Goodness-of-fit tests with nuisance regression and scale

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1 Introduction

Robust estimators are constructed in such a way that they are insensitive to small deviations from the assumed distribution of the model errors; for instance, the Huber estimator of the location or regression parameters is minimaximally robust over a family of contaminated normal distributions.

Before using a robust estimator, likely operating well in a neighborhood of some distribution, we can try to verify a hypothesis on the shape of this distribution; otherwise, we can start with a suitable goodness-of-fit test. The χ^2 and the Kolmogorov-Smirnov tests are probably most well-known. Many various other tests can be found in the literature (e.g., Huber-Carol et al. (2002)).

These tests work well in the simplest situation, when our observations $Y_1, \ldots, Y_n$ are independent and identically distributed with a distribution function F, and we want to verify the hypothesis $\mathbf{H}_0: \ F \equiv F_0$, where F_0 is a fully specified distribution function.

However, the hypothetical distribution function F_0 is often specified only up to several unknown parameters, e.g., up to the location, scale or regression parameters. This is a typical situation: our observations can follow a linear regression model, whose parameters we want to estimate by a suitable robust estimator, and an approximate knowledge of the shape of the distribution of errors would lead to a good choice of the score function. This situation is more realistic, but the standard goodness-of-fit tests then lose their simplicity.

Taking these facts into account, we want to offer some goodness-of-fit tests on the shape of the distribution in the presence of nuisance regression and scale parameters.

2 Tests of normality of the Shapiro-Wilk type with nuisance regression and scale parameters

If the distribution seems to have a symmetric unimodal density, then the first natural idea is to test for its normality. A highly intuitive goodness-of-fit test of normality with nuisance location and scale parameters was proposed by Shapiro and Wilk (1965). Their test has received considerable attention in the literature; its asymptotic null distribution was later studied by de Wet and Venter (1973), and recently by Sen (2002). Because we often also have a nuisance regression, we shall describe an extension of the Shapiro-Wilk test of normality to the situation with nuisance regression and scale parameters, constructed by Sen, Jurečková and Picek (2003). Their test is based on the pair of two estimators of the standard deviation of errors in the linear regression model, namely on the maximum likelihood estimator and on an L-estimator. Similar to the Shapiro-Wilk test, the asymptotic equivalence of these estimators is a characteristic property of the normal distribution of the errors, i.e., it is true only under the normality, and thus provides a test.

Let $Y_1, \ldots, Y_n$ be independent observations following the linear model

$$Y_i = \theta + \mathbf{x}_i'\boldsymbol{\beta} + \sigma e_i, \;\; i = 1, \ldots, n \tag{2.1}$$

where $\mathbf{x}_i \in I\!\!R^p, \;\; i = 1, \ldots, n$ are given regressors, not all equal, $\theta \in I\!\!R^1, \;\; \boldsymbol{\beta} \in I\!\!R^p$ and $\sigma > 0$ are unknown intercept, regression and scale parameters, and the errors e_i are independent and identically distributed according to a continuous distribution function F with location 0 and scale parameter 1.

We want to test the hypothesis

$$\mathbf{H}_0 : \;\; F \equiv \Phi, \qquad \text{against} \qquad \mathbf{H}_1 : \;\; F \equiv F_1 \neq \Phi \tag{2.2}$$

where Φ is the standard normal distribution function, F_1 is a general nonnormal distribution function, and $\theta, \;\boldsymbol{\beta}$, and σ are treated as nuisance parameters.

For the special location-scale model (i.e., when $\boldsymbol{\beta} = \mathbf{0}$), Shapiro and Wilk (1965) proposed a goodness-of-fit test based on two estimators of σ: L_n, the BLUE (best linear estimator) under $\mathbf{H}_0$, and $\hat{\sigma}_n$, the maximum likelihood estimator (MLE) under $\mathbf{H}_0$.

Suppose that $Y_1, \ldots, Y_n$ are *i.i.d.* observations with the distribution $\mathcal{N}(\mu, \sigma^2)$. Then the MLE of σ is $\hat{\sigma}_n$, where

$$\hat{\sigma}_n^2 = n^{-1} \sum_{i=1}^{n} (Y_i - \bar{Y}_i)^2 \tag{2.3}$$

The best linear unbiased estimate (BLUE) L_n of σ has the form

$$L_n = \sum_{i=1}^{n} a_{ni} Y_{ni} \tag{2.4}$$

where

$$\mathbf{a}' = (a_1, \ldots, a_n) = (\mathbf{M}_n' \mathbf{V}_n^{-1} \mathbf{M}_n)^{-1} (\mathbf{M}_n' \mathbf{V}_n^{-1}), \quad \mathbf{a}_n' \mathbf{1}_n = 0 \tag{2.5}$$

and where $\mathbf{M}_n = \mathbf{M}$ denotes the vector of expected values of order statistics and $\mathbf{V}_n = \mathbf{V}$ is their variance matrix. Shapiro and Wilk (1965) modified the BLUE of σ to $L_{n0} = \sum_{i=1}^{n} a_{ni,0} Y_{ni}$ where $(a_{n1,0}, \ldots, a_{nn,0})' = \mathbf{a}_{n0}$ is such that

$$\mathbf{a}_{n0}' = \frac{\mathbf{M}'\mathbf{V}^{-1}}{(\mathbf{M}'\mathbf{V}^{-1}\mathbf{V}^{-1}\mathbf{M})^{1/2}} \tag{2.6}$$

then $\mathbf{a}_n' \mathbf{1}_n = 0$ and $\mathbf{a}_{n0}' \mathbf{a}_{n0} = 1$. L_{n0} is asymptotically equivalent to

$$T_n = \frac{1}{n} \sum_{i=1}^{n} \Phi^{-1} \left(\frac{i}{n+1} \right) Y_{ni} \tag{2.7}$$

(see, e.g., Serfling (1980)). Let us write the Shapiro-Wilk criterion in the form

$$W_n = n\left(1 - \frac{L_{n0}^2}{\hat{\sigma}_n^2}\right) \tag{2.8}$$

Two scale estimators L_{n0} and $\hat{\sigma}_n$ are asymptotically equivalent if and only if $F \equiv \Phi$, i.e., if the hypothesis of normality is true, while under nonnormal alternative F_1 with the finite second moment, the sequence $\sqrt{n}\left(1 - \frac{L_{n0}^2}{\hat{\sigma}_n^2}\right)$ has a nondegenerate asymptotic (normal) distribution. It means that the test criterion is consistent with respect to the non-normal alternatives.

We propose the goodness-of-fit test of the hypothesis (2.2) of the normality, based on the observations $Y_1, \ldots, Y_n$, following the linear regression model (2.1) with unknown θ, $\boldsymbol{\beta}$ and σ. The test criterion is

$$\widehat{W}_n = n\left(1 - \frac{\hat{L}_n^2}{\hat{s}_n^2}\right) \tag{2.9}$$

where

$$\begin{aligned} \hat{s}_n^2 &= \frac{1}{n}\sum_{i=1}^{n}(Y_i - \bar{Y}_n - \widehat{\boldsymbol{\beta}}'\mathbf{x}_i)^2 \\ \hat{L}_n &= \sum_{i=1}^{n} a_{ni}^0 r_{n:i} \end{aligned} \tag{2.10}$$

are the residual variance and the linear estimator of σ with $a_{ni,0},\ i = 1, \ldots, n$ defined in (2.6) and the $r_{n:i}$ are the order statistics corresponding to the residuals

$$r_{ni} = Y_i - \bar{Y}_n - \widehat{\boldsymbol{\beta}}'\mathbf{x}_i,\ i = 1, \ldots, n \tag{2.11}$$

We assume that the $n \times p$ matrix $\mathbf{X}_n = [\mathbf{x}_1, \ldots, \mathbf{x}_n]'$ satisfies

$$\begin{aligned} &\mathbf{X}'\mathbf{1}_n = \mathbf{0}, \quad \text{Rank}(\mathbf{X}_n) = p < n - 1 \\ \text{and} \quad & \\ &\max_{1\le i\le n} h_{n,ii} = \mathcal{O}(n^{-1}) \ \text{ as } \ n \to \infty \ \text{ (the balanced design)} \end{aligned} \tag{2.12}$$

where $h_{n,ij} = \mathbf{x}_i(\mathbf{X}'\mathbf{X})^{-1}\mathbf{x}_j,\ i, j = 1, \ldots, n$. The MLE of parameters $\theta,\ \boldsymbol{\beta},\ \sigma$ under normal Φ have the form

$$\begin{aligned} \hat{\theta}_n &= \bar{Y}_n = n^{-1}\mathbf{1}_n'\mathbf{Y}_n = \theta + \bar{\mathbf{e}}_n,\ \bar{\mathbf{e}}_n = n^{-1}\mathbf{1}_n'\mathbf{e}_n \\ \widehat{\boldsymbol{\beta}}_n &= (\mathbf{X}_n'\mathbf{X}_n)^{-1}\mathbf{X}_n'\mathbf{Y}_n = \boldsymbol{\beta} + \sigma(\mathbf{X}_n'\mathbf{X}_n)^{-1}\mathbf{e}_n \\ \hat{\sigma}_n^2 &= n^{-1}\sum_{i=1}^{n}(Y_i - \hat{\theta}_n - \mathbf{x}_i'\widehat{\boldsymbol{\beta}}_n)^2 \end{aligned} \tag{2.13}$$

Sen, Jurečková and Picek (2003) proved that the asymptotic null distribution of $\widehat{W}_n$ coincides with that of W_n; hence, the test rejects the hypothesis of the normality on the asymptotic significance level α provided

$$\widehat{W}_n \ge \tau_\alpha \tag{2.14}$$

where τ_α is the asymptotic critical value of the Shapiro-Wilk test of normality with nuisance location and scale. The coefficients $a_{ni,0},\ i = 1,\dots,n$ and the critical values of the original Shapiro-Wilk test for $n \leq 50$ are tabulated in Shapiro and Wilk (1965). The critical values for $n > 50$ we have calculated by a Monte Carlo procedure.

3 Goodness-of-fit tests for general distribution with nuisance regression and scale

Consider again the linear model (2.1), but this time we have another distribution of errors $e_1,\dots,e_n$, in mind, and we want to test the hypothesis $\mathbf{H}_0 :\ F(e) \equiv F_0(e/\sigma)$, for a specified distribution function F_0. This is not necessarily normal, but for simplicity we assume that $F_0 \in \mathcal{F}$, the class of distribution functions is symmetric around 0 and possessing a positive density, finite variance and finite Fisher's information. Similarly as in the test of normality, our proposed test of $\mathbf{H}_0$ is based on the ratio of two scale statistics: the first one is based on regression rank scores $\hat{a}_{ni}(\alpha),\ i = 1,\dots,n,\ 0 \leq \alpha \leq 1$, introduced in Section 4.6, and the second one is an extension of the interquartile range to the regression quantiles.

If $F_0 \in \mathcal{F}$, we may choose the score generating function $\varphi_0(u) = F_0^{-1}(u)$ or $-f_0'(F_0^{-1}(u))/f_0(F_0^{-1}(u)),\ 0 < u < 1$ (if F_0 is strongly unimodal with finite Fisher information), because $F(e) = F_0(e/\sigma),\ F^{-1}(u) = \sigma F_0^{-1}(u)$ under $\mathbf{H}_0$. Our proposed test is based on the statistic

$$T_n^* = n^{\frac{1}{2}} \left\{ \log\ \frac{S_{n0}}{S_{n1}} - \log\ \xi(F_0) \right\} \tag{3.15}$$

where

$$S_{n0} = S_{n0}(\mathbf{Y}) = n^{-1} \sum_{i=1}^{n} Y_i \hat{b}_{ni} = n^{-1} \mathbf{Y}' \hat{\mathbf{b}}_n \tag{3.16}$$

and $\hat{\mathbf{b}}_n = (\hat{b}_{n1},\dots,\hat{b}_{nn})$ are the *regression scores* generated by φ_0 in the following way:

$$\hat{b}_{ni} = -\int_0^1 \varphi_0(u) d\hat{a}_{ni}(u), \quad i = 1,\dots,n \tag{3.17}$$

The second scale statistic S_{n1} will be based on the regression quantiles; note that the regression quantiles and regression rank scores are asymptotically independent. For simplicity, we recommend the regression interquartile range,

$$S_{n1} = \hat{\beta}_1(\tfrac{3}{4}) - \hat{\beta}_1(\tfrac{1}{4}) \tag{3.18}$$

where $\hat{\beta}_1(\alpha)$ is the first component of the α-regression quantile. Moreover, denote

$$\begin{aligned} S_0(F) &= \int_0^1 \varphi_0(u) F^{-1}(u) du \\ S_1(F) &= F^{-1}(\tfrac{3}{4}) - F^{-1}(\tfrac{1}{4}) = 2F^{-1}(\tfrac{3}{4}) \\ \xi(F) &= \frac{S_0(F)}{S_1(F)}, \quad F \in \mathcal{F} \end{aligned} \tag{3.19}$$

Note that $\xi(F_0) = \frac{S_0(F_0)}{S_1(F_0)}$ is a completely known function, because it depends on the chosen φ_0 and on the hypothetical F_0, and does not depend on σ.

Jurečková, Picek and Sen (2003) proved that the asymptotic (null) distribution of the criterion T_n^* under the hypothesis $\mathbf{H}_0$ is normal

$$T_n^* \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \tau_{01}^{*2}\right) \tag{3.20}$$

where

$$\tau_{01}^{*2} = \frac{1}{S_0^2(F_0)}\left\{\gamma_{00}^0 - 2\xi(F_0)\gamma_{01}^0 + \xi^2(F_0)\gamma_{11}^0\right\} \tag{3.21}$$

where

$$\begin{aligned}
\gamma_{00}^0 &= \tfrac{1}{4}(\mu_4 - \mu_2^2) \\
\gamma_{01}^0 &= \gamma_{10}^0 = \frac{-1}{2f_0(F_0^{-1}(\frac{3}{4}))}\left(\int_{F_0^{-1}(\frac{1}{4})}^{F_0^{-1}(\frac{3}{4})} e^2 dF_0(e) - \tfrac{1}{2}\mu_2\right) \\
\gamma_{11}^0 &= \frac{q_{11}}{4f_0^2(F_0^{-1}(\frac{3}{4}))} \qquad\qquad (3.22) \\
\mu_2 &= \int_R e^2 dF_0(e), \quad \mu_4 = \int_R e^4 dF_0(e)
\end{aligned}$$

q_{11} is the first diagonal element of the matrix $\mathbf{D}^{-1}$ where $\mathbf{D} = \lim_{n\to\infty} \mathbf{D_n} = \lim_{n\to\infty} n^{-1}\mathbf{X}'\mathbf{X}$.

Then τ_{01}^* does not depend on σ and is positive unless $\frac{Sn0}{Sn1} \equiv \xi(F_0)$ (what happens with probability 0).

We are almost ready to formulate the critical region of the test; however, we should think over alternative distributions F against which we wish to have the test consistent.

We shall introduce the following one- and two-sided alternatives of $\mathbf{H}_0$. For a pair (F_0, F) of distributions, let

$$A(F_0, F) = S_0(F)\frac{S_1(F_0)}{S_1(F)} - S_0(F_0) \tag{3.23}$$

and set the partial ordering

$$F \succ F_0 \;\text{ or }\; F \prec F_0 \;\text{ accordingly }\; A(F_0, F) \;\text{ is }\; > \;\text{ or }\; < 0$$

This partial ordering is linked to Hájek's (1969) interpretation of F *having heavier or lighter tails than* F_0. Consider the following alternatives to $\mathbf{H}_0$:

$$\mathbf{H}_1^{\succ} : \; F \succ F_0, \qquad \mathbf{H}_1^{\prec} : \; F \prec F_0, \qquad \mathbf{H}_1^{\neq} : \; \mathbf{H}_1^{\succ} \cup \mathbf{H}_1^{\prec}$$

Then

- we reject $\mathbf{H}_0$ in favor of $\mathbf{H}_1^{\succ}$ on the asymptotic significance level α if

$$\frac{T_n^*}{\tau_{01}^*} \geq u_\alpha$$

- we reject $\mathbf{H}_0$ in favor of $\mathbf{H}_1^{\prec}$ on the asymptotic significance level α if

$$\frac{T_n^*}{\tau_{01}^*} \leq -u_\alpha$$

- we reject $\mathbf{H}_0$ in favor of $\mathbf{H}_1^{\neq}$ on the asymptotic significance level α if

$$\left|\frac{T_n^*}{\tau_{01}^*}\right| \geq u_{\frac{\alpha}{2}}$$

where $u_\alpha = \Phi^{-1}(1-\alpha)$ and Φ is the standard normal distribution function.

4 Numerical illustration

4.1 Comparison of tests for testing normality

Let us illustrate the performance of the proposed test on the simulated regression model. Concerning the design matrix, we generate three columns as independent identically distributed random variables with uniform distribution on $(-10, 10)$ with the first column $\mathbf{1}_n$ added; $\boldsymbol{\beta} = (2,\ -2,\ 1, -1)'$ and consider 25 rows for it.

The errors were generated from the following densities:

normal $\mathcal{N}(0,1)$:	$f(x) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{x^2}{2}}$
normal $\mathcal{N}(0,4)$:	$f(x) = \frac{1}{4\sqrt{2\pi}}\mathrm{e}^{-\frac{x^2}{32}}$
logistic $(0,1)$:	$f(x) = \frac{\mathrm{e}^{-x}}{(1+\mathrm{e}^{-x})^2}$
logistic $(0,4)$:	$f(x) = \frac{\mathrm{e}^{-x/4}}{(1+\mathrm{e}^{-x/4})^2}$
Laplace $(0,1)$:	$f(x) = \frac{1}{2}\mathrm{e}^{-\lvert x\rvert}$
Laplace $(0,4)$:	$f(x) = \frac{4}{2}\mathrm{e}^{-4\lvert x\rvert}$
Cauchy:	$f(x) = \frac{1}{\pi(1+x^2)}$.

In order to gain insight into larger sample size behavior for our proposed tests, we also generate the design matrix of 100, 250 and 500 rows, respectively; in each case the errors e_i are generated to insure independence.

1000 replications were simulated for each case. Based on these data, we calculated the test statistics

$$\widehat{W}_n = n\left(1 - \frac{\hat{L}_n^2}{\hat{s}_n^2}\right) = n\left(1 - \frac{\left(\sum_{i=1}^n a_{ni,0} r_{n:i}\right)^2}{\sum_{i=1}^n r_{ni}^2}\right) \tag{4.24}$$

where

$$r_{ni} = D_n^{-1/2}\left(Y_i - \hat{\theta}_n - \mathbf{x}_i'\widehat{\boldsymbol{\beta}}_n\right)$$

and

$$\mathbf{a}'_{n0} = (a_{n1,0}, \ldots, a_{nn,0}) = \frac{\mathbf{M}'\mathbf{V}^{-1}}{(\mathbf{M}'\mathbf{V}^{-1}\mathbf{V}^{-1}\mathbf{M})^{1/2}} \tag{4.25}$$

Because the asymptotic null distributions of the test statistics of the Shapiro-Wilk type test $\widehat{W}_n$ are not known for $n > 50$, they were approximated by the following simple Monte Carlo procedure:
For a fixed n, a random sample of size n from the normal distribution was generated and $\widehat{W}_n$ was computed, and this random experiment was repeated 100, 000 times.

For the sake of comparison, the nonparametric test of Section 7.3 was performed on the same data for testing the normality.
Tables 1–4 give the numbers of rejections of $\mathbf{H}_0$ (among 1000 tests) for both statistics described above.

	α=0.01		α=0.05		α=0.1	
Distribution of errors	$\widehat{W}_n$	T_n^*	$\widehat{W}_n$	T_n^*	$\widehat{W}_n$	T_n^*
Normal $N(0,1)$	21	13	42	61	105	142
Normal $N(0,4)$	9	12	54	57	90	118
Logistic $(0,1)$	42	22	136	76	175	139
Logistic $(0,4)$	42	19	131	76	182	130
Laplace $(0,1)$	141	56	275	127	365	212
Laplace $(0,4)$	148	54	255	120	320	216
Cauchy $(0,1)$	838	624	900	720	920	765

Table 1: *Numbers of rejections of* $\mathbf{H}_0$ *among 1000 cases on level* α *for matrix (25×4)*

	α=0.01		α=0.05		α=0.1	
Distribution of errors	$\widehat{W}_n$	T_n^*	$\widehat{W}_n$	T_n^*	$\widehat{W}_n$	T_n^*
Normal $N(0,1)$	8	12	46	60	104	143
Normal $N(0,4)$	8	12	47	57	96	119
Logistic $(0,1)$	51	22	119	76	174	137
Logistic $(0,4)$	44	18	112	76	158	131
Laplace $(0,1)$	333	63	499	127	581	210
Laplace $(0,4)$	354	60	539	125	616	224
Cauchy $(0,1)$	1000	620	1000	730	1000	773

Table 2: *Numbers of rejections of* $\mathbf{H}_0$ *among 1000 cases on level* α *for matrix (100×4)*

	α=0.01		α=0.05		α=0.1	
Distribution of errors	$\widehat{W}_n$	T_n^*	$\widehat{W}_n$	T_n^*	$\widehat{W}_n$	T_n^*
Normal $N(0,1)$	10	9	49	56	96	107
Normal $N(0,4)$	10	13	50	57	100	113
Logistic $(0,1)$	52	88	96	206	147	302
Logistic $(0,4)$	44	86	98	197	146	289
Laplace $(0,1)$	546	669	721	836	782	890
Laplace $(0,4)$	563	687	719	832	796	891
Cauchy $(0,1)$	1000	1000	1000	1000	1000	1000

Table 3: *Numbers of rejections of* $\mathbf{H}_0$ *among 1000 cases on level* α *for matrix (250×4)*

	α=0.01		α=0.05		α=0.1	
Distribution of errors	$\widehat{W}_n$	T_n^*	$\widehat{W}_n$	T_n^*	$\widehat{W}_n$	T_n^*
Normal $N(0,1)$	10	15	56	53	102	107
Normal $N(0,4)$	9	10	56	51	100	101
Logistic $(0,1)$	56	199	96	389	142	495
Logistic $(0,4)$	49	185	95	369	136	489
Laplace $(0,1)$	869	967	937	991	960	995
Laplace $(0,4)$	856	963	926	990	939	995
Cauchy $(0,1)$	1000	1000	1000	1000	1000	1000

Table 4: *Numbers of rejections of* $\mathbf{H}_0$ *among 1000 cases on level* α *for matrix (500×4)*

4.2 Testing for nonnormal distributions

We used the test of Section 3 for verifying the following three null hypotheses:

(i) $\mathbf{H}_0: \ F \equiv \textit{logistic}$
Logistic scores for S_{n0} :

$$\varphi_0(u) = \log u - \log(1-u), \ 0 < u < 1$$

(ii) $\mathbf{H}_0: \ F \equiv \textit{normal}$
Normal scores for S_{n0} :

$$\varphi_0(u) = \Phi^{-1}(u), \ 0 < u < 1$$

where Φ is the standard normal distribution function.

(iii) $\mathbf{H}_0: \ F \equiv \textit{Laplace}$
Laplace scores for S_{n0} :)

$$\varphi_0(u) = \begin{cases} \log 2u & 0 < u < 0.5 \\ -\log(2(1-u)) & u \geq 0.5 \end{cases}$$

The errors were generated by sampling from the hypothetical F_0 (normal, logistic, Laplace), and from the following alternative densities:

normal $\mathcal{N}(0,1)$: $f(x) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{x^2}{2}}$

normal $\mathcal{N}(0,4)$: $f(x) = \frac{1}{4\sqrt{2\pi}} \mathrm{e}^{-\frac{x^2}{32}}$

logistic $(0,1)$: $f(x) = \frac{\mathrm{e}^{-x}}{(1+\mathrm{e}^{-x})^2}$

logistic $(0,4)$: $f(x) = \frac{\mathrm{e}^{-x/4}}{(1+\mathrm{e}^{-x/4})^2}$

Laplace $(0,1)$: $f(x) = \frac{1}{2} \mathrm{e}^{-|x|}$

Laplace $(0,4)$: $f(x) = \frac{4}{2} \mathrm{e}^{-4|x|}$

Cauchy: $f(x) = \frac{1}{\pi(1+x^2)}$

1000 replications were simulated for each case. Based on these data, we calculated the test statistics

$$T_n^* = n^{\frac{1}{2}} \left\{ \log \frac{S_{n0}}{S_{n1}} - \log \ \xi(F_0) \right\}$$

for the hypothesis $\mathbf{H}_0: \ F \equiv F_0, \ (\boldsymbol{\beta}, \sigma$ unspecified).

We took the regression interquartile range $\hat{\beta}_1(\frac{3}{4}) - \hat{\beta}_1(\frac{1}{4})$ in the role of S_{n1}.

Tables 5–7 give the numbers of rejections of $\mathbf{H}_0$ (among 1000 tests) for all above cases.

I. $\mathbf{H}_0 : F \equiv$ LOGISTIC (0,1) (i.e., we used logistic scores).

Distribution	n=27			n=108		
	α=0.01	α=0.05	α=0.1	α=0.01	α=0.05	α=0.1
Normal $N(0,1)$	19	124	213	45	163	279
Normal $N(0,4)$	18	92	197	45	187	286
Logistic $(0,1)$	24	93	169	12	61	127
Logistic $(0,4)$	18	86	181	10	62	114
Laplace $(0,1)$	21	87	149	84	199	286
Laplace $(0,4)$	22	100	169	88	210	298
Cauchy $(0,1)$	513	632	687	999	1000	999

Table 5: *Numbers of rejections of* $\mathbf{H}_0$ *among 1000 cases on level* α *for matrix (25×4) and for matrix* (100×3)

II. $\mathbf{H}_0 : F \equiv$ NORMAL (0,1) (i.e., we used van der Waerden scores).

Distribution	n=27			n=108		
	α=0.01	α=0.05	α=0.1	α=0.01	α=0.05	α=0.1
Normal $N(0,1)$	11	60	142	12	52	116
Normal $N(0,4)$	11	59	129	11	53	112
Logistic $(0,1)$	22	77	136	40	116	183
Logistic $(0,4)$	18	77	132	38	122	192
Laplace $(0,1)$	55	129	212	337	516	611
Laplace $(0,4)$	53	124	216	331	509	607
Cauchy $(0,1)$	626	722	765	1000	1000	1000

Table 6: *Numbers of rejections of* $\mathbf{H}_0$ *among 1000 cases on level* α *for matrix (25×4) and for matrix* (100×3)

III. $\mathbf{H}_0 : F \equiv$ LAPLACE (0,1) (i.e., we used Laplace scores).

	n=27			n=108		
Distribution	α=0.01	α=0.05	α=0.1	α=0.01	α=0.05	α=0.1
Normal $N(0,1)$	43	259	424	366	689	806
Normal $N(0,4)$	42	251	378	365	687	802
Logistic $(0,1)$	37	189	319	153	378	516
Logistic $(0,4)$	44	203	337	144	375	511
Laplace $(0,1)$	19	121	189	11	69	135
Laplace $(0,4)$	24	131	231	9	73	145
Cauchy $(0,1)$	342	452	515	971	991	1000

Table 7: *Numbers of rejections of* $\mathbf{H}_0$ *among 1000 cases on level* α *for for matrix (25×4) and for matrix* (100×3)

References:

C. Huber-Carol, N. Balakrishnan, M. S. Nikulin, M. Mesbah (eds.) (2002). *Goodness-of-Fit Tests and Model Validity.* Birkhäuser, Boston.

J. Jurečková and J. Picek (2005): *Robust Statistical Methods with R.* Chapman & Hall/CRC.

J. Jurečková, J. Picek and P. K. Sen (2003). Goodness-of-fit test with nuisance regression and scale. *Metrika* 58, 235–258.

J. Jurečková and P. K. Sen (1996). *Robust Statistical Procedures: Asymptotics and Interrelations.* John Wiley & Sons, New York.

P. Royston (1982a). An extension of Shapiro and Wilk's W test for normality to large samples. *Appl. Statistics* 31, 115–124.

P. Royston (1982b). Algorithm AS 181: The W test for normality. *Appl. Statistics* 31, 176–180.

P. Royston (1995). A remark on algorithm AS 181: The W test for normality. *Appl. Statistics* 44, 547–551.

P. K. Sen (2002). Shapiro-Wilk type goodness-of-fit tests for normality: Asymptotics revisited. *Goodness-of-Fit Tests and Model Validity* (C. Huber-Carol et al., eds.), pp. 73–88. Birkhäuser, Boston.

P. K. Sen, J. Jurečková and J. Picek (2003). Goodness-of-fit test of Shapiro-Wilk type with nuisance regression and scale. *Austrian J. of Statist.* 32, No. 1 & 2, 163–177.

S. S. Shapiro and M. B. Wilk (1965). An analysis of variance for normality (complete samples). *Biometrika* 52, 591–611.