(a) $A \sim B \qquad \exists\, A_0, \ldots, A_\ell$

$A = A_0, \quad \ldots \quad, A_\ell = B \qquad B = R \cdot A$

$A_i \rightsquigarrow A_{i+1}$ πομαί Ε.λ.ι.γ. $\qquad A = R^{-1} B$

$A_{i+1} = E_i A_i \qquad 0 \le i \le \ell - 1$

$B = A_\ell = E_{\ell-1} \cdot A_{\ell-1} = \cdots \cdots$

$= \cdots\cdots = \underbrace{E_{\ell-1} \cdots E_0}_{R} A_0$

$\Leftarrow \quad P = E_1 \ldots E_n$

$A = E_1 \ldots (E_n B)$

$B \sim E_1 B \sim E_2 (E_n B) \sim \ldots$

$\sim A \quad \Uparrow$

$A = FB$

$$\rho(A) \geq \rho(P \cdot A)$$

$$\rho(A) = \rho(P^{-1} \cdot (PA)) \leq \rho(PA)$$

$$\rho(A) = \rho(AQ) = \rho(PAQ)$$

$(I) \Rightarrow (II)$ $\qquad \mathcal{B} = (v_1, \ldots, v_m)$

$$(f(x))_\alpha = (f)_{\alpha,\mathcal{B}} (x)_\mathcal{B} \qquad \begin{pmatrix}0\\ \vdots\\ 0\\ 1\\ \vdots\\ 0\end{pmatrix}_\alpha \leftarrow j$$

$$f = id_V \qquad (x)_\alpha = P_{\alpha,\mathcal{B}} \cdot (x)_\mathcal{B}$$

$(II) \Rightarrow (III)$ $\qquad (v_j)_\alpha = P\,(v_j)_\mathcal{B}$

$$w_j = \alpha(v_j) = \alpha \cdot (P\,(v_j)_\mathcal{B}) = \alpha\, a_j(P)$$

$$\mathcal{B} = \alpha \cdot P$$

(iii) $\Rightarrow$ (ii) $\qquad \left((v_j)_\alpha \right)_{j=1}^{m} =: \mathbb{P}$

$$D_j(\mathbb{P}) = (v_j)_\alpha$$

$$\alpha \cdot \mathbb{P} = \beta$$

$$\underline{\underline{\alpha \, D_j(\mathbb{P})}} = v_j$$

$$\alpha = (u_1, \dots, u_n)$$

$$P_{\alpha,\alpha} = \left((u_j)_\alpha \right) = \left(\begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \right) = \underline{\underline{I_n}}$$

$$\underline{\underline{P_{\beta,\alpha}}}\,(x)_\alpha = (x)_\beta \qquad \Big|\cdot \left(P_{\beta,\alpha}\right)^{-1}$$

$$(x)_\alpha = P_{\beta,\alpha}^{-1}\,(x)_\beta$$

$$\begin{matrix} \parallel \\ P_{\alpha,\beta} \end{matrix}$$

$$\alpha(P_{\alpha\beta} P_{\beta\gamma}) = \underbrace{(\alpha P_{\alpha,\beta})}_{\beta} P_{\beta\gamma} =$$

$$= \beta P_{\beta\gamma} = \gamma$$

$$\alpha P_{\alpha\gamma} = \gamma$$

$$P_{\beta_2\alpha_2} \cdot (\varphi)_{\alpha_2,\alpha_1} \underbrace{P_{\alpha_1,\beta_1} (\pi)_{\beta_1}} =$$

$$= P_{\beta_2\alpha_2} \overbrace{(\varphi)_{\alpha_2,\alpha_1} (\pi)_{\alpha_1}} =$$

$$= P_{\beta_2\alpha_2} (\varphi(\pi))_{\alpha_2} =$$

$$= (\varphi(\pi))_{\beta_2} = (\varphi)_{\beta_2,\beta_1} (\pi)_{\beta_1}$$

$$P_{\beta_2,\alpha_2} (\varphi)_{\alpha_2,\alpha_1} P_{\alpha_1,\beta_1} =$$

$$= (\mathrm{id})_{\beta_2,\alpha_2} (\varphi)_{\alpha_2,\alpha_1} (\mathrm{id})_{\alpha_1,\beta_1} =$$

$$= (\mathrm{id}_{V_2} \circ \varphi \circ \mathrm{id}_{V_1})_{\beta_2,\beta_1} = (\varphi)_{\beta_2,\beta_1}$$

(I) $\Rightarrow$ (II) $\qquad A = (\varphi)_{\alpha_1, \beta_1}$

$B = (\varphi)_{\alpha_2, \beta_2}$

$$B = P_{\alpha_2, \alpha_1} \cdot A \; P_{\beta_1, \beta_2}$$

$$\quad\ \| \qquad\qquad\quad \|$$

$$\quad\ P \qquad\qquad\quad Q$$

$$B = PAQ$$

$(II) \Rightarrow (III)$ $R(A) = R(B)$ więc!

$(III) \Rightarrow (I)$ $A \sim C$ n.o.l.

CSD $\quad RAR' = D = SBS'$

$$D = \begin{pmatrix} \boxed{\begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix}} & 0 \\ 0 & 0 \end{pmatrix}$$

$B = S^{-1} R A R' S'^{-1} = PAQ$

(M). $\alpha = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 0 & 2 & 1 \end{pmatrix}$ $P_{\alpha,\beta} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$

$P_{\sigma,\alpha} = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

α, β, σ

$\sigma \, P_{\sigma,\alpha} = \alpha$

$\alpha \cdot P_{\alpha,\beta} = \beta$

$$\begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 2 \\ 7 & 5 & 7 \\ 5 & 2 & 5 \end{pmatrix}$$

$$\delta = \partial \, P_{012}^{-1} \qquad \delta = \begin{pmatrix} -1 & 0 & 1 \\ -3 & -1 & 4 \\ 0 & -1 & 2 \end{pmatrix}$$

$$\delta^T = \left(P_{012}^T\right)^{-1} \partial^T$$

$$\left(\begin{array}{ccc|ccc} 0 & 2 & 1 & 1 & 2 & 0 \\ 1 & 0 & 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 & 3 & 1 \end{array}\right) \sim_{I_{1-2}}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 & 3 & 1 \\ 0 & 0 & -1 & -1 & -4 & -2 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -3 & 0 \\ 0 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 4 & 2 \end{array}\right)$$

12.

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 3 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}$$

$\varepsilon^4, \varepsilon^3$

$$= \underline{\underline{\begin{pmatrix} 2 & 3 & 4 \\ 0 & -1 & -2 \\ 5 & 9 & 15 \\ 1 & 4 & 9 \end{pmatrix}}} \overset{!!}{=} B$$

$(\varphi)_{\mathcal{B}_1\mathcal{B}_2} = \alpha^{-1} B$

$\alpha\left(\cancel{x}\right)_{b} = \cancel{x}$

$\alpha\,(\varphi)_{\mathcal{B}_1\mathcal{B}_2} = B$

$$\left(\begin{array}{cccc|ccc} 0 & 1 & 0 & 1 & 2 & 3 & 4 \\ 1 & -1 & 1 & -1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 & 5 & 9 & 15 \\ 0 & 0 & 0 & 1 & 1 & 4 & 9 \end{array}\right) \quad \begin{pmatrix} -3 \\ 5 \\ -2 \\ 3 \end{pmatrix}$$

$$\left(\begin{array}{cccc|ccc} 1 & -1 & 1 & -1 & 0 & -1 & -2 \\ 0 & 1 & 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 2 & 2 & 3 & 6 & 11 \\ 0 & 0 & 0 & 1 & 1 & 4 & 9 \end{array}\right) \sim \begin{pmatrix} 5 \\ -3 \\ 5 \\ 3 \end{pmatrix} \qquad \begin{pmatrix} \frac{5}{2} \\ -6 \\ -\frac{1}{2} \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ -6 \\ -1 \\ 3 \end{pmatrix} \left(\begin{array}{cccc|ccc} 1 & 0 & 1 & 0 & 2 & +2 & 2 \\ 0 & 1 & 0 & 0 & 1 & -1 & -5 \\ 0 & 0 & 2 & 0 & 1 & -2 & -7 \\ 0 & 0 & 0 & 1 & 1 & 4 & 9 \end{array}\right) \sim \left(\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & \frac{3}{2} & 3 & \frac{11}{2} \\ 0 & 1 & 0 & 0 & 1 & -1 & -5 \\ 0 & 0 & 1 & 0 & \frac{1}{2} & -1 & -\frac{7}{2} \\ 0 & 0 & 0 & 1 & 1 & 4 & 9 \end{array}\right)$$

$$\begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} \longmapsto \begin{pmatrix} -3 \\ 5 \\ 2 \\ 3 \end{pmatrix}$$

$(0\ 1\ 2\ 3)$

$-6 - 1 + 9 = 2$

$$\begin{pmatrix} \frac{5}{2} \\ -6 \\ -\frac{1}{2} \\ 3 \end{pmatrix}$$

$(0\ 1\ 0\ 1) \to -3$

$1\ -1\ 1\ -1$

$\frac{5}{2} + 6 - \frac{1}{2} - 3 = 5$

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 3 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \\ 3 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$