CVIČENÍ

Taylorova věta: $f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + R_n(x)$,

kde $R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$, ξ leží mezi x_0 a x.

Pokud vynecháme $R_n(x)$, získáme Taylorův polynom. Pokud v Taylorově větě položíme $x_0 = 0$, získáme McLaurinův rozvoj.

1) Napište Tayl. polynom pro $n=4$, $x_0 = 1$, $f(x) = x \ln x$

$$x \ln x = 0 + \frac{1 \cdot \ln x_0 + x_0 \frac{1}{x_0}}{1!}(x-1) + \frac{\frac{1}{x_0} + 0}{2!}(x-1)^2 + \frac{-\frac{1}{x_0^2}}{3!}(x-1)^3 + \frac{\frac{2}{x_0^3}}{4!}(x-1)^4 =$$

$$= (x-1) + \frac{1}{2}(x-1)^2 - \frac{(x-1)^3}{6} + \frac{1}{12}(x-1)^4$$

2) Vyjádřete $\sin \frac{x\pi}{4}$ pomocí mocnin $(x-2)$, (tj. po Tayl. polynom v bodě 2)

$$\sin \frac{x\pi}{4} = \sin \frac{x_0\pi}{4} + \frac{\cos \frac{x_0\pi}{4} \cdot \frac{\pi}{4}}{1!}(x-2) + \frac{-\sin \frac{x_0\pi}{4}\left(\frac{\pi}{4}\right)^2}{2!}(x-2)^2 - \frac{\cos \frac{x_0\pi}{4}\left(\frac{\pi}{4}\right)^3}{3!}(x-2)^3 + \dots =$$

$$= 1 + \frac{\pi}{4} \cdot 0 - \left(\frac{\pi}{4}\right)^2 \frac{1}{2!}(x-2)^2 + 0 + \dots$$

3) Najděte McLaurinův rozvoj pro $f(x) = e^x$

$$e^x = e^0 + \frac{e^0}{1!}(x-0) + \frac{e^0}{2!}(x)^2 + \frac{e^0}{3!}x^3 + \frac{e^0}{4!}x^4 + \dots + \frac{e^0}{n!}x^n + \frac{e^{\xi}}{(n+1)!}x^{n+1} =$$ ξ leží mezi 0 a x

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots + \frac{x^n}{n!} + \frac{e^{\xi}}{(n+1)!}x^{n+1}$$

4) Napište Taylorův polynom $n = 2$, $x_0 = 1$, $f(x) =$
$= \operatorname{arctg} x$ a odhadněte chybu pro $x \in (0{,}9;\ 1{,}1)$

$$\operatorname{arctg} x = \operatorname{arctg} 1 + \frac{\frac{1}{1+x_0^2}}{1!}(x-1) + \frac{\frac{-2x_0}{(1+x_0^2)^2}}{2!}(x-1)^2 + \frac{\frac{6(\xi^2 - \frac{1}{3})}{(1+\xi^2)^3}}{3!}(x-1)^3$$

$$= \frac{\pi}{4} + \frac{1}{2}(x-1) - \frac{1}{4}(x-1)^2 + \frac{6\xi^2 - 2}{6(1+\xi^2)^3}(x-1)^3$$

$$\left(\frac{-2x}{(1+x^2)^2}\right)' = \frac{-2(1+x^2)^2 + 2x\,2(1+x^2)\cdot 2x}{(1+x^2)^4} =$$

$$= \frac{-2(1+x^2)^2 + 8x^2(1+x^2)}{(1+x^2)^4} =$$

$$= \frac{-2(1+2x^2+x^4) + 8x^2 + 8x^4}{(1+x^2)^4} =$$

$$= \frac{-2 - 4x^2 - 2x^4 + 8x^2 + 8x^4}{(1+x^2)^4} = \frac{6x^4 + 4x^2 - 2}{(1+x^2)^4} =$$

$$= \frac{6(x^2 - \frac{1}{3})(x^2+1)}{(1+x^2)^4} = \frac{6(x^2 - \frac{1}{3})}{(1+x^2)^3}$$

$$6a^2 + 4a - 2 = 0$$

$$16 - 4\cdot 6\cdot(-2) = 16 + 48 = 64 \quad \sqrt{\ } = 8$$

$$\frac{-4 \pm 8}{12} = \left\langle \begin{matrix} \frac{1}{3} \\ -1 \end{matrix} \right.$$

$$R_2 = \frac{6\xi^2 - 2}{6(1+\xi^2)^3}(x-1)^3, \quad 0{,}9 < \xi < 1{,}1$$

Odhad čitatele: $|6\xi^2 - 2| \leq 6|\xi|^2 + 2 < 6\cdot(1{,}1)^2 + 2 =$

$= 9{,}26$

Odhad jmenovatele: $|6(\xi^2+1)^3| = 6(\xi^2+1)^3$

$$\xi^2 + 1 \geq 1$$

$$6(\xi^2+1)^3 \geq 6$$

$$\Rightarrow |R_2| = \left| \frac{6\xi^2 - 2}{6(\xi^2+1)^3} (x-1)^3 \right| \leq \frac{|6\xi^2 - 2|}{6(\xi^2+1)^3} |x-1|^3 \leq$$

$$\leq \frac{9{,}26}{6} |x-1|^3 \leq 1{,}54 \cdot (1{,}1-1)^3 = 0{,}00154$$

c) Vypočtěte konstantu e s chybou menší než 0,001

$$e^x \doteq 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + R_n$$

$$e^1 \doteq 1 + 1 + \frac{1}{2} + \dots + \frac{1}{n!} + \frac{e^\xi}{(n+1)!}$$

$$\frac{e^\xi}{(n+1)!} < 0{,}001 \qquad \xi \in (0;\ 1)$$

$$e^\xi < e < 3$$

$$\frac{e^\xi}{(n+1)!} < \frac{3}{(n+1)!} < 0{,}001$$

$$3000 < (n+1)!$$

$$\left.\begin{array}{l} 7! = 5040 \\ 6! = 720 \end{array}\right\} \Rightarrow n > 5\ [(n+1) > 6] \Rightarrow n = 6$$

$$e = 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} = 2{,}71805556$$

6) Vypočti $\sin 20°$ s odhadem chyby.

$$T(\sin x) = x - \frac{x^3}{3!} + R_5$$

$$R_5 = \frac{\cos \xi}{5!} x^5 \qquad \xi \in (0; x)$$

$$20° = \frac{\pi}{9} \Rightarrow \sin 20° = \frac{\pi}{9} - \frac{\left(\frac{\pi}{9}\right)^3}{3!} = 0{,}3420$$

$$R_5 = \frac{\cos \xi}{5!} x^5, \quad \xi \in \left(0; \frac{\pi}{9}\right)$$

$$|R_5| = \left| \frac{\cos \xi}{5!} x^5 \right| \leq \frac{|\cos \xi| \cdot |x^5|}{5!} \leq \frac{|x^5|}{5!} \leq$$

$$\leq \frac{\left(\frac{\pi}{9}\right)^5}{5!} < 10^{-4}$$

7) Pro jaké hodnoty x platí $\cos x = 1 - \frac{x^2}{2}$ s přesností $0{,}0001$?

$$\cos x = 1 - \frac{x^2}{2!} + R_3$$

$$R_3 = \frac{\cos \xi}{4!} x^4, \quad \xi \in (0; x)$$

$$|R_3| \leq \frac{|x^4|}{4!} = \frac{x^4}{24}$$

$$\frac{x^4}{24} \leq 0{,}0001$$

$$x^4 \leq 0{,}0024 \Rightarrow x \in \langle -0{,}22 ; 0{,}22 \rangle$$

$$|x| < 0{,}22 = 12°30'$$