$$\|x\| = \sqrt{\langle x, x\rangle}$$

$$\|x\| = 0 \iff \langle x, x\rangle = 0 \iff x = 0$$

$$x \neq 0 \Rightarrow \langle x, x\rangle > 0$$

$$\|c\,x\| = \sqrt{\langle c\,x, c\,x\rangle} = \sqrt{c\langle x, c\,x\rangle} =$$

$$= \sqrt{c^2\langle x, x\rangle} = |c|\sqrt{\langle x, x\rangle} = |c|\,\|x\|$$

$$\|x+y\| \le \|x\| + \|y\| \qquad (\;)^2$$

$$\|x+y\|^2 = \langle x+y,\, x+y\rangle = \langle x, x\rangle + \langle x, y\rangle$$

$$+ \langle y, x\rangle + \langle y, y\rangle = \|x\|^2 + 2\langle x, y\rangle$$

$$+ \|y\|^2 \qquad \text{C.B.S}$$

$$(\|x\| + \|y\|)^2 = \|x\|^2 + 2\|x\|\|y\| + \|y\|^2$$

$$\|x+y\|^2 = \|x\|^2 + 2\underline{\langle x, y \rangle} + \|y\|^2 =$$

$$\cos\alpha = \frac{\langle x, y \rangle}{\|x\| \|y\|}$$

$x \neq 0$
$y \neq 0$

$$= \|x\|^2 + 2\cos\alpha \|x\| \|y\| + \|y\|^2$$

$$\underline{\cos\alpha} = \underline{\underline{0}}$$

P.V.

$$\|x+y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2$$

$$\|x-y\|^2 = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2$$

$+$

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

$$\|x\| = \|y\|$$

$$\langle x+y, x-y\rangle = \cancel{\langle x, x\rangle} + \cancel{\langle y, x\rangle} - \cancel{\langle x, y\rangle} - \cancel{\langle y, y\rangle}$$

$$C$$

$$\langle u_1, \sum_{i=1}^{k} \lambda_i w_i \rangle = \sum_{i=1}^{k} \lambda_i \underset{\substack{\shortparallel \\ 0}}{\langle u_1, w_i \rangle} = 0$$

$u_1, \ldots, u_k \quad 0 \quad \mathsf{G} \quad u_j \neq 0$

$$\sum_{j=1}^{k} \lambda_j \cdot u_j = 0 \Rightarrow \lambda_1 = \ldots = \lambda_k = 0$$

$$\langle \sum \lambda_j u_j , u_1 \rangle = 0 = \langle 0, u_1 \rangle$$

$$\sum_{j=1}^{k} x_j \langle u_j, u_1 \rangle = 0$$

$\langle u_j, u_1 \rangle = 0$

$j \neq 1$

$$x_1 \langle u_1, u_1 \rangle = 0$$

$x_1 = 0$

$\langle u_1, u_1 \rangle > 0$

$k = 1 \qquad [u_1] = [e_1] \qquad e_1 = u_1$

$[u_1, \ldots, u_{k-1}] = [e_1, \ldots, e_{k-1}]$

$[u_1, \ldots, u_k] \overset{?}{=} [e_1, \ldots, e_k]$

$q_k = \sum_{j=1}^{k-1} p_j e_j + u_k \qquad | \; e_i$

$\langle q_k, e_i \rangle = p_i \langle e_i, e_i \rangle + \langle u_k, e_i \rangle$

0

$u_1 = (0, 1, 2, 1)$, $u_2 = (-1, 1, 1, 1)$, $u_3 = (1, 0, 1, 0)$

$e_1 = u_1$

$e_2 = \lambda_1 e_1 + u_2 \mid \odot e_1 \qquad \lambda_1 = -\frac{2}{3}$

$0 = \lambda_1 \langle e_1, e_1 \rangle + \langle u_2, e_1 \rangle$

$\langle e_1, e_1 \rangle = 0 + 1 \cdot 1 + 4 + 1 \cdot 1 = 6$

$\langle u_2, e_1 \rangle = 0 \cdot \cancel{(-1)} + \underline{1 \cdot 1} + \underline{2 \cdot 1} + \underline{1 \cdot 1} = \underline{\underline{4}}$

$e_2 = -\frac{2}{3}(0, 1, 2, 1) + (-1, 1, 1, 1) = \left(-1, \frac{1}{3}, -\frac{1}{3}, \frac{1}{3}\right)$

$$e_1 = (0, 1, 2, 1), \quad e_2 = \left(-1, \tfrac{1}{3}, -\tfrac{1}{3}, \tfrac{1}{3}\right)$$

$$e_3 = r_1 e_1 + r_2 e_2 + (1, 0, 1, 0) \qquad | \, e_1$$

$$0 = r_1 \langle e_1, e_1 \rangle + 0 + \langle (1,0,1,0), (0,1,2,1) \rangle$$

$$0 = 6 r_1 + 2 \qquad r_1 = -\frac{1}{3} \qquad | \, e_2$$

$$0 = 0 + r_2 \langle e_2, e_2 \rangle + \langle (1,0,1,0), e_2 \rangle$$

$$0 = r_2 \left(1 + \frac{1}{9} + \frac{1}{9} + \frac{1}{9}\right) + \left(-\frac{4}{3}\right)$$

$$\frac{4}{3} = \frac{4}{3} r_2 \qquad r_2 = 1$$

$h_1 = -\frac{1}{3}$, $h_2 = 1$

$e_3 = (1, 0, 1, 0) - \frac{1}{3}(0, 1, 2, 1) +$

$+ 1\left(-1, \frac{1}{3}, -\frac{1}{3}, \frac{1}{3}\right) =$

$= (0, 0, 0, 0)$

$[u_1, u_2, u_3] = [e_1, e_2]$

$x \in W, \; y \in W^{\perp}, \quad c, d \in \mathbb{R}$

$\overset{?}{\Longrightarrow} \; c\,x + d\,x \in W^{\perp}$

$x \in W^{\perp} \Leftrightarrow c\langle x, w\rangle = 0 \quad \forall w \in W$

$d\langle y, w\rangle = 0$

$$\frac{\qquad}{\langle cx + dy, w\rangle = \underline{0}}$$

$\in W^{\perp}$

W podpr.

$$W \oplus W^{\perp} = V$$

$w_1, \ldots, w_k$ j ON bazą W

$$W \oplus W^{\perp} = V$$

V

$\underline{\underline{w_1 \ldots w_k}}, \underline{\underline{w_{k+1}, \ldots w_n}}$

$w_{k+1}, \ldots w_n \perp W$ $\quad \underline{\underline{W^{\perp}}}$

$\oplus$

$$\dim W + \dim W^{\perp} = \dim (W \oplus W^{\perp}) \leq n$$

$$k \qquad n-k \leq \qquad \qquad \qquad \parallel$$

$$n$$

$$n$$

$$W \oplus W^{\perp} = V$$

$$W \oplus W^{\perp} = V$$

$$\dim W = \dim W^{\perp\perp}$$

$$W^{\perp} \oplus W^{\perp\perp} = V$$

$$W \subseteq W^{\perp\perp}$$

$$(W+S)^{\perp} = W^{\perp} \cap S^{\perp}$$

$$x \in (W+S)^{\perp} \Leftrightarrow \forall\, z \in W+S$$

$$\langle x, z \rangle = 0 \iff \forall\, w, \forall\, s \quad \in W \quad \in S$$

$$\underline{\langle x, w \rangle = 0}, \ \underline{\langle x, s \rangle = 0}$$

$$\Leftrightarrow x \in W^{\perp} \wedge x \in S^{\perp} \Leftrightarrow x \in W^{\perp} \cap S^{\perp}$$

$$\left((W \cap S)^{\perp}\right)^{\perp} = \left(W^{\perp} + S^{\perp}\right)^{\perp}$$

$$W \cap S = \left(W^{\perp} + S^{\perp}\right)^{\perp} = W^{\perp\perp} \cap S^{\perp\perp}$$

$$= W \cap S$$