$$\varphi(0) = \varphi(0 \cdot 0) = 0 \cdot \varphi(0) = \underline{\underline{0}}$$

$$\varphi(-x) = \varphi((-1)x) = (-1)\varphi(x) = -\varphi(x)$$

$$a \cdot x = \varphi_a(x)$$

$$\varphi_a(x+y) = a(x+y) = ax + ay =$$

$$= \varphi_a(x) + \varphi_a(y)$$

$$\varphi_a(cx) = a(cx) = (ac)x = (ca)x$$

$$= c(ax) = c\,\varphi_a(x)$$

$(I) \Rightarrow (II)$

$\varphi(\underline{ax} + \underline{by}) = \varphi(ax) + \varphi(by) =$

$a\varphi(x) + b\varphi(y)$

$(II) \Rightarrow (III)$ $\quad \underline{\underline{u = 0}}$

$\varphi(0) = 0$

$\varphi(0) = \underbrace{0 \cdot \varphi(0)}_{0} + \underbrace{0\,\varphi(0)}_{0} = 0$

$(n) \Rightarrow (n+1)$

$$\varphi\left(\sum_{i=1}^{n+1} c_i x_i\right) = \varphi\Big(1\cdot\underline{\sum_{i=1}^{n} c_i x_i} + \underline{c_{n+1} x_{n+1}}\Big)$$

$$= 1\,\varphi\left(\sum_{i=1}^{n} c_i x_i\right) + c_{n+1}\,\varphi(x_{n+1})$$

$$\overset{IP}{=} \sum_{i=1}^{n} c_i\,\varphi(x_i) + c_{n+1}\varphi(x_{n+1}) = \sum_{i=1}^{n+1} c_i\,\varphi(x_i)$$

(iii) $\Rightarrow$ (ii) $\quad \varphi(cx) = c\,\varphi(x)$

$\underline{\underline{n=1}}$

$\varphi(1 \cdot x + 1 \cdot y) = 1\,\varphi(x) + 1 \cdot \varphi(y) \qquad \underline{\underline{n=2}}$

Dk. $\psi : W \to V$, $\varphi : V \to U$ LZ

$\varphi \circ \psi : W \to U$? LZ

$$(\varphi \circ \psi)(ax + by) = \varphi(\underline{\psi(ax + by)}) =$$

$$= \varphi(a\psi(x) + b\psi(y)) =$$

$$= a\varphi(\psi(x)) + b\varphi(\psi(y)) =$$

$$= a(\varphi \circ \psi)(x) + b(\varphi \circ \psi)(y)$$

$\varphi : V \to U$, $S \subseteq V$, pod.

? $\varphi(S)$ je VP

$x, y \in \varphi(S)$, $a, b \in K \overset{?}{\Rightarrow}$

$a \underset{\varphi(v_1)}{x} + b \underset{\varphi(v_2)}{y} \in \varphi(S)$

$\exists\, v_1 \in V$, $\varphi(v_1) = x$, $v_1 \in S$

v_2 $\qquad \varphi(v_2) = y$, $v_2 \in S$

$a v_1 + b v_2 \in S \qquad \varphi(\underbrace{a v_1 + b v_2}_{\in S}) = \in \varphi(S)$

$\varphi: V \to U$, $T \subseteq U$ VP

$\overset{?}{\Longrightarrow}$ $\varphi^{-1}(T)$ VP

$v, w \in \varphi^{-1}(T)$, $a, b \in K$ $\overset{?}{\Longrightarrow}$

$av + bw \in \varphi^{-1}(T)$

$\varphi(v) \in T$, $\varphi(w) \in T$

$a\varphi(v) + b\varphi(w) \in T$

$\parallel$

$\varphi(av + bw)$

$av + bw \in \varphi^{-1}(T)$

$\parallel$

$= \{x \in V : \varphi(x) \in T\}$

$X = \{1, 2\}$ $\quad x = 1$

$V^X = V \times V$

$\pi_1 : V \times V \to V \quad (x, y) \mapsto x$

$\pi_2 : V \times V \to V \quad (x, y) \mapsto y$

$X = \{1, 2, 3\}$, $Y = \{2, 3\}$

$(x, y, \pi z) \longmapsto (y, \pi z)$

(a) φ inj $\overset{?}{\Longrightarrow}$ $\operatorname{Ker}\varphi = \{0\}$

$x \in \operatorname{Ker}\varphi \Rightarrow \varphi(x) = 0 = \varphi(0)$

$\Rightarrow \underline{\underline{x = 0}}$.

$\operatorname{Ker}\varphi = \{0\} \overset{?}{\Longrightarrow} \varphi$ inj

$$\frac{\varphi(x) = \varphi(y)}{\varphi(x) - \varphi(y) = 0} \Longrightarrow x = y$$

$\Rightarrow \varphi(x-y) = 0$

$\cancel{x - y} = 0 \Rightarrow x = y$

V kon. $\Rightarrow$ $\mathrm{Ker}\,\varphi \subseteq V$

$\vec{v}_1, \ldots, \vec{v}_k$ báze $\mathrm{Ker}\,\varphi$

$\varphi(\vec{v}_1) = \vec{o} = \ldots = \varphi(\vec{v}_k)$

$\vec{v}_1, \ldots, \vec{v}_k, \vec{v}_{k+1}, \ldots, \vec{v}_n$

? $\varphi(\vec{v}_{k+1}), \ldots, \varphi(\vec{v}_n)$ báze $\mathrm{Im}\,\varphi$

$\underset{\vec{o}}{\underset{\|}{\varphi(\vec{v}_1)}}, \varphi(\vec{v}_2), \ldots \underset{\vec{o}}{\underset{\|}{\varphi(\vec{v}_k)}}, \underline{\varphi(\vec{v}_{k+1}), \ldots, \varphi(\vec{v}_n)}$

$$\boxed{c_{k+1}\varphi(v_{k+1}) + \cdots + c_n\varphi(v_n)} = 0$$

$$\overset{?}{\Longrightarrow} \quad c_{k+1} = \cdots = c_n = 0$$

$$\varphi(\underbrace{c_{k+1}v_{k+1} + \cdots + c_n v_n}_{\in \operatorname{Ker}\varphi}) = 0$$

$$\Rightarrow c_1 = \cdots = c_k = c_{k+1} = \cdots = c_n = 0$$

$$c_1 v_1 + \cdots + c_k v_k = c_{k+1} v_{k+1} + \cdots + c_n v_n$$

$$-c_1 v_1 - \cdots - c_k v_k + c_{k+1} v_{k+1} + \cdots + c_n v_n = 0$$

φ je prosté $\iff \operatorname{Ker}\varphi = \{0\} \iff$

$\dim \operatorname{Ker}\varphi = 0$

$\dim V = \dim \operatorname{Ker}\varphi + \dim \operatorname{Im}\varphi$

$\iff \dim V = \dim \operatorname{Im}\varphi$

$\operatorname{Im}\varphi \subseteq V$

$\iff \operatorname{Im}\varphi = V \iff \varphi$ je na.

$$\varphi^{-1}(c u_1 + d u_2) \overset{?}{=} c\,\varphi^{-1}(u_1) + d\,\varphi^{-1}(u_2)$$

$$\varphi(\varphi^{-1}(c u_1 + d u_2)) = c u_1 + d u_2$$

$$\varphi\left(c\,\varphi^{-1}(u_1) + d\,\varphi^{-1}(u_2)\right) =$$

$$= c\,\underbrace{\varphi\varphi^{-1}(u_1)}_{u_1} + d\,\underbrace{\varphi(\varphi^{-1}(u_2))}_{u_2} =$$

$$= c u_1 + d u_2$$

$V \cong U \qquad \exists\, \varphi : V \to U$

lin. izom.

$$\dim V = \dim \operatorname{Ker} \varphi + \dim \operatorname{Im} \varphi$$

$$\operatorname{Ker}\varphi = 0$$

$\operatorname{Im} \varphi = U$

$\dim \operatorname{Im} \varphi = \dim U \qquad \Rightarrow \dim V = \dim U$

$\dim V = \dim U = n$

$V \cong K^n \cong U \implies V \cong U$

$$\varphi : \mathbb{K}^n \to \mathbb{K}^m$$

$$\begin{pmatrix} 1 & 0 & & 0 \\ 0 & \vdots & \ddots & \vdots \\ \vdots & 0 & & 1 \end{pmatrix} \qquad \left(\varphi(e_1), \ldots, \varphi(e_n) \right) = A$$

$$e_1 \quad e_2 \quad e_n \qquad \in \mathbb{K}^m \qquad \in \mathbb{K}^m$$

$$m \times n$$

$x \in \mathbb{R}^n$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1 e_1 + \quad + x_n e_n$$

$$\varphi(x) = x_1 \varphi(e_1) + \ldots + x_n \varphi(e_n) =$$

$$= \underline{A x}$$

V, W $\quad \varphi : V \to W$

$\qquad n \qquad m$

$\mathcal{B} = (v_1, \ldots, v_n)$ $\quad \mathcal{A} = (w_1, \ldots, w_m)$

$(\varphi(v_1))_{\mathcal{A}} \in W \quad , \ldots \quad (\varphi(v_n))_{\mathcal{A}}$

m Zeilen $\quad A = \left((\varphi(v_j))_{\mathcal{A}} \right)_{j=1}^{n}$

$m \times n$

$$A(x)_B = \left((\varphi(\mathfrak{X}_j))_2 \right)_{j=1}^{m} \begin{pmatrix} c_1 \\ c_m \end{pmatrix} =$$

$$x = c_1 \mathfrak{X}_1 + \quad + c_m \mathfrak{X}_m$$

$$(x)_B = \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix}$$

$$x$$
$$\shortparallel$$

$$= \sum_{j=1}^{m} c_j (\varphi(\mathfrak{X}_j))_2 = ((\varphi(\Sigma c_j \mathfrak{X}_j))_2 =$$

$$= (\varphi(x))_2$$

$$\big(\varphi(x)\big)_\alpha = A(x)_\beta = (\varphi_{\alpha\beta}(x))_\beta$$

$$\varphi \mapsto A = (\varphi)_{\alpha,\beta}$$

β, α

$$id_V : V \to V$$

$$\alpha = (v_1, \ldots, v_n)$$

$$\underline{(id_V)_{\alpha, \alpha}}$$

$$id_V(v_1), \ldots, id_V(v_n)$$

$$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = (v_1)_\alpha, \ldots, (v_n)_\alpha = \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}$$

$$\left(\varphi(x)\right)_\alpha = A \cdot (x)_\beta$$

?

$$\Rightarrow \quad A = (\varphi)_{\alpha,\beta}$$

$$B = (v_1, \ldots, v_n)$$

$$\underline{\underline{\left(\varphi(v_1)\right)_\alpha}} \qquad A \cdot (v_1)_\beta = A \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} =$$

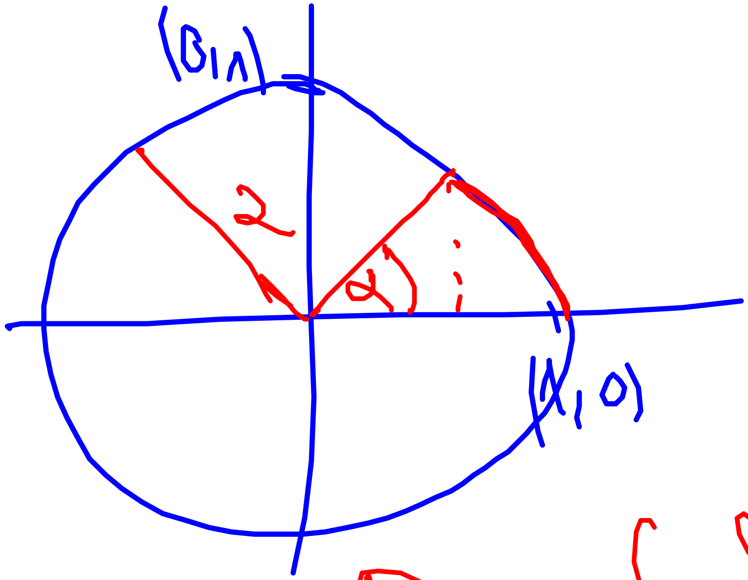
$$= s_1(A)$$

$$\big((\varphi \circ \psi)_{\alpha,\sigma}(w)\big)_\sigma = \underline{\big((\varphi \circ \psi)(w)\big)_\alpha}$$

$$(\psi)_{\beta,\sigma}(w)_\sigma = (\psi(w))_\beta$$

$$\overline{\big((\varphi)_{\alpha,\beta}(\psi)_{\beta,\sigma}(w)\big)_\sigma} =$$

$$(\varphi)_{\alpha,\beta}(\psi(w))_\beta = \underline{\underline{(\varphi(\psi(w)))_\alpha}}$$



$$\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} \quad \begin{pmatrix} -\sin\alpha \\ \cos\alpha \end{pmatrix}$$

$$R_\alpha(e_1) \qquad R_\alpha(e_2)$$

$$R_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

$$R_\alpha \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\alpha - y\sin\alpha \\ x\sin\alpha + y\cos\alpha \end{pmatrix}$$

$$\begin{pmatrix} \cos^2\alpha - \sin^2\alpha, & 2\cos\alpha\sin\alpha \\ 2\cos\alpha\sin\alpha, & \sin^2\alpha - \cos^2\alpha \end{pmatrix} R_\alpha \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} R_{-\alpha} =$$

$$= \begin{pmatrix} \cos^2 & \\ & \end{pmatrix}$$

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix}$$

$$\begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix} \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}$$