Hence $\mathbf{v}' = ie\mathbf{E}/m\omega$. The displacement $\mathbf{r}$ of the electron due to the field is given by $\dot{\mathbf{r}} = \mathbf{v}'$, and therefore $\mathbf{r} = -e\mathbf{E}/m\omega^2$. The polarization $\mathbf{P}$ of the body is the dipole moment per unit volume. Summing over all electrons, we find $\mathbf{P} = \Sigma e\mathbf{r} = -e^2 N\mathbf{E}/m\omega^2$, where N is the number of electrons in all the atoms in unit volume of the substance. By the definition of the electric induction, we have $\mathbf{D} = \varepsilon\mathbf{E} = \mathbf{E} + 4\pi\mathbf{P}$. We thus have the formula

$$\varepsilon(\omega) = 1 - 4\pi Ne^2/m\omega^2. \tag{78.1}$$

The range of frequencies over which this formula is applicable begins, in practice, at the far ultra-violet for light elements and at the X-ray region for heavier elements.

If $\varepsilon(\omega)$ is to retain the significance which it has in Maxwell's equations, the frequency must also satisfy the condition $\omega \ll c/a$. We shall see later (§124), however, that the expression (78.1) can be allotted a certain physical significance even at higher frequencies.

§79. The dispersion of the magnetic permeability

Unlike $\varepsilon(\omega)$, the magnetic permeability $\mu(\omega)$ ceases to have any physical meaning at relatively low frequencies. To take account of the deviation of $\mu(\omega)$ from unity would then be an unwarrantable refinement. To show this, let us investigate to what extent the physical meaning of the quantity $\mathbf{M} = (\mathbf{B} - \mathbf{H})/4\pi$, as being the magnetic moment per unit volume, is maintained in a variable field. The magnetic moment of a body is, by definition, the integral

$$\frac{1}{2c}\int \mathbf{r}\times\overline{\rho\mathbf{v}}\,\mathrm{d}V. \tag{79.1}$$

The mean value of the microscopic current density is related to the mean field by equation (75.7):

$$\mathbf{curl}\,\mathbf{B} = \frac{4\pi}{c}\overline{\rho\mathbf{v}} + \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}. \tag{79.2}$$

Subtracting the equation $\mathbf{curl}\,\mathbf{H} = (1/c)\,\partial\mathbf{D}/\partial t$, we obtain

$$\overline{\rho\mathbf{v}} = c\,\mathbf{curl}\,\mathbf{M} + \partial\mathbf{P}/\partial t. \tag{79.3}$$

The integral (79.1) can, as shown in §29, be put in the form $\int\mathbf{M}\,\mathrm{d}V$ only if $\overline{\rho\mathbf{v}} = c\,\mathbf{curl}\,\mathbf{M}$ and $\mathbf{M} = 0$ outside the body.

Thus the physical meaning of $\mathbf{M}$, and therefore of the magnetic susceptibility, depends on the possibility of neglecting the term $\partial\mathbf{P}/\partial t$ in (79.3). Let us see to what extent the conditions can be fulfilled which make this neglect permissible.

For a given frequency, the most favourable conditions for measuring the susceptibility are those where the body is as small as possible (to increase the space derivatives in $\mathbf{curl}\,\mathbf{M}$) and the electric field is as weak as possible (to reduce $\mathbf{P}$). The field of an electromagnetic wave does not satisfy the latter condition, because $E \sim H$. Let us therefore consider a variable magnetic field, say in a solenoid, with the body under investigation placed on the axis. The electric field is due only to induction by the variable magnetic field, and the order of magnitude of E inside the body can be obtained by estimating the terms in the equation $\mathbf{curl}\,\mathbf{E} = -(1/c)\partial\mathbf{B}/\partial t$, whence $E/l \sim \omega H/c$ or $E \sim (\omega l/c)H$, where l is the dimension of the body. Putting $\varepsilon - 1 \sim 1$, we have $\partial P/\partial t \sim \omega E \sim \omega^2 lH/c$. For the space derivatives of the magnetic moment $\mathbf{M} = \chi\mathbf{H}$ we have $c\,\mathbf{curl}\,\mathbf{M} \sim c\chi\mathbf{H}/l$. If $|\partial\mathbf{P}/\partial t|$ is small compared with $|c\,\mathbf{curl}\,\mathbf{M}|$, we must have

$$l^2 \ll \chi c^2/\omega^2. \tag{79.4}$$

It is evident that the concept of magnetic susceptibility can be meaningful only if this inequality allows dimensions of the body which are (at least) just macroscopic, i.e. if it is compatible with the inequality $l \gg a$, where a is the atomic dimension. This condition is certainly not fulfilled for the optical frequency range; for such frequencies, the magnetic susceptibility is always $\sim v^2/c^2$, where v is the electron velocity in the atom;† but the optical frequencies themselves are $\sim v/a$, and therefore the right-hand side of the inequality (79.4) is $\sim a^2$.

Thus there is no meaning in using the magnetic susceptibility from optical frequencies onward, and in discussing such phenomena we must put $\mu = 1$. To distinguish between **B** and **H** in this frequency range would be an over-refinement. Actually, the same is true for many phenomena even at frequencies well below the optical range.‡

The presence of a considerable dispersion of the permeability makes possible the existence of quasi-steady oscillations of the magnetization in ferromagnetic bodies. In order to exclude the possible influence of the conductivity, we shall consider ferrites, which are non-metallic ferromagnets.

The term "quasi-steady" means, as usual (§58), that the frequency is assumed to satisfy the condition $\omega \ll c/l$, where l is the characteristic dimension of the body (or the "wavelength" of the oscillation). In addition, we shall neglect the exchange energy related to the inhomogeneity of the magnetization resulting from the oscillations; that is, the spatial dispersion (§103) of the permeability is assumed to be unimportant. For this, the dimensions l must be much greater than the characteristic length for the inhomogeneity energy: $l \gg \sqrt{\alpha}$, where α is of the order of the coefficients in (43.1).

We can put **H** and **B** in the forms $\mathbf{H} = \mathbf{H}_0 + \mathbf{H}'$, $\mathbf{B} = \mathbf{B}_0 + \mathbf{B}'$, where $\mathbf{H}_0$ and $\mathbf{B}_0$ are the field and induction in the statically magnetized body, $\mathbf{H}'$ and $\mathbf{B}'$ the variable parts in the oscillations. When the displacement current is neglected, these variable parts satisfy the equations

$$\mathbf{curl}\,\mathbf{H}' = 0, \quad \operatorname{div}\mathbf{B}' = 0, \tag{79.5}$$

which differ from the magnetostatic equations only in that the permeability is now (for a monochromatic field $\propto e^{-i\omega t}$) a function of the frequency, not a constant.§ A ferromagnetic medium is magnetically anisotropic, and its permeability is therefore a tensor $\mu_{ik}(\omega)$, which determines the linear relation between the variable parts of the induction and the field.

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† This estimate relates to the diamagnetic susceptibility; the relaxation times of any paramagnetic or ferromagnetic processes are certainly long compared with the optical periods. It must be emphasized, however, that the estimates are made for an isotropic body, and must be used with caution when applied to ferromagnets. In particular, the gyrotropic terms in the tensor μ_{ik} which decrease only slowly (as $1/\omega$) with increasing frequency (see Problem 1) may be important even at fairly high frequencies.

‡ This is discussed from a somewhat different standpoint in §103 below; see the second footnote to that section.

§ These oscillations are therefore called *magnetostatic oscillations*. The theory has been given by C. Kittel (1947) for homogeneous (see below) magnetostatic oscillations and by L. R. Walker (1957) for inhomogeneous ones.

The first equation (79.5) shows that the magnetic field has a potential: $\mathbf{H}' = -\mathbf{grad}\,\psi$. Substituting in the second equation $B'_i = \mu_{ik}H'_k = -\mu_{ik}\,\partial\psi/\partial x_k$, we then obtain an equation for the potential within the body:

$$\mu_{ik}(\omega)\partial^2\psi/\partial x_i\,\partial x_k = 0. \tag{79.6}$$

Outside the body, the potential satisfies Laplace's equation $\triangle\psi = 0$; on the surface, $\mathbf{H}_t$ and B'_n must as usual be continuous. The first condition is equivalent to the continuity of the potential ψ itself; the second implies the continuity of $\mu_{ik}n_i\partial\psi/\partial x_k$, where $\mathbf{n}$ is a unit vector along the normal to the surface. Far from the body, we must have $\psi \to 0$.

The problem thus formulated has non-trivial solutions only for certain values of the μ_{ik} regarded as parameters. Equating the functions $\mu_{ik}(\omega)$ to these, we find the natural oscillation frequencies of the magnetization of the body, called the *inhomogeneous ferromagnetic resonance* frequencies.

The simplest type of magnetostatic oscillation of a uniformly magnetized ellipsoid consists in oscillations which maintain the uniformity, the magnetization oscillating as a whole. To find their frequencies, it is not necessary to obtain a new solution of the field equations; they can be derived directly from the relations (29.14):

$$H_i + n_{ik}(B_k - H_k) = \mathfrak{H}_i, \tag{79.7}$$

where n_{ik} is the demagnetizing factor tensor of the ellipsoid; $\mathbf{H}$ and $\mathbf{B}$ relate to the field within it, and $\mathfrak{H}$ is the external magnetic field. The latter is assumed to be uniform; in $\mathbf{H}$ and $\mathbf{B}$, we again separate the oscillatory parts $\mathbf{H}'$ and $\mathbf{B}'$, which are now uniform throughout the body. For these we have

$$H'_i + n_{ik}(B'_k - H'_k) = 0$$

or

$$(\delta_{ik} + 4\pi n_{il}\chi_{lk})H'_k = 0,$$

with the magnetic susceptibility tensor $\chi_{ik}(\omega)$ defined by $\mu_{ik} = \delta_{ik} + 4\pi\chi_{ik}$. Equating to zero the determinant of this system of linear homogeneous equations, we find

$$\det|\delta_{ik} + 4\pi n_{il}\chi_{lk}(\omega)| = 0, \tag{79.8}$$

the roots of which give the natural oscillation frequencies. These are called the *homogeneous ferromagnetic resonance* frequencies.

PROBLEMS

PROBLEM 1. Using the macroscopic equation of motion of the magnetic moment (the Landau–Lifshitz equation; see *SP* 2, (69.9)), derive the magnetic permeability tensor for a uniformly magnetized uniaxial ferromagnet of the easy-axis type, in the absence of dissipation (L. D. Landau and E. M. Lifshitz, 1935).

SOLUTION. The equation of motion of the magnetization in a ferromagnet is

$$\dot{\mathbf{M}} = \gamma(\mathbf{H} + \beta M_z \boldsymbol{\nu}) \times \mathbf{M},$$

where $\gamma = g|e|/2mc$ (g being the gyromagnetic ratio), $\beta > 0$ the anisotropy coefficient, and $\boldsymbol{\nu}$ a unit vector along the axis of easy magnetization (the z-axis). We write $\mathbf{H} = \mathbf{H}_0 + \mathbf{H}'$, where $\mathbf{H}'$ is a small variable field in any direction, and $\mathbf{H}_0$ a constant field which we take to be along the z-axis.† The transverse magnetization M_x, M_y due

† This field is used here with a view to applying the results in the subsequent Problems.