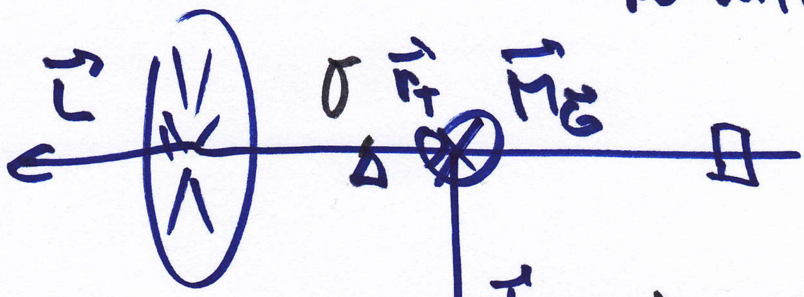
16. přednáška

13. 11. 09

Dobrý den!

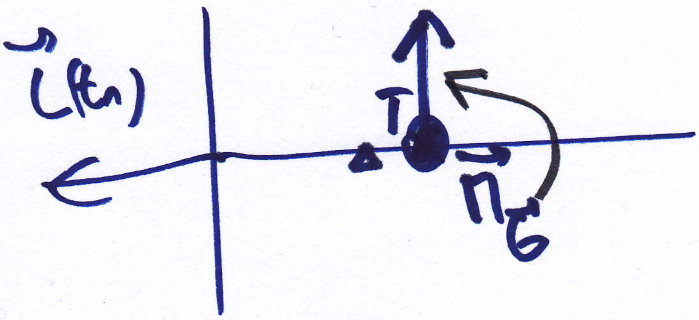
... pokračování

a)



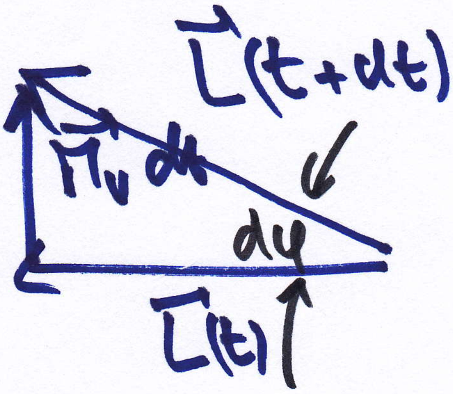
$$\dot{\vec{L}} = \frac{d\vec{L}}{dt} = \vec{M}_v^{ext}$$

pohled „shora“:

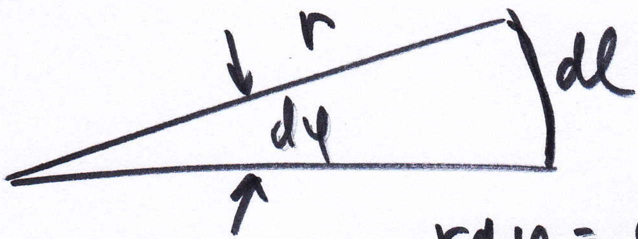


$$d\vec{L} = \vec{M}_v\,dt$$

$$\vec{L}(t+dt) - \vec{L}(t) = \vec{M}_v\,dt$$



$$\Omega = \frac{d\varphi}{dt}$$



$$r\,d\varphi = dl$$

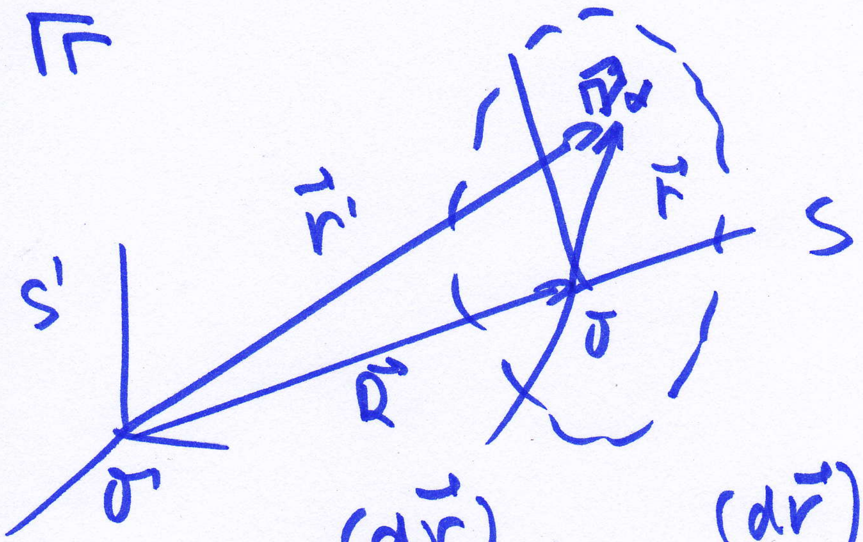
$$L\,d\varphi = M_v\,dt$$

$$I_{obruče}\,\omega\,d\varphi = Mg\,r_T\,dt$$

$$\frac{d\varphi}{dt} = \Omega = \frac{Mg\,r_T}{I_{ob}\,\omega}$$

nepovinné:

ΠΓ



$(d\vec{r})_{rot}$ $(d\vec{r})_{fix}$

$$\vec{v} = \vec{V} + \vec{v}_r + \vec{\omega}\times\vec{r}$$

tuhé těleso: $\vec{v}_r = \vec{0}$ $\qquad \vec{r}'_\alpha = \vec{R} + \vec{r}_\alpha$

$$\vec{L} = \sum_\alpha \vec{l}_\alpha = \sum_\alpha m_\alpha \vec{r}'_\alpha \times \vec{v}'_\alpha =$$

$$\sum_\alpha m_\alpha (\vec{R} + \vec{r}_\alpha) \times \vec{v}'_\alpha =$$

$$\dot{\vec{r}}'_\alpha = \dot{\vec{R}} + \dot{\vec{r}}_\alpha$$

$$= \sum_\alpha m_\alpha (\vec{R} + \vec{r}_\alpha) \times (\dot{\vec{R}} + \dot{\vec{r}}_\alpha) =$$

$$= \sum_\alpha m_\alpha \left[\underline{(\vec{R}\times\dot{\vec{R}})} + (\vec{r}_\alpha \times \dot{\vec{r}}_\alpha) + (\vec{R}\times\vec{v}_\alpha) + \underline{(\vec{R}\times\dot{\vec{r}}_\alpha)} \right] =$$

$$= \underbrace{\left(\sum_\alpha m_\alpha\right)}_{M} \vec{R}\times\dot{\vec{R}} + \sum_\alpha m_\alpha \vec{R}\times\dot{\vec{r}}_\alpha$$

$$\sum_\alpha m_\alpha \left[(\vec R \times \dot{\vec R}) + \underbrace{(\vec R \times \dot{\vec r}_\alpha) + (\vec r_\alpha \times \dot{\vec R})}_{2.} + (\vec r_\alpha \times \dot{\vec r}_\alpha) \right] =$$

$$= \left(\sum_\alpha m_\alpha\right) \vec R \times \dot{\vec R} + \sum_\alpha m_\alpha \vec r_\alpha \times \dot{\vec r}_\alpha$$

$$= \vec R \times \underbrace{M \dot{\vec R}}_{\vec P} + \underbrace{\sum_\alpha \vec r_\alpha \times m_\alpha \vec v_\alpha}_{\vec L_{\text{vůči S (těžišti)}}}$$

$$\boxed{\vec L_0 = \vec R \times \vec P + \vec L_T}$$

. . . .

$$\sum_\alpha m_\alpha \vec R \times \dot{\vec r}_\alpha = \vec R \times \sum_\alpha m_\alpha \dot{\vec r}_\alpha \;..\; \vec 0$$

$$\sum_\alpha m_\alpha \vec r_\alpha \times \dot{\vec R} = \left(\sum_\alpha m_\alpha \vec r_\alpha\right) \times \dot{\vec R} \;..\; \vec 0$$

$$\frac{d}{dt}\left(\sum_\alpha m_\alpha \vec R \times \vec r_\alpha\right) = \underbrace{\sum_\alpha m_\alpha \dot{\vec R} \times \vec r_\alpha}_{\dot{\vec R} \times \underbrace{\sum_\alpha m_\alpha \vec r_\alpha}_{0}} + \sum_\alpha m_\alpha \vec R \times \dot{\vec r}_\alpha$$

kinetická energie soustavy hmotných bodů

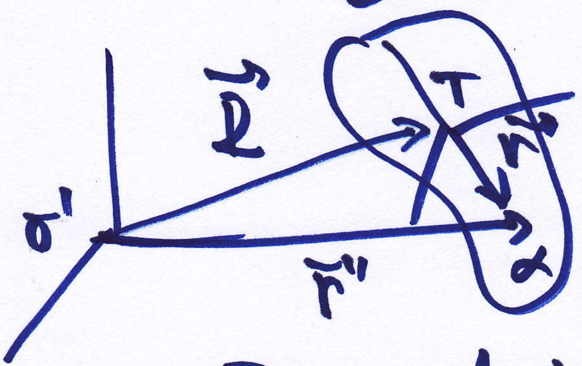
<u>TUHÉ</u> těleso:

$$E_k = \sum_\alpha \frac{1}{2} m_\alpha v_\alpha^2 = \frac{1}{2}\sum_\alpha m_\alpha (\vec{V} + \vec{\omega}\times\vec{r}_\alpha)^2$$

$$= \frac{1}{2}\sum_\alpha m_\alpha (\vec{V} + \vec{\omega}\times\vec{r}_\alpha)\cdot(\vec{V} + \vec{\omega}\times\vec{r}_\alpha)$$

$$= \frac{1}{2}\underbrace{\sum_\alpha m_\alpha}_{M} V^2 + \frac{1}{2}\sum_\alpha m_\alpha 2\,\vec{V}\cdot(\vec{\omega}\times\vec{r}_\alpha) +$$

$$+ \frac{1}{2}\sum_\alpha m_\alpha (\vec{\omega}\times\vec{r}_\alpha)^2$$



těžiště

$$E_k = \frac{1}{2}MV^2 + \frac{1}{2}\cdot 2\cdot\vec{V}\cdot\vec{\omega}\times\overbrace{\left(\sum_\alpha m_\alpha \vec{r}_\alpha\right)}^{\vec{0}}$$

$$+ \frac{1}{2}\sum_\alpha m_\alpha (\vec{\omega}\times\vec{r}_\alpha)^2$$

$$= E_{k,\text{transl}} + E_{k,\text{rotační}}$$

$$(\vec{A}\times\vec{B})\cdot(\vec{A}\times\vec{B}) = A^2B^2 - (\vec{A}\cdot\vec{B})^2$$

$$E_{k,rot} = \frac{1}{2}\sum m_\alpha (\vec{\omega}\times\vec{r_\alpha})^2 =$$

$$= \frac{1}{2}\sum_\alpha m_\alpha \left[\omega^2 r_\alpha^2 - (\vec{\omega}\cdot\vec{r_\alpha})^2\right]$$

$\vec{\omega} = (\omega_1, \omega_2, \omega_3)$ $\quad \vec{r_\alpha} = (r_{\alpha,1}; r_{\alpha,2}; r_{\alpha,3})$

$$\omega^2 = \omega_1\cdot\omega_1 + \omega_2\omega_2 + \omega_3\omega_3 = \sum_i \omega_i^2$$

$$r_\alpha^2 = \sum_k r_{\alpha,k}^2$$

$$\vec{\omega}\cdot\vec{r_\alpha} = \sum_i \omega_i r_{\alpha,i}$$

$$(\vec{\omega}\cdot\vec{r_\alpha})^2 = \sum_i \omega_i r_{\alpha,i} \cdot \sum_j \omega_j r_{\alpha,j}$$

$$E_{k,rot} = \frac{1}{2}\sum m_\alpha \left[\left(\sum_i \omega_i^2\right)\left(\sum_k r_{\alpha,k}^2\right) - \sum_i \omega_i r_{\alpha,i} \sum_j \omega_j r_{\alpha,j}\right]$$

$$\omega_i^2 = \omega_i \omega_j \delta_{ij}$$

$$= \frac{1}{2}\sum_\alpha m_\alpha \left[\sum_i \underline{\omega_i \omega_j} \delta_{ij} \sum_k r_{\alpha,k}^2 - \sum_{i,j} \underline{\omega_i \omega_j} r_{\alpha,i} r_{\alpha,j}\right]$$

$$= \frac{1}{2}\sum_{i,j} \omega_i \omega_j \sum m_\alpha \left(\delta_{ij}\sum_k r_{\alpha,k}^2 - r_{\alpha,i} r_{\alpha,j}\right)$$

$$I_{ij} = \sum_\alpha m_\alpha \left(\delta_{ij}\sum_k r_{\alpha,k}^2 - r_{\alpha,i} r_{\alpha,j}\right)$$

$$E_{k,rot} = \frac{1}{2}\sum_{i,j} I_{ij}\,\omega_i \omega_j$$

$\vec{L} = \{I\}\cdot\vec{\omega}$ matica 3x3

$E_{k,rot} = \frac{1}{2}\vec{\omega}\cdot\vec{L} = \frac{1}{2}\vec{\omega}\cdot\{I\}\cdot\vec{\omega}$

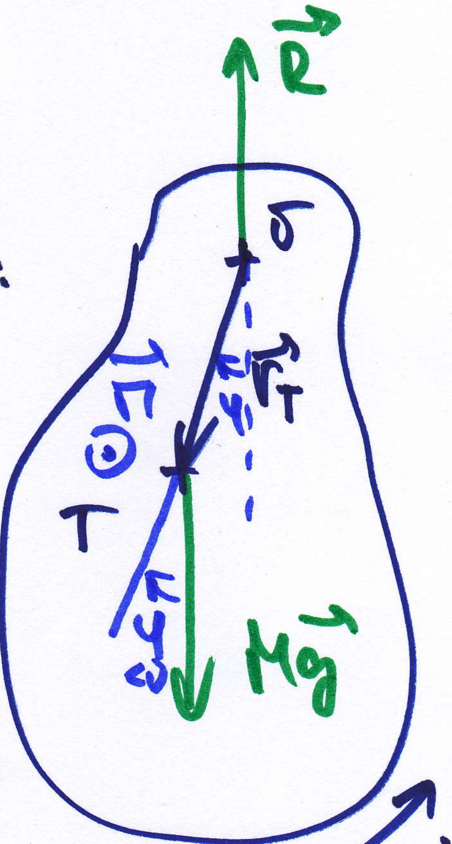
vektor

konec „nepovinné" $\Downarrow$

KYVADLA:

fyzické:

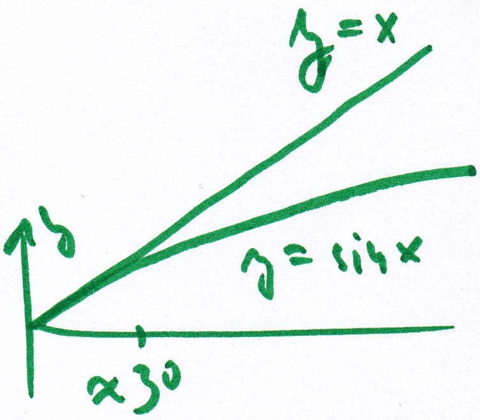
$\downarrow\vec{g}$



$\vec{M}_V^{ext} = I\vec{\varepsilon}$

$\vec{r}_T \times M\vec{g} = I\vec{\varepsilon}$

$-r_T M g \sin\varphi\,(\vec{k}) = I\varepsilon\vec{k} = I\ddot{\varphi}\vec{k}$



$I\ddot{\varphi} + r_T M g \sin\varphi = 0$

$\ddot{\varphi}(t) + \frac{r_T M g}{I}\sin\varphi(t) = 0$

malé uhly

$\ddot{\varphi} + \Box^2\varphi = 0$

$\varphi(t) = A\sin(\Box t + \alpha)$

$\frac{2\pi}{T} = \omega = \Box \Rightarrow T = 2\pi\ldots$

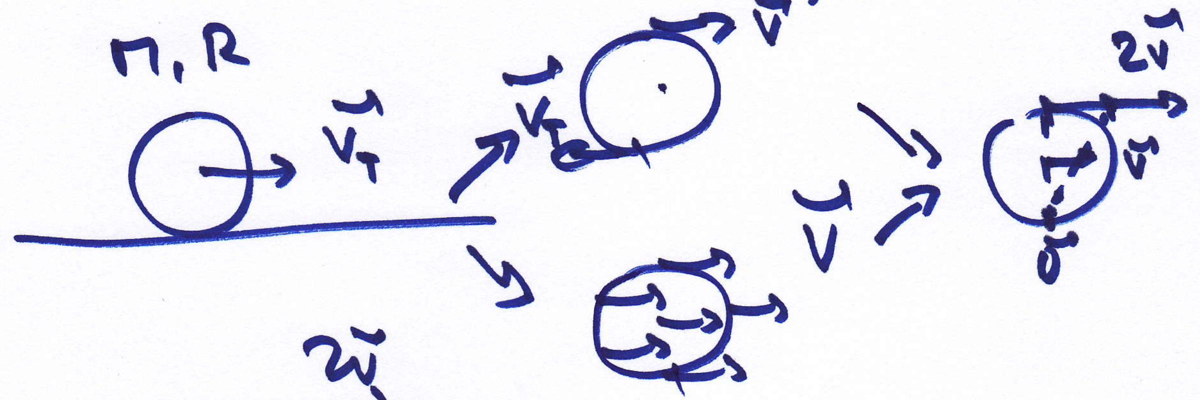
$T = 2\pi\sqrt{\frac{I}{r_T M g}}$

Matematické kyvadlo: DÚ

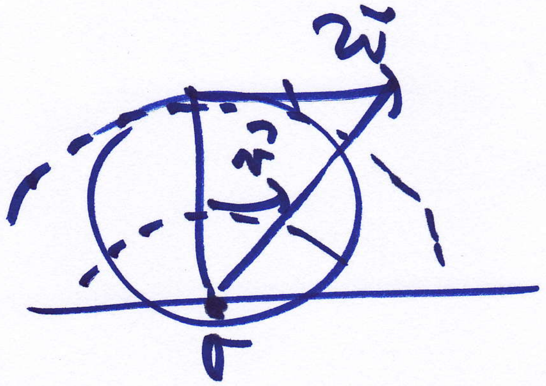
Rotace & valení

Pohyb
- „čistá" rotace + „čistá" translace i)
- rotace kolem ok. osy otáčení ii)

i)



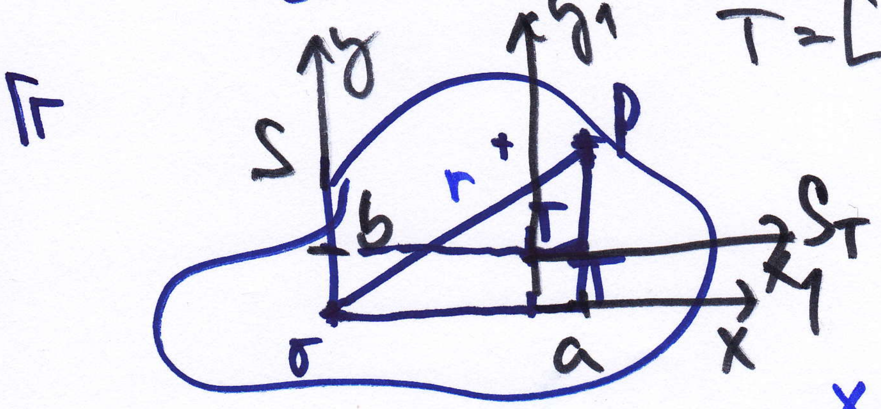
ii)



$$\omega = \frac{2v}{2R} = \frac{v}{R}$$

i):
$$E_k = \frac{1}{2} I_T \omega^2 + \frac{1}{2} M v^2 = \frac{1}{2}\left(\frac{1}{2} M \cancel{R^2}\right)\frac{v^2}{\cancel{R^2}} + \frac{1}{2} M v^2$$
$$= \underline{\underline{\frac{3}{4} M v^2}}$$

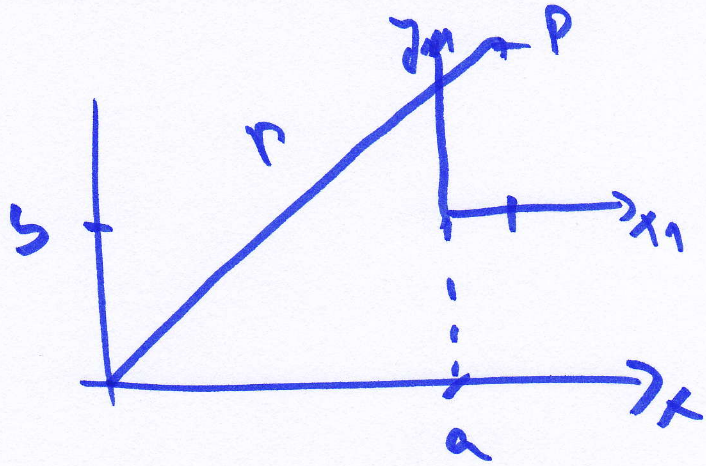
ii)
$$E_k = \frac{1}{2} I_O \omega^2 = \frac{1}{2}\left(\frac{1}{2} M R^2 + M R^2\right)\omega^2 = \underline{\underline{\frac{3}{4} M v^2}}$$



$T = [a, b]$

$$I_O = \int_M r^2 \, dm = \int_M (x^2 + y^2)\, dm$$

$x_1 = x - a \quad y_1 = y - b$



$x_1 = x - a$

$y_1 = y - a$

$$\int_M (x^2 + y^2)\,dm = \int_M x^2\,dm + \int y^2\,dm$$

$$= \int (x_1 + a)^2\,dm + \int (y_1 + b)^2\,dm =$$

$$= \int x_1^2\,dm + \int 2x_1 a\,dm + \int a^2\,dm +$$

$$\int y_1^2\,dm + 2\int y_1 b\,dm + \int b^2\,dm =$$

T ťažisti:

$$\int x_1\,dm = \int y_1\,dm = 0$$

$$\int (x_1^2 + y_1^2)\,dm = \int r_1^2\,dm = I_T \Downarrow$$

$$= I_T + \underbrace{(a^2 + b^2)}_{d^2} M$$

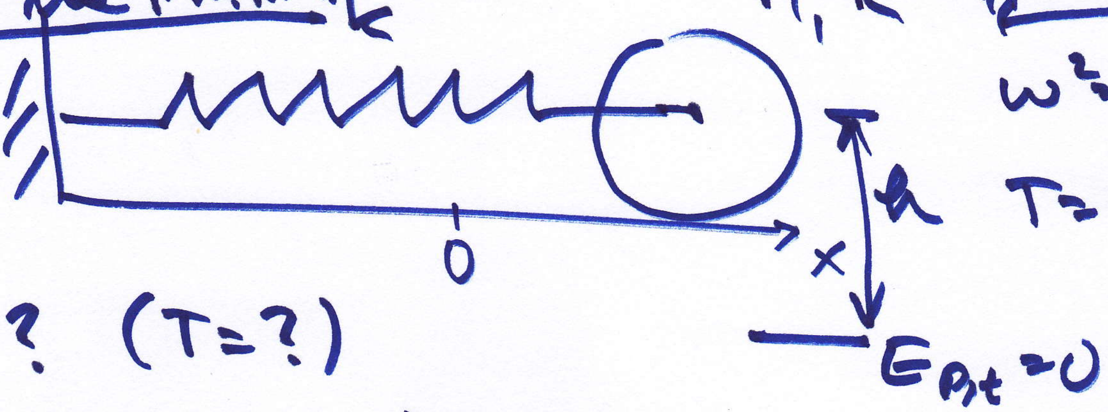
d^2 .. vzdialenosť os J a T

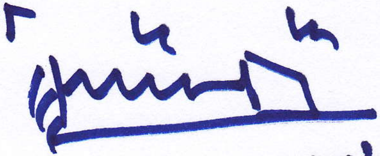
Steinerova veta

$$\boxed{I_J = I_T + md^2}$$

„čertovný" příklad

válec na pružině





$\omega^2 = \frac{k}{m} = \frac{4\pi^2}{T^2}$

$T = 2\pi\sqrt{\frac{m}{k}}$

$\omega = ?$ $(T = ?)$

$$E_M(x,\dot{x}) = mgh + \frac{1}{2}kx^2 + \frac{3}{4}M\dot{x}^2 = \text{konst.}$$

$$\frac{dE_M(x,\dot{x})}{dt} = 0 = \frac{\partial E_M}{\partial x}\frac{dx}{dt} + \frac{\partial E_M}{\partial \dot{x}}\frac{d\dot{x}}{dt}$$

$$kx\dot{x} + \frac{3}{2}M\dot{x}\ddot{x} = 0$$

$$\dot{x}\left(kx + \frac{3}{2}M\ddot{x}\right) = 0$$

$\dot{x} = 0 \Rightarrow x = \text{konst}$

$kx + \frac{3}{2}M\ddot{x} = 0$

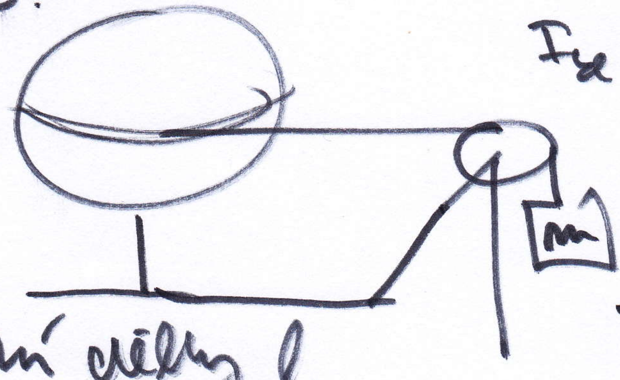
$$\ddot{x} + \frac{2k}{3M}x = 0 \Rightarrow \omega = \sqrt{\frac{2}{3}\frac{k}{M}} = \frac{2\pi}{T} \ldots$$

další „čertovný" příklad

slupka: $v_0 = 0$

$I = \frac{2}{3}MR_{sl}^2$

$I_{kl} = \frac{1}{2}M_{kl}R_{kl}^2$

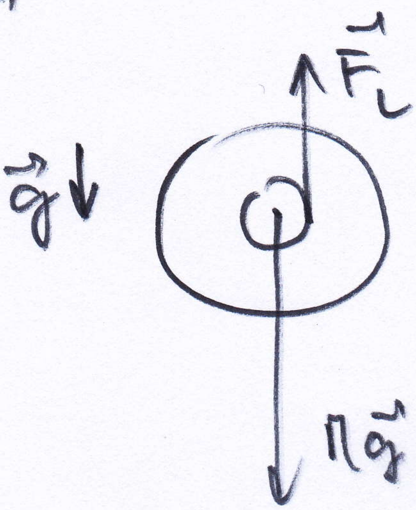


$d\vec{g}$

$\vec{v} = ?$ po uražení délky l

DÚ (77ME)

„jojo"



r_1, r_2

$\downarrow x$

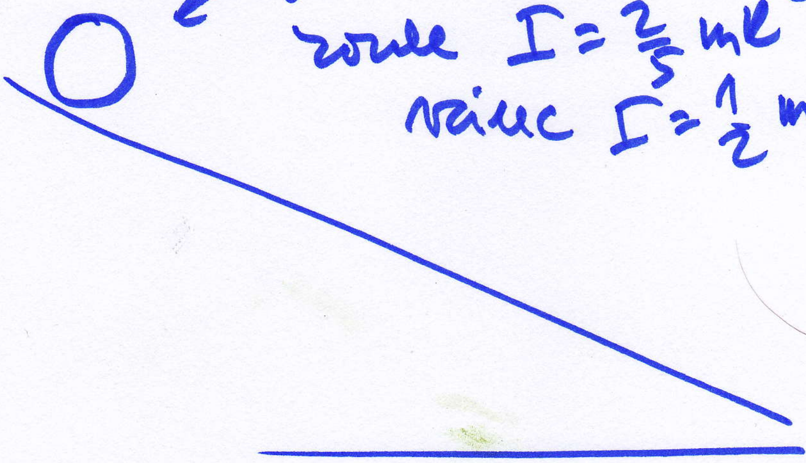
$\ddot{x} = ?$

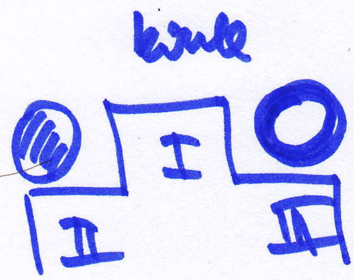
„závody" těles

obruč $I = mR^2$

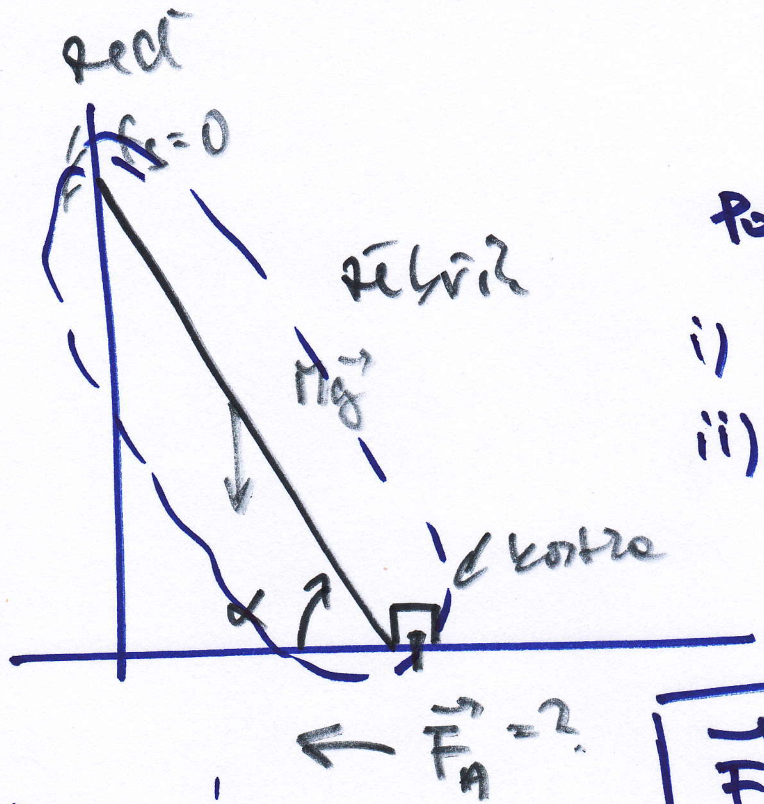
koule $I = \frac{2}{5} mR^2$

válec $I = \frac{1}{2} mR^2$





STATICKÁ ROVNOVÁHA TĚLES



Podmínky:

$$\boxed{\begin{array}{lll} \text{i)} & \vec{a}_T = \vec{0} & (\vec{v}_T(0) = \vec{0}) \\ \text{ii)} & \vec{\varepsilon} = \vec{0} & (\vec{\omega}_0 = \vec{0}) \end{array}}$$

$$\boxed{\begin{array}{l} \vec{F}_V^{ext} = \vec{0} \\ \vec{M}_V^{ext} = \vec{0} \end{array}} \begin{array}{l} \dots M\vec{a}_T \\ \dots I\vec{\varepsilon} \end{array}$$

$$\vec{N}_2 + M\vec{g} + \vec{N}_1 + \vec{F}_A = \vec{0}$$

$$\vec{0} \times \vec{F}_A + \vec{0} \times \vec{N}_2 + \vec{l}/2 \times M\vec{g} + \vec{l} \times \vec{N}_1 = \vec{0}$$

DÚ ... dořešit!