# Cold atoms

Lecture 5.

14th November, 2007

# Preliminary plan/reality in the fall term

| | | |
|---|---|---|
| Lecture 1<br>… | Something about everything (see next slide)<br>The textbook version of BEC in extended systems | Oct 4 |
| Lecture 2<br>… | thermodynamics, grand canonical ensemble, extended gas; atomic clouds in the traps – independent bosons. | Oct 11 |
| Lecture 3<br>… | atomic clouds in the traps – interactions, GP equation at zero temperature, variational prop., chem. potential | Oct 17 |
| Lecture 4<br>… | Infinite systems: Bogolyubov theory | Oct 31 |
| Lecture 5<br>… | ODLRO; BEC and symmetry breaking, coherent states | Nov 14 |

# *Recapitulation*

# Operators

## Additive observable

$$X = \sum X_j \quad \rightarrow \quad X = \iint \mathrm{d}^3\boldsymbol{r}\,\mathrm{d}^3\boldsymbol{r}'\,\psi^\dagger(\boldsymbol{r})\langle \boldsymbol{r}|X|\boldsymbol{r}'\rangle \psi(\boldsymbol{r}')$$

## General definition of the one particle density matrix – OPDM

$$\langle X \rangle = \left\langle \iint \mathrm{d}^3\boldsymbol{r}\,\mathrm{d}^3\boldsymbol{r}'\,\psi^\dagger(\boldsymbol{r})\langle \boldsymbol{r}|X|\boldsymbol{r}'\rangle \psi(\boldsymbol{r}') \right\rangle = \iint \mathrm{d}^3\boldsymbol{r}\,\mathrm{d}^3\boldsymbol{r}'\,\langle \boldsymbol{r}|X|\boldsymbol{r}'\rangle \underbrace{\langle \psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r}')\rangle}_{\langle \boldsymbol{r}'|\rho|\boldsymbol{r}\rangle}$$

$$\equiv \iint \mathrm{d}^3\boldsymbol{r}\,\mathrm{d}^3\boldsymbol{r}'\,\langle \boldsymbol{r}|X|\boldsymbol{r}'\rangle\langle \boldsymbol{r}'|\rho|\boldsymbol{r}\rangle = \mathrm{Tr}\,X\rho$$

## Particle number

$$N = \sum 1_{\mathrm{OP},j} \quad \rightarrow \quad N = \int \mathrm{d}^3\boldsymbol{r}\,\psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})$$

$$N = \sum a_\kappa^\dagger a_\kappa$$

## Momentum

$$\boldsymbol{P} = \sum \boldsymbol{p}_j \quad \rightarrow \quad \boldsymbol{P} = \int \mathrm{d}^3\boldsymbol{r}\,\psi^\dagger(\boldsymbol{r})\left(-\mathrm{i}\,\mathrm{h}\nabla\right)\psi(\boldsymbol{r})$$

$$\boldsymbol{P} = \sum \mathrm{h}\boldsymbol{k}\cdot a_\kappa^\dagger a_\kappa$$

## Particle density

$$n_{\mathrm{OP}}(\boldsymbol{r}) = \sum \delta(\boldsymbol{r}-\boldsymbol{r}_j) \quad \rightarrow \quad n_{\mathrm{OP}}(\boldsymbol{r}) = \psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})$$

$$n_{\mathrm{OP}}(\boldsymbol{r}) = \frac{1}{V}\sum_{\boldsymbol{q}} \mathrm{e}^{\mathrm{i}\boldsymbol{qr}} \sum_{\boldsymbol{k}} a_{\boldsymbol{k}-\boldsymbol{q}/2}^\dagger a_{\boldsymbol{k}+\boldsymbol{q}/2} \equiv \frac{1}{V}\sum_{\boldsymbol{q}} \mathrm{e}^{\mathrm{i}\boldsymbol{qr}}\, n_{\boldsymbol{q}}$$

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# Hamiltonian

$$H = \sum_a \frac{1}{2m} p_a^2 + V(\boldsymbol{r}_a) \quad \text{single-particle additive}$$

$$+ \frac{1}{2} \sum_{a \neq b} \sum U(\boldsymbol{r}_a - \boldsymbol{r}_b) \quad \text{two-particle binary}$$

$$\rightarrow \int \mathrm{d}^3\boldsymbol{r}\, \psi^\dagger(\boldsymbol{r}) \left( -\tfrac{\hbar^2}{2m} \Delta + V(\boldsymbol{r}) \right) \psi(\boldsymbol{r})$$

$$+ \tfrac{1}{2} \iint \mathrm{d}^3\boldsymbol{r}\, \mathrm{d}^3\boldsymbol{r}'\, \psi^\dagger(\boldsymbol{r}) \psi^\dagger(\boldsymbol{r}') U(\boldsymbol{r} - \boldsymbol{r}') \psi(\boldsymbol{r}') \psi(\boldsymbol{r})$$

# Hamiltonian

$$H = \sum_a \frac{1}{2m} p_a^2 + V(\boldsymbol{r}_a)$$ single-particle additive

acts in the $N$-particle sub-space $\mathbf{H}_N \subset \mathbf{F}$

$$+\frac{1}{2} \sum_{a \neq} \sum_{b} U(\boldsymbol{r}_a - \boldsymbol{r}_b)$$ two-particle binary

$$\to \int \mathrm{d}^3\boldsymbol{r}\, \psi^\dagger(\boldsymbol{r}) \Big( -\tfrac{\hbar^2}{2m} \Delta + V(\boldsymbol{r}) \Big) \psi(\boldsymbol{r})$$

acts in the whole Fock space $\mathbf{F}$

$$+\tfrac{1}{2} \iint \mathrm{d}^3\boldsymbol{r}\, \mathrm{d}^3\boldsymbol{r}'\, \psi^\dagger(\boldsymbol{r}) \psi^\dagger(\boldsymbol{r}') U(\boldsymbol{r} - \boldsymbol{r}') \psi(\boldsymbol{r}') \psi(\boldsymbol{r})$$

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$$+\tfrac{1}{2}\iint \mathrm{d}^3\boldsymbol{r}\,\mathrm{d}^3\boldsymbol{r}'\, \psi^\dagger(\boldsymbol{r})\psi^\dagger(\boldsymbol{r}')U(\boldsymbol{r}-\boldsymbol{r}')\psi(\boldsymbol{r}')\psi(\boldsymbol{r})$$

but K

Particle number conservation

$$[H, N] = 0$$

Equilibrium density operators and the ground state (*ergodic property*)

$$P = P(H), \quad [N, P] = 0$$

# On symmetries and conservation laws

# Hamiltonian of a homogeneous gas

$$H = \sum_a \frac{1}{2m} p_a^2 + V + \frac{1}{2} \sum_{a \neq b} \sum U(\boldsymbol{r}_a - \boldsymbol{r}_b), \qquad \boxed{V = \text{const.}}$$

$$= \int \mathrm{d}^3\boldsymbol{r}\, \psi^\dagger(\boldsymbol{r}) \left( -\tfrac{\hbar^2}{2m} \Delta + V \right) \psi(\boldsymbol{r}) + \tfrac{1}{2} \iint \mathrm{d}^3\boldsymbol{r}\, \mathrm{d}^3\boldsymbol{r}'\, \psi^\dagger(\boldsymbol{r}) \psi^\dagger(\boldsymbol{r}') U(\boldsymbol{r} - \boldsymbol{r}') \psi(\boldsymbol{r}') \psi(\boldsymbol{r})$$

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- conserves the particle number

$$[\mathcal{H}, \mathcal{N}] = 0$$

- conserves the total momentum

$$[\mathcal{H}, \mathcal{P}_{x,y,z}] = 0$$

# Summary: two symmetries compared

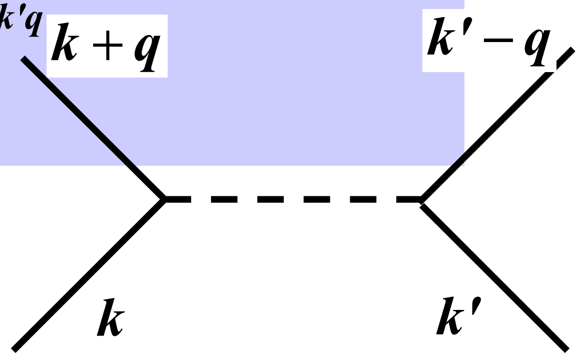
| Gauge invariance of the 1st kind | Translational invariance |
|---|---|
| universal for atomic systems | specific for homogeneous systems |
| $\mathcal{O}^\dagger(\varphi)\mathcal{H}\mathcal{O}(\varphi)=\mathcal{H},\quad \varphi\in\langle 0,2\pi\rangle$<br>$\mathcal{O}(\varphi)=\mathrm{e}^{-\mathrm{i}\mathcal{N}\varphi}$ | $\mathcal{T}^\dagger(\boldsymbol{a})\mathcal{H}\mathcal{T}(\boldsymbol{a})=\mathcal{H},\quad \boldsymbol{a}\in R_3$<br>$\mathcal{T}(\boldsymbol{a})=\mathrm{e}^{-\mathrm{i}\mathcal{P}\boldsymbol{a}/\hbar}$ |
| global phase shift of the wave function | global shift in the configuration space |
| $[\mathcal{H},\mathcal{N}]=0$<br>particle number conservation | $[\mathcal{H},\mathcal{P}_{x,y,z}]=0$<br>total momentum conservation |
| $[N,P]=0 \Leftrightarrow \mathrm{e}^{-\mathrm{i}N\varphi}P\,\mathrm{e}^{\mathrm{i}N\varphi}=P$<br>for equilibrium states | $[\mathcal{P},P]=0 \Leftrightarrow \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\mathcal{P}\boldsymbol{a}}P\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\mathcal{P}\boldsymbol{a}}=P$<br>for equilibrium states |
| selection rules<br>$\langle\psi\psi\ldots\psi^\dagger\rangle=0$<br>unless there are as many $\psi^\dagger$ as $\psi$ . | selection rules<br>$\langle a_{\boldsymbol{k}}a_{\boldsymbol{k}'}\ldots a^\dagger_{\boldsymbol{k}''}\rangle=0$<br>unless the total momentum transfer $-\boldsymbol{k}-\boldsymbol{k}'\ldots+\boldsymbol{k}''=0$ is zero |

# Hamiltonian of the homogeneous gas

In the momentum representation

$$H = \sum \frac{\hbar^2}{2m} \boldsymbol{k}^2 a_{\boldsymbol{k}}^{\dagger} a_{\boldsymbol{k}} + \frac{1}{2} V^{-1} \sum_{\boldsymbol{k}\boldsymbol{k}'\boldsymbol{q}} U_{\boldsymbol{q}} a_{\boldsymbol{k}+\boldsymbol{q}}^{\dagger} a_{\boldsymbol{k}'-\boldsymbol{q}}^{\dagger} a_{\boldsymbol{k}'} a_{\boldsymbol{k}}$$

$$U_{\boldsymbol{k}} = \int \mathrm{d}^3 \boldsymbol{r}\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\boldsymbol{r}} U(\boldsymbol{r})$$




Momentum conservation

$$(\boldsymbol{k}+\boldsymbol{q}) + (\boldsymbol{k}'-\boldsymbol{q}) - \boldsymbol{k} - \boldsymbol{k}' = 0$$

Particle number conservation

$$a^{\dagger} a^{\dagger} a \; a$$

# Hamiltonian of the homogeneous gas

In the momentum representation

For the Fermi pseudopotential

$$U_{\boldsymbol{q}} = U_0 \equiv U \; (= g)$$

$$H = \sum \frac{\hbar^2}{2m} \boldsymbol{k}^2 a^{\dagger}_{\boldsymbol{k}} a_{\boldsymbol{k}} + \tfrac{1}{2} V^{-1} \sum_{\boldsymbol{k}\boldsymbol{k}'\boldsymbol{q}} U_{\boldsymbol{q}} a^{\dagger}_{\boldsymbol{k}+\boldsymbol{q}} a^{\dagger}_{\boldsymbol{k}'-\boldsymbol{q}} a_{\boldsymbol{k}'} a_{\boldsymbol{k}}$$

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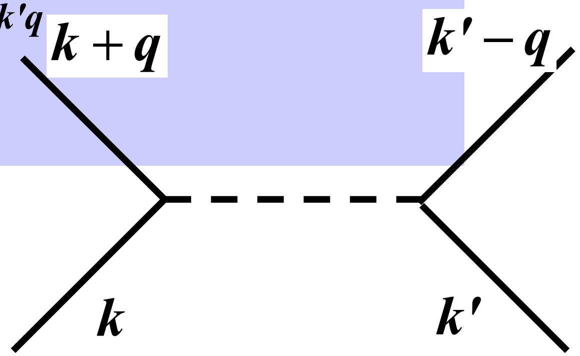


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Particle number conservation

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E5F
E55555F

# Bogolyubov method

Originally, intended and conceived for extended (rather *infinite*) homogeneous system.

Reflects the 'Paradoxien der Unendlichen'

# Basic idea

**Bogolyubov method**
is devised for boson quantum fluids with weak interactions – at *T*=0 now

no interaction

$$g = 0$$

$$N = N_{\mathrm{BE}} = \left\langle a_0^\dagger a_0 \right\rangle ?\ \ 1$$

weak interaction

$$g \neq 0$$

$$N = N_{\mathrm{BE}} + \sum_{\boldsymbol{k} \neq 0} \left\langle a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k}} \right\rangle \approx N_{\mathrm{BE}}\ ?\ \ 1$$

The condensate dominates,
but some particles are kicked out
**by the interaction** (*not thermally*)

# Basic idea

**Bogolyubov method**
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| no interaction | weak interaction |
| --- | --- |
| $g = 0$ | $g \neq 0$ |
| $N = N_{\mathrm{BE}} = \left\langle a_0^\dagger a_0 \right\rangle ?\ 1$ | $N = N_{\mathrm{BE}} + \sum_{\boldsymbol{k} \neq 0} \left\langle a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k}} \right\rangle \approx N_{\mathrm{BE}}\ ?\ 1$ |

The condensate dominates,
but some particles are kicked out
**by the interaction** (*not thermally*)

Strange idea introduced by Bogolyubov

$$N_0 = \left\langle a_0^\dagger a_0 \right\rangle ?\ 1 \Rightarrow \left\langle a_0^\dagger a_0 \right\rangle ?\ a_0^\dagger a_0 - a_0 a_0^\dagger = 1 \Rightarrow \text{like } c\text{-numbers}$$

$$\boxed{a_0 \approx \sqrt{N_0}, \quad a_0^\dagger \approx \sqrt{N_0}}$$

$$N = N_0 + \sum_{\boldsymbol{k} \neq 0} a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k}} \quad \text{... mixture of } c\text{-numbers and } q\text{-numbers}$$

# Approximate Hamiltonian

Keep at most two particles out of the condensate …. use

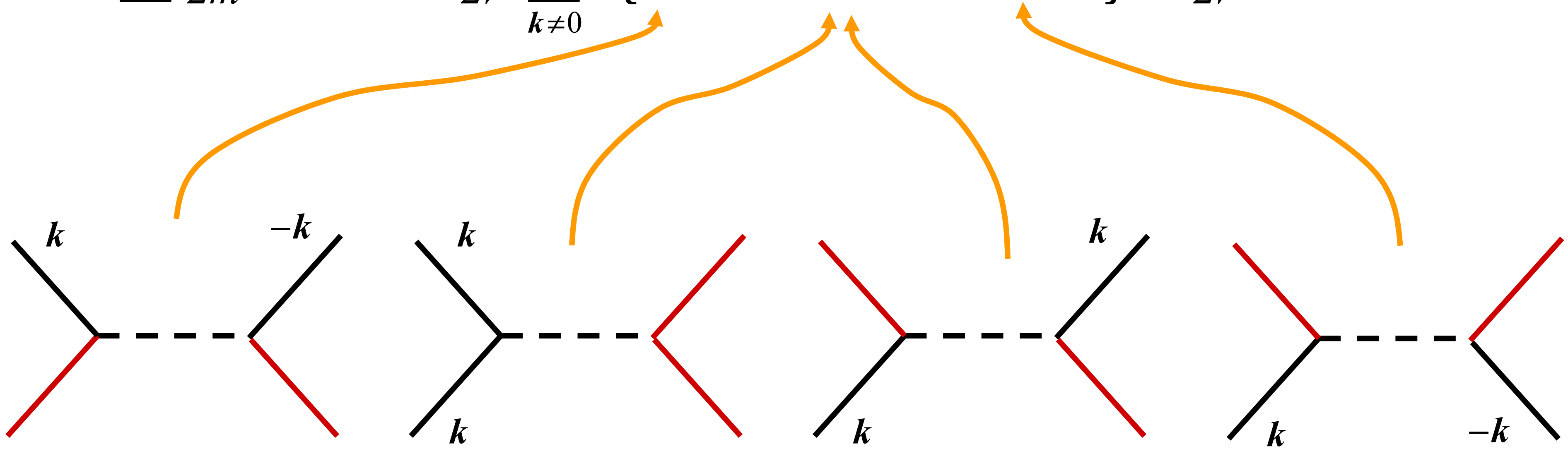
$$H = \sum \frac{\hbar^2}{2m} \boldsymbol{k}^2 a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k}} + \frac{1}{2} V^{-1} \sum_{\boldsymbol{k}\boldsymbol{k}'\boldsymbol{q}} U \; a^\dagger_{\boldsymbol{k}+\boldsymbol{q}} a^\dagger_{\boldsymbol{k}'-\boldsymbol{q}} a_{\boldsymbol{k}'} a_{\boldsymbol{k}}$$

$$a_0 \approx \sqrt{N_0}, \quad a_0^\dagger \approx \sqrt{N_0}$$

$$\to \sum \frac{\hbar^2}{2m} \boldsymbol{k}^2 a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k}} + \frac{UN_0}{2V} \sum_{\boldsymbol{k}\neq 0} \left\{ a_{\boldsymbol{k}}^\dagger a_{-\boldsymbol{k}}^\dagger + 4 a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k}} + a_{\boldsymbol{k}} a_{-\boldsymbol{k}} \right\} + \frac{UN_0^2}{2V}$$

$$N_0 = N - \sum_{\boldsymbol{k}\neq 0} a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k}}$$

$$= \sum \frac{\hbar^2}{2m} \boldsymbol{k}^2 a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k}} + \frac{UN}{2V} \sum_{\boldsymbol{k}\neq 0} \left\{ a_{\boldsymbol{k}}^\dagger a_{-\boldsymbol{k}}^\dagger + 2 a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k}} + a_{\boldsymbol{k}} a_{-\boldsymbol{k}} \right\} + \frac{UN^2}{2V}$$




condensate particle

# Bogolyubov transformation

Last rearrangement

$$H = \frac{1}{2}\sum \underbrace{\left(\frac{\hbar^2}{2m}\boldsymbol{k}^2 + gn\right)}_{\text{mean field}} \left\{ a_{\boldsymbol{k}}^{\dagger} a_{\boldsymbol{k}} + a_{-\boldsymbol{k}}^{\dagger} a_{-\boldsymbol{k}} \right\} + \frac{gn}{2} \sum_{\boldsymbol{k}} \underbrace{\left\{ a_{\boldsymbol{k}}^{\dagger} a_{-\boldsymbol{k}}^{\dagger} + a_{\boldsymbol{k}} a_{-\boldsymbol{k}} \right\}}_{\text{anomalous}} + \frac{gN^2}{2V}$$

**Conservation properties: momentum … YES, particle number … NO**

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NEW FIELD OPERATORS  notice momentum conservation!!

$$b_{\boldsymbol{k}} = u_{\boldsymbol{k}} a_{\boldsymbol{k}} + v_{\boldsymbol{k}} a_{-\boldsymbol{k}}^\dagger$$
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**requirements**

❶ New operators should satisfy the boson commutation rules

$$\left[b_{\boldsymbol{k}}, b_{\boldsymbol{k}'}^\dagger\right] = \delta_{\boldsymbol{k}\boldsymbol{k}'}, \quad \left[b_{\boldsymbol{k}}, b_{\boldsymbol{k}'}\right] = 0, \quad \left[b_{\boldsymbol{k}}^\dagger, b_{\boldsymbol{k}'}^\dagger\right] = 0$$

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$$\begin{aligned} b_{\boldsymbol{k}} &= u_{\boldsymbol{k}} a_{\boldsymbol{k}} + v_{\boldsymbol{k}} a_{-\boldsymbol{k}}^\dagger \\ b_{-\boldsymbol{k}}^\dagger &= v_{\boldsymbol{k}} a_{\boldsymbol{k}} + u_{\boldsymbol{k}} a_{-\boldsymbol{k}}^\dagger \end{aligned} \quad \Bigg| \quad \begin{aligned} a_{\boldsymbol{k}} &= u_{\boldsymbol{k}} b_{\boldsymbol{k}} - v_{\boldsymbol{k}} b_{-\boldsymbol{k}}^\dagger \\ a_{-\boldsymbol{k}}^\dagger &= -v_{\boldsymbol{k}} b_{\boldsymbol{k}} + u_{\boldsymbol{k}} b_{-\boldsymbol{k}}^\dagger \end{aligned}$$

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$$\begin{array}{l|l} b_{\boldsymbol{k}} = u_{\boldsymbol{k}} a_{\boldsymbol{k}} + v_{\boldsymbol{k}} a_{-\boldsymbol{k}}^\dagger & a_{\boldsymbol{k}} = u_{\boldsymbol{k}} b_{\boldsymbol{k}} - v_{\boldsymbol{k}} b_{-\boldsymbol{k}}^\dagger \\ b_{-\boldsymbol{k}}^\dagger = v_{\boldsymbol{k}} a_{\boldsymbol{k}} + u_{\boldsymbol{k}} a_{-\boldsymbol{k}}^\dagger & a_{-\boldsymbol{k}}^\dagger = -v_{\boldsymbol{k}} b_{\boldsymbol{k}} + u_{\boldsymbol{k}} b_{-\boldsymbol{k}}^\dagger \end{array}$$

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$$\textbf{iff} \quad u_{\boldsymbol{k}}^2 - v_{\boldsymbol{k}}^2 = 1$$

❷ In terms of the new operators, the anomalous terms in the Hamiltonian have to vanish

# Bogolyubov transformation

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$$H = \frac{1}{2}\sum\left(\frac{\hbar^2}{2m}\boldsymbol{k}^2 + gn\right)\left\{a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k}} + a_{-\boldsymbol{k}}^\dagger a_{-\boldsymbol{k}}\right\} + \frac{gn}{2}\sum_{\boldsymbol{k}}\left\{a_{\boldsymbol{k}}^\dagger a_{-\boldsymbol{k}}^\dagger + a_{\boldsymbol{k}} a_{-\boldsymbol{k}}\right\} + \frac{gN^2}{2V}$$

$$= \sum\left(\frac{\hbar^2}{2m}\boldsymbol{k}^2 + gn\right)\left\{u_{\boldsymbol{k}}^2 b_{\boldsymbol{k}}^\dagger b_{\boldsymbol{k}} + v_{\boldsymbol{k}}^2 b_{\boldsymbol{k}} b_{\boldsymbol{k}}^\dagger + u_{\boldsymbol{k}} v_{\boldsymbol{k}}(b_{\boldsymbol{k}}^\dagger b_{-\boldsymbol{k}}^\dagger + b_{\boldsymbol{k}} b_{-\boldsymbol{k}})\right\}$$

$$+ \frac{gn}{2}\sum_{\boldsymbol{k}} 2u_{\boldsymbol{k}} v_{\boldsymbol{k}}\left\{(b_{\boldsymbol{k}}^\dagger b_{\boldsymbol{k}} + b_{\boldsymbol{k}} b_{\boldsymbol{k}}^\dagger) + (u_{\boldsymbol{k}}^2 + v_{\boldsymbol{k}}^2)(b_{\boldsymbol{k}}^\dagger b_{-\boldsymbol{k}}^\dagger + b_{\boldsymbol{k}} b_{-\boldsymbol{k}})\right\}$$

$$+ \frac{gN^2}{2V}$$

$$\Rightarrow \left(\frac{\hbar^2}{2m}\boldsymbol{k}^2 + gn\right)u_{\boldsymbol{k}} v_{\boldsymbol{k}} + \frac{gn}{2}(u_{\boldsymbol{k}}^2 + v_{\boldsymbol{k}}^2) = 0$$

$$u_{\boldsymbol{k}}^2 - v_{\boldsymbol{k}}^2 = 1$$

$$u_{\boldsymbol{k}}^2 = \left(\frac{\hbar^2}{2m}\boldsymbol{k}^2 + gn + \varepsilon(\boldsymbol{k})\right)/2\varepsilon(\boldsymbol{k}) \qquad v_{\boldsymbol{k}}^2 = \left(\frac{\hbar^2}{2m}\boldsymbol{k}^2 + gn - \varepsilon(\boldsymbol{k})\right)/2\varepsilon(\boldsymbol{k})$$

$$\varepsilon(\boldsymbol{k}) = \sqrt{\left(\frac{\hbar^2}{2m}\boldsymbol{k}^2 + gn\right)^2 - (gn)^2}$$

# Bogolyubov transformation – result

Without quoting the transformation matrix

$$H = \tfrac{1}{2}\sum \underbrace{\varepsilon(\boldsymbol{k})\left\{b_{\boldsymbol{k}}^{\dagger} b_{\boldsymbol{k}} + b_{-\boldsymbol{k}}^{\dagger} b_{-\boldsymbol{k}}\right\}}_{\text{independent quasiparticles}} + \underbrace{\frac{gN^2}{2V} + \text{higher order constant}}_{\text{ground state energy } E}$$

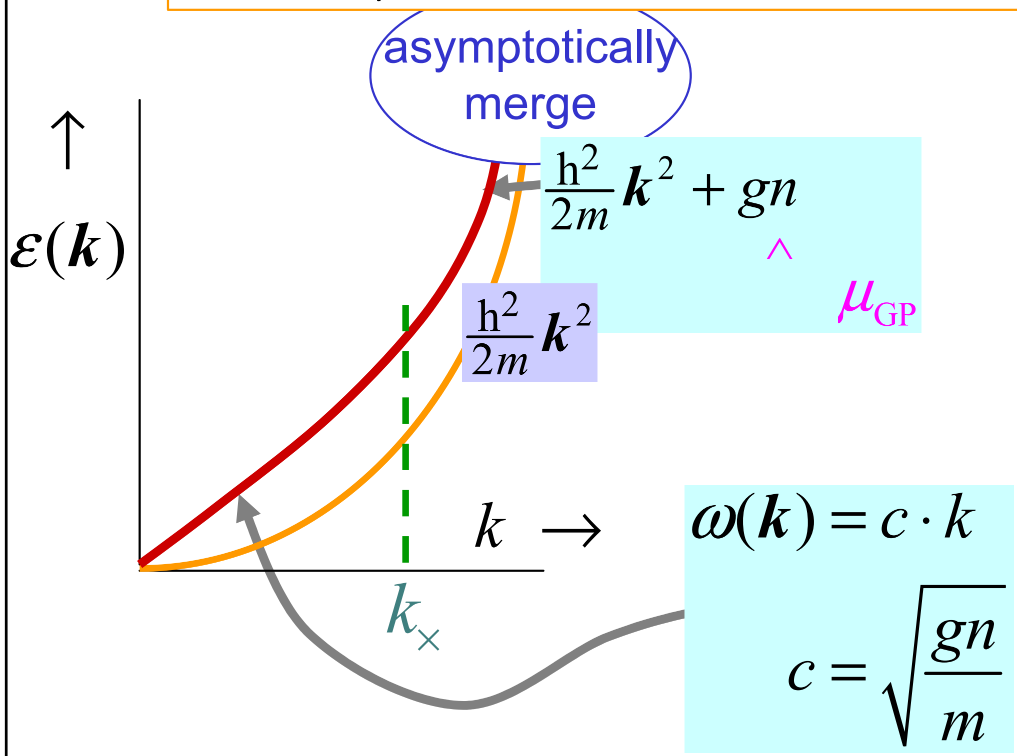
$$\varepsilon(\boldsymbol{k}) = \sqrt{\left(\frac{\hbar^2}{2m}\boldsymbol{k}^2 + gn\right)^2 - (gn)^2} = \sqrt{\frac{\hbar^2}{2m}\boldsymbol{k}^2}\sqrt{\frac{\hbar^2}{2m}\boldsymbol{k}^2 + 2gn}$$

# Bogolyubov transformation – result

Without quoting the transformation matrix

$$H = \tfrac{1}{2}\sum \underbrace{\varepsilon(\boldsymbol{k})\, b_k^\dagger b_k}_{\text{ind. quasi-particles}} + \underbrace{\frac{gN^2}{2V} + \text{higher order constant}}_{\text{ground state energy } E}$$

$$\varepsilon(\boldsymbol{k}) = \sqrt{\left(\frac{\hbar^2}{2m}\boldsymbol{k}^2 + gn\right)^2 - (gn)^2} = \sqrt{\frac{\hbar^2}{2m}\boldsymbol{k}^2}\sqrt{\frac{\hbar^2}{2m}\boldsymbol{k}^2 + 2gn}$$




**high energy region**

quasi-particles are nearly just particles

**sound region**

quasi-particles are collective excitations

**cross-over**

$$k_\times = \sqrt{\frac{4mgn}{\hbar^2}}$$

defines scale for $k$

# More about the sound part of the dispersion law

$$\omega(\boldsymbol{k}) = c \cdot k$$

$$c = \sqrt{\frac{gn}{m}}$$

- Entirely dependent on the interactions, both the magnitude of the velocity and the linear frequency range determined by $g$
- Can be shown to really be a sound:

a
$$c = \text{"}\sqrt{\frac{\kappa}{\rho}}\text{"} = \sqrt{\frac{V\partial_{VV}E}{m \cdot n}}\,, \qquad E = \frac{gN^2}{2V} + \ldots$$

b
$$b_{\boldsymbol{k}}^{\dagger} = u_{\boldsymbol{k}} a_{\boldsymbol{k}}^{\dagger} + v_{\boldsymbol{k}} a_{-\boldsymbol{k}} \xrightarrow{k \to 0} u_{\boldsymbol{k}} (a_{\boldsymbol{k}}^{\dagger} + a_{-\boldsymbol{k}})$$

$$n_{\boldsymbol{k}} = \sum_{\boldsymbol{q}} a_{\boldsymbol{q}-\boldsymbol{k}/2}^{\dagger} a_{\boldsymbol{q}+\boldsymbol{k}/2} \approx (a_{\boldsymbol{k}}^{\dagger} + a_{-\boldsymbol{k}})$$

- Even a weakly interacting gas exhibits superfluidity; the ideal gas does not.
- The phonons are actually Goldstone modes corresponding to a broken symmetry
- The dispersion law has no **roton** region, contrary to the reality in $^4$He
- The dispersion law bends upwards $\Rightarrow$ quasi-particles are unstable, can decay

# Particles and quasi-particles

At zero temperature, there are no quasi-particles, just the condensate.

Things are different with the true particles. Not all particles are in the condensate, but they are not thermally agitated in an incoherent way, they are a part of the fully coherent ground state

$$\left\langle a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}} \right\rangle = \left\langle \left( -v_{\mathbf{k}} b_{\mathbf{k}} + u_{\mathbf{k}} b_{-\mathbf{k}}^{\dagger} \right) \left( u_{\mathbf{k}} b_{-\mathbf{k}} - v_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} \right) \right\rangle = v_{\mathbf{k}}^{2} \neq 0$$

The total fraction of particles outside of the condensate is

$$\frac{N - N_0}{N} \approx \frac{8}{3\sqrt{\pi}} a_s^{3/2} n^{1/2}$$

$$\sqrt{a_s^3 n}$$

square root of the gas parameter is the expansion variable

# What is the Bogolyubov approximation about

The results for various quantities are

$$N_0 \approx N \times \left(1 - \frac{8}{3\sqrt{\pi}} a_s^{3/2} n^{1/2}\right)$$

$$E \approx \frac{gn}{2} N \times \left(1 + \frac{128}{15\sqrt{\pi}} a_s^{3/2} n^{1/2}\right)$$

$$\mu \approx gn \times \left(1 + \frac{32}{3\sqrt{\pi}} a_s^{3/2} n^{1/2}\right)$$

square root of the gas parameter $\sqrt{a_s^3 n}$ is the expansion variable

general pattern

$$[\mathrm{BG}] \approx [\mathrm{GP}] \times \left(1 + \frac{\mathrm{L}}{\mathrm{L}\ \sqrt{\pi}} a_s^{3/2} n^{1/2}\right)$$

The Bogolyubov theory is the lowest order correction to the mean field (Gross-Pitaevskii) approximation

It provides thus the criterion for the validity of the mean field results

It is the simplest genuine field theory for quantum liquids with a condensate

# Trying to understand the Bogolyubov method

# Notes to the contents of the Bogolyubov theory

- The first consistent microsopic theory of the ground state and elementary excitations (quasi-particles) for a quantum liquid (1947)
- The quantum condensate turns into the classical order parameter in the thermodynamic limit $\mathcal{N} \to \infty, \quad \mathcal{V} \to \infty, \quad \mathcal{N} / \mathcal{V} = n = \text{ const.}$
- The Bogolyubov transformation became one of the standard technical means for treatment of "anomalous terms" in many body Hamiltonians (…de Gennes)
- Central point of the theory is the assumption

$$\boxed{a_0 \approx \sqrt{N_0}, \quad a_0^\dagger \approx \sqrt{N_0}}$$

- Its introduction and justification intuitive, surprisingly lacks mathematical rigor. Two related problems:

lowering operator $\longleftarrow$ **?** $\longrightarrow$ gauge symmetry, s. rule

$$a_0 \left|G, N\right\rangle \in \mathrm{H}_{N-1} \qquad \left\langle G, N \middle| a_0 \middle| G, N \right\rangle = \sqrt{N_0} \qquad \left\langle a_0 \right\rangle = 0$$

- Additional assumptions: something of a crutch/bar to study of finite systems
  - homogeneous system
  - infinite system

  *Infinity as a problem: philosophical, mathematical, physical*

# *What next ???*

- Off-diagonal long range order and the Bogolyubov ground state
- Coherent state as the GP vacuum
- Spontaneous symmetry breaking

# Off-Diagonal Long Range Order

Analysis of BEC on the one-particle level

ODLRO as a measure of coherence in the system

# Coherence in BEC: OPDM for non-interacting bosons

### ***O**ff-**D**iagonal Long Range Order*

Without field-theoretical means, the coherence of the condensate may be studied using the **one-particle density matrix.**

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**Definition of OPDM** for **non**-interacting particles: Take an additive observable, like local density, or current density. Its average value for the whole assembly of atoms in a given equilibrium state $\left|\{n_\alpha\}\right\rangle$:

$$\left\langle X \right\rangle = \sum_\alpha \left\langle \alpha \middle| X \middle| \alpha \right\rangle \left\langle n_\alpha \right\rangle \quad \text{double average, quantum and thermal}$$

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$$= \mathrm{Tr}\,\rho X \qquad\qquad \rho = \sum_\alpha |\alpha\rangle\langle n_\alpha\rangle\langle\alpha|$$

# OPDM for homogeneous systems

In coordinate representation

$$\rho(\boldsymbol{r},\boldsymbol{r}') = \left\langle \boldsymbol{r} \middle| \sum_{\boldsymbol{k}} \middle| \boldsymbol{k} \right\rangle \left\langle n_{\boldsymbol{k}} \right\rangle \left\langle \boldsymbol{k} \middle| \boldsymbol{r}' \right\rangle = \sum_{\boldsymbol{k}} \left\langle \boldsymbol{r} \middle| \boldsymbol{k} \right\rangle \left\langle n_{\boldsymbol{k}} \right\rangle \left\langle \boldsymbol{k} \middle| \boldsymbol{r}' \right\rangle$$

$$= \frac{1}{V} \sum_{\boldsymbol{k}} \mathrm{e}^{\mathrm{i}\boldsymbol{k}(\boldsymbol{r}-\boldsymbol{r}')} \left\langle n_{\boldsymbol{k}} \right\rangle$$

- depends only on the relative position (transl. invariance)
- Fourier transform of the occupation numbers
- isotropic … provided thermodynamic limit is allowed
- in systems without condensate, the *momentum distribution* is smooth and the density matrix has a finite range.

CONDENSATE    lowest orbital with $\boldsymbol{k}_0$

# OPDM for homogeneous systems: ODLRO

CONDENSATE   lowest orbital with $\boldsymbol{k}_0 = O(V^{-\frac{1}{3}}) \approx 0$

$$\rho(\boldsymbol{r}-\boldsymbol{r}') = \underbrace{\frac{1}{V}\mathrm{e}^{\mathrm{i}\boldsymbol{k}_0(\boldsymbol{r}-\boldsymbol{r}')}\langle n_0\rangle}_{\text{coherent across the sample}} + \underbrace{\frac{1}{V}\sum_{\boldsymbol{k}\neq\boldsymbol{k}_0}\mathrm{e}^{\mathrm{i}\boldsymbol{k}(\boldsymbol{r}-\boldsymbol{r}')}\langle n_{\boldsymbol{k}}\rangle}_{FT\text{ of a smooth function has a finite range}}$$

$$\equiv \rho_{\mathrm{BE}}(\boldsymbol{r}-\boldsymbol{r}') \quad + \rho_{\mathrm{G}}(\boldsymbol{r}-\boldsymbol{r}')$$

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DIAGONAL ELEMENT $\boldsymbol{r} = \boldsymbol{r}'$

$$\rho(\boldsymbol{0}) = \rho_{\mathrm{BE}}(\boldsymbol{0}) \qquad + \rho_{\mathrm{G}}(\boldsymbol{0})$$

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DISTANT OFF-DIAGONAL ELEMENT $|\boldsymbol{r}-\boldsymbol{r}'| \to \infty$

$$\rho_{\mathrm{BE}}(\boldsymbol{r}-\boldsymbol{r}') \xrightarrow{|\boldsymbol{r}-\boldsymbol{r}'|\to\infty} n_{\mathrm{BE}}$$

$$\rho_{\mathrm{G}}(\boldsymbol{r}-\boldsymbol{r}') \xrightarrow{|\boldsymbol{r}-\boldsymbol{r}'|\to\infty} 0$$

$$\rho(\boldsymbol{r}-\boldsymbol{r}') \xrightarrow{|\boldsymbol{r}-\boldsymbol{r}'|\to\infty} n_{\mathrm{BE}}$$

Off-Diagonal Long Range Order

ODLRO

# From OPDM towards the macroscopic wave function

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$$= \underbrace{\Psi(\boldsymbol{r})\Psi^*(\boldsymbol{r}')}_{\text{dyadic}} + \frac{1}{V}\sum_{\boldsymbol{k}\neq\boldsymbol{k}_0}\mathrm{e}^{\mathrm{i}\boldsymbol{k}(\boldsymbol{r}-\boldsymbol{r}')}\langle n_{\boldsymbol{k}}\rangle$$

MACROSCOPIC WAVE FUNCTION

$\Psi(\boldsymbol{r}) = \sqrt{n_{BE}} \cdot \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_0\boldsymbol{r}+\varphi)}, \quad \varphi \ldots$ an arbitrary phase

- expresses ODLRO in the density matrix
- measures the condensate density
- appears like a pure state in the density matrix, but macroscopic
- expresses the notion that the condensate atoms are in the same state
- is the order parameter for the BEC transition

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MACROSCOPIC WAVE FUNCTION

$\Psi(\boldsymbol{r}) = \sqrt{n_{BE}}\cdot\mathrm{e}^{\mathrm{i}(\boldsymbol{k}_0\boldsymbol{r}+\varphi)}$,   $\varphi$ K   an arbitrary phase     ? **why bother?**

- expresses ODLRO in the density matrix ✓
- measures the condensate density ✓
- appears like a pure state in the density matrix, but macroscopic ✓
- expresses the notion that the condensate atoms are in the same state ? **how?**
- is the order parameter for the BEC transition ? **what is it?**

# ODLRO for interacting bosons

Basic expressions for the OPDM for a homogeneous system

$$\langle \boldsymbol{r} | \rho | \boldsymbol{r'} \rangle = \langle \psi^\dagger(\boldsymbol{r'}) \psi(\boldsymbol{r}) \rangle = V^{-1} \left\langle \sum \mathrm{e}^{-\mathrm{i}\boldsymbol{k'r'}} a^\dagger_{\boldsymbol{k'}} \cdot \sum a_{\boldsymbol{k}} \, \mathrm{e}^{\mathrm{i}\boldsymbol{kr}} \right\rangle$$ by definition

$$= V^{-1} \sum_{\boldsymbol{k},\boldsymbol{k}'} \mathrm{e}^{\mathrm{i}\boldsymbol{kr}} \, \mathrm{e}^{-\mathrm{i}\boldsymbol{k'r'}} \left\langle a^\dagger_{\boldsymbol{k'}} a_{\boldsymbol{k}} \right\rangle = V^{-1} \sum_{\boldsymbol{k},\boldsymbol{k}'} \mathrm{e}^{\mathrm{i}\boldsymbol{kr}} \, \mathrm{e}^{-\mathrm{i}\boldsymbol{k'r'}} \left\langle a^\dagger_{\boldsymbol{k}} a_{\boldsymbol{k}} \right\rangle \delta_{\boldsymbol{kk'}}$$ transl. invariance

Off-diagonal long range order

# ODLRO for interacting bosons

One particle density matrix

## Basic expressions for the OPDM for a homogeneous system

$$\langle \boldsymbol{r} | \rho | \boldsymbol{r}' \rangle = \langle \psi^\dagger(\boldsymbol{r}')\psi(\boldsymbol{r}) \rangle = V^{-1} \langle \sum \mathrm{e}^{-\mathrm{i}\boldsymbol{k}'\boldsymbol{r}'} a^\dagger_{\boldsymbol{k}'} \cdot \sum a_{\boldsymbol{k}}\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\boldsymbol{r}} \rangle \qquad \text{by definition}$$

$$= V^{-1} \sum_{\boldsymbol{k},\boldsymbol{k}'} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\boldsymbol{r}}\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}'\boldsymbol{r}'} \langle a^\dagger_{\boldsymbol{k}'} a_{\boldsymbol{k}} \rangle = V^{-1} \sum_{\boldsymbol{k},\boldsymbol{k}'} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\boldsymbol{r}}\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}'\boldsymbol{r}'} \langle a^\dagger_{\boldsymbol{k}} a_{\boldsymbol{k}} \rangle \delta_{\boldsymbol{k}\boldsymbol{k}'} \qquad \text{transl. invariance}$$

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$$\rho(\boldsymbol{r},\boldsymbol{r}') = V^{-1} \sum_{\boldsymbol{k}} \mathrm{e}^{\mathrm{i}\boldsymbol{k}(\boldsymbol{r}-\boldsymbol{r}')} \left\langle n_{\boldsymbol{k}} \right\rangle$$

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just like for non-interacting bosons

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transl. invariance

General expression for the one particle density matrix with condensate

$$\rho(\boldsymbol{r},\boldsymbol{r}') = V^{-1} \sum_{\boldsymbol{k}} \mathrm{e}^{\mathrm{i}\boldsymbol{k}(\boldsymbol{r}-\boldsymbol{r}')} \langle n_{\boldsymbol{k}} \rangle$$

$$\boldsymbol{k}_0 \to 0, \langle n_0 \rangle = N_0$$

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$$\Psi(\boldsymbol{r}) = \sqrt{\frac{N_0}{V}}\cdot \mathrm{e}^{\mathrm{i}\varphi}\cdot \mathrm{e}^{\mathrm{i}\boldsymbol{k}_0\boldsymbol{r}}$$

$\varphi$ K an arbitrary phase

# ODLRO in the Bogolyubov theory

## Basic expressions for the OPDM for a homogeneous system

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by definition

$$= V^{-1}\sum_{\boldsymbol{k},\boldsymbol{k}'} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\boldsymbol{r}}\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}'\boldsymbol{r}'} \langle a^\dagger_{\boldsymbol{k}'} a_{\boldsymbol{k}}\rangle = V^{-1}\sum_{\boldsymbol{k},\boldsymbol{k}'} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\boldsymbol{r}}\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}'\boldsymbol{r}'} \langle a^\dagger_{\boldsymbol{k}} a_{\boldsymbol{k}}\rangle \delta_{\boldsymbol{k}\boldsymbol{k}'}$$

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## General expression for the one particle density matrix with condensate

$$\rho(\boldsymbol{r},\boldsymbol{r}') = V^{-1}\sum_{\boldsymbol{k}} \mathrm{e}^{\mathrm{i}\boldsymbol{k}(\boldsymbol{r}-\boldsymbol{r}')} \langle n_{\boldsymbol{k}}\rangle$$

$$\boldsymbol{k}_0 \to 0,\ \langle n_0\rangle = N_0$$

just like for non-interacting bosons

$$= \underbrace{V^{-1}\mathrm{e}^{\mathrm{i}\boldsymbol{k}_0(\boldsymbol{r}-\boldsymbol{r}')}\langle n_0\rangle}_{\text{coherent across the sample}} + \underbrace{V^{-1}\sum_{\boldsymbol{k}\neq\boldsymbol{k}_0} \mathrm{e}^{\mathrm{i}\boldsymbol{k}(\boldsymbol{r}-\boldsymbol{r}')}\langle n_{\boldsymbol{k}}\rangle}_{FT\text{ of a smooth function has a finite range}}$$

$$= \underbrace{\Psi(\boldsymbol{r})\Psi^*(\boldsymbol{r}')}_{\text{dyadic}} + V^{-1}\sum_{\boldsymbol{k}\neq\boldsymbol{k}_0} \mathrm{e}^{\mathrm{i}\boldsymbol{k}(\boldsymbol{r}-\boldsymbol{r}')}\langle n_{\boldsymbol{k}}\rangle$$

$$\Psi(\boldsymbol{r}) = \sqrt{\frac{N_0}{V}}\cdot \mathrm{e}^{\mathrm{i}\varphi}$$

$\varphi$ … an arbitrary phase

**Interpretation in the Bogolyubov theory – at zero temperature**

$$\rho(\boldsymbol{r},\boldsymbol{r}') = V^{-1/2}\langle a_0\rangle \cdot V^{-1/2}\langle a_0^\dagger\rangle + V^{-1}\sum_{\boldsymbol{k}\neq\boldsymbol{k}_0} \mathrm{e}^{\mathrm{i}\boldsymbol{k}(\boldsymbol{r}-\boldsymbol{r}')} v_{\boldsymbol{k}}^2$$

**Rich microscopic content hinging on the Bogolyubov assumption**

# ODLRO in the Bogolyubov theory

Basic expressions for the OPDM for a homogeneous system

$$\langle \boldsymbol{r}|\rho|\boldsymbol{r}'\rangle = \langle \psi^\dagger(\boldsymbol{r}')\psi(\boldsymbol{r})\rangle = V^{-1}\left\langle \sum \mathrm{e}^{-\mathrm{i}\boldsymbol{k}'\boldsymbol{r}'} a^\dagger_{\boldsymbol{k}'} \cdot \sum a_{\boldsymbol{k}}\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\boldsymbol{r}} \right\rangle \qquad \text{by definition}$$

$$= V^{-1}\sum_{\boldsymbol{k},\boldsymbol{k}'} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\boldsymbol{r}}\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}'\boldsymbol{r}'} \langle a^\dagger_{\boldsymbol{k}'} a_{\boldsymbol{k}}\rangle = V^{-1}\sum_{\boldsymbol{k},\boldsymbol{k}'} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\boldsymbol{r}}\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}'\boldsymbol{r}'} \langle a^\dagger_{\boldsymbol{k}} a_{\boldsymbol{k}}\rangle \delta_{\boldsymbol{k}\boldsymbol{k}'} \qquad \text{transl. invariance}$$

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# Three methods of reformulating the Bogolyubov theory

In the original BEC theory … no need for non-zero averages of linear field operators

Why so important? … microscopic view of the condensate phase
quasi-particles and superfluidity
basis for a perturbation treatment of Bose fluids

We shall explore three approaches having a common basic idea:

⌘ relax the particle number conservation ⌘ work in the thermodynamic limit

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| | | |
|---|---|---|
| I | explicit construction of the classical part of the field operators | *Pitaevski in* LL IX (1978) |
| II | the condensate represented by a coherent state | *Cummings & Johnston* (1966)<br>*Langer, Fisher & Ambegaokar* (1967 – 1969) |
| III | spontaneous symmetry breakdown, particle number conservation violated | *Bogolyubov* (1960)<br>*Hohenberg&Martin* (1965)<br>*P W Anderson* (1983 – book) |

# I.
# explicit construction of the classical part of the field operators

# Quotation from Landau-Lifshitz IX



ψ-операторов, которая меняет на 1 число частиц в конденсате, имеем, таким образом, по определению,

$$\hat{\Xi}|m, N+1\rangle = \Xi|m, N\rangle, \quad \hat{\Xi}^{+}|m, N\rangle = \Xi^{*}|m, N+1\rangle,$$

где символы $|m, N\rangle$ и $|m, N+1\rangle$ обозначают два «одинаковых» состояния, отличающихся только числом частиц в системе, а $\Xi$ — некоторое комплексное число. Эти утверждения справедливы строго в пределе $N \to \infty$. Поэтому определение величины $\Xi$ следует записать в виде

$$\lim_{N\to\infty} \langle m, N|\hat{\Xi}|m, N+1\rangle = \Xi,$$

$$\lim_{N\to\infty} \langle m, N+1|\hat{\Xi}^{+}|m, N\rangle = \Xi^{*}; \qquad (26,3)$$

переход к пределу совершается при заданном конечном значении плотности жидкости $N/V$.

Если представить ψ-операторы в виде

$$\hat{\Psi} = \hat{\Xi} + \hat{\Psi}', \quad \hat{\Psi}^{+} = \hat{\Xi}^{+} + \hat{\Psi}'^{+}, \qquad (26,4)$$

то остальная («надконденсатная») их часть переводит состояние $|m, N\rangle$ в ортогональные ему состояния, т. е. матричные элементы [1])

$$\lim_{N\to\infty} \langle m, N|\hat{\Psi}'|m, N+1\rangle = 0, \quad \lim_{N\to\infty} \langle m, N+1|\hat{\Psi}'^{+}|m, N\rangle = 0. \qquad (26,5)$$

В пределе $N \to \infty$ разница между состояниями $|m, N\rangle$ и $|m, N+1\rangle$ исчезает вовсе, и в этом смысле величина $\Xi$ становится средним значением оператора $\hat{\Psi}$ по этому состоянию. Подчеркнем, что характерным для системы с конденсатом является именно конечность этого предела.

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… that part of the $\Psi$ operators, which changes the condensate particle number by 1, we have, then, by definition

$$\hat{\Xi}|m, N+1\rangle = \Xi|m, N\rangle, \quad \hat{\Xi}^+|m, N\rangle = \Xi^*|m, N+1\rangle,$$

the symbols $|m, N\rangle$ и $|m, N+1\rangle$ denoting two "identical" states, differing only by the number of the particles in the system, and $\Xi$ is a complex number. These statements are strictly valid in the limit $N \longrightarrow \infty$. The definition of the quantity $\Xi$ has thus to be written in the form

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the limiting transition is to be performed at a given fixed value of the liquid density $N/V$.

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If the $\Psi$ operators are represented in the form

$$\hat{\Psi} = \hat{\Xi} + \hat{\Psi}', \quad \hat{\Psi}^+ = \hat{\Xi}^+ + \hat{\Psi}'^+, \quad (26,4)$$

then their remaining ("supercondensate") parts transform the state $|m, N\rangle$ to states which are orthogonal to it, that is, the matrix elements are

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In the limit $N \longrightarrow \infty$, the difference between the states $|m, N\rangle$ and $|m, N+1\rangle$ vanishes entirely and in this sense the quantity $\Xi$ becomes the mean value of the operator $\hat{\Psi}$ over this state.

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**Qui capere potest, capiat**

**Let anyone accept this who can**

**Kdo můžeš pochopiti, pochop**

# II.
# the condensate represented by a coherent state

# Reformulation of the Bogolyubov requirements

Bogolyubov himself and his faithful followers never speak of the many particle wave function. Looks like he wanted

$$a_0 \left| \Psi \right\rangle = \Lambda \left| \Psi \right\rangle, \quad \Lambda = \sqrt{N_0}\, e^{i\phi}\,, \text{ so that}$$

**The ground state**

$$\left\langle a_0 \right\rangle = \Lambda$$

This is in contradiction with the selection rule, $\left\langle a_0 \right\rangle = 0$

The above eigenvalue equation is known and defines the "ground" state ≡ Bogolyubov condensate state to be a **coherent state with the parameter** $\Lambda$

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HISTORICAL REMARK

⌘ The coherent states (not their name) discovered by Schrödinger as the minimum uncertainty wave packets, obtained by shifting the ground state of a harmonic oscillator.

⌘ These states were introduced into the quantum theory of the coherence of light by Roy Glauber (NP 2005). Hence the name.

⌘ The uses of the coherent states in the many body theory and quantum field theory have been manifold.

# About the coherent states

**OUR BASIC DEFINITION**

$$a_0 |\Psi\rangle = \Lambda |\Psi\rangle, \quad \Lambda = \sqrt{N_0}\, \mathrm{e}^{\mathrm{i}\phi}, \quad \langle a_0 \rangle = \Lambda$$

If a particle is removed from a coherent state, it remains unchanged (*cf.* the Pitaevskii requirement above). It has a rather uncertain particle number, but a reasonably well defined phase

| General coherent state | Condensate |
|---|---|
| $\lvert\Lambda\rangle = \mathrm{e}^{-\lvert\Lambda\rvert^2/2} \cdot \mathrm{e}^{\Lambda a_0^\dagger} \lvert \mathrm{vac} \rangle$ | $= \lvert\Psi\rangle$ |
| $\langle\Lambda \lvert a_0 \rvert \Lambda\rangle = \Lambda$ | $= \sqrt{N_0}\, \mathrm{e}^{\mathrm{i}\phi}$ |
| $\langle\Lambda \lvert a_0^\dagger a_0 \rvert \Lambda\rangle = \lvert\Lambda\rvert^2$ | $\langle n_0 \rangle = N_0$ |
| $\langle\Lambda \lvert a_0^\dagger a_0 a_0^\dagger a_0 \rvert \Lambda\rangle = \lvert\Lambda\rvert^4 + \lvert\Lambda\rvert^2$ | $\langle n_0^2 \rangle = N_0^2 + N_0$ |
| $\Delta n_0 = \lvert\Lambda\rvert$ | $\Delta n_0 = \sqrt{N_0} = N_0$ |

# New vacuum and the shifted field operators

Does all that make sense? Try to work in the full Fock space F rather in its fixed $N$ sub-space $\mathrm{H}_N$ This implies using the "grand Hamiltonian"

$$H - \mu N$$

# **L1:** *Thermodynamics*: which environment to choose?

THE ENVIRONMENT IN THE THEORY SHOULD CORRESPOND TO THE EXPERIMENTAL CONDITIONS

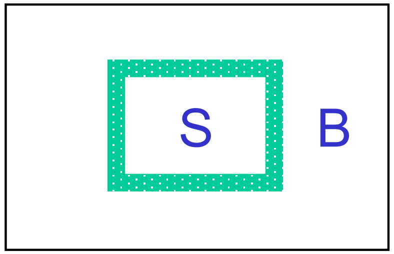
*… a truism difficult to satisfy*

❶ For large systems, this is not so sensitive for two reasons

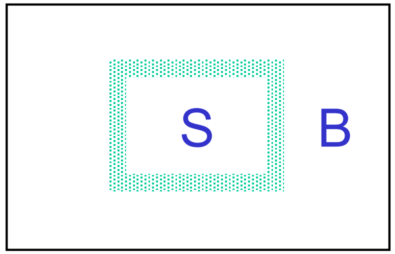
- System serves as a thermal bath or particle reservoir all by itself
- Relative fluctuations (distinguishing mark) are negligible

❷

| | Adiabatic system | Real system | Isothermal system |
|---|---|---|---|
| SB heat exchange – | the slowest | medium fast process | the fastest |




- temperature lag
- interface layer




❸ Atoms in a trap: ideal model … isolated. In fact: unceasing energy exchange (laser cooling). A small number of atoms may be kept (one to, say, 40). With $10^7$, they form a bath already. Besides, they are cooled by evaporation and they form an open (albeit non-equilibrium) system.

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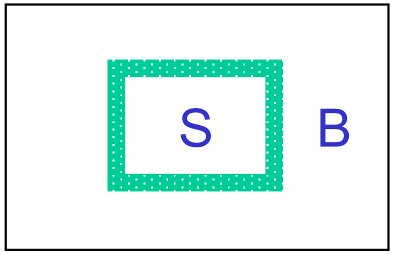


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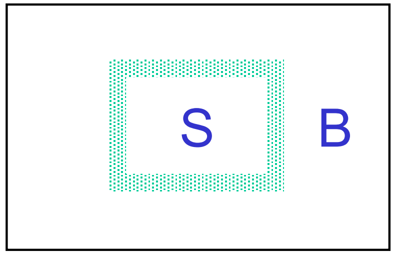


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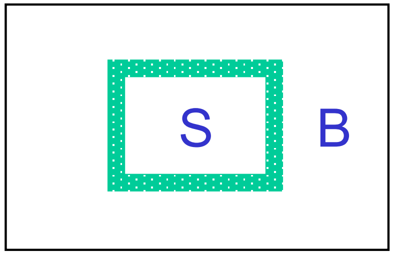


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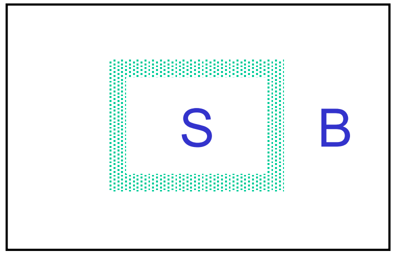


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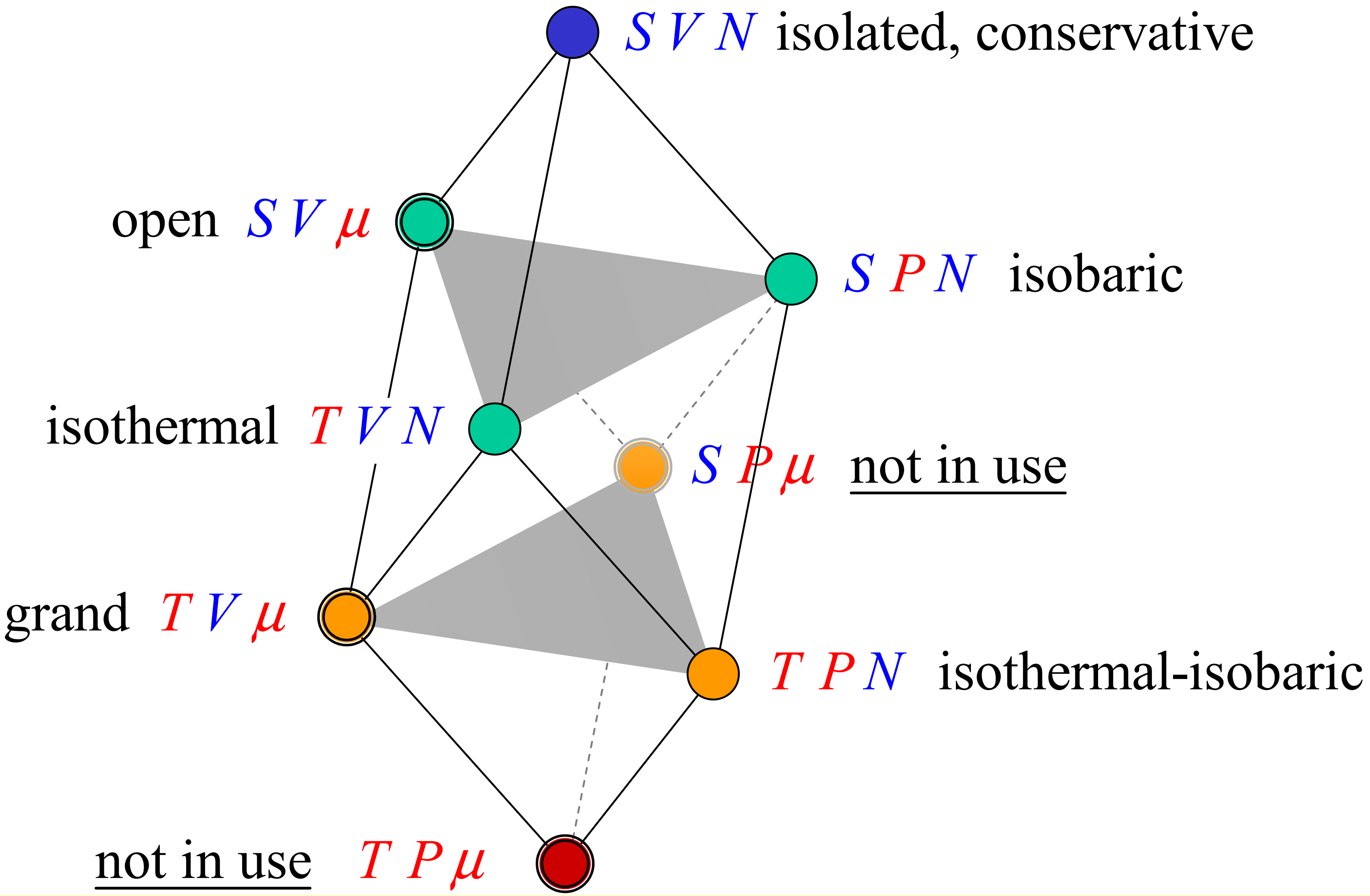
## L1: Homogeneous one component phase: boundary conditions (environment) and state variables

$S\ V\ N$ additive variables, have densities $s = S/V \quad n = N/V$ "extensive"

b b b

$T\ P\ \mu$ dual variables, intensities "intensive"



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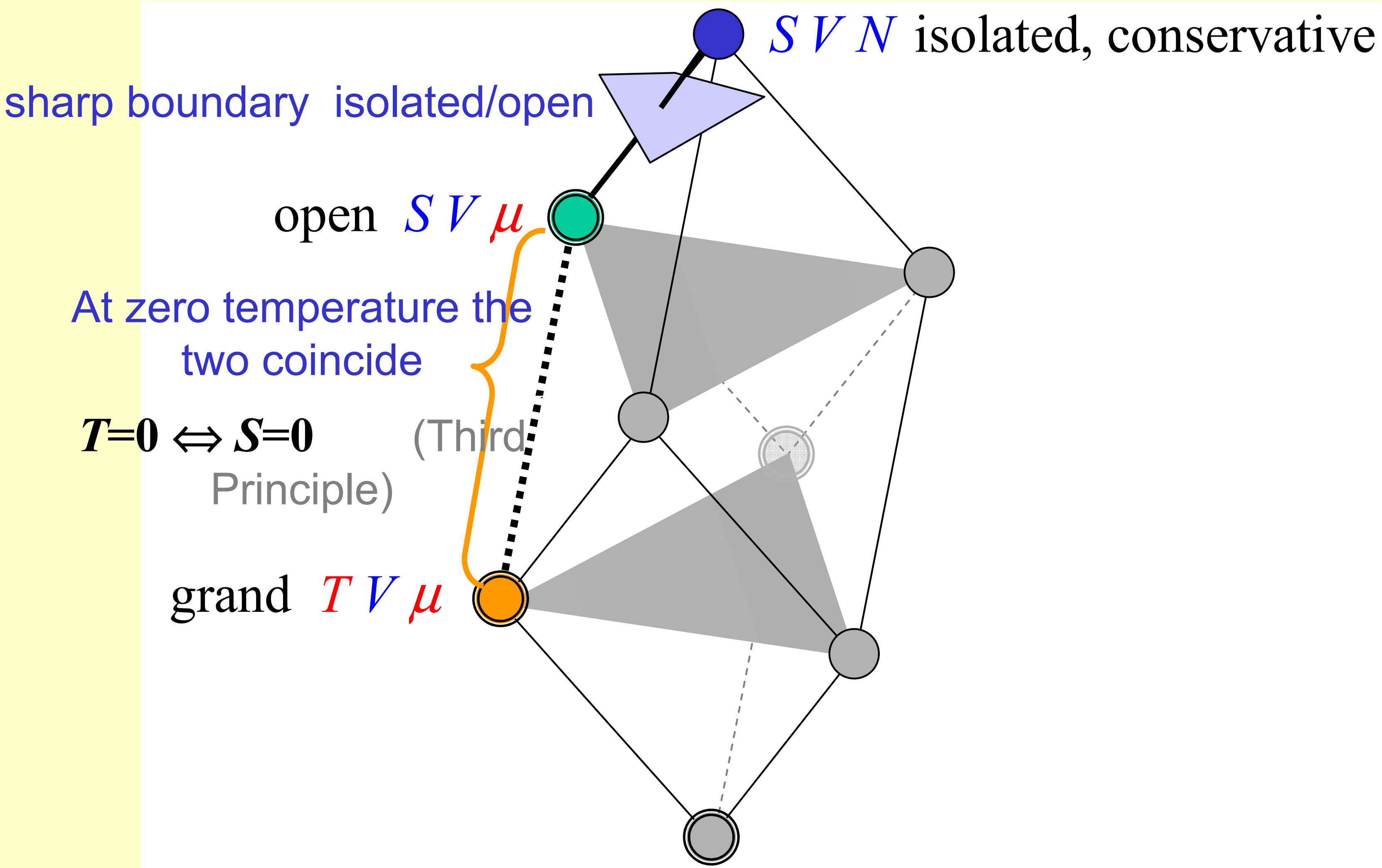

# New vacuum and the shifted field operators

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Let us define the shifted field operator

$$b_0 = a_0 - \Lambda, \quad b_0^\dagger = a_0^\dagger - \Lambda^*$$

$$\left[ b_0, b_0^\dagger \right] = 1, \quad b_0 \left| \Psi \right\rangle = 0$$ ... new vacuum

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*What next? … is this coherent state able to represent the condensate?*

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$$b_0 = a_0 - \Lambda, \quad b_0^\dagger = a_0^\dagger - \Lambda^*$$

$$\left[ b_0, b_0^\dagger \right] = 1, \quad b_0 \left| \Psi \right\rangle = 0 \quad \ldots \text{ new vacuum} \quad \left| \Psi \right\rangle = \mathrm{e}^{-\left|\Lambda\right|^2/2} \cdot \mathrm{e}^{\Lambda a_0^\dagger} \left| \mathrm{vac} \right\rangle$$

*What next? … is this coherent state able to represent the condensate?*

Test example: ideal Bose gas – limit of a BE system without interactions

$$(H - \mu N) \left| \Psi \right\rangle = \sum \left( \frac{\mathrm{h}^2}{2m} \boldsymbol{k}^2 - \mu \right) a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k}} \left| \Psi \right\rangle$$

$$= -\mu a_0^\dagger a_0 \left| \Psi \right\rangle = 0 \quad \text{for} \quad \mu = 0$$

# New vacuum and the shifted field operators

Does all that make sense? Yes: work in the full Fock space F rather than in its fixed $N$ sub-space $H_N$ This implies using the "grand Hamiltonian"

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$|\Psi\rangle$… a true eigenstate with $\mathcal{E} = 0$, $\mu$ the same as for the particle number conserving state $|B\rangle = |N_0, 0, 0, \ldots, 0, \ldots\rangle = (N_0!)^{-\frac{1}{2}} \left(a_0^\dagger\right)^{N_0} |\mathrm{vac}\rangle$

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**Two different, but macroscopically equivalent possibilities.**

# General case: the approximate vacuum

$$H = \int \mathrm{d}^3 \boldsymbol{r}\, \psi^\dagger(\boldsymbol{r})\left(-\tfrac{\hbar^2}{2m}\Delta + V(\boldsymbol{r})\right)\psi(\boldsymbol{r}) + \tfrac{1}{2}\iint \mathrm{d}^3\boldsymbol{r}\,\mathrm{d}^3\boldsymbol{r}'\,\psi^\dagger(\boldsymbol{r})\psi^\dagger(\boldsymbol{r}')U(\boldsymbol{r}-\boldsymbol{r}')\psi(\boldsymbol{r}')\psi(\boldsymbol{r})$$

$$g\delta(\boldsymbol{r}-\boldsymbol{r}')$$

✓ Trial function … a coherent state

$$\psi(\boldsymbol{r})|\Psi\rangle = \Psi(\boldsymbol{r})|\Psi\rangle$$

✓ We should minimize the average grand energy

$$\langle\Psi|H-\mu N|\Psi\rangle = \int \mathrm{d}^3\boldsymbol{r}\,\Psi^*(\boldsymbol{r})\left(-\tfrac{\hbar^2}{2m}\Delta + V(\boldsymbol{r}) - \mu\right)\Psi(\boldsymbol{r})$$
$$+\tfrac{1}{2}\iint \mathrm{d}^3\boldsymbol{r}\,\mathrm{d}^3\boldsymbol{r}'\,\Psi^*(\boldsymbol{r})\Psi(\boldsymbol{r})U(\boldsymbol{r}-\boldsymbol{r}')\Psi^*(\boldsymbol{r}')\Psi(\boldsymbol{r}')$$

This is precisely the energy functional of the Hartree type we met already

and the ✓ Euler-Lagrange equation is the good old Gross-Pitaevski equation

$$\left(-\frac{\hbar^2}{2m}\Delta + V(\boldsymbol{r}) + g\left|\Psi(\boldsymbol{r})\right|^2\right)\Psi(\boldsymbol{r}) = \mu\Psi(\boldsymbol{r})$$

with the normalization condition

$$N[n] = N = \int \mathrm{d}^3\boldsymbol{r}\,\left|\Psi(\boldsymbol{r})\right|^2$$

# More about the approximate vacuum

Approximate vacuum … a coherent state

$$\psi(\boldsymbol{r})|\Psi\rangle = \mathit{\Psi}(\boldsymbol{r})|\Psi\rangle$$

What is the OPDM?

$$\langle\Psi|\psi(\boldsymbol{r})|\Psi\rangle = \langle\Psi|\mathit{\Psi}(\boldsymbol{r})|\Psi\rangle = \mathit{\Psi}(\boldsymbol{r}), \quad \langle\Psi|\psi^\dagger(\boldsymbol{r})|\Psi\rangle = \langle\Psi|\mathit{\Psi}^*(\boldsymbol{r})|\Psi\rangle = \mathit{\Psi}^*(\boldsymbol{r})$$

$$\langle\boldsymbol{r}|\rho|\boldsymbol{r}'\rangle = \left\langle\psi^\dagger(\boldsymbol{r}')\psi(\boldsymbol{r})\right\rangle = \langle\Psi|\psi^\dagger(\boldsymbol{r}')\psi(\boldsymbol{r})|\Psi\rangle = \mathit{\Psi}(\boldsymbol{r})\mathit{\Psi}^*(\boldsymbol{r}')$$

Full ODLRO with the normalization condition $\int \mathrm{d}^3\boldsymbol{r}\,|\mathit{\Psi}(\boldsymbol{r})|^2 = N$

Explicit form of the coherent state

$$|\Psi\rangle = \exp\left\{-\tfrac{1}{2}\int \mathrm{d}^3\boldsymbol{r}\,|\mathit{\Psi}(\boldsymbol{r})|^2\right\}\exp\left\{\int \mathrm{d}^3\boldsymbol{r}\,\mathit{\Psi}(\boldsymbol{r})\psi^\dagger(\boldsymbol{r})\right\}|\mathrm{vac}\rangle$$

$$\langle\Psi|\Psi\rangle = 1$$

**NOTE**: this is not a unitary transformation

# General case: the Bogolyubov transformation

Define the shifed field operators and the condensate as the new vacuum

$$\eta(\boldsymbol{r}) = \psi(\boldsymbol{r}) - \Psi(\boldsymbol{r}), \quad \eta^\dagger(\boldsymbol{r}) = \psi^\dagger(\boldsymbol{r}) - \Psi^*(\boldsymbol{r})$$

$$\left[\eta(\boldsymbol{r}), \eta^\dagger(\boldsymbol{r}')\right] = \delta(\boldsymbol{r} - \boldsymbol{r}'), \quad \eta(\boldsymbol{r})|\Psi\rangle = 0$$

If we keep only the terms not more than quadratic in the new operators, the approximate quadratic Hamiltonian becomes

$$H = \int \mathrm{d}^3\boldsymbol{r}\, \eta^\dagger(\boldsymbol{r})\left(-\tfrac{\hbar^2}{2m}\Delta + V(\boldsymbol{r}) - \mu\right)\eta(\boldsymbol{r})$$

$$+\tfrac{g}{2}\int \mathrm{d}^3\boldsymbol{r}\, n_{\mathrm{BE}}(\boldsymbol{r})\left\{\eta^\dagger(\boldsymbol{r})\eta^\dagger(\boldsymbol{r}) + 4\eta^\dagger(\boldsymbol{r})\eta(\boldsymbol{r}) + \eta(\boldsymbol{r})\eta(\boldsymbol{r})\right\}$$

Now eliminate the anomalous terms by the Bogolyubov transformation.

It is required that, in terms of the new field operators,

$$H = \sum_\alpha \varepsilon_\alpha b_\alpha^\dagger b_\alpha + E_0$$

$$\left[b_\alpha, b_{\alpha'}^\dagger\right] = \delta_{\alpha\alpha'}, \quad \left[b_\alpha, b_{\alpha'}\right] = 0, \quad \left[b_\alpha^\dagger, b_{\alpha'}^\dagger\right] = 0$$

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This is achieved by the Bogolyubov transformation

$$\eta(\boldsymbol{r}) = \sum_\alpha u_\alpha(\boldsymbol{r}) b_\alpha + v_\alpha(\boldsymbol{r}) b_\alpha^\dagger$$

$$\eta^\dagger(\boldsymbol{r}) = \sum_\alpha v_\alpha^*(\boldsymbol{r}) b_\alpha + u_\alpha^*(\boldsymbol{r}) b_\alpha^\dagger$$

with $\int \mathrm{d}^3 \boldsymbol{r} \left\{ |u_\alpha|^2 - |v_\alpha|^2 \right\} = 1$

For $u(\boldsymbol{r})$ and $v(\boldsymbol{r})$ … coupled Schrödinger-like Bogolyubov-de Gennes eqs.

$$\left( -\frac{\hbar^2}{2m} \Delta + V(\boldsymbol{r}) + 2g \left| \Psi(\boldsymbol{r}) \right|^2 - \mu \right) u(\boldsymbol{r}) + g \left[ \Psi(\boldsymbol{r}) \right]^2 v^*(\boldsymbol{r}) = +\varepsilon \cdot u(\boldsymbol{r})$$

$$\left( -\frac{\hbar^2}{2m} \Delta + V(\boldsymbol{r}) + 2g \left| \Psi(\boldsymbol{r}) \right|^2 - \mu \right) v(\boldsymbol{r}) + g \left[ \Psi^*(\boldsymbol{r}) \right]^2 u(\boldsymbol{r}) = -\varepsilon \cdot v(\boldsymbol{r})$$

# Detail: the mean-field for a homogeneous system

Before: minimize the energy functional with fixed particle number $N$, find the chemical potential $\mu$ afterwards

Now: minimize the grand energy functional with fixed chemical potential, find the average particle number in the process

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Homogeneous system:

order parameter $\Psi(\boldsymbol{r}) \equiv \Psi = \text{const.} = \sqrt{N_0/V} \equiv \sqrt{n}$

$$\langle \Psi | H - \mu N | \Psi \rangle = \int \mathrm{d}^3 \boldsymbol{r}\, \Psi^*(\boldsymbol{r}) \left( -\tfrac{\hbar^2}{2m} \Delta + V(\boldsymbol{r}) - \mu \right) \Psi(\boldsymbol{r})$$

$$+ \tfrac{1}{2} \iint \mathrm{d}^3 \boldsymbol{r}\, \mathrm{d}^3 \boldsymbol{r}'\, \Psi^*(\boldsymbol{r}) \Psi(\boldsymbol{r})\, g\, \delta(\boldsymbol{r} - \boldsymbol{r}')\, \Psi^*(\boldsymbol{r}') \Psi(\boldsymbol{r})$$

$$= V \times \underbrace{\left( -\mu |\Psi|^2 + \tfrac{1}{2} g |\Psi|^4 \right)}_{\in(\Psi)}$$

energy per unit volume $\in(\Psi)$

$$\langle \Psi | N | \Psi \rangle = \int \mathrm{d}^3 \boldsymbol{r}\, \Psi^*(\boldsymbol{r}) \Psi(\boldsymbol{r}) = V \times \underbrace{|\Psi|^2}_{n(\Psi)}$$

average particle density $n(\Psi)$

# Detail: the mean-field for a homogeneous system

The GP equation reduces from differential to an algebraic one:

$$\frac{\partial}{\partial x}\in(x)=0,\quad |\Psi|\equiv x$$

$$-2\mu x+\tfrac{1}{2}g\cdot 4x^3=0,\quad |\Psi|_{\max}=0,\quad \boxed{|\Psi|_{\min}=\sqrt{\frac{\mu}{g}},}\quad \in_{\min}=-\tfrac{1}{2}g|\Psi|^4_{\min}=-\frac{\mu^2}{2g}$$

$$\Rightarrow\quad \boxed{n=|\Psi|^2_{\min}=\frac{\mu}{g}}\ ✓$$

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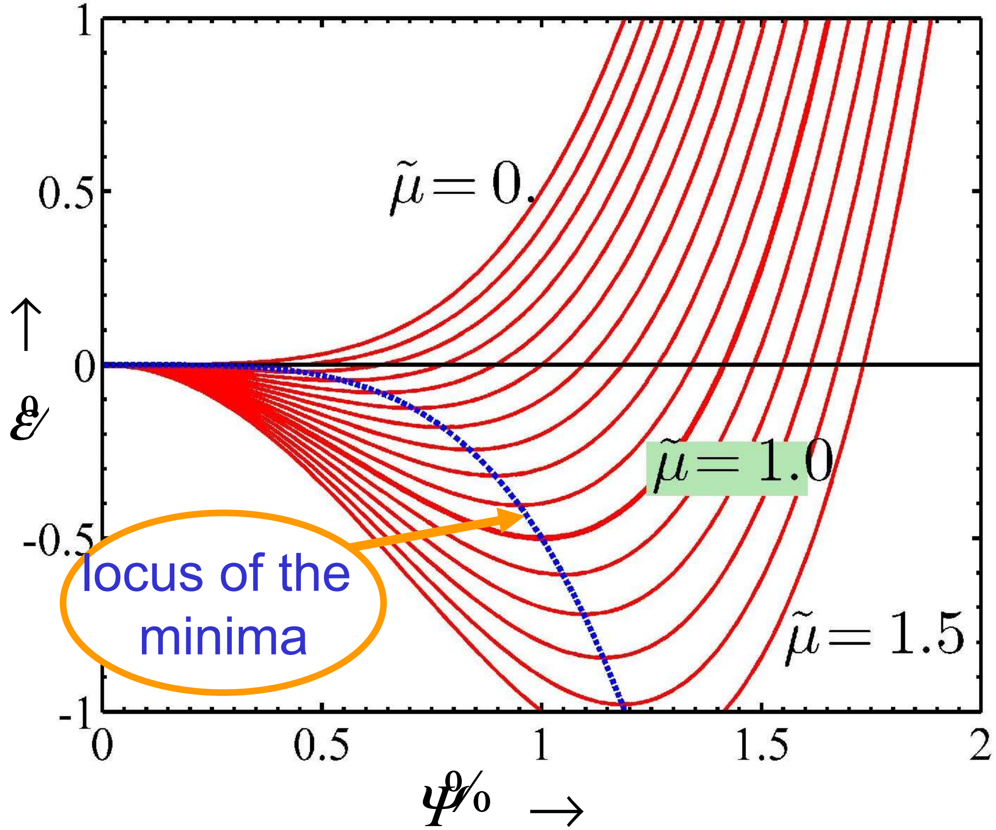


Plot in relative units

*choose* $\mu_{\text{ref}}$; $|\Psi_{\text{ref}}|=\sqrt{\mu_{\text{ref}}/g}$

$\mu=\tilde{\mu}\,\mu_{\text{ref}}$ $\quad |\Psi|=\tilde{\Psi}\,|\Psi_{\text{ref}}|$

$\in=\tilde{\mathcal{E}}\,g\,|\Psi_{\text{ref}}|^4$

# III.
# broken symmetry and quasi-averages

# Zero temperature limit of the grand canonical ensemble

$$\mathcal{P} = Z^{-1}\,\mathrm{e}^{-\beta(\mathcal{H}-\mu\mathcal{N})}$$

$$= Z^{-1}\sum |\alpha N\rangle \mathrm{e}^{-\beta(E_{\alpha N}-\mu N)}\langle \alpha N|$$

$$\to Z^{-1}\sum |0N\rangle \mathrm{e}^{-\beta(E_{0N}-\mu N)}\langle 0N| \propto \sum |0N\rangle\langle 0N|$$

Picks up the correct ground state energy,

all ground states are taken with equal statistical weight

**SYSTEMS**

- **NORMAL** — NON-DEGENERATE GROUND STATE
- **"ANOMALOUS"** — DEGENERATE GROUND STATE

# Degenerate ground state

**SYSTEMS**

- **NORMAL** — NON-DEGENERATE GROUND STATE
- **"ANOMALOUS"** — DEGENERATE GROUND STATE

Characterized by a classical order parameter … **macroscopic quantity**
Typical cause: a symmetry degeneracy
Everything depends on the system characteristic parameters
Ginsburg – Landau phenomenological model

$$\mathcal{E}(\Psi) = a\Psi^2 + b\Psi^4$$

| $a > 0$ | $a < 0$ |
| --- | --- |
| stable equilibrium | metastable equilibrium |
| non-degenerate | degenerate |

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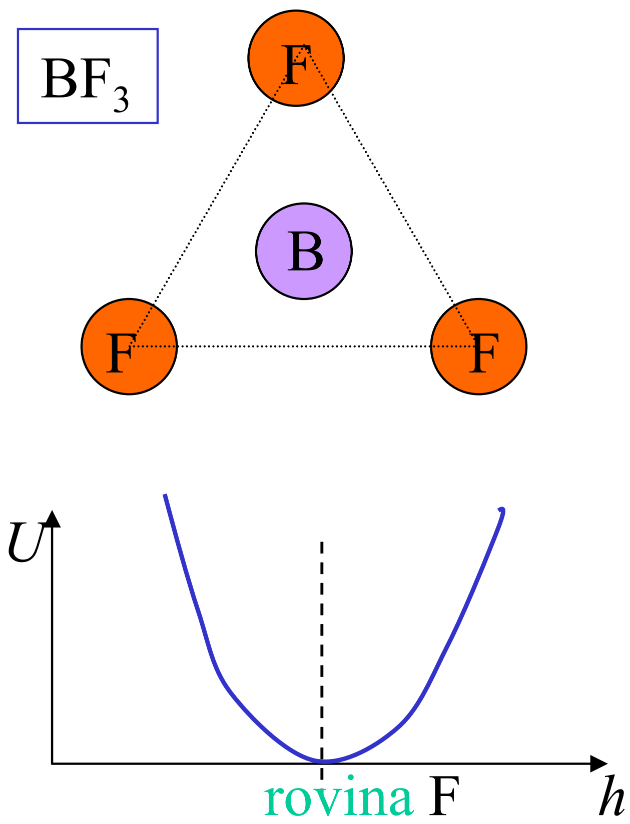
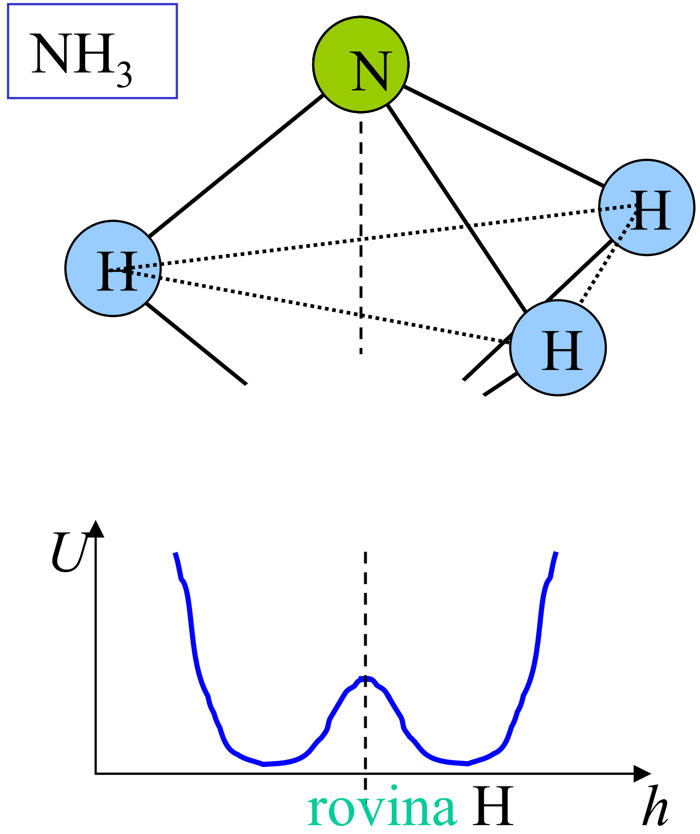
**spontaneous symmetry breaking**

$a > 0$
stable equilibrium
non-degenerate

$a < 0$
metastable equilibrium
degenerate

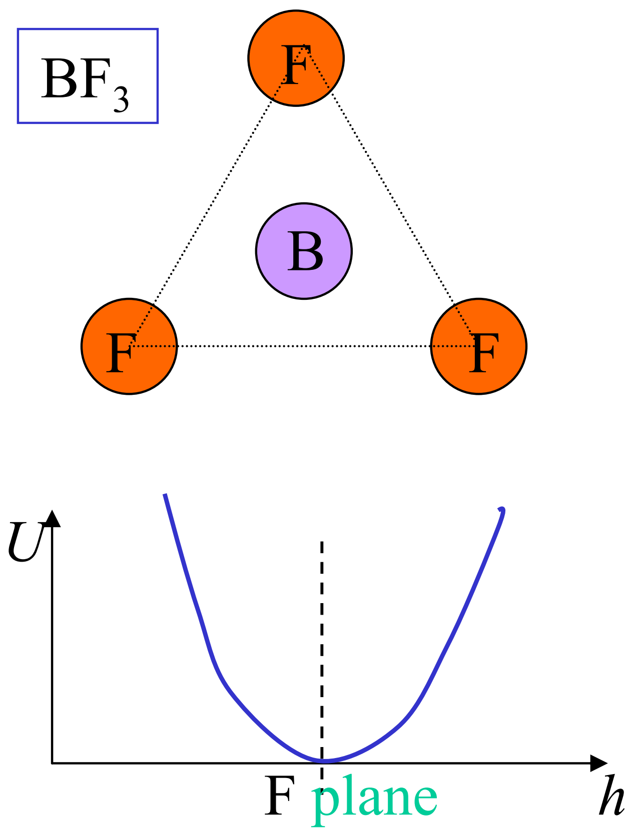
# Rovnovážná struktura molekul $AB_3$

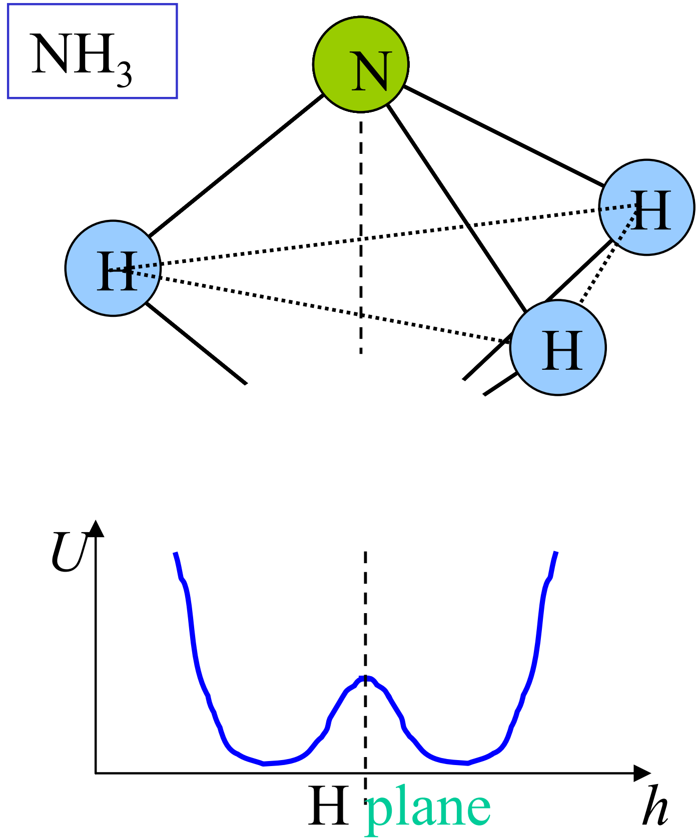



$U$ adiabatická potenciální energie

# Equilibrium structure of the $AB_3$ molekules




$U$ adiabatic potential energy

**stable equilibrium**
**non-degenerate**
**ground state**

**metastable equilibrium**
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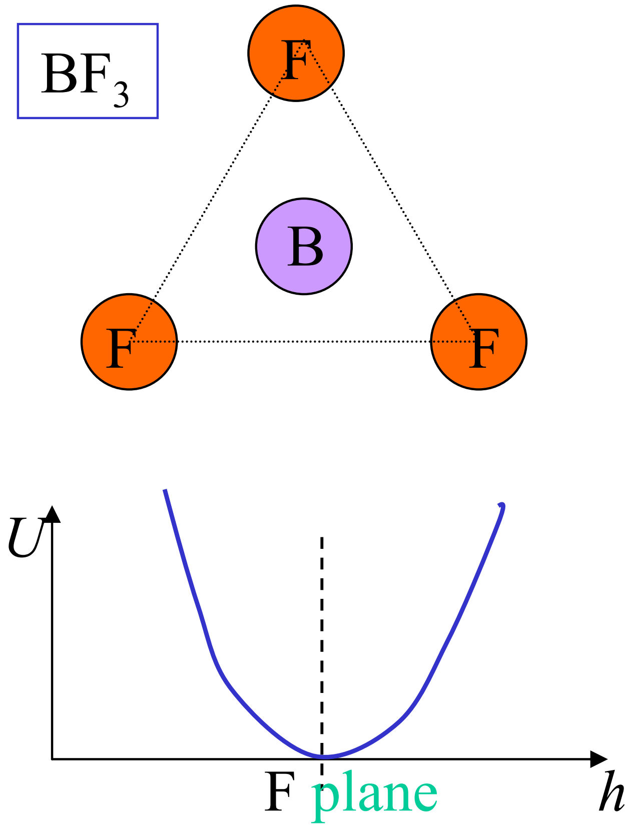


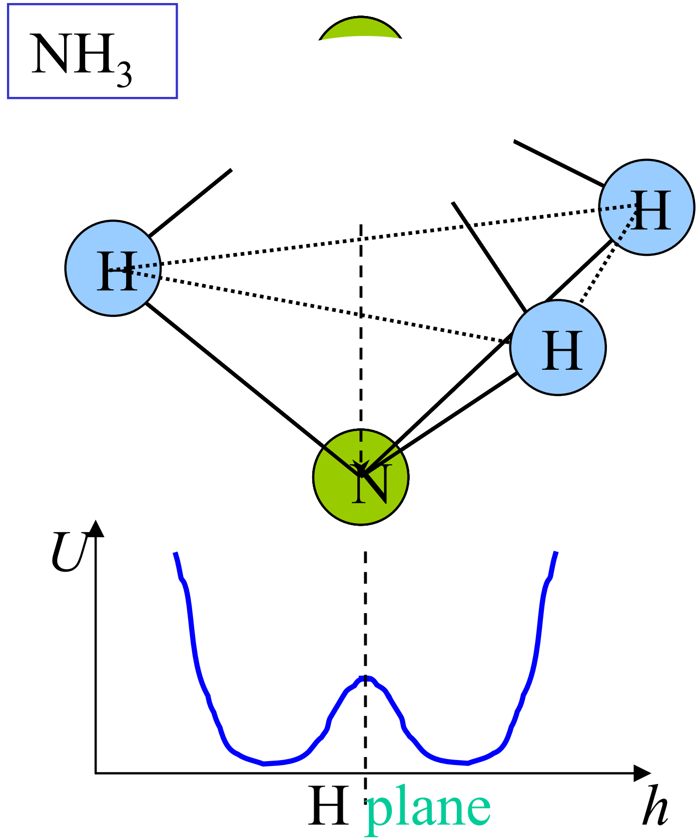


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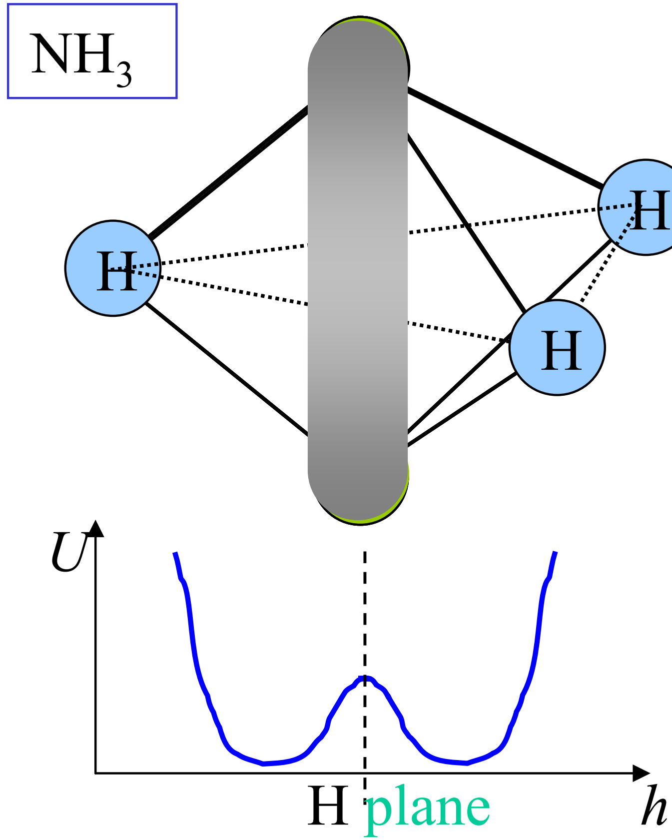
# Equilibrium structure of the $AB_3$ molekules

<u>Ammonia molecule</u>
***pyramidal molecule***.
***two minima*** of
potential energy
separated by a ***barrier***.

<u>Different from a typical extended system:</u>

⌘ **Small system:**
quantum *barrier & tunneling*

⌘ **Discrete symmetry broken:**
discrete set of equivalent equilibria states



# Broken continouous symmetries in extended systems

Three popular cases

| System | Isotropic ferromagnet | Atomic crystal lattice | Bosonic gas/liquid |
| --- | --- | --- | --- |
| Hamiltonian | Heisenberg spin Hamiltonian | Distinguishable atoms with int. | Bosons with short range interactions |
| Symmetry | 3D rotational in spin space | Translational | Global gauge invariance |
| Order parameter | homogeneous magnetization | periodic particle density | macroscopic wave function |
| Symmetry breaking field | external magnetic field | "empty lattice" potential | particle source/drain |
| Goldstone modes | magnons | acoustic phonons | sound waves |

For a nearly exhaustive list see the PWA book of 1983

# Bose condensate - degeneracy of the ground state

## The coherent ground state

mean field energy $E(\Psi) = \left(-\mu|\Psi|^2 + \tfrac{1}{2} g |\Psi|^4\right)$

order parameter $\Psi = \sqrt{\langle N_0 \rangle} \cdot e^{i\phi}$ ^ any from $(0, 2\pi)$

mf ground state $|\Psi\rangle = e^{-\frac{1}{2}|\Psi|^2} \cdot e^{\sqrt{N_0} \cdot e^{i\phi} a_0} |\text{vac}\rangle$

**degeneracy**

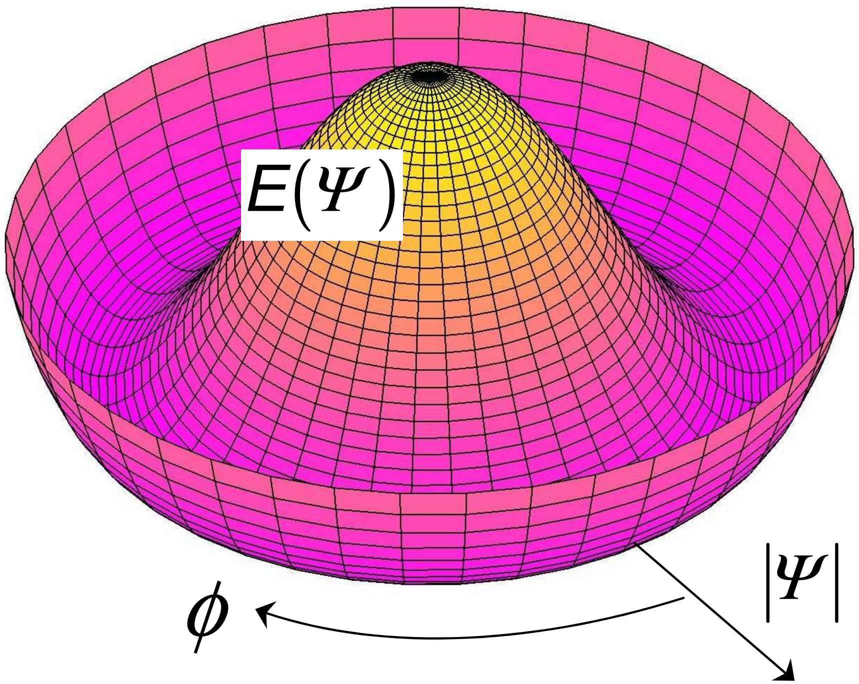
genuinely different for different $\phi$

## Selection rule

$$\langle a_0 \rangle_\phi = |\Psi| e^{i\phi} \neq 0$$

$$\langle a_0 \rangle = \int d\phi \langle a_0 \rangle_\phi = 0$$

average over all degenerate states




"Mexican hat"

# Symmetry breaking – removal of the degeneracy

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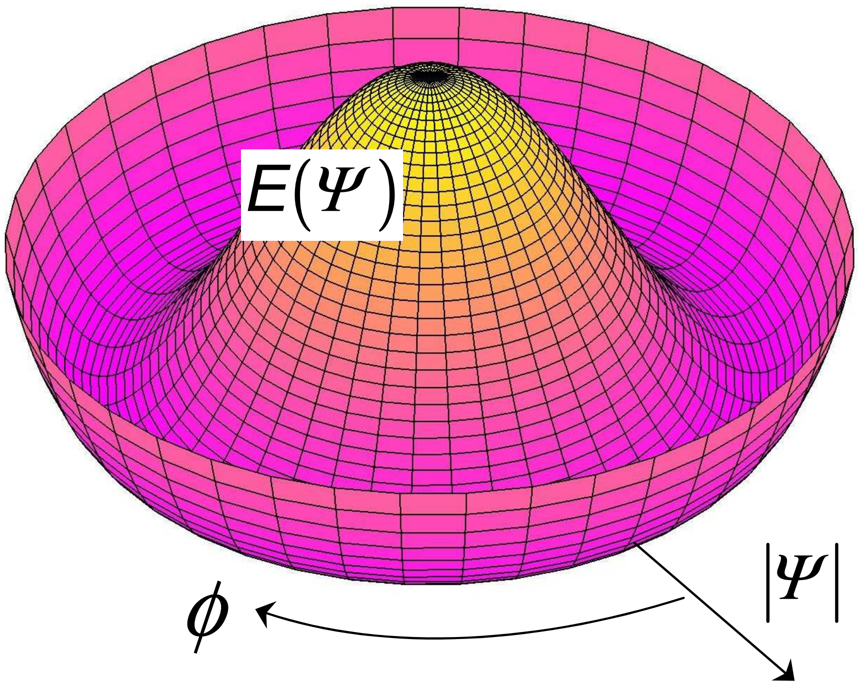
genuinely different for different $\phi$

Symmetry broken by a small perturbation picking up one $\phi$

$$\mathcal{H} - \mu\mathcal{N} \rightarrow$$

$$\mathcal{H} - \mu\mathcal{N} - \lambda\left(a_0^\dagger \mathrm{e}^{\mathrm{i}\phi} + a_0 \mathrm{e}^{-\mathrm{i}\phi}\right)$$

particle number NOT conserved

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Symmetry broken by a small perturbation picking up one $\phi$

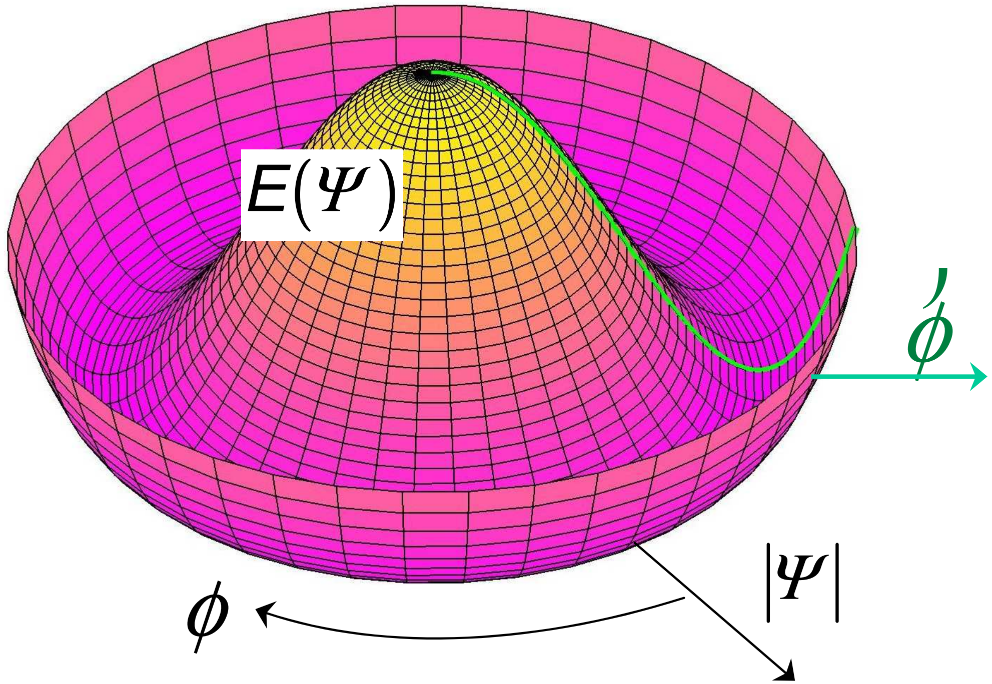
$$\mathcal{H} - \mu\mathcal{N} \to$$

$$\mathcal{H} - \mu\mathcal{N} - \lambda\left(a_0^\dagger \, \mathrm{e}^{\mathrm{i}\phi} + a_0 \, \mathrm{e}^{-\mathrm{i}\phi}\right)$$

particle number NOT conserved

For $\lambda \to 0$

one particular phase selected




"Mexican hat"

# How the symmetry breaking works – ideal BE gas

Without interactions, the ground level is uncoupled from the excited levels:

$$\mathcal{H} - \mu\mathcal{N} - \lambda\left(a_0^\dagger\,\mathrm{e}^{\mathrm{i}\phi} + a_0\,\mathrm{e}^{-\mathrm{i}\phi}\right)$$

$$= -\mu a_0^\dagger a_0 - \lambda\left(a_0^\dagger\,\mathrm{e}^{\mathrm{i}\phi} + a_0\,\mathrm{e}^{-\mathrm{i}\phi}\right) + \sum_{\boldsymbol{k}\neq 0}\frac{\mathrm{h}^2}{2m}\left(\boldsymbol{k}^2 - \mu\right)a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k}}$$

$$\to -\mu a_0^\dagger a_0 - \lambda\left(a_0^\dagger\,\mathrm{e}^{\mathrm{i}\phi} + a_0\,\mathrm{e}^{-\mathrm{i}\phi}\right)$$

The control parameter is the chemical potential μ, but it will be adjusted to yield a fixed average particle number in the condensate.

Transformation:

$$-\mu a_0^\dagger a_0 - \lambda\left(a_0^\dagger\,\mathrm{e}^{\mathrm{i}\phi} + a_0\,\mathrm{e}^{-\mathrm{i}\phi}\right) = -\mu\left(a_0^\dagger a_0 + \frac{\lambda}{\mu}\mathrm{e}^{\mathrm{i}\phi}\,a_0^\dagger + \frac{\lambda}{\mu}\mathrm{e}^{-\mathrm{i}\phi}\,a_0\right)$$

$$\equiv -\mu\left(a_0^\dagger a_0 - \Lambda a_0^\dagger - \Lambda^* a_0\right) = -\mu\left(\left(a_0^\dagger - \Lambda^*\right)\left(a_0 - \Lambda\right) - \Lambda^*\Lambda\right)$$

$$\equiv -\mu\left(b_0^\dagger b_0 - \Lambda^*\Lambda\right)$$

# How the symmetry breaking works – ideal BE gas

Now we determine the many-body ground state

$$-\mu\left(b_0^\dagger b_0 - \Lambda^* \Lambda\right)\left|\Psi\right\rangle = \mathcal{E}\left|\Psi\right\rangle, \qquad \boxed{b_0 = a_0 - \Lambda, \quad \Lambda = -\lambda\mu^{-1}\,\mathrm{e}^{\mathrm{i}\breve{\phi}}, \quad \mu \leq 0}$$

The lowest energy corresponds to

$$b_0\left|\Psi\right\rangle = 0, \text{ i.e. } a_0\left|\Psi\right\rangle = \Lambda\left|\Psi\right\rangle \quad \text{... coherent state}$$

$$\Lambda^*\Lambda = \left\langle\Psi\right|a_0^\dagger a_0\left|\Psi\right\rangle = N_0, \quad \Lambda = \sqrt{N_0}\,\mathrm{e}^{\mathrm{i}\breve{\phi}}$$

$$\mathcal{E} = \mu\Lambda^*\Lambda = \mu N_0 \qquad \mu = -\lambda/\sqrt{N_0}$$

The control parameter is the chemical potential μ, but it will be adjusted to yield a fixed average particle number in the condensate.

Infinitesimal symmetry breaking field $\lambda \to 0$

$\lambda \to 0$ with $N_0$ fixed:

$$\mu \to 0 - 0$$

$$\mathcal{E} \to 0$$

$$\Lambda = \sqrt{N_0}\,\mathrm{e}^{\mathrm{i}\breve{\phi}}, \quad \left|\Psi\right\rangle \text{ fixed}$$

# How the symmetry breaking works – ideal BE gas

Now we determine the many-body ground state

$$-\mu\left(b_0^\dagger b_0 - \Lambda^*\Lambda\right)|\Psi\rangle = \mathcal{E}|\Psi\rangle, \qquad \boxed{b_0 = a_0 - \Lambda, \quad \Lambda = -\lambda\mu^{-1}\,\mathrm{e}^{\mathrm{i}\breve{\phi}}, \quad \mu \le 0}$$

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$$\Lambda^*\Lambda = \langle\Psi|a_0^\dagger a_0|\Psi\rangle = N_0, \quad \Lambda = \sqrt{N_0}\,\mathrm{e}^{\mathrm{i}\breve{\phi}}$$

$$\mathcal{E} = \mu\Lambda^*\Lambda = \mu N_0 \qquad \mu = -\lambda/\sqrt{N_0}$$

The control parameter is the chemical potential μ, but it will be adjusted to yield a fixed average particle number in the condensate.

Infinitesimal symmetry breaking field $\lambda \to 0$

$\lambda \to 0$ with $N_0$ fixed:

$$\mu \to 0-0$$

$$\mathcal{E} \to 0$$

$$\Lambda = \sqrt{N_0}\,\mathrm{e}^{\mathrm{i}\breve{\phi}}, \quad |\Psi\rangle \text{ fixed}$$

- The coherent state is the exact ground state for the ideal BE gas
- The order parameter picks up the phase from the perturbing field
- The order of limits: first $\lambda \to 0$, only then the thermodynamic limit $N_0 \to \infty$

The end