tilde{\gamma} \end{bmatrix} \qquad \tilde{\epsilon}_t = Z \tilde{\Lambda}^{-1} \tilde{\Gamma}_4 \epsilon_t \tag{13}$$

and the inverse

$$\tilde{\Lambda}^{-1} = \begin{bmatrix} \Lambda_{11}^{-1} & -\Lambda_{11}^{-1} \Lambda_{12} \\ 0 & I \end{bmatrix} \tag{14}$$

We have to make the assumption here that Λ_{11} is nonsingular. However, the diagonal elements will generally be nonzero, so it is very likely that the matrix is nonsingular. With equation (11) we have removed the RE from the control model.

The steady state of the LQ framework can be obtained by solving the algebraic matrix Riccati equation and tracking equation[2]. The algebraic matrix Riccati equation has the form for the control model in equations (1)-(2), see Amman, Kendrick and Neudecker (1996),

$$X = \hat{W} + \beta \hat{A}' X \hat{A} - (\beta \hat{A}' X \hat{B} + \hat{F})(R' + \beta \hat{B}' X \hat{B})^{-1} (\beta \hat{B}' X \hat{A} + \hat{F}') \tag{15}$$

and the tracking equation

$$\begin{aligned} p = & - (\beta \hat{A}' X \hat{B} + \hat{F})(\beta \hat{B}' X \hat{B} + R')^{-1} (\beta \hat{B}' (X \hat{C} \hat{z} + p) - R \bar{u}) \\ & + \beta \hat{A}' (X \hat{C} \hat{z} + p) - \hat{W} \bar{x} \end{aligned} \tag{16}$$

where X is the Riccati matrix and p the Riccati vector both for the steady state. $\hat{W}$ and $\hat{F}$ are the penalty matrices adjusted to conformable size of the augmented system in equation (5). In Amman and Neudecker (1997) a simple method is described for solving the Riccati matrix using a newton or quasi-newton solution method.

Once we have derived the solution for X and p, it is easy to compute the steady state of the system. The optimal control is computed by the feedback equation

[2] Alternatively, one can use a Lagrangian procedure for the case that $\beta = 1$ - see Appendix A.

$$u_\infty = Gx_\infty + g \tag{17}$$

with

$$G = -(\beta \hat{B}' X \hat{B} + R')^{-1}(\hat{F}' + \beta \hat{B}' X \hat{A}) \tag{18}$$
$$g = -(\beta \hat{B}' X \hat{B} + R')^{-1}(\beta \hat{B}'(X\hat{C}\hat{z} + p) - R\bar{u}) \tag{19}$$

In absence of random shocks we can compute the steady state from

$$x_\infty = \hat{A}x_\infty + \hat{B}u_\infty + \hat{C}\hat{z} \tag{20}$$

With the help of the above equations, and knowing that γ depends on u_∞ we can set up a simple iterative scheme to compute the steady state solution of the control vector. The algorithm is

Step 0. Set the iteration counter $\nu = 0$ and set the instruments u_∞^ν, to an initial value.

Step 1. Compute γ^ν using equation (9).

Step 2. Compute X, p, G, g, $x_\infty^{\nu+1}$ and $u_\infty^{\nu+1}$ as described above.

Step 3. Set $\nu = \nu + 1$ and goto Step 0 until convergence is reached.

3. An example

In this section we will present a simple example that be checked by hand to illustrate the algorithm. Consider a simple macro model with output, x_t, consumption, c_t, investment, i_t, government expenditures, g_t, and taxes τ_t. The problem can then be stated as:

Find the steady state solution of the control vector u_∞ and the corresponding state vector x_∞ that minimizes the welfare loss function

$$J = \frac{1}{2}\sum_{t=0}^{\infty} 0.90^t\{(x_t - 1600)^2 + g_t^2\} \tag{21}$$

for the model

$$x_{t+1} = c_{t+1} + i_{t+1} + g_{t+1} \tag{22}$$
$$c_{t+1} = 0.8(x_t - \tau_t) + 200 \tag{23}$$
$$i_{t+1} = 0.2E_t x_{t+2} + 100 + \epsilon_t \tag{24}$$
$$g_{t+1} = u_t \tag{25}$$
$$\tau_{t+1} = 0.25x_{t+1} \tag{26}$$

If we reduce the above model to one equation for output we get

$$x_{t+1} = 0.6x_t + u_t + 0.2E_t x_{t+2} + 300 + \epsilon_t \tag{27}$$

which leads to the augmented system

$$\begin{bmatrix} 1 & -0.2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_{t+1} \\ E_t x_{t+2} \end{bmatrix} = \begin{bmatrix} 0.6 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_t \\ x_{t+1} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_t + \begin{bmatrix} 300 \\ 0 \end{bmatrix} z_t + \begin{bmatrix} \epsilon_t \\ 0 \end{bmatrix} \tag{28}$$

Apply the QZ factorization, Coleman and van Loan (1988), to compute the generalized eigenvalues of the model gives us the time invariant solution[3]

$$\Lambda = \begin{bmatrix} 1.0822 & -0.9136 \\ 0 & 0.1848 \end{bmatrix} \quad \Omega = \begin{bmatrix} 0.7546 & 0.3979 \\ 0 & 0.7951 \end{bmatrix} \tag{29}$$

$$Z = \begin{bmatrix} 0.8202 & -0.5719 \\ 0.5719 & 0.8203 \end{bmatrix} \quad Q = \begin{bmatrix} 0.6523 & 0.7580 \\ -0.7580 & 0.6523 \end{bmatrix} \tag{30}$$

so the eigenvalues are $\{0.7546/1.0822, 0.7952/0.1848\} = \{0.6972, 4.328\}$ and the ordering of the system is such that the unstable root 4.328 is in the lower right corner, so no reordering is required. The other components are

$$\tilde{A} = \begin{bmatrix} 0.2966 & 0.5745 \\ 0.2068 & 0.4006 \end{bmatrix} \qquad \tilde{B} = \begin{bmatrix} 0.2944 \\ 0.3447 \end{bmatrix} \tag{31}$$

$$\tilde{C} = \begin{bmatrix} 148.3243 & 0.7580 & 0.1205 \\ 103.4153 & 0.5285 & 1.3031 \end{bmatrix} \tag{32}$$

The adjust penalty matrices are in this case

$$\tilde{W} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad R = [1] \qquad \tilde{F} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{33}$$

The Riccati matrix and the tracking vector have the solution

$$X = \begin{bmatrix} 1.0938 & 0.1818 \\ 0.1818 & 0.3520 \end{bmatrix} \qquad p = \begin{bmatrix} -2116.7492 \\ -1000.8060 \end{bmatrix} \tag{34}$$

leading to a steady state of $x_\infty = 1585.66$ and $u_\infty = 17.13$.

4. Summary

In this paper we have presented a method for solving the steady state solution of the Linear-Quadratic control model augmented with rational expectations. Our solution method is based on Sims's method of generalized eigenvalues. By using an iterative scheme, the reduced model can be fitted into a standard linear-quadratic framework that allows us to derive the optimal policy instruments for the model with rational expectations.

[3] As Γ_0 is invertible we could have used a *Schur decomposition* too in thid example.

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Appendix A
An Alternate Method

This appendix contains an alternative method which relies on Lagrangian method rather than on the Riccati equation approach, which can be applied when $\beta = 1$. The outside loop in the iteration is the same but the optimization inside the loop has been replaced with the Lagrangian method.

Begin with the system equations

$$x_{t+1} = Ax_t + Bu_t + Cz \tag{A-1}$$

and solve this for the steady state, i.e.

$$x^* = Ax^* + Bu^* + Cz \tag{A-2}$$

or

$$x^* = (I - A)^{-1}Bu^* + (I - A)^{-1}Cz \tag{A-3}$$

or

$$x^* = Hu^* + h \tag{A-4}$$

where

$$H = (I - A)^{-1}B \tag{A-5}$$
$$h = (I - A)^{-1}Cz \tag{A-6}$$

The criterion function (with F = 0 and $\beta = 1$) for this model in the steady state is

$$J = \frac{1}{2}(x^* - \bar{x})'W(x^* - \bar{x}) + \frac{1}{2}(u^* - \bar{u})'R(u^* - \bar{u}) \tag{A-7}$$

Thus the maximization of J subject to equation (A-4) in Lagarangian form is

$$J = \frac{1}{2}(x^* - \bar{x})'W(x^* - \bar{x}) + \frac{1}{2}(u^* - \bar{u})'R(u^* - \bar{u}) + \lambda'(x^* - Hu^* - h) \tag{A-8}$$

The first order conditions for the optimization are then

$$\frac{\partial J}{\partial x^*} = W(x^* - \bar{x}) + \lambda = 0 \tag{A-9}$$

$$\frac{\partial J}{\partial u^*} = R(u^* - \bar{u}) - H'\lambda = 0 \tag{A-10}$$

From equation (A-9) one can obtain

$$\lambda = -W(x^* - \bar{x}) \tag{A-11}$$

and substitution of this into equation (A-10) yields

$$R(u^* - \bar{u}) + H'W(x^* - \bar{x}) = 0 \tag{A-12}$$

Solving this equation for u^* yields

$$u^* = \bar{u} - R^{-1}H'W(x^* - \bar{x}) \tag{A-13}$$

or

$$u^* = \bar{u} - R^{-1}H'Wx^* + R^{-1}H'W\bar{x} \tag{A-14}$$

Then substitution of equation (A-4) into equation (A-14) yields

$$u^* = \bar{u} - R^{-1}H'W(Hu^* + h) + R^{-1}H'W\bar{x} \tag{A-15}$$

or

$$(I + R^{-1}H'WH)u^* = \bar{u} - R^{-1}H'Wh + R^{-1}H'W\bar{x} \tag{A-16}$$

or

$$(I + R^{-1}H'WH)u^* = \bar{u} + R^{-1}H'W(\bar{x} - h) \tag{A-17}$$

Solving this equation for u^* yields

$$u^* = (I + R^{-1}H'WH)^{-1}(\bar{u} + R^{-1}H'W(\bar{x} - h)) \tag{A-18}$$

Recall that

$$h = (I - A)^{-1}Cz \tag{A-19}$$

and that z is a function of γ^{ν}.

Thus equation (A-18) can be embedded in an iterative loop to solve for the steady state solution. This algorithm is like the one outlined in Section 2 in the body of the paper except that the γ^{ν} term effects the h variable in equation (A-18) on each iteration. Also, Step 2 in the algorithm is changed to compute u^{*} and x^{*} using equation (A-18). Matlab code written for this algorithm iterates to the same solution for u^{*} as does the Riccati algorithm used in the body of the paper.

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