

$[X]$

$c_1 u_1 +$

$+ c_2 u_2 = 0$

(I)  $\Rightarrow$  (III)

$u_1, \dots,$

$u_m$

$\mathbb{R}^2$

(II)  $\Rightarrow$  (III)

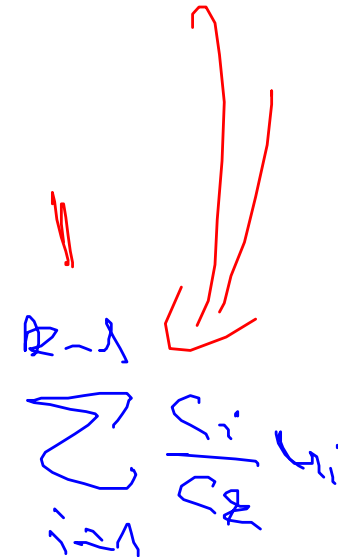
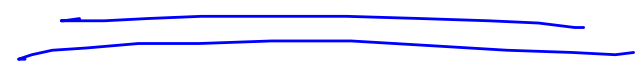
$c_1 u_1 + c_2 u_2 + \dots + c_m u_m = 0$

$\exists (c_1, \dots, c_m) \neq 0$

$R = \{ \alpha \} : c_\alpha \neq 0$

$\delta \in R \Rightarrow c_\delta = 0$

$c_\delta = - \sum_{i \in S} c_i u_i$



$$(III) \Rightarrow (IIII) \quad \text{JAS ME}$$

$$(IIII) \Rightarrow (I)$$

$$u_R = \sum_{\substack{i=1 \\ i \neq R}}^3 c_i u_i$$

$$0 = c_1 u_1 + c_2 u_2 + c_3 u_3$$

$$c_2 = -1$$

$$\neq$$

$$0$$

$$\Rightarrow x \in [u_1, \dots, u_m]$$

$$x = \sum_{i=1}^m c_i u_i = \sum_{i=1}^m d_i u_i$$



$$0 = \sum_{i=1}^m (c_i - d_i) u_i$$

$c_i = d_i \quad \text{if} \quad \underbrace{\quad}_{= 0}$



$$c_1 u_1 + \dots + c_m u_m = 0$$

$$\begin{pmatrix} \vdots \\ c_1 \\ \vdots \\ c_m \end{pmatrix} \begin{pmatrix} | & & | \\ u_1 & \dots & u_m \\ | & & | \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$0 u_1 + \dots + 0 u_m = 0$$

$$c_1 = c_2 = \dots = c_m = 0$$

$$u_1, \dots, u_m, \mathbb{1}$$

LN

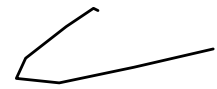
$$\{u_1, \dots, u_m\} \ni \mathbb{1}$$

$$(i) \Rightarrow (ii)$$

$$\mathbb{1} \in \{u_1, \dots, u_m\}$$

$$\mathbb{1} = \sum_{i=1}^m c_i u_i$$

$$u_1, \dots, u_m, \mathbb{1}$$



(13)  $\Rightarrow$  (iii)

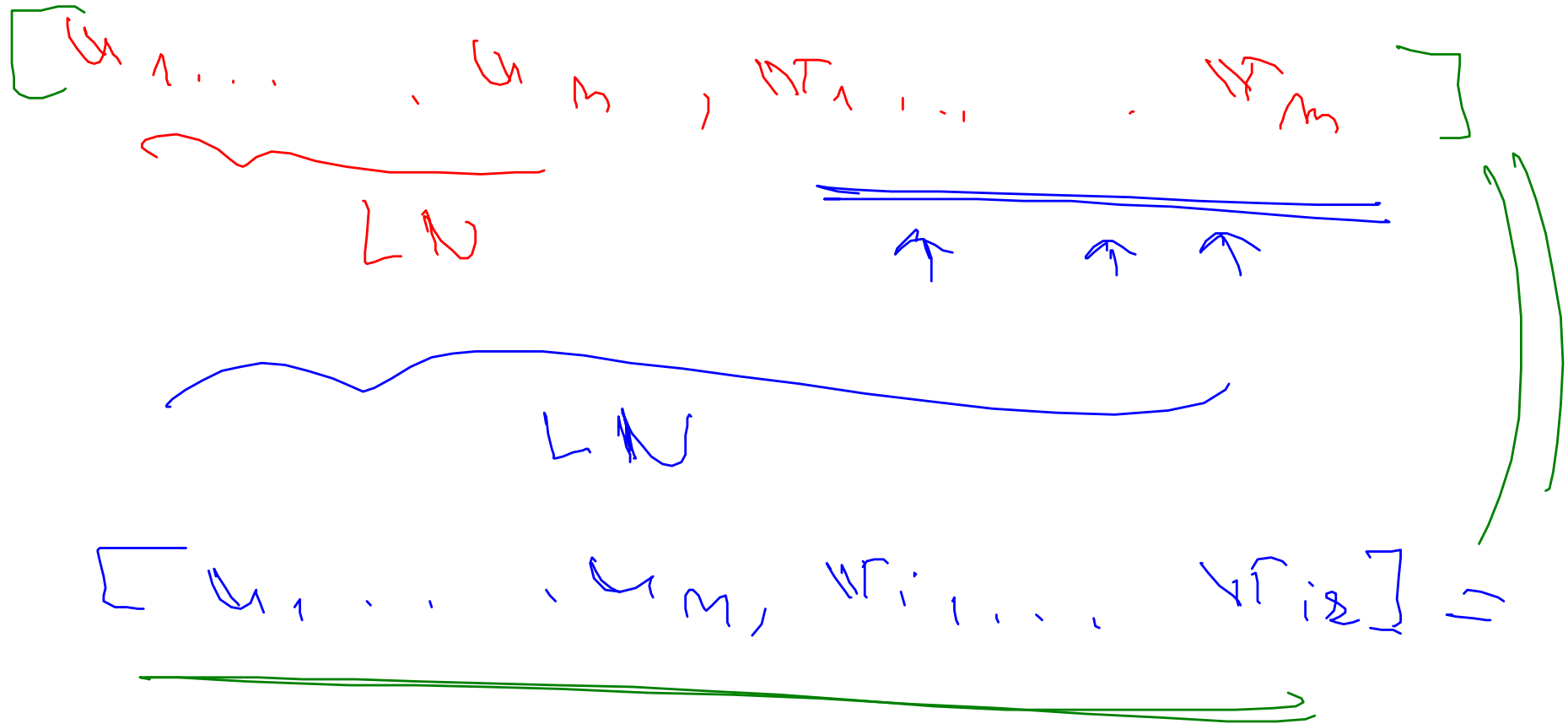
$$[u_1, \dots, u_n] \subseteq [u_1, \dots, u_n, v]$$

$$x = \sum_{i=1}^n c_i u_i + c v$$

$$x = \sum_{i=1}^n d_i u_i$$

$$c_i + c d_i = d_i$$

$$\Rightarrow x \in [u_1, \dots, u_n]$$



$$u_1, \dots, u_m \quad \underline{L \cap Z}$$

$$u_1, \dots, u_m, N_1 \quad L \cap Z \quad L \cap Z$$

$$i_1 = 1 \quad V_0 = [u_1, \dots, u_m]$$

$$R = 1 \quad V_1 = [u_1, \dots, u_m, N_1]$$

$$V_0 = V_1 \quad \left. \begin{array}{l} N \cap C \\ N \cap E \end{array} \right\} \begin{array}{l} N \cap P \text{ R O V E D U } \\ \Sigma \text{ R } \text{ R A D I } \text{ M} \\ \text{inde } 1 \text{ de } J \end{array}$$



$$R = J \quad V_j = [ [V_{j-1}] \cup \{A_j\} ]$$

$$V_j = V_{j-1} \quad \text{Nie}$$

$$V_j \neq V_{j-1} \quad \text{dominacja } \underline{J} \text{ do } \underline{J}$$

$$J = \{ i_1, \dots, i_k \}$$

$$u_1, \dots, u_m, A_1, \dots, A_k \quad \underline{LN}$$

$$\begin{array}{c}
 \left( \begin{array}{cccc|cc}
 1 & 0 & 3 & 0 & 3 & 1 \\
 1 & 1 & 1 & 0 & 5 & 1 \\
 -1 & 1 & 0 & 1 & -2 & 1 \\
 -1 & 1 & 5 & 2 & 1 & 1
 \end{array} \right) \sim \left( \begin{array}{cccc|cc}
 1 & 0 & 3 & 0 & 3 & 1 \\
 0 & 1 & -2 & 0 & 2 & 0 \\
 0 & 0 & 0 & 1 & 1 & 2 \\
 0 & 0 & 0 & 2 & 1 & 2
 \end{array} \right) \\
 \sim \left( \begin{array}{cccc|cc}
 1 & 0 & 3 & 0 & 3 & 1 \\
 0 & 1 & -2 & 0 & 2 & 0 \\
 0 & 0 & 0 & 1 & 1 & 2 \\
 0 & 0 & 0 & 2 & 1 & 2
 \end{array} \right) \sim \left( \begin{array}{cccc|cc}
 1 & 0 & 3 & 0 & 3 & 1 \\
 0 & 1 & -2 & 0 & 2 & 0 \\
 0 & 0 & 0 & 1 & 1 & 2 \\
 0 & 0 & 0 & 0 & 0 & 0
 \end{array} \right)
 \end{array}$$

$\mathbb{R} \neq [x_1, \dots, x_4]$   
 $\neq \in [x_1, \dots, x_4]$

$c_4 = 1$   
 $c_2 = 2 + 2c_3$   
 $c_1 = 3 - 3c_3$

$$X, Y \in \mathbb{K}^{m \times n}$$

$$X'$$

$$X \sim Y$$

$$[D_{j_1}(X) \dots D_{j_r}(X)] \sim Y'$$

$$X_{j_i} = D_{j_i}(X)$$

A.2.

$$j_1, \dots, j_r$$

LNZ

$$X_{j_1}, \dots$$

$$X_{j_r}$$

LNZ



$$X' \cdot C = 0$$

$\Leftrightarrow$

$$Y' \cdot R = 0$$

$$j \in \{j_1, \dots, j_r\} \quad X \sim Y$$

$$X_j = \sum_{i=1}^r c_{ji} X_{j_i}$$

$$r = \max \{j_1, \dots, j_r\} \leq n \quad \text{Rst}$$

$$\begin{pmatrix} D_{j_1}(X) & \dots & D_{j_r}(X) & X_j \end{pmatrix} \sim \begin{pmatrix} Y' & Y' \end{pmatrix}$$

$$X'c = X_j \iff Y'c = d' \quad \begin{pmatrix} 1 & \\ 0 & 0 \end{pmatrix}!$$

$$\underline{\underline{[x_{j_1} \dots x_{j_2}]} = \underline{\underline{[x_1 \dots x_m]}}$$

$$\begin{pmatrix} 1 & 1 & 3 & -1 & 1 \\ 2 & 0 & 2 & 1 & 3 \\ 1 & 1 & 3 & 2 & 4 \\ 2 & 0 & 2 & 0 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 3 & -1 & 1 \\ 2 & 0 & 2 & 1 & 3 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & -1 & -1 \end{pmatrix}$$
  

$$\sim \begin{pmatrix} 1 & 1 & 3 & -1 & 1 \\ 0 & -2 & -4 & 3 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$x_1, x_2, x_4,$

(ii)  $\Rightarrow$  (iii)

$$\mathbb{N} \in [X]$$

$$\exists x_1, \dots, x_m \in X$$

$$c_1, \dots, c_m \in \mathbb{K}$$

$$\mathbb{N} = \sum_{i=1}^m c_i x_i$$

$$c_1 x_1 + \dots + c_m x_m - \mathbb{N} = 0$$

$$\{x_1, \dots, x_m, \mathbb{N}\}$$

$u_1, \dots, u_m \in \mathbb{R}^n$

$u_j \in \text{span}\{u_1, \dots, u_m\}$

$$C = (c_{ij})$$

$$u_j = \sum_{i=1}^m c_{ij} u_i$$

$(u_1, \dots, u_m)$

$$\begin{pmatrix} c_{1j} \\ \vdots \\ c_{mj} \end{pmatrix}$$

$= u_j$

$$(u_1, \dots, u_m) C = (u_1, \dots, u_m)$$


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$$\begin{aligned}
 & (a_1, \dots, a_m) \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = 0 \\
 & m \geq m \\
 & m < m \quad C \quad m \times m \quad \text{SPOR} \\
 & \exists x \quad \underline{C \neq 0}, \quad \text{X} \neq 0 \\
 & \underline{(a_1, \dots, a_m) \cdot C \neq 0} = 0 \Rightarrow \underline{C \neq 0} \Rightarrow \underline{X \neq 0}
 \end{aligned}$$

(I)  $\Rightarrow$  (II)  $\Rightarrow$   $X \subseteq V$   $[X] = V$   
 $X_m$

$\exists$  x. matr. lin. no. m.  $(Y)$

$\delta_1 \dots \delta_m$   $\dots$   $\delta_{m+1} \dots$

$\text{rank}(X) = m$

$\delta_1 \dots \delta_{m+1}$  LN2

$\in [X]$

$\in [X] = U$

$m+1 \leq m$

S.P.U.R

$$(III) \Rightarrow (i)$$

$$y_1 \in V, \quad M_1 \neq \emptyset$$

$$[y_1] = V \neq \emptyset$$

$$\exists y_2 \in V - [y_1]$$

$$y_1, y_2$$

$$V = [y_1, y_2]$$

$$y_1, y_2, y_3$$

$$\frac{y_2}{y_1}$$

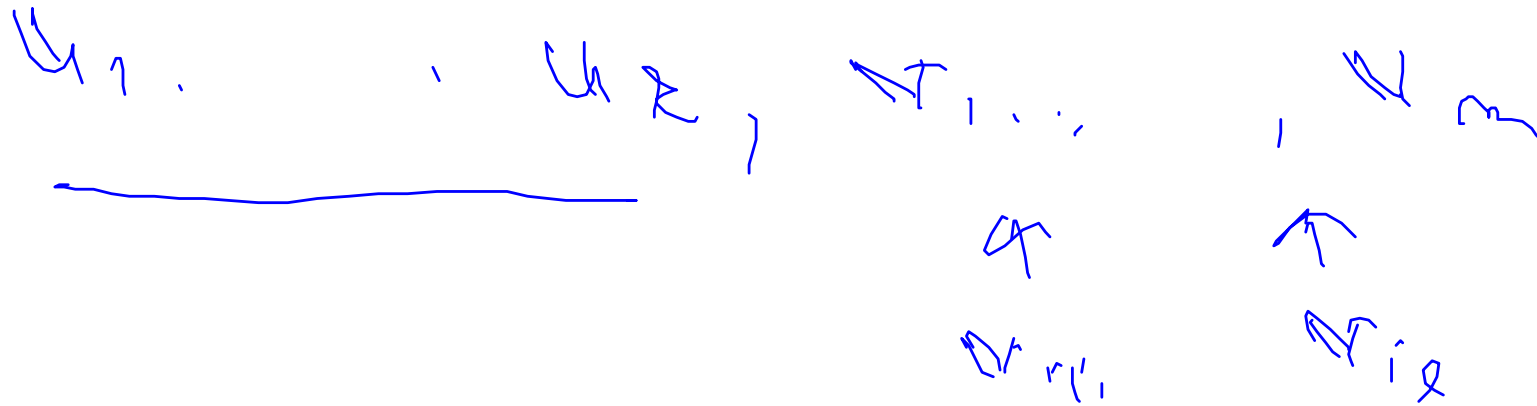
$$[y_1, \dots, y_m] = V$$

$$Y = [X] \quad X \text{ ?..}$$

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad \text{dim}$$

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = [X] = Y$$

LUZ



$$[u_1, \dots, u_n, \dots, u_m] = v_1$$

$L \cup Z$

$$[u_1, \dots, u_n, \dots, u_m] = v_2$$

$\mathcal{L}_1, \dots, \mathcal{L}_m$       $\mathcal{L}_m$       $\text{base} \checkmark \checkmark$

$\mathcal{L}_1, \dots, \mathcal{L}_m$

$\mathcal{L}_2$

gl.

$[\mathcal{L}_1, \dots, \mathcal{L}_m] = \checkmark \checkmark$

$$\underline{\mathcal{L}_3 \leq \mathcal{L} \leq \mathcal{L}_3}$$

$$\underline{\Rightarrow \mathcal{L} = \mathcal{L}_3}$$

$\mathbb{C}$  mod  $\mathbb{R}$

$$\mathbb{Z} = x + iy = \underbrace{x}_{\in \mathbb{R}} \cdot 1 + \underbrace{y}_{\in \mathbb{R}} \cdot (i)$$

$\{1, i\}$

$\{1, i\}$

$\dim_{\mathbb{R}} \mathbb{C} = 2$

but:

$\mathbb{C}$  mod  $\mathbb{C}$

$\mathbb{C}$

$$\mathbb{Z} = \mathbb{Z} \cdot 1$$

$$\dim_{\mathbb{C}} \mathbb{C} = 1$$