$M \nparallel N \qquad \underline{M \cap N = \emptyset} \qquad \underbrace{\mathrm{Dim}\, M}_{= S} \cap \underbrace{\mathrm{Dim}\, N}_{= T} \neq \{0\}$

$\underline{\dim(S \cap T)} \geq 1$

$S \not\subseteq T,\ T \not\subseteq S$

$T \cap S \subsetneq T \ ,\ T \cap S \subsetneq S$

$\Rightarrow \dim T \geq 2,\ \dim S \geq 2$

$\underline{S + T = V} \Rightarrow M \cap N \neq \emptyset \qquad \Rightarrow S+T \subsetneq V$

$\dim V \geq 4 \qquad \Leftarrow$

$$\dim(S+T) = \underbrace{\dim S}_{\geq 2} + \underbrace{\dim T}_{\geq 2} - \underbrace{\dim(S\cap T)}_{\geq}$$

$$\dim(S\cap T) \leq \min(\dim(S), \dim T) - 1$$

$$-\dim(S\cap T) \geq 1 - \min(\dim S, \dim T)$$

$$\geq \cancel{\dim S} + \cancel{\dim T} + 1 - \min\cancel{(\dim S, \dim T)}$$

$$2 + 1 = 3$$

$$\dim(S+T) \geq 3$$

$\Leftarrow$ y anm!

$\Rightarrow$ $n = 0$ $\sim$ evident

$\lambda_0 = 1$ $f(1 p_0) = f(p_0) = 1 \cdot f(p_0)$

$n \Rightarrow n+1$

$$f\left(\sum_{j=0}^{n+1} \lambda_j p_j\right) = f\left(\lambda_0 p_0 + (1-\lambda_0)\sum_{j=1}^{n+1} \frac{\lambda_j}{1-\lambda_0} p_j\right) =$$

$\exists j_0 \;\; \lambda_{j_0} \neq 1 \Rightarrow \lambda_0 \neq 1$

$$= \lambda_0 f(P_0) + (1-\lambda_0)\, f\left(\sum_{j=1}^{n+1} \frac{\lambda_j}{1-\lambda_0} P_j\right)$$

$$= \lambda_0 f(P_0) + \cancel{(1-\lambda_0)} \sum_{j=1}^{n+1} \frac{\lambda_j}{\cancel{1-\lambda_0}} f(P_j)$$

$$\lambda_0 = \lambda_1 = \qquad = \lambda_{n+1} = 1$$

analogiczne jako przy $\underline{AP}$.

$\underline{\text{Bk.}}$ f AZ $f(0) = u$

$\varphi := f - u \Rightarrow f = \varphi + u$

$\varphi(cx_1 + dx_2) = f(cx_1 + dx_2) - u$ $= c\,(f(x_1) - u) + d\,(f(x_2) - u)$ $\overset{\|}{\varphi(x_1)}$ $\overset{\|}{\varphi(x_2)}$

$c, d \in K,\ x_1, x_2 \in V$

$= f(cx_1 + dx_2 + (1-c-d)\,\underline{0}) - u$

$= c\,f(x_1) + d\,f(x_2) + (1-c-d)\,u - u =$

$$1 = \alpha + (1-\alpha)$$

$$f = \varphi + u, \quad \varphi \ \mathrm{LZ} \quad \overset{?}{\Longrightarrow} \quad f \ \mathrm{AZ}$$

$$f(\alpha p + (1-\alpha) q) = \varphi(\alpha p + (1-\alpha) q) +$$

$$+ u = \alpha \varphi(p) + (1-\alpha) \varphi(q) + 1 \cdot u =$$

$$= \alpha(\varphi(p) + u) + (1-\alpha)(\varphi(q) + u) =$$

$$= \alpha f(p) + (1-\alpha) f(q)$$

$$f = \varphi + u, \quad g = \psi + v$$

$$(f \circ g)(\pi) = ((\varphi + u) \circ (\psi + v))(\pi)$$

$$= (\varphi + u)\left(\underline{\psi(\pi) + v}\right) =$$

$$= \varphi(\psi(\pi) + v) + u =$$

$$= \underline{\underline{\varphi(\psi(\pi))}} + \varphi(v) + u$$

$$f = \varphi + u \qquad\qquad M = P + S$$

$$f(M) = \varphi(M) + u =$$

$$= \varphi(P + S) + u = \varphi(P) + \varphi(S) + u$$

$$= \underbrace{\varphi(S)}_{\substack{\| \\ \text{vett. } P}} + (\underbrace{\varphi(P) + u}_{\text{Pnt}}) \qquad \text{ij } AP$$

$f^{-1}(N) = \emptyset$ good!

$f^{-1}(N) \neq \emptyset$

$f^{-1}(N) = \{p \in V : f(p) \in N\} =$

$\neq \emptyset \quad \exists p_0 \quad f(p_0) \in N$

$N = f(p_0) + T$

$= \{p \in V : f(p) = f(p_0) + h,\ h \in T\}$

$$\alpha p_1 + (1-\alpha) p_2 \overset{?}{\in} f^{-1}(N)$$

$$p_1, p_2 \in f^{-1}(N)$$

$$f(p_1) = f(p_0) + h_1 |_{p_0}, \quad h_1 \in T$$

$$f(p_2) = f(p_0) + h_2 |_{p_0} \quad h_2 \in T$$

$$f(\underbrace{\alpha p_1 + (1-\alpha) p_2}_{\in f^{-1}(N)}) = f(p_0) + \underbrace{\alpha h_1 + (1-\alpha) h_2}_{\in T}$$

$$\dim V = \dim \operatorname{Ker} \varphi + \dim \operatorname{Im} \varphi$$

$$f = \varphi + u$$

$$\dim \operatorname{Im} \varphi = \dim \operatorname{Im} f \quad //$$

$$\dim \operatorname{Ker} \varphi = \dim f^{-1}(y), \quad y \in \operatorname{Im} f$$

$$\operatorname{Im} f = f(U + V) = (\varphi + u)(U + V)$$

$$= \varphi(V) + u = \underline{\underline{\operatorname{Im} \varphi + u}} \quad \checkmark$$

$$\dim \operatorname{Ker} \varphi = \dim f^{-1}(y) \qquad \underbrace{f^{-1}(y)}_{=\, z + \operatorname{Ker}\varphi}$$

$$\operatorname{Ker} \varphi = \{ v : \varphi(v) = 0 \}$$

$$\forall y \in \operatorname{Im} f \quad \exists z \in V$$

$$f(z) = y \implies f(z) = \varphi(z) + u$$

$$f(z + \operatorname{Ker}\varphi) = (\varphi + u)(z + \operatorname{Ker}\varphi) =$$

$$= \varphi(z + \operatorname{Ker}\varphi) + u = \varphi(z) + \underbrace{\varphi(\operatorname{Ker}\varphi)}_{=\,0} + u$$

$$y, r \in \mathcal{R}(A|b) \overset{?}{\Rightarrow} y - r \in \mathcal{R}(A)$$

$$Ay = b$$

$$Ar = b$$

$$A(y - r) = 0 \quad \Rightarrow \quad y - r \in \underline{\mathcal{R}(A)}$$

$$Ar = b$$

$$Ax = 0$$

$$A(r + x) = b \qquad r + x \in \mathcal{R}(A|b)$$

$$R(A|B) = \pi\mathbb{Z} + R(A)$$

$$\pi\mathbb{Z} + R(A) \subseteq R(A|B)$$

$$R(A|B) - \pi\mathbb{Z} \subseteq R(A)$$

$$R(A|B) \subseteq R(A) + \pi\mathbb{Z}$$

$Ax = b$ có nghiệm $\Leftrightarrow$

$$\left(c_1(A), \ldots, c_n(A)\right)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = b$$

$$\Leftrightarrow \sum_{i=1}^{n} x_i c_i(A) = b \Leftrightarrow$$

$$b \in [c_1(A), \ldots, c_n(A)] \Leftrightarrow$$

$$\Leftrightarrow r(A) = r(A|b)$$

$$A\underline{x} = \underline{b} \Leftrightarrow \exists\, \underline{t} \quad B\underline{x} = C\underline{t} + D :$$

$$(B \mid C \mid D) \sim \left(\begin{array}{c|c|c} A' & D & b' \\ \hline A & 0 & b \end{array}\right)$$

$$(\underline{x}, \underline{t}, 1)$$

$$A\underline{x} = \underline{b}, \quad A'\underline{x} + D\underline{t} = \underline{b}'$$

$$\Leftarrow$$

$Ax = b$ mà vô nghiệm

$A'x + Dt \overset{?}{=} b'$

$Dt = b' - A'x$