Cold atoms

Lecture 2.

10. October 2007

BEC for independent particles

Two basic models: BEC in an ideal gas *vs.* in a trapped atomic cloud

Problems with thermodynamic limit

BEC for independent particles

LAST TIME Ideální kvantové plyny

$$\langle n \rangle = \mathrm{e}^{-\beta(\varepsilon-\mu)}$$ Boltzmannovo rozdělení

vysoké teploty, zředěný plyn

fermiony — N

FD

$$\langle n \rangle = \frac{1}{\mathrm{e}^{\beta(\varepsilon-\mu)}+1}$$

$T \to 0$

Áufbau princip

$$|F\rangle = |1, 1, \mathrm{K}, 1, 0, \mathrm{K}\rangle$$

bosony — N

BE

$$\langle n \rangle = \frac{1}{\mathrm{e}^{\beta(\varepsilon-\mu)}-1}$$

$T \to 0$

diskontinuita
fázový přechod

BEC

$$|B\rangle = |N, 0, 0, \mathrm{K}, 0, \mathrm{K}\rangle$$

$$\langle n \rangle = \frac{1}{\mathrm{e}^{\beta\varepsilon}-1}$$

$T \to 0$

vymrzání

$$|\mathrm{vac}\rangle$$

Ideal quantum gases at a finite temperature

mean occupation number of a one-particle state with energy ε

$$\langle n \rangle = \mathrm{e}^{-\beta(\varepsilon-\mu)}$$ Boltzmann distribution

high temperatures, dilute gases

fermions

N

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bosons

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$T \to 0$

BEC

$$|B\rangle = |N, 0, 0, \mathrm{K}, 0, \mathrm{K}\rangle$$

$$\langle n \rangle = \frac{1}{\mathrm{e}^{\beta\varepsilon}-1}$$

$T \to 0$

freezing out

$$|\mathrm{vac}\rangle$$

Ideal quantum gases at a finite temperature

$\langle n \rangle = \mathrm{e}^{-\beta(\varepsilon-\mu)}$ Boltzmann distribution

high temperatures, dilute gases

fermions

bosons

N

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BE

$$\langle n \rangle = \frac{1}{\mathrm{e}^{\beta(\varepsilon-\mu)}-1}$$

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T –

Equation for the chemical potential closes the equilibrium problem:

$$N = \mathcal{N}(T,\mu) = \sum_j \langle n(\varepsilon_j) \rangle = \sum_j \frac{1}{\mathrm{e}^{\beta(\varepsilon_j-\mu)}-1}$$

ing out

$|F\rangle = |1,1,K,1,0,K\rangle$

$|B\rangle = |N,0,0,K,0,K\rangle$

$|\mathrm{vac}\rangle$

A gas with a fixed average number of atoms

Ideal boson gas (macroscopic system)

atoms: mass m, dispersion law $\varepsilon(p) = \frac{p^2}{2m}$

system as a whole:

volume V, particle number N, density $n=N/V$, temperature T.

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Always $\mu < 0$. At high temperatures, in the thermodynamic limit, the continuum approximation can be used:

$$N \approx V\int_0^\infty \mathrm{d}\varepsilon \frac{1}{\mathrm{e}^{\beta(\varepsilon-\mu)}-1}\mathcal{D}(\varepsilon) \equiv \mathcal{N}^0(T,\mu)$$

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It holds

$$\mathcal{N}^0(T,\mu<0) < \mathcal{N}^0(T,0) < \infty$$

For each temperature, we get a critical number of atoms the gas can accommodate. Where will go the rest?

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This will be shown in a while

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For each temperature, we get a critical number of atoms the gas can accommodate. Where will go the rest? To the condensate

Condensate concentration

The integral is doable:

$$\mathcal{N}_0(T,0) = V\int_0^\infty \mathrm{d}\varepsilon \frac{1}{\mathrm{e}^{\beta\varepsilon}-1}\mathcal{D}(\varepsilon)$$

use the
general formula

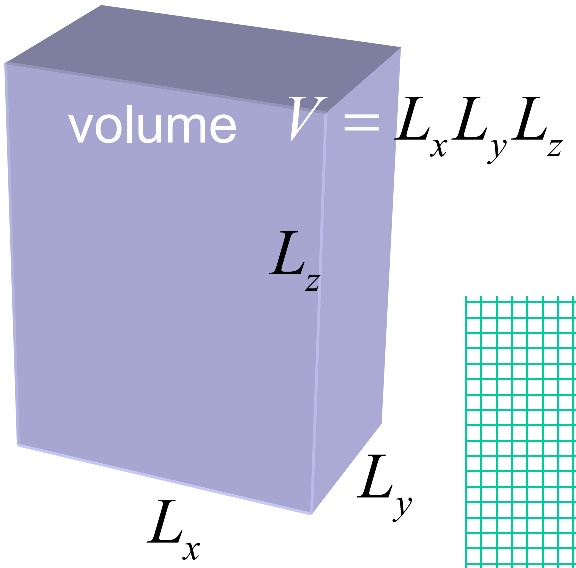
Plane waves in a cavity

Plane wave in classical terms and its quantum transcription

$$X = X_0 \mathrm{e}^{-\mathrm{i}(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})}, \quad \omega = \omega(k), \quad \lambda = 2\pi / k$$

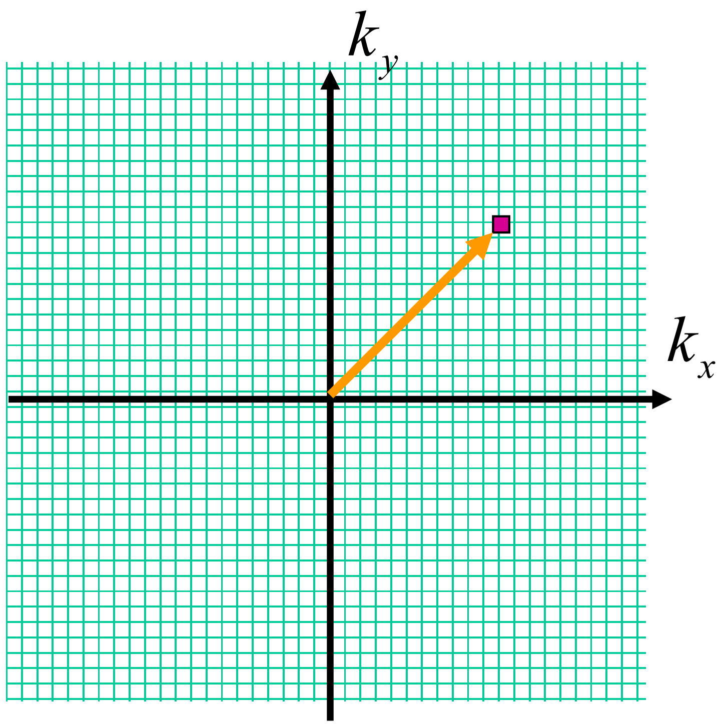
$$\varepsilon = \hbar\omega, \quad \boldsymbol{p} = \hbar\boldsymbol{k}, \quad \varepsilon = \varepsilon(p), \quad \lambda = h / p \text{ de Broglie wavelength}$$

Discretization ("quantization") of wave vectors in the cavity



periodic boundary conditions

$$\begin{cases} k_{x1} = \dfrac{2\pi}{L_x}\cdot 1, \\ k_{ym} = \dfrac{2\pi}{L_x}\cdot m, \\ k_{zn} = \dfrac{2\pi}{L_x}\cdot n \end{cases}$$



Cell size (per ***k*** vector)

$$\Omega_k = (2\pi)^d / V$$

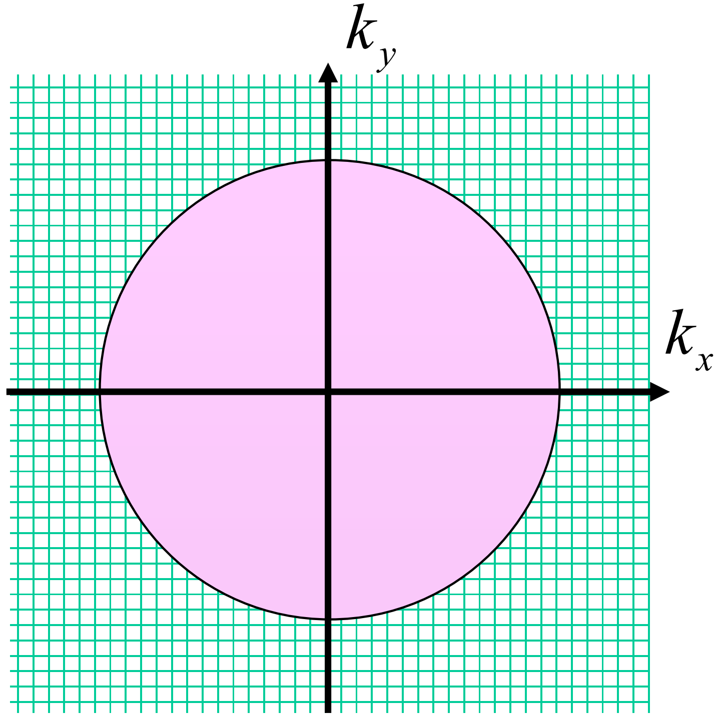
Cell size (per ***p*** vector)

$$\Omega_p = h^d / V$$

In the (***r, p***)-phase space

$$\hbar^d \Omega_k V = h^d$$

Density of states



IDOS Integrated Density Of States:

How many states have energy less than ε

Invert the dispersion law

$$\varepsilon(p) \quad \square \quad p(\varepsilon)$$

Find the volume of the *d*-sphere in the *p*-space

$$\Omega_d(p) = C_d \cdot p^d$$

Divide by the volume of the cell

$$\Gamma(\varepsilon) = \Omega_d(p(\varepsilon)) / \Omega_p = V \cdot \Omega_d(p(\varepsilon)) / h^d$$

$$C_d = \frac{2\pi^{d/2}}{(d/2-1)!}$$

DOS Density Of States:

How many states are around ε per unit energy per unit volume

$$\boxed{\mathcal{D}(\varepsilon) = \frac{1}{V}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Gamma(\varepsilon)}$$

$$= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Omega_d(p(\varepsilon)/h)^d = dC_d h^{-1} \cdot (p(\varepsilon)/h)^{d-1}\frac{\mathrm{d}\,p(\varepsilon)}{\mathrm{d}\varepsilon}$$

Condensate concentration

The integral is doable:

$$\mathcal{D}(\varepsilon) = 2\pi \left(\frac{2m}{h^2}\right)^{\frac{3}{2}} \cdot \sqrt{\varepsilon}$$

$$\mathcal{N}^0(T,0) = V \int_0^\infty \mathrm{d}\,\varepsilon \frac{1}{\mathrm{e}^{\beta\varepsilon} - 1} \mathcal{D}(\varepsilon)$$

$$= V\, 4\pi \left(\frac{2mk_B T}{h^2}\right)^{\frac{3}{2}} \Gamma(\tfrac{3}{2}) \zeta(\tfrac{3}{2})$$

$\Gamma(\tfrac{3}{2}) = \sqrt{\pi}/2$

Riemann function

$$= V \left(2\pi \frac{2mk_B T}{h^2}\right)^{\frac{3}{2}} \cdot 2{,}612$$

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CRITICAL TEMPERATURE

the lowest temperature at which all atoms are still accomodated in the gas:

$$\mathcal{N}'(T_c,0) = N$$

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atomic mass

$$T_c = \frac{h^2}{4\pi m k_B}\cdot\left(\frac{N}{2,612V}\right)^{\frac{2}{3}} = 0,52725\frac{h^2}{4\pi u k_B}\cdot\frac{n^{\frac{2}{3}}}{M} = 1,6061\times10^{-18}\cdot\frac{n^{\frac{2}{3}}}{M}$$

From a gas to an inhomogeneous system

Physical interpretation of BEC

Where are the condensate Bosons?

LAST TIME *Digression*: simple interpretation of T_C

Rearranging the formula for critical temperature

$$T_c = \frac{h^2}{4\pi m k_B} \cdot \left(\frac{N}{2{,}612 V} \right)^{\frac{2}{3}}$$

we get

$$\left(\frac{V}{N} \right)^{\frac{1}{3}} : \frac{h}{\sqrt{m k_B T_c}}$$

mean interatomic distance

thermal de Broglie wavelength

The quantum breakdown sets on when

the wave clouds of the atoms start overlapping

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NICE, BUT TOO SPECIFIC FOR A GAS

LAST TIME What is the nature of BEC?

With lowering the temperature, the atoms of the gas lose their energy and drain down to the lowest energy states. There is less and less of these:

$$\mathcal{N}(E < k_B T) = \mathrm{const} \times T^{3/2}$$

A given amount N of the atoms becomes too large starting from a critical temperature.

Their excess precipitates to the lowest level, which becomes *macroscopically occupied,* i.e., it holds a finite fraction of all atoms.

This is the BE condensate.

At the zero temperature, all atoms are in the condensate.

Einstein was the first to realize that and to make an exact calculation of the integrals involved.

$$\mathcal{N}^{0}_{G}(T) = V \times 4\pi \left(\frac{2mk_B T}{h^2} \right)^{\frac{3}{2}} \Gamma(\tfrac{3}{2})\zeta(\tfrac{3}{2}) \equiv BT^{\frac{3}{2}}$$

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Where are the condensate atoms?

ANSWER: <u>On the lowest one-particle energy level</u>

For understanding, return to the discrete levels.

$$N = \mathcal{N}(T,\mu) = \sum_j \left\langle n(\varepsilon_j)\right\rangle = \sum_j \frac{1}{\mathrm{e}^{\beta(\varepsilon_j-\mu)}-1}$$

There is a sequence of energies

$$\mu < \varepsilon_0 = \varepsilon(\vec{0}) = 0 < \varepsilon_1 < \varepsilon_2 \mathrm{L}$$

For very low temperatures, $\beta(\varepsilon_1 - \varepsilon_0)\;?\;1$

all atoms are on the lowest level, so that

$$n_0 = N - O(\mathrm{e}^{-\beta(\varepsilon_1-\varepsilon_0)})$$ all atoms are in the condensate

$$N \approx \frac{1}{\mathrm{e}^{\beta(\varepsilon_0-\mu)}-1}$$ connecting equation

$$\mu \approx \varepsilon_0 - \frac{k_B T}{N}$$ chemical potential is zero on the gross energy scale

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Where are the condensate atoms? Continuation

ANSWER: On the lowest one-particle energy level

For temperatures below T_C

all condensate atoms are on the lowest level, so that

$$n_0 = N_{BE}$$ all condensate atoms remain on the lowest level

$$N_{BE} \approx \frac{1}{e^{\beta(\varepsilon_0 - \mu)} - 1}$$ connecting equation

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ANSWER: On the lowest one-particle energy level

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$$\mu \approx \varepsilon_0 - \frac{k_B T}{N_{BE}}$$ chemical potential keeps zero on the gross energy scale

question … what happens with the occupancy of the next level now?
Estimate:

$$\varepsilon_1 - \varepsilon_0 : \left(h^2/m\right)\cdot V^{-\frac{2}{3}}$$

$$n_0 = \frac{k_B T}{\varepsilon_0 - \mu} = O(V), \quad n_1 = \frac{k_B T}{\varepsilon_1 - \mu} = O(V^{\frac{2}{3}})$$ .... much slower growth

Where are the condensate atoms? Summary

ANSWER: On the lowest one-particle energy level

The final balance equation for $T < T_C$ is

$$N = \mathcal{N}(T,\mu) = \frac{1}{e^{\beta(\varepsilon_0-\mu)}-1} + V\int_0^\infty \mathrm{d}\varepsilon \frac{1}{e^{\beta(\varepsilon-\mu)}-1}\mathcal{D}(\varepsilon)$$

LESSON:

be slow with making the thermodynamic limit (or any other limits)

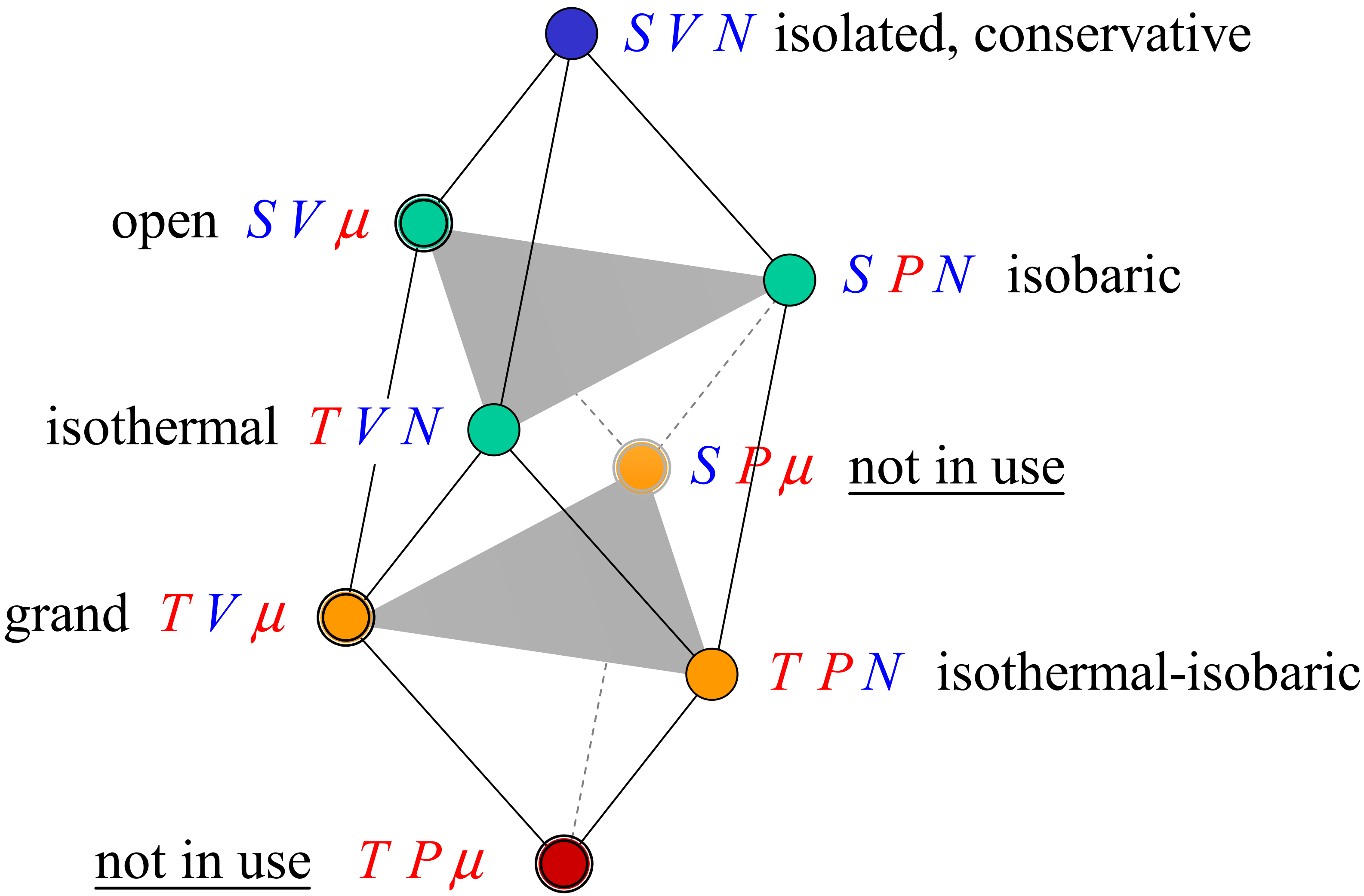
Thermodynamics of BEC

Capsule on thermodynamics
Grand canonical ensemble
Thermodynamic functions of an ideal gas
BEC in an ideal gas

Homogeneous one component phase: boundary conditions (environment) and state variables

$S\ V\ N$ additive variables, have densities $s = S/V \quad n = N/V$ "extensive"
b b b

$T\ P\ \mu$ dual variables, intensities "intensive"



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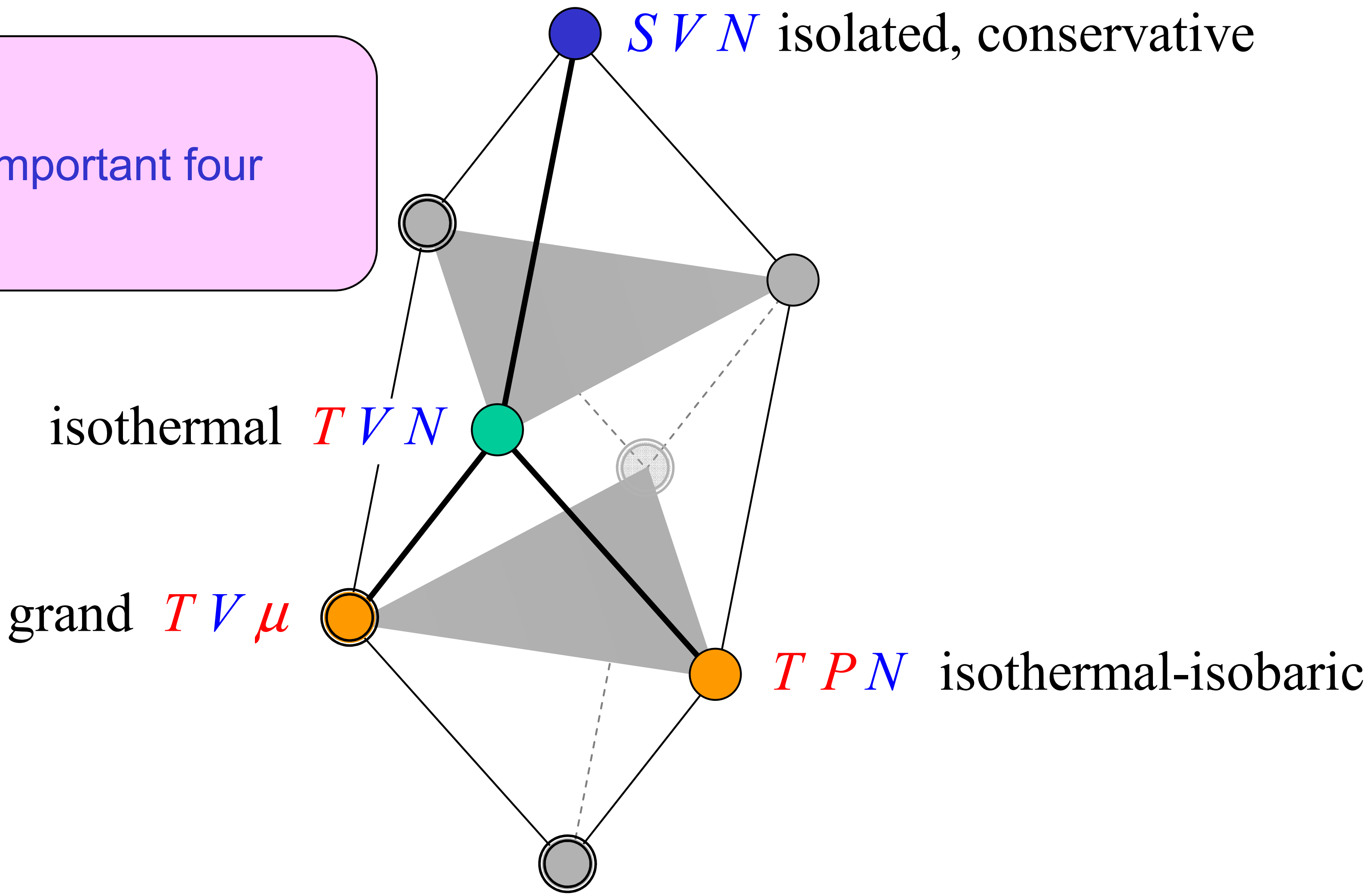
The important four



Digression: which environment to choose?

THE ENVIRONMENT IN THE THEORY SHOULD CORRESPOND
TO THE EXPERIMENTAL CONDITIONS

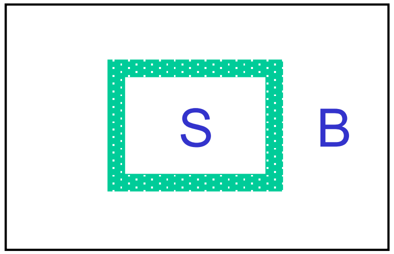
… a truism difficult to satisfy

❶ For large systems, this is not so sensitive for two reasons

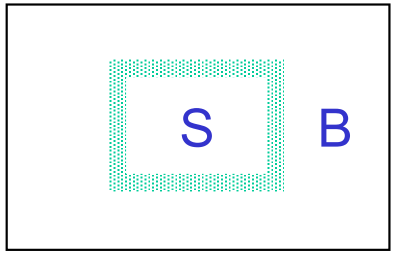
- System serves as a thermal bath or particle reservoir all by itself
- Relative fluctuations (distinguishing mark) are negligible

❷

	Adiabatic system	Real system	Isothermal system
SB heat exchange –	the slowest	medium fast process	the fastest



- temperature lag
- interface layer



❸ Atoms in a trap: ideal model … isolated. In fact: unceasing energy exchange (laser cooling). A small number of atoms may be kept (one to, say, 40). With 10^7, they form a bath already. Besides, they are cooled by evaporation and they form an open (albeit non-equilibrium) system.

❹ Some people, notably *Leggett*, insist on using clouds with a fixed number of atoms. This changes the physics of BEC substantially!

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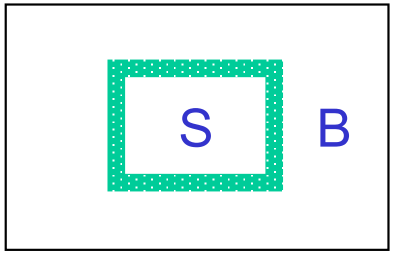

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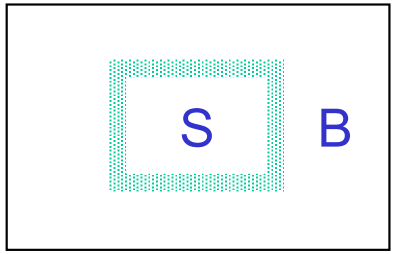

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Grand canonical ensemble

Definition following Gibbs

General treatment for independent particles

Thermodynamic functions of an ideal gas

Grand canonical ensemble - definition

Grand canonical ensemble admits both energy and particle number exchange between the system and its environment.

The statistical operator (many body density matrix) $\hat{\rho}$ acts in the Fock space

External variables are T, V, μ. They are specified by the conditions

$$\left\langle \hat{H} \right\rangle \equiv \mathrm{Tr}\,\hat{\rho}\hat{H} = U \qquad V = \text{ sharp} \qquad \left\langle \hat{N} \right\rangle \equiv \mathrm{Tr}\,\hat{\rho}\hat{N} = N$$

$$S = -k_{\mathrm{B}} \cdot \mathrm{Tr}\,\hat{\rho}\ln\hat{\rho} = \max$$

Grand canonical statistical operator has the Gibbs' form

$$\boxed{\hat{\rho} = Z^{-1}\,\mathrm{e}^{-\beta(\hat{H}-\mu\hat{N})}}$$

$$Z(\beta,\mu,V) = \mathrm{Tr}\,\mathrm{e}^{-\beta(\hat{H}-\mu\hat{N})} \equiv \mathrm{e}^{-\beta\Omega(\beta,\mu,V)} \qquad \text{statistical sum}$$

$$\Omega(\beta,\mu,V) = -k_{\mathrm{B}}T\ln Z(\beta,\mu,V) \qquad \text{grand canonical potential}$$

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volume … for an extended homogeneous system

V … generic for generalized coordinates of external fields whose change is connected with the mechanical work done by the system

Fluctuations I. – global quantities

Fluctuations of the total number of particles around the mean value

First derivative of the grand potential

$$\frac{\partial \Omega}{\partial \mu} = \frac{\partial}{\partial \mu}\left(-k_{\mathrm{B}} T \ln Z\right) = -k_{\mathrm{B}} T \frac{\partial}{\partial \mu} \ln \mathrm{Tr}\, \mathrm{e}^{-\beta(\hat{H}-\mu\hat{N})} = -\frac{\mathrm{Tr}\,\hat{N}\,\mathrm{e}^{-\beta(\hat{H}-\mu\hat{N})}}{\mathrm{Tr}\,\mathrm{e}^{-\beta(\hat{H}-\mu\hat{N})}} = -\left\langle \hat{N} \right\rangle$$

Second derivative of the grand potential

$$\frac{\partial^2 \Omega}{\partial \mu^2} = -\frac{\partial}{\partial \mu}\left\langle \hat{N} \right\rangle = -\frac{\partial}{\partial \mu} \frac{\mathrm{Tr}\,\hat{N}\,\mathrm{e}^{-\beta(\hat{H}-\mu\hat{N})}}{\mathrm{Tr}\,\mathrm{e}^{-\beta(\hat{H}-\mu\hat{N})}} =$$

$$= -\frac{\mathrm{Tr}\,\hat{N}^2\,\mathrm{e}^{-\beta(\hat{H}-\mu\hat{N})}}{\mathrm{Tr}\,\mathrm{e}^{-\beta(\hat{H}-\mu\hat{N})}} + \frac{\left(\mathrm{Tr}\,\hat{N}\,\mathrm{e}^{-\beta(\hat{H}-\mu\hat{N})}\right)^2}{\left(\mathrm{Tr}\,\mathrm{e}^{-\beta(\hat{H}-\mu\hat{N})}\right)^2} = -\left\langle \hat{N}^2 \right\rangle + \left\langle \hat{N} \right\rangle^2$$

Final estimate for the relative fluctuation

$$\frac{\left\langle \hat{N}^2 \right\rangle - \left\langle \hat{N} \right\rangle^2}{\left\langle \hat{N} \right\rangle^2} = \frac{\frac{\partial}{\partial \mu}\left\langle \hat{N} \right\rangle}{\left\langle \hat{N} \right\rangle^2} = O\left(\left\langle \hat{N} \right\rangle^{-1}\right)$$

Grand canonical statistical sum for independent bosons

Recall

$$Z(\beta,\mu,V)=\mathrm{Tr}\,\mathrm{e}^{-\beta(\hat{H}-\mu\hat{N})}\equiv \mathrm{e}^{-\beta\Omega(\beta,\mu,V)}$$ statistical sum

$$=\sum_{1}\mathrm{e}^{-\beta(E_1-\mu N_1)}$$ 1K eigenstate label $1\equiv\{n_\alpha\}$ with $\sum_\alpha n_\alpha = N_1$

$$=\sum_{\{n_\alpha\}}\mathrm{e}^{-\beta\sum_\alpha(\varepsilon_\alpha-\mu)n_\alpha}=\sum_{\{n_\alpha\}}\prod_\alpha\left(\mathrm{e}^{-\beta(\varepsilon_\alpha-\mu)}\right)^{n_\alpha}$$ up to here trivial

TRICK!! $$=\prod_\alpha\sum_{n_\alpha}\left(\mathrm{e}^{-\beta(\varepsilon_\alpha-\mu)}\right)^{n_\alpha}=\prod_\alpha\frac{1}{1-\mathrm{e}^{-\beta(\varepsilon_\alpha-\mu)}}$$

$\mathrm{e}^{\beta\mu}\equiv z$ activity, fugacity

$$Z(\beta,\mu,V)=\prod_\alpha\frac{1}{1-\mathrm{e}^{-\beta(\varepsilon_\alpha-\mu)}}\equiv\prod_\alpha\frac{1}{1-z\,\mathrm{e}^{-\beta\varepsilon_\alpha}}$$

$$\Omega(\beta,\mu,V)=-k_\mathrm{B}T\ln Z(\beta,\mu,V)$$ grand canonical potential

$$=+k_\mathrm{B}T\sum_\alpha\ln\left(1-\mathrm{e}^{-\beta(\varepsilon_\alpha-\mu)}\right)$$

$$=+k_\mathrm{B}T\sum_\alpha\ln\left(1-z\,\mathrm{e}^{-\beta\varepsilon_\alpha}\right)$$

Grand canonical statistical sum for independent bosons

Recall

$$Z(\beta,\mu,V) = \mathrm{Tr}\,\mathrm{e}^{-\beta(\hat{H}-\mu\hat{N})} \equiv \mathrm{e}^{-\beta\Omega(\beta,\mu,V)}$$ statistical sum

$$= \sum_{1} \mathrm{e}^{-\beta(E_1-\mu N_1)}$$ 1K eigenstate label $1 \equiv \{n_\alpha\}$ with $\sum_\alpha n_\alpha = N_1$

$$= \sum_{\{n_\alpha\}} \mathrm{e}^{-\beta\sum_\alpha(\varepsilon_\alpha-\mu)n_\alpha} = \sum_{\{n_\alpha\}}\prod_\alpha\left(\mathrm{e}^{-\beta(\varepsilon_\alpha-\mu)}\right)^{n_\alpha}$$ up to here trivial

TRICK!! $$= \prod_\alpha\sum_{n_\alpha}\left(\mathrm{e}^{-\beta(\varepsilon_\alpha-\mu)}\right)^{n_\alpha} = \prod_\alpha \frac{1}{1-\mathrm{e}^{-\beta(\varepsilon_\alpha-\mu)}}$$

$\mathrm{e}^{\beta\mu} \equiv z$ activity / fugacity

$$Z(\beta,\mu,V) = \prod_\alpha \frac{1}{1-\mathrm{e}^{-\beta(\varepsilon_\alpha-\mu)}} \equiv \prod_\alpha \frac{1}{1-z\,\mathrm{e}^{-\beta\varepsilon_\alpha}}$$

symbolic control parameter

$$\Omega(\beta,\mu,V) = -k_\mathrm{B}T\ln Z(\beta,\mu,V)$$ grand canonical potential

$$= +k_\mathrm{B}T\sum_\alpha \ln\left(1-\mathrm{e}^{-\beta(\varepsilon_\alpha-\mu)}\right)$$

$$= +k_\mathrm{B}T\sum_\alpha \ln\left(1-z\,\mathrm{e}^{-\beta\varepsilon_\alpha}\right)$$

valid for
- extended "infinite" gas
- parabolic traps

just the same

Non-interacting bosons in a trap

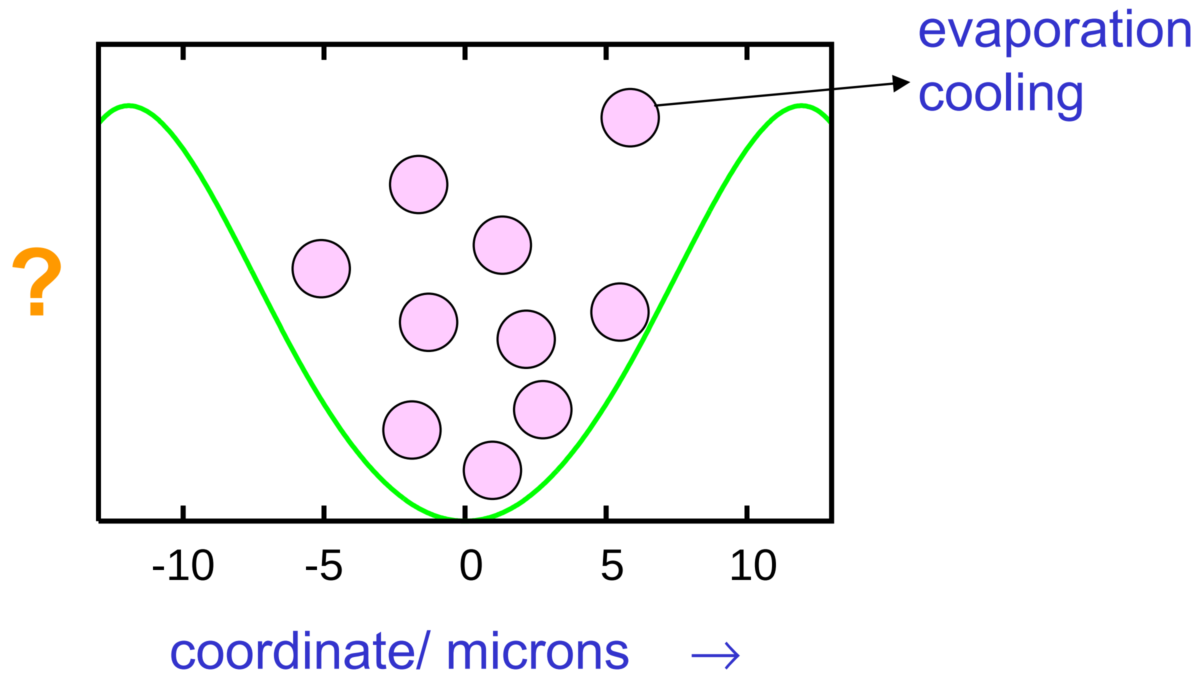
Useful digression: energy units

energy	1K	1eV	s^{-1}
1K	k_B/J	k_B/e	k_B/h
1eV	e/k_B	e/J	e/h
s^{-1}	h/k_B	h/e	h/J

energy	1K	1eV	s^{-1}
1K	1.38×10^{-23}	8.63×10^{-05}	$2.08\times10^{+10}$
1eV	$1.16\times10^{+04}$	1.60×10^{-19}	$2.41\times10^{+14}$
s^{-1}	4.80×10^{-11}	4.14×10^{-15}	6.63×10^{-34}

LAST TIME Trap potential

Typical profile



This is just one direction

Presently, the traps are mostly 3D

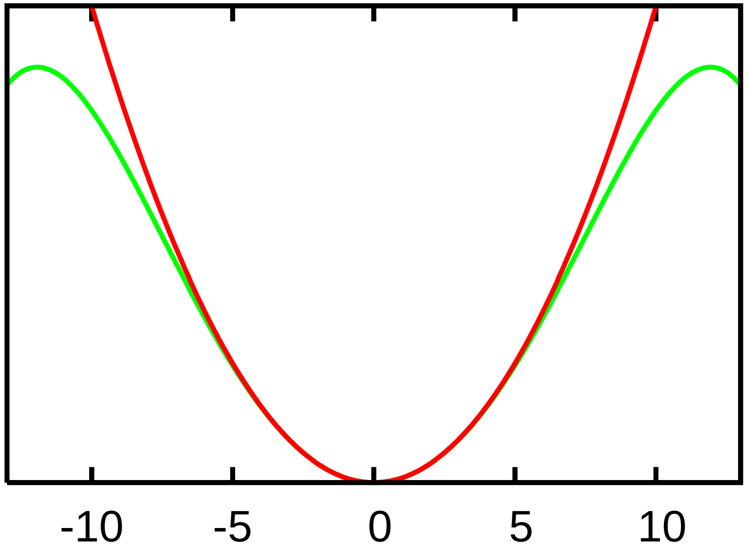
The trap is clearly from the real world, the atomic cloud is visible almost by a naked eye

LAST TIME Trap potential

Parabolic approximation

in general, an anisotropic harmonic oscillator

usually with axial symmetry



$$H = \frac{1}{2m}\boldsymbol{p}^2 + \frac{1}{2}m\omega_x^2 x^2 + \frac{1}{2}m\omega_y^2 y^2 + \frac{1}{2}m\omega_z^2 z^2$$

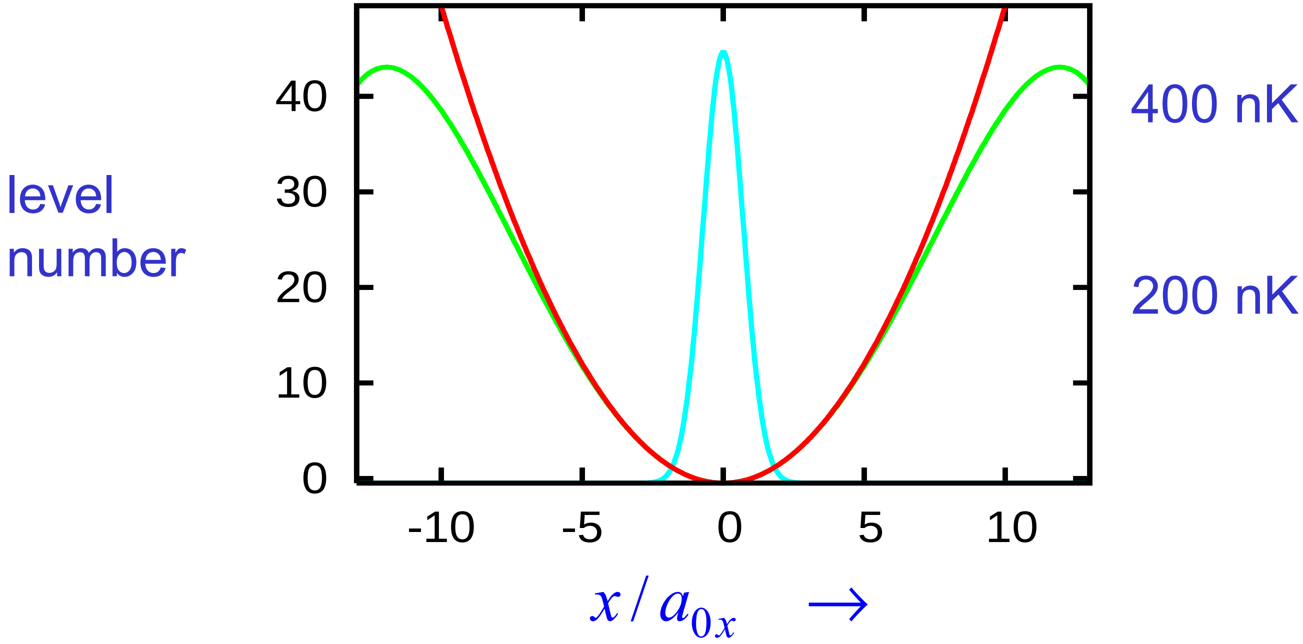
$$= H_x + H_y + H_z$$

1D

2D

3D

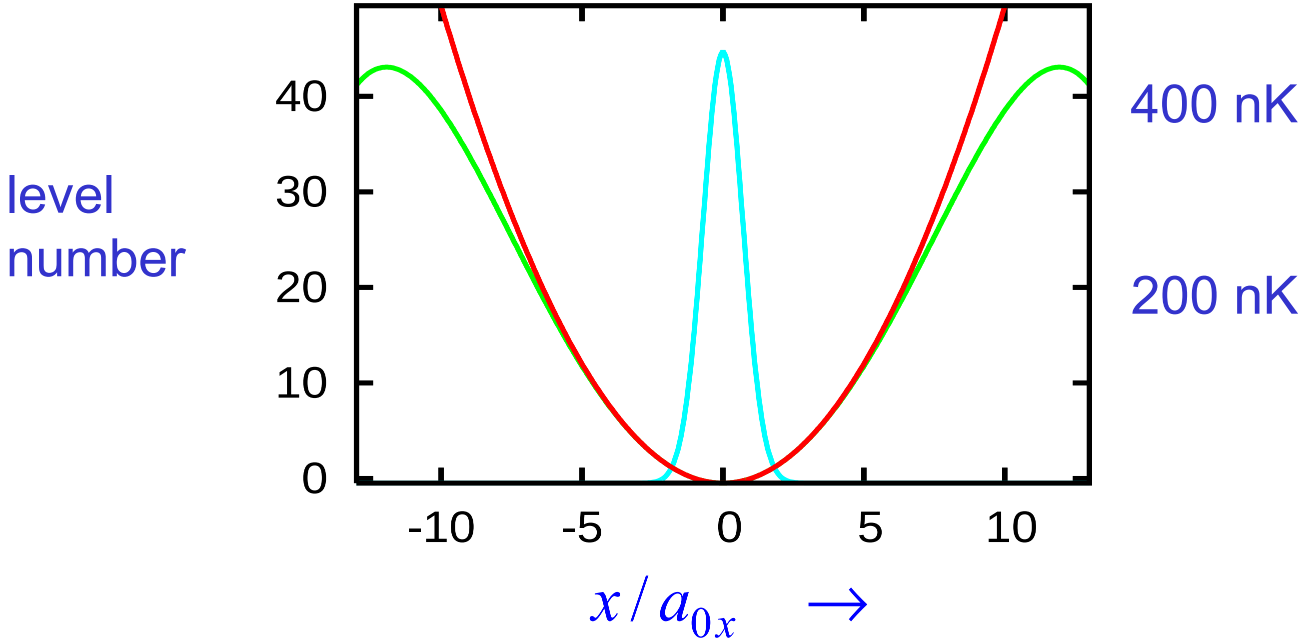
LAST TIME Ground state orbital and the trap potential



$$\psi_0(x,y,z) = \phi_{0x}(x)\,\phi_{0y}(y)\,\phi_{0z}(z)$$

$$\phi_0(u) = \frac{1}{\sqrt{a_0\pi}}\,\mathrm{e}^{-\frac{u^2}{2a_0^2}}, \quad \boxed{a_0 = \sqrt{\frac{\hbar}{m\omega}}}, \quad E_0 = \frac{1}{2}\hbar\omega = \frac{1}{2}\cdot\frac{\hbar^2}{ma_0^2} = \frac{1}{2}\cdot\frac{\hbar^2}{Mu_m a_0^2}$$

LAST TIME Ground state orbital and the trap potential



^{87}Rb
$a_0 = 1\mu$m
$\hbar\omega = 10$ nK
$N \sim 10^6$ at.

$$\psi_0(x,y,z) = \phi_{0x}(x)\phi_{0y}(y)\phi_{0z}(z)$$

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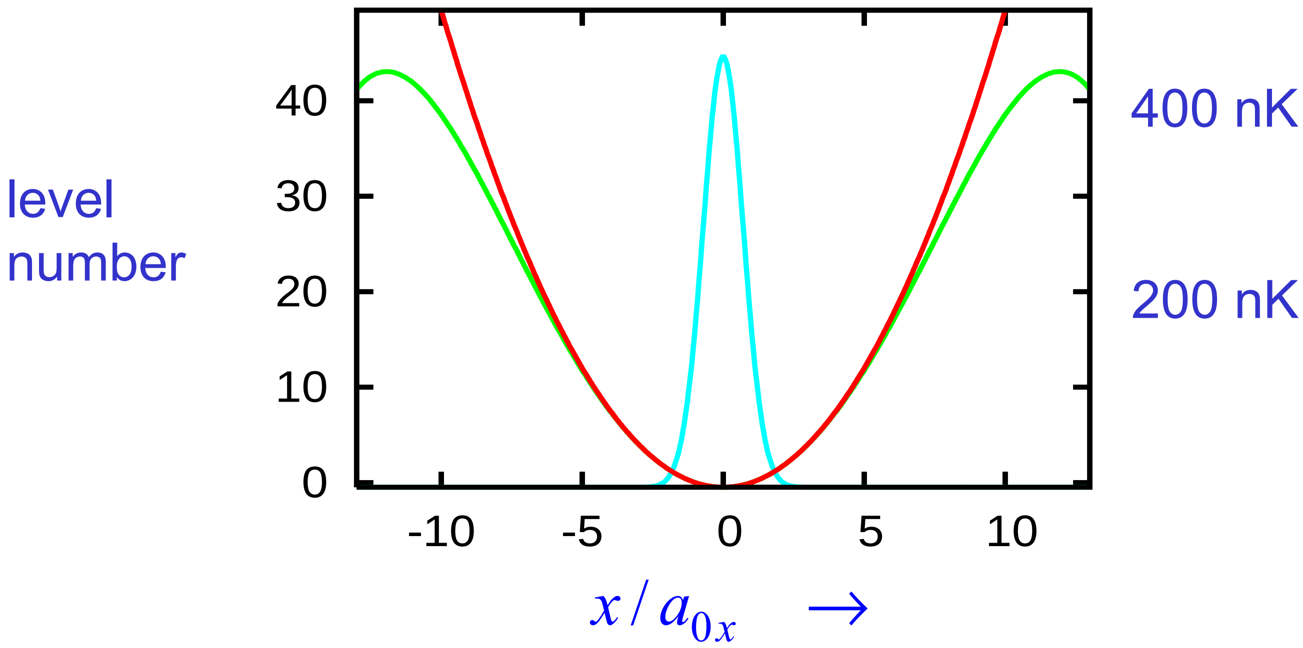

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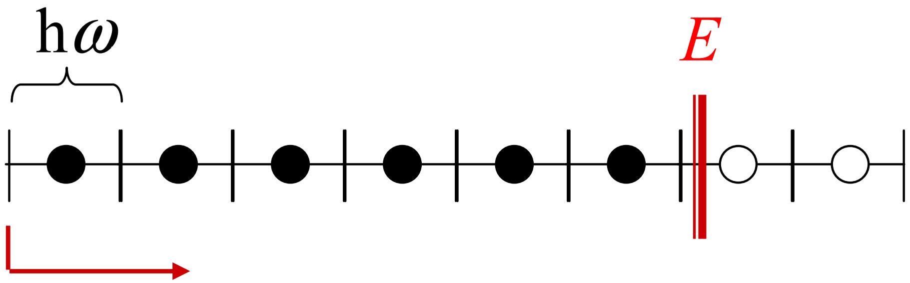
$$\boxed{V(u) = \frac{1}{2} m\omega^2 u^2 = \frac{1}{2}\hbar\omega\left(\frac{u}{a_0}\right)^2}$$

- characteristic energy
- characteristic length

Filling the trap with particles: IDOS, DOS

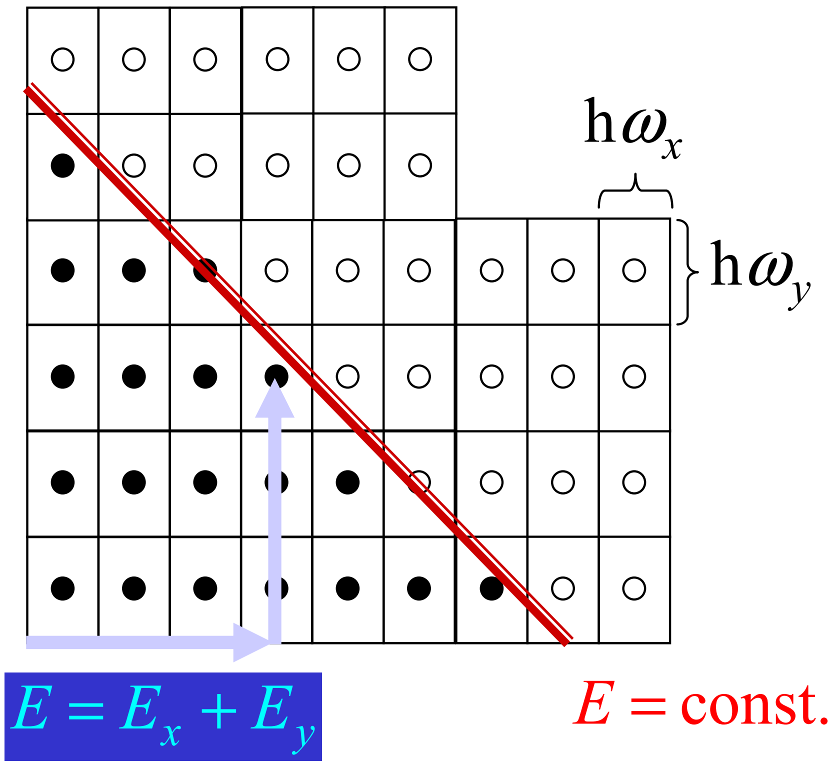


1D

$$\Gamma(E) = \mathrm{int}(E/\hbar\omega) \sim E/\hbar\omega$$

$$\mathcal{D}(E) = \Gamma'(E) = \left(\hbar\omega\right)^{-1}$$

For the finite trap, unlike in the extended gas, $\mathcal{D}(E)$ is **not** divided by volume



2D

$$\Gamma(E) \sim \tfrac{1}{2}E^2/(\hbar\omega_x \cdot \hbar\omega_y)$$

$$\mathcal{D}(E) = \Gamma'(E) = E/(\hbar\omega_x \cdot \hbar\omega_y)$$

"thermodynamic limit"

only approximate … finite systems

better for small $\hbar\omega$

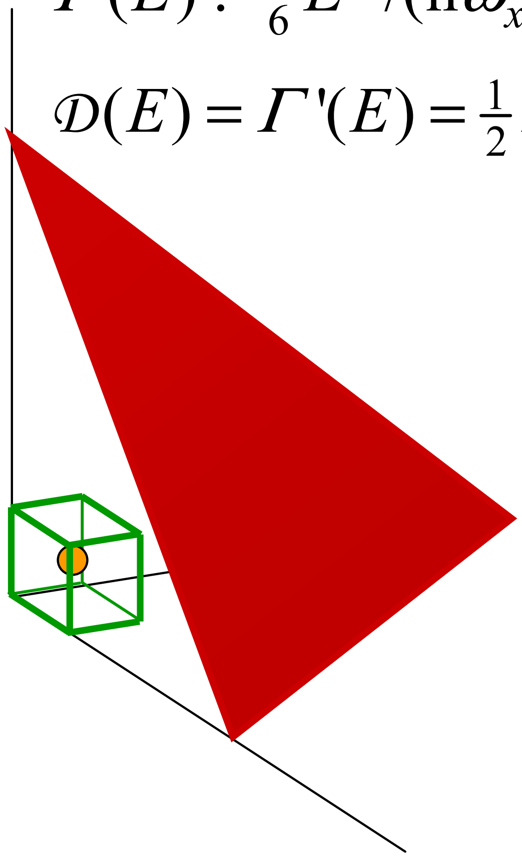
meaning wide trap potentials

Filling the trap with particles

3D

$$\Gamma(E) \sim \tfrac{1}{6} E^3 / (\hbar\omega_x \cdot \hbar\omega_y \cdot \hbar\omega_z)$$

$$\mathcal{D}(E) = \Gamma'(E) = \tfrac{1}{2} E^2 / (\hbar\omega_x \cdot \hbar\omega_y \cdot \hbar\omega_z)$$



Estimate for the transition temperature

particle number comparable with
the number of states in the thermal shell

$$N \approx \Gamma\left(k_B T\right)$$

2D $\quad T_c \approx \hbar\bar{\omega} / k_B \cdot N^{\frac{1}{2}} \qquad \bar{\omega} = (\omega_x \cdot \omega_y)^{\frac{1}{2}}$

3D $\quad T_c \approx \hbar\bar{\omega} / k_B \cdot N^{\frac{1}{3}} \qquad \bar{\omega} = (\omega_x \cdot \omega_y \cdot \omega_z)^{\frac{1}{3}}$

For 10^6 particles,

characteristic energy

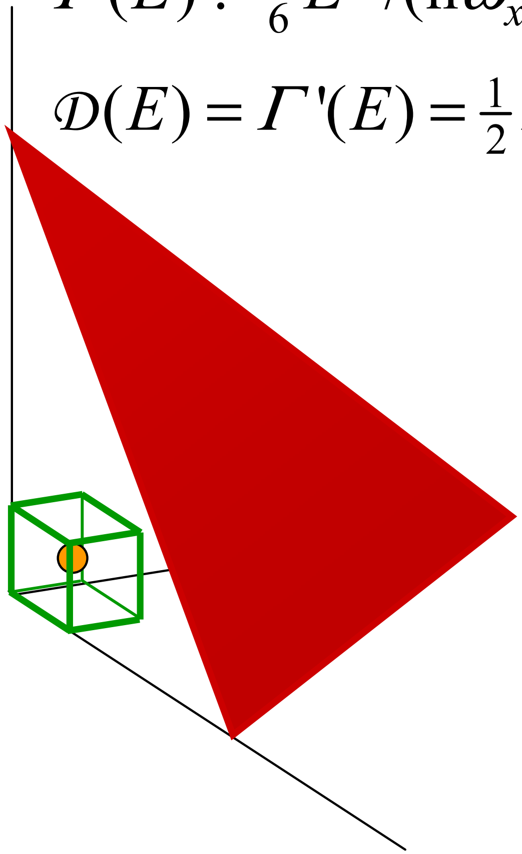
$$k_B T_c \approx 10^2 \hbar\bar{\omega}$$

Filling the trap with particles

3D

$$\Gamma(E) \sim \tfrac{1}{6}E^3/(\hbar\omega_x \cdot \hbar\omega_y \cdot \hbar\omega_z)$$

$$\mathcal{D}(E) = \Gamma'(E) = \tfrac{1}{2}E^2/(\hbar\omega_x \cdot \hbar\omega_y \cdot \hbar\omega_z)$$



Estimate for the transition temperature

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For 10^6 particles,

• characteristic energy

$k_B T_c \approx 10^2\,\hbar\bar{\omega} \gg \hbar\bar{\omega}$ important for therm. limit

Exact expressions for critical temperature etc.

The general expressions are the same like for the homogeneous gas.

Working with discrete levels, we have

$$N = \mathcal{N}(T,\mu) = \sum_j \langle n(\varepsilon_j) \rangle = \sum_j \frac{1}{\mathrm{e}^{\beta(\varepsilon_j-\mu)}-1}$$

and this can be used for numerics without exceptions.

In the approximate thermodynamic limit, the old equation holds, only the volume V does not enter as a factor:

$$N = \mathcal{N}(T,\mu) = \frac{1}{\mathrm{e}^{\beta(\varepsilon_0-\mu)}-1} + \cancel{V}\int_0^\infty \mathrm{d}\varepsilon \frac{1}{\mathrm{e}^{\beta(\varepsilon-\mu)}-1}\mathcal{D}(\varepsilon)$$

$\mu \to 0 \quad \text{for} \quad T \le T_C$

In 3D,

$$T_c = (\zeta(3))^{-\frac{1}{3}} \hbar\bar{\omega}/k_\mathrm{B} \cdot N^{\frac{1}{3}} = 0.94\hbar\bar{\omega}/k_\mathrm{B} \cdot N^{\frac{1}{3}}$$

$$N_\mathrm{BE} = N \cdot \left(1-(T/T_c)^3\right), \quad T < T_c$$

How good is the thermodynamic limit

1D illustration (almost doable)

$$N = \sum_j \frac{1}{e^{\beta(\hbar\omega \times j - \mu)} - 1} \overset{?}{=} \frac{1}{e^{-\beta\mu} - 1} + \int_0^\infty d\varepsilon \frac{1}{e^{\beta(\varepsilon - \mu)} - 1} \frac{1}{\hbar\omega}$$

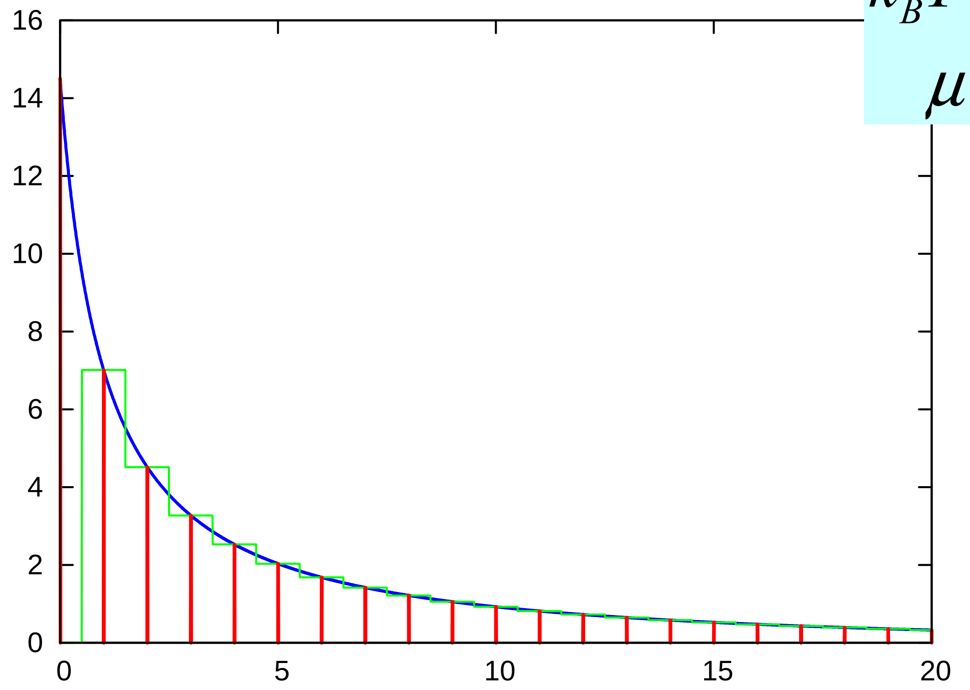
How good is the thermodynamic limit

1D illustration

$$N=\sum_j \frac{1}{e^{\beta(\hbar\omega\times j-\mu)}-1} \overset{?}{=} \frac{1}{e^{-\beta\mu}-1}+\int_0^\infty d\varepsilon \frac{1}{e^{\beta(\varepsilon-\mu)}-1}\frac{1}{\hbar\omega}$$

$$k_B T = 15\hbar\omega$$

$$\mu = -\hbar\omega$$



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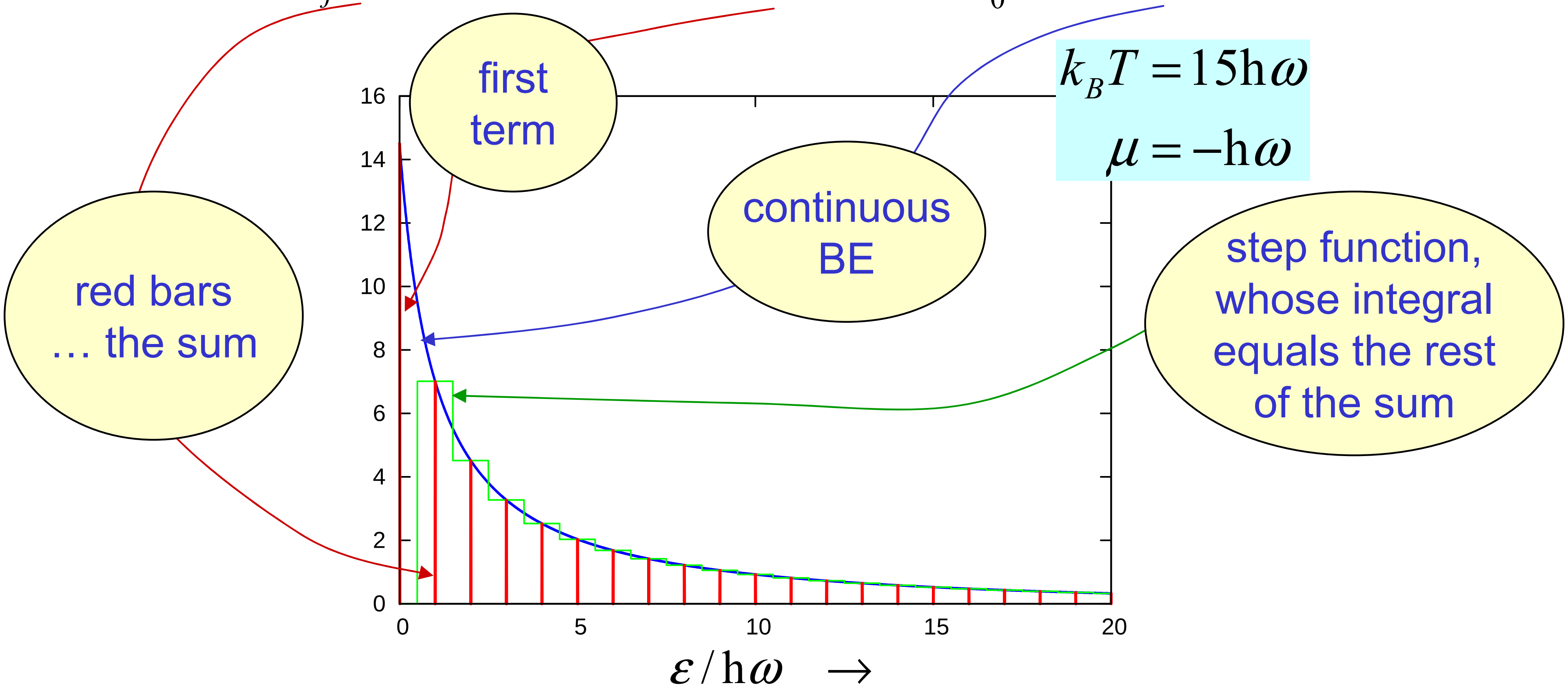

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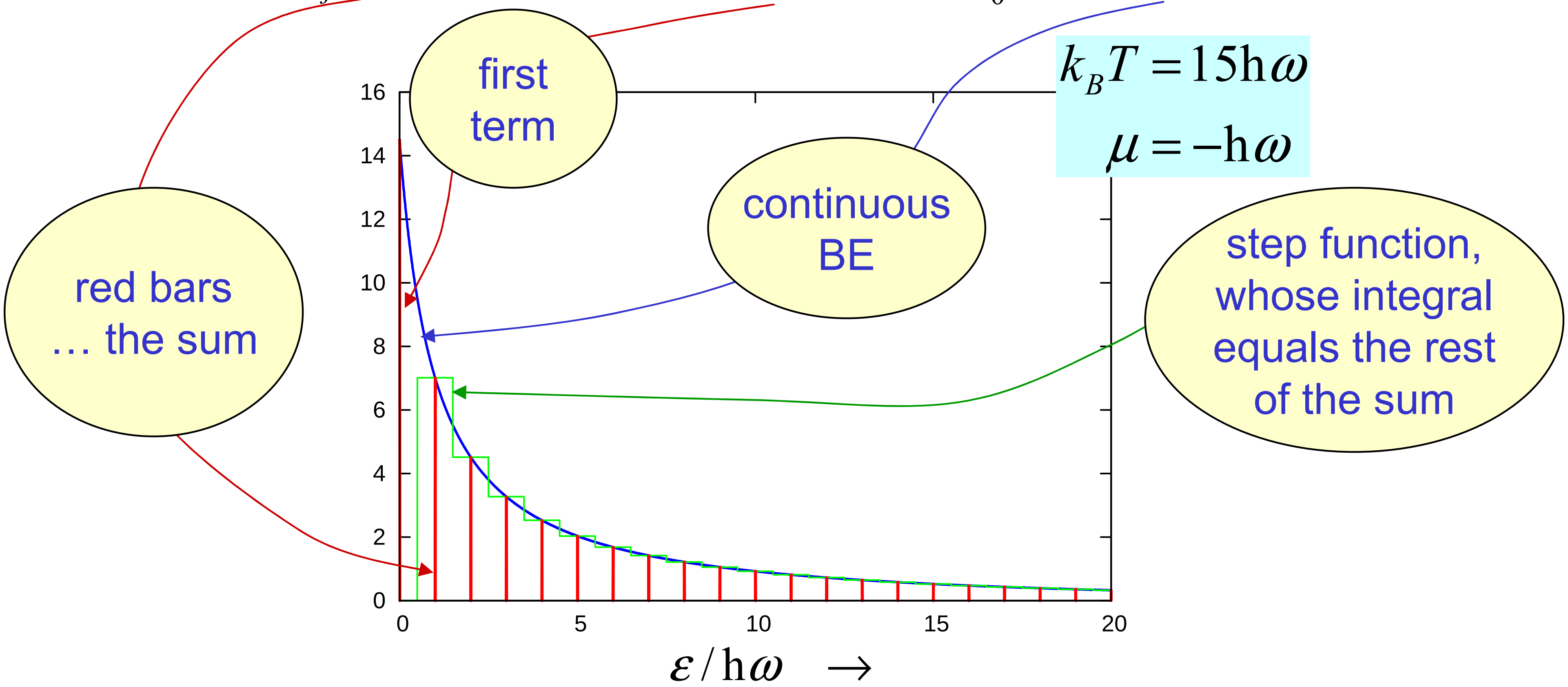
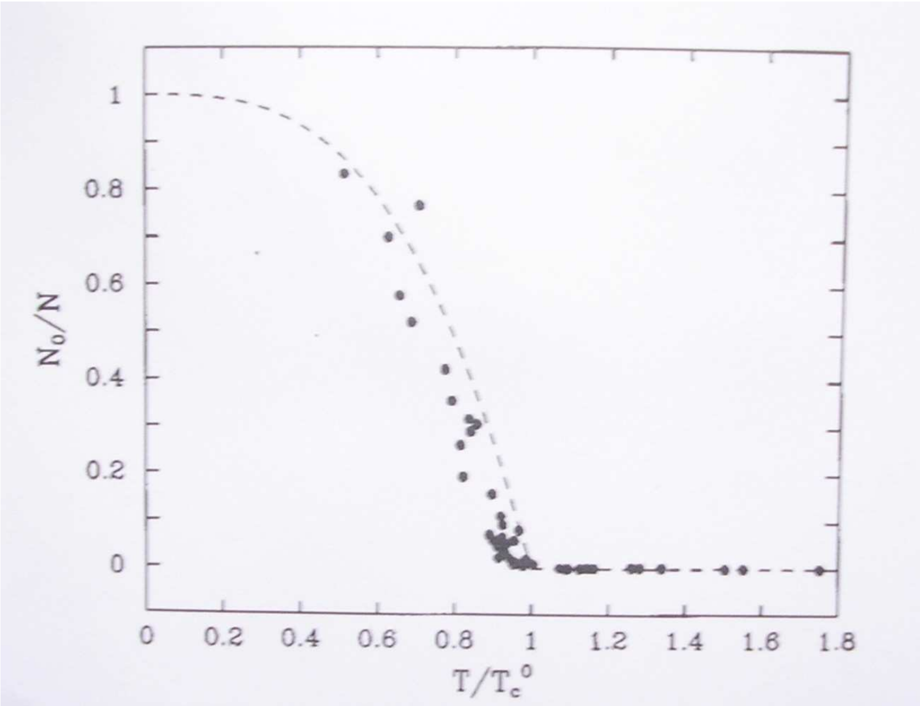


The quantitative criterion for the thermodynamic limit

$$\frac{k_B T_C}{\hbar\omega} \; ? \; 1$$

How sharp is the transition



These are experimental data

fitted by the formula

$$N_{\mathrm{BE}} = N \cdot \left(1 - (T / T_c)^3\right), \quad T < T_c$$

The rounding is apparent,

but not really an essential feature

The end