$$\varphi : \mathbb{V} \longrightarrow U \qquad [\varphi(x)]_\beta = (\varphi)_{\alpha,\beta} \langle x \rangle_\beta$$

$$\beta \qquad \alpha$$

$$m \qquad m$$

$$\mathbb{C} = A \langle x \rangle_\beta$$

$$\varphi(\alpha_j)_\alpha \in \mathbb{K}^{m \times 1}$$

$$A \longmapsto \varphi_A$$

$$(\cdot$$

$$v \mapsto 0 \in \mathcal{L}(V, U)$$

$$\varphi_1, \varphi_2 \in \mathcal{L}(V, U), \quad a, b \in K$$

$$\overset{?}{\Longrightarrow} \quad \underline{a\varphi_1 + b\varphi_2} \in \mathcal{L}(V, U)$$

$$v_1, v_2 \in V, \quad c, d \in K$$

$$(a\varphi_1 + b\varphi_2)(cv_1 + dv_2) \overset{?}{=}$$

$$c(a\varphi_1 + b\varphi_2)(v_1) + d(a\varphi_1 + b\varphi_2)(v_2)$$

$$(a\varphi_1 + b\varphi_2)(c\psi_1 + d\psi_2) =$$

$$= (a\varphi_1)(c\psi_1 + d\psi_2) + (b\varphi_2)(\text{—}\,||\,\text{—})$$

$$= a(\varphi_1(c\psi_1 + d\psi_2)) + b(\varphi_2(\text{—}||\text{—}))$$

$$= a(c\varphi_1(\psi_1) + d\varphi_1(\psi_2)) +$$
$$b(c\varphi_2(\psi_1) + d\varphi_2(\psi_2)) =$$

$$= c((a\varphi_1 + b\varphi_2)(\psi_1)) + d((a\varphi_1 + b\varphi_2)(\psi_2))$$

$x \mapsto \hat{x}$

$\eta(x) = \hat{x}$

$\hat{x} \in V^{**}$

$\hat{x}(\varphi) = \varphi(x)$
$\in \mathbb{K}$

$\hat{x} \in \mathbb{K}^{V^*}$

$$\underline{\underline{\hat{x}(c\varphi_1 + d\varphi_2)}} = (c\varphi_1 + d\varphi_2)(x) =$$

$$= c\varphi_1(x) + d\varphi_2(x) = c\,\underline{\underline{\hat{x}(\varphi_1) +}}$$

$$\underline{\underline{+\, d\,\hat{x}(\varphi_2)}}$$

$\hat{x} \in \mathcal{L}(V^*, \mathbb{K}) = V^{**}$

$$\eta(a x_1 + b x_2) = a\,\eta(x_1) + b\,\eta(x_2)$$

$$\parallel$$

$$\overbrace{(a \hat{x}_1 + b \hat{x}_2)}(\varphi) = \varphi(a x_1 + b x_2)$$

$$(a \hat{x}_1 + b \hat{x}_2)(\varphi) = (a \hat{x}_1)(\varphi) + (b \hat{x}_2)(\varphi)$$

$$= a\,\varphi(x_1) + b\,\varphi(x_2)$$

$\eta(v) = 0$ $\qquad$ $\varphi_v(v) = \underline{\underline{1}}$

$v \neq 0$ $\qquad$ $\langle v \rangle$

$\varphi_v(x) = \begin{cases} k & x = kv \\ 0 & x \notin \langle v \rangle \end{cases}$

$x, y \in \langle v \rangle$ $\qquad$ $x + y = (k_1 + k_2)v$

$x = k_1 v, \quad y = k_2 v$

$$\hat{v}^*(\varphi_v) = \varphi_v(v) = 1$$

$$\eta(v) = \hat{v}^* = 0 \qquad \text{SPOR}$$

(b) $V^* \cong V$ $\quad \dim V^{**} = \dim V = n$

$V^{**} \cong V$

$$n = \dim V = \dim \operatorname{Ker} \eta + \dim \operatorname{Im} \eta = n$$

$$\eta : V \to V^{**}$$

$$[r_1(A) \dots \quad r_i(A), \quad r_m(A)]$$

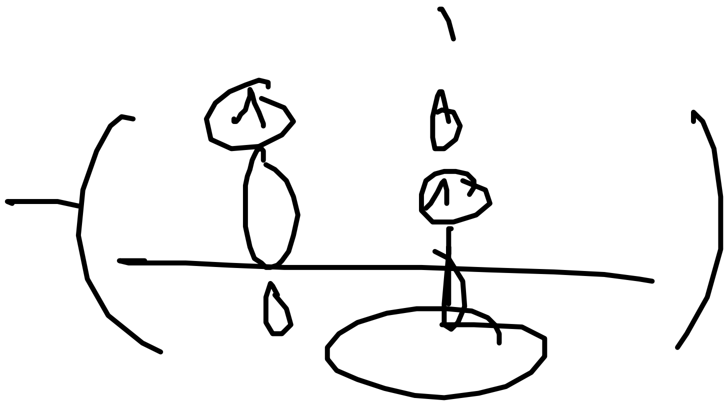
$$[r_1(A) \quad \dots \quad c\, r_i(A) \quad r_m(A)]$$

$c \neq 0$ $\Longleftarrow$

$$r_j(A) + c\, r_i(A)$$

$\Longleftarrow$

$A \sim \qquad\qquad \sim B \mod I.$



$D_n(B) = D_0(B)$

$D_0(B) = D_0(A)$

$\dim [\, D_n(A), \ .$

$\qquad\qquad \| $

$\qquad\quad D_0(B)$

$D_n(A)] = D_0(A)$

$$R_n(B) = R_D(B) = R_D(A)$$

$$\dim_m [\, n_n(A), \ldots, n_m(A) \,] =$$

$$= \dim [\, n_n(B), \qquad , n_m(B) \,]$$

$$R_n(B) = R_n(A)$$

$$A \mapsto \varphi(x) = \underset{m\times n}{A}\ \underset{n\times 1}{x} \qquad R_0(A) = R(\varphi)$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \dim \operatorname{Im} \varphi$$

$$\varphi : \mathbb{K}^{n\times 1} \to \mathbb{K}^{m\times 1}$$

$$B \mapsto \psi(y) = \underset{m\times n}{B}\ \underset{n\times 1}{y} \qquad R_0(B) = R(\psi) =$$

$$\underset{m\times 1}{} \qquad\qquad\qquad\qquad\qquad = \dim \operatorname{Im} \psi$$

$$\psi : \mathbb{K}^{n\times 1} \to \mathbb{K}^{m\times 1}$$

$$A \cdot B \longmapsto \varphi \circ \psi(y) =$$

$$= \varphi(\psi(y)) = A \cdot (B \cdot y)$$

$$R_0(A \cdot B) = \dim \operatorname{Im}(\varphi \circ \psi) \leq$$

$$\leq \dim \operatorname{Im} \varphi = R_0(A) \qquad R(B)$$

$$\underline{R(A \cdot B)} = R((A \cdot B)^T) = R(B^T \cdot A^T) \leq R(B^T)$$

$$\left(\varphi\right)_{\alpha,\beta} \cdot \left(\varphi^{-1}\right)_{\beta,\alpha} = \left(\mathrm{id}_{U}\right)_{\alpha,\alpha} \qquad \left(\varphi^{-1}\right)_{\beta,\alpha} \cdot \left(\varphi\right)_{\alpha,\beta} = \left(\mathrm{id}_{V}\right)_{\beta,\beta}$$

$$\underline{A \quad B} = I_n \qquad \underline{B \cdot A} = I_n$$

A, B

$$\varphi^{-1} : U \to V$$

$$\left(\varphi^{-1}(u)\right)_\beta = A^{-1} \cdot (u)_\alpha$$

$$A A^{-1} = I_n \qquad | \; A$$

$$A A^{-1} A = A$$

$$r(A^{-1} A) \le \min(r(A), r(A^{-1}))$$

$$\| \\ n$$

$$n \le r(A) \le n$$

$R(A) = n \qquad \varphi(x) = Ax$

$\varphi : \mathbb{K}^n \to \mathbb{K}^n$

$R(\varphi) = \dim \operatorname{Im} \varphi = R(A) = n$

φ je surjektivna $\Rightarrow$ injektivna $\Rightarrow$ bijektivna

izom.

$\exists (\varphi^{-1})_{e,e} = A^{-1}$

$A \mapsto \varphi(x) = Ax$ $\qquad$ $\mathbb{K}^n$ $\quad \exists\, \varphi^{-1}, \psi^{-1}$

$B \mapsto \psi(y) = By$ $\qquad$ $\psi^{-1} \circ \varphi^{-1} = \underline{\mathrm{id}}$

φ $\Downarrow$ szürjektív

$\underset{\overset{\|}{I_n}}{A \cdot B} \mapsto \varphi \circ \psi = \mathrm{id}$ $\quad \nearrow \quad$ $BA = I_n$

φ is $\underline{\text{szürjektív}}$, ψ is $\underset{\Downarrow}{\text{injektív}}$

$n = \underset{\overset{\|}{0}}{\dim \operatorname{Ker} \varphi} + \underbrace{\dim \operatorname{Im} \varphi}_{n}$ $\qquad$ ψ is az

$$A^{-1} \cdot A = I_n = A \cdot A^{-1}$$

$$(A^{-1})^{-1} = A$$

$$A \cdot A^{-1} = I_n \qquad B \cdot B^{-1} = \underline{B^{-1} B} = I_n$$

$$\underline{(B^{-1} A^{-1})} \cdot (A \cdot B) =$$

$$= B^{-1} (\underbrace{A^{-1} \cdot A}_{I_n}) \cdot B = B^{-1} B = I_n$$

$$(A \cdot A^{-1})^T = I_n^T = (A^{-1} A)^T$$

$$(A^{-1})^T A^T = I_n = A^T \cdot (A^{-1})^T$$

$$(A^T)^{-1} = (A^{-1})^T$$

$$\begin{array}{c} i \\ j \end{array} \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \qquad E_{i,j} = \begin{pmatrix} 1 & & & & & & 0 \\ & \ddots & & & & & \\ & & 0 & \cdots & 1 & & \\ & & \vdots & \ddots & \vdots & & \\ & & 1 & \cdots & 0 & & \\ & & & & & \ddots & \\ 0 & & & & & & 1 \end{pmatrix}$$

$E_{i,j} A$

k-te Zeile E_{ij} $\qquad k \neq i, j$

$k = i \quad \vee \quad k = j$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ & 2 & & 0 \\ 0 & & 3 & \\ & & & 4 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \qquad \begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & 1 & & \\ & 0 & & c & \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix}$$

$x\,y = 1 \Rightarrow yx = 1$

$$\boxed{x\,y = yx = 1}$$

$x\,y = 1$

$(y\,x)\,\cancel{y} = \cancel{y}$ $\quad yx = 1$