Global analysis. Exercises 5

1) Let V be a vector space of dimension n, $A \in \Lambda^2 V$, $B \in V$. Find the components of the tensor $A \wedge B$.

2)Let V and W be vector spaces of dimensions n and m, and with the bases $e_1, ..., e_n$ and $f_1, ..., f_m$. Let $\mathcal{A} : V \to V$ and $\mathcal{B} : W \to W$ be linear maps. Define the linear map $\mathcal{A} \otimes \mathcal{B} : V \otimes W \to V \otimes W$ by $\mathcal{A} \otimes \mathcal{B}(v \otimes w) = \mathcal{A}(v) \otimes \mathcal{B}(w)$. Prove that the matrix of $\mathcal{A} \otimes \mathcal{B}$ in the basis $e_1 \otimes f_1, e_1 \otimes f_2, ..., e_1 \otimes f_m, e_2 \otimes f_1, e_2 \otimes f_2, ..., e_2 \otimes f_m, ..., e_n \otimes f_1, e_n \otimes f_2, ..., e_n \otimes f_m$ of $V \otimes W$ has the form

$$\begin{pmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \dots & \dots & \dots & \dots \\ a_{n1}B & a_{n2}B & \dots & a_{nn}B \end{pmatrix},$$

where $A = (a_{ij})$ and $B = (b_{ij})$ are the matrices of the linear maps $\mathcal{A}$ and $\mathcal{B}$.

3) Prove that $V \otimes \mathbb{R} \simeq V$.

4) Let V be a vector space, T a tensor of type $(1,0)$ on V and S a tensor of type $(0,1)$ on V. What gives the contraction of the tensor product T and S.

5) Let $A \in \otimes^2 \mathbb{R}^2 \otimes (\mathbb{R}^2)^*$ be the tensor with the components:

$A^{11}_1 = 3, \quad A^{11}_2 = 0, \quad A^{12}_1 = 2, \quad A^{12}_2 = 1,$

$A^{21}_1 = 0, \quad A^{21}_2 = 1, \quad A^{22}_1 = 0, \quad A^{11}_1 = 5.$

Find the contractions of the first and the second upper indices with the down index.

6) Let V be a vector space, $A \in \otimes^r V \bigotimes \otimes^s V^*$. Let $e_1, ..., e_n$ and $e_{1'}, ..., e_{n'}$ be bases of V and let $d^1, ..., d^n$ and $d^{1'}, ..., d^{n'}$ be the dual bases. Find the relation between the components $A^{i_1 \dots i_r}_{j_1, \dots, j_s}$ and $A^{i'_1 \dots i'_r}_{j'_1, \dots, j'_s}$ of the tensor A in these bases.