Global analysis. Exercises 10

1) Prove that if the connections with the Christoffel symbols Γ_{ij}^k and $\tilde{\Gamma}_{ij}^k$ have equal geodesics, then the connection with the Christoffel symbols $\alpha\Gamma_{ij}^k + \beta\tilde{\Gamma}_{ij}^k$ $(\alpha + \beta = 1)$ has the same geodesics.

2) Solve the equation of the parallel displacement on the sphere with the metric $(d\theta)^2 + \sin^2\theta(d\varphi)^2$ in the spherical coordinates:

- along a parallel $(\theta = \theta_0 = const)$;
- along a meridian $(\varphi = \varphi_0 = const)$.

3) Find the angle between a tangent vector to the sphere and its image under the parallel displacement along the parallel.

4) Let M be a manifold with a torsion-free affine connection ∇. Prove that if X and Y are parallel vector fields (i.e. $\nabla_Z X = \nabla_Z Y = 0$ for all vector fields Z), then $[X, Y] = 0$.

5) Let M be a manifold with a torsion-free affine connection ∇. Prove that any parallel distribution on M is involutive (a distribution is called parallel if $\nabla_Y X$ belongs to this distribution for all X from this distribution and all vector fields Y on M).