Inženýrská matematika
Diferenciální rovnice
Metody řešení a příklady

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Obsah

Část I

DR prvního řádu

1 DR se separovanými proměnnými

Definice (DR se separovanými proměnnými). Diferenciální rovnice tvaru

$$y' = f(x)g(y), \tag{S}$$

kde f a g jsou funkce spojité na (nějakých) otevřených intervalech se nazývá *obyčejná diferenciální rovnice se separovanými proměnnými.*

DR se separovanými proměnnými

$$y' = f(x)g(y)$$

Rovnice se separovanými proměnnými.

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$$y' = f(x)g(y)$$

Rovnice má konstatní řešení tvaru $y = y_i$, kde y_i jsou řešeními rovnice $g(y_i) = 0$.

Nejdřív najdeme konstantní řešení.

DR se separovanými proměnnými

$$y' = f(x)g(y)$$

Rovnice má konstatní řešení tvaru $y = y_i$, kde y_i jsou řešeními rovnice $g(y_i) = 0$. Dále budeme hledat nekonstantní řešení.

$$\frac{dy}{dx} = f(x)g(y)$$

napíšeme derivaci jako podíl diferenciálů $\frac{dy}{dx}$.

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Rovnice má konstatní řešení tvaru $y = y_i$, kde y_i jsou řešeními rovnice $g(y_i) = 0$. Dále budeme hledat nekonstantní řešení.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y)$$

$$\frac{\mathrm{d}y}{g(y)} = f(x)\,\mathrm{d}x$$

Vynásobíme rovnici jmenovateli zlomků a odseparujeme tak proměnné.

DR se separovanými proměnnými

$$y' = f(x)g(y)$$

Rovnice má konstatní řešení tvaru $y = y_i$, kde y_i jsou řešeními rovnice $g(y_i) = 0$. Dále budeme hledat nekonstantní řešení.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y)$$

$$\frac{\mathrm{d}y}{g(y)} = f(x)\,\mathrm{d}x$$

$$\int \frac{\mathrm{d}y}{g(y)} = \int f(x)\,\mathrm{d}x + C$$

Zintegrujeme obě strany rovnice. Použijeme jenom jednu integrační konstantu. Získáme obecné řešení.

DR se separovanými proměnnými

$$y' = f(x)g(y)$$

Rovnice má konstatní řešení tvaru $y = y_i$, kde y_i jsou řešeními rovnice $g(y_i) = 0$. Dále budeme hledat nekonstantní řešení.

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$$\int \frac{\mathrm{d}y}{g(y)} = \int f(x)\,\mathrm{d}x + C$$

Je-li zadána počáteční podmínka, najdeme nejprve konstantu C pro kterou je počáteční podmínka splněna.

Najděte funkci $y(x)$ splňující $\boxed{y' = y\cos x}$ a $\boxed{y(0) = 0.1}$

Najděte funkci $y(x)$ splňující $y' = y \cos x$ a $y(0) = 0.1$

$$\frac{dy}{dx} = y \cdot \cos x$$

Napíšeme derivaci y' jako podíl diferenciálů $\frac{dy}{dx}$

Najděte funkci $y(x)$ splňující $y' = y\cos x$ a $y(0) = 0.1$

$$\frac{dy}{dx} = y \cdot \cos x$$

$$\frac{1}{y}\,dy = \cos x\,dx$$

- Násobením rovnice výrazy ve jmenovatelích odseparuje proměnné.
- Z podmínky $y(0) = 0.1$ je zřejmé, že funkce není rovna nule (alespoň v nějakém okolí bodu $x = 0$).

Najděte funkci $y(x)$ splňující $\boxed{y' = y\cos x}$ a $\boxed{y(0) = 0.1}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y \cdot \cos x$$

$$\int \frac{1}{y}\,\mathrm{d}y = \int \cos x \ \mathrm{d}x$$

Připíšeme integrály. Vlevo integrujeme podle y, vpravo podle x.

Najděte funkci $y(x)$ splňující $\boxed{y' = y\cos x}$ a $\boxed{y(0) = 0.1}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y \cdot \cos x$$

$$\int \frac{1}{y}\,\mathrm{d}y = \int \cos x \;\mathrm{d}x$$

$$\ln y = \sin x + C$$

- Vypočteme integrály. Funkce y je kladná (alespoň v nějakém okolí bodu $x = 0$). Uvažujeme jenom jednu integrační konstantu.
- Získáváme rovnici popisující **všechna řešení** rovnice $y' = y \cdot \cos x$.

Najděte funkci $y(x)$ splňující $y' = y\cos x$ a $y(0) = 0.1$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y \cdot \cos x$$

$$\int \frac{1}{y}\,\mathrm{d}y = \int \cos x \,\mathrm{d}x$$

$$\ln y = \sin x + C$$

$$\ln 0.1 = \sin 0 + C$$

Použijeme počáteční podmínku $y(0) = 0.1$ pro nalezení integrační konstanty.

Najděte funkci $y(x)$ splňující $\boxed{y' = y\cos x}$ a $\boxed{y(0) = 0.1}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y \cdot \cos x$$

$$\int \frac{1}{y}\,\mathrm{d}y = \int \cos x \;\mathrm{d}x$$

$$\ln y = \sin x + C$$

$$\ln 0.1 = \sin 0 + C$$

$$C = \ln 0.1$$

Vypočteme C.

Najděte funkci $y(x)$ splňující $\boxed{y' = y\cos x}$ a $\boxed{y(0) = 0.1}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y \cdot \cos x$$

$$\int \frac{1}{y}\,\mathrm{d}y = \int \cos x \;\mathrm{d}x$$

$$\ln y = \sin x + C$$

$$\ln 0.1 = \sin 0 + C$$

$$C = \ln 0.1$$

$$\ln y = \sin x + \ln 0.1$$

Dosadíme za C a získáme **partikulární řešení** zadané **počáteční úlohy**. Toto řešení je zatím v implicitním tvaru.

Najděte funkci $y(x)$ splňující $\boxed{y' = y\cos x}$ a $\boxed{y(0) = 0.1}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y \cdot \cos x$$

$$\int \frac{1}{y}\,\mathrm{d}y = \int \cos x\ \mathrm{d}x$$

$$\ln y = \sin x + C$$

$$\ln 0.1 = \sin 0 + C$$

$$C = \ln 0.1$$

$$\ln y = \sin x + \ln 0.1$$

$$\ln y - \ln 0.1 = \sin x$$

Převedeme logaritmy na jednu stranu.

Najděte funkci $y(x)$ splňující $\boxed{y' = y\cos x}$ a $\boxed{y(0) = 0.1}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y \cdot \cos x$$

$$\int \frac{1}{y}\,\mathrm{d}y = \int \cos x\ \mathrm{d}x$$

$$\ln y = \sin x + C$$

$$\ln 0.1 = \sin 0 + C$$

$$C = \ln 0.1$$

$$\ln y = \sin x + \ln 0.1$$

$$\ln y - \ln 0.1 = \sin x$$

$$\ln \frac{y}{0.1} = \sin x$$

Odečteme logaritmy.

Najděte funkci $y(x)$ splňující $y' = y\cos x$ a $y(0) = 0.1$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y \cdot \cos x$$

$$\int \frac{1}{y}\,\mathrm{d}y = \int \cos x \;\mathrm{d}x$$

$$\ln y = \sin x + C$$

$$\ln 0.1 = \sin 0 + C$$

$$C = \ln 0.1$$

$$\ln y = \sin x + \ln 0.1$$

$$\ln y - \ln 0.1 = \sin x$$

$$\ln \frac{y}{0.1} = \sin x$$

$$\frac{y}{0.1} = e^{\sin x}$$

Odstraníme logaritmus použitím inverzní funkce.

Najděte funkci $y(x)$ splňující $y' = y\cos x$ a $y(0) = 0.1$

$$\frac{dy}{dx} = y \cdot \cos x$$
$$\int \frac{1}{y}\,dy = \int \cos x \; dx$$
$$\ln y = \sin x + C$$
$$\ln 0.1 = \sin 0 + C$$
$$C = \ln 0.1$$
$$\ln y = \sin x + \ln 0.1$$

$$\ln y - \ln 0.1 = \sin x$$
$$\ln \frac{y}{0.1} = \sin x$$
$$\frac{y}{0.1} = e^{\sin x}$$
$$y = 0.1 \cdot e^{\sin x}$$

Osamostatníme y. Získáme řešení v explicitním tvaru.

Najděte funkci $y(x)$ splňující $y' = y\cos x$ a $y(0) = 0.1$

$$\frac{dy}{dx} = y \cdot \cos x$$

$$\int \frac{1}{y}\,dy = \int \cos x \, dx$$

$$\ln y = \sin x + C$$

$$\ln 0.1 = \sin 0 + C$$

$$C = \ln 0.1$$

$$\ln y = \sin x + \ln 0.1$$

$$\ln y - \ln 0.1 = \sin x$$

$$\ln \frac{y}{0.1} = \sin x$$

$$\frac{y}{0.1} = e^{\sin x}$$

$$y = 0.1 \cdot e^{\sin x}$$

Označení:
diferenciální rovnice + počáteční podmínka = *počáteční úloha*,
obecné řešení, partikulární řešení

Řešte $y^2 - 1 + yy'(x^2 - 1) = 0, \quad y(0) = 2.$

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$$yy'(1 - x^2) = y^2 - 1$$

Osamostatníme y'.

Řešte $y^2 - 1 + yy'(x^2 - 1) = 0, \quad y(0) = 2.$

$$yy'(1 - x^2) = y^2 - 1$$

$$y' = \frac{y^2 - 1}{y} \frac{1}{1 - x^2}$$

Rovnice má separované proměnné a má smysl pro $y \neq 0$ a $x \neq \pm 1$.

Řešte $y^2 - 1 + yy'(x^2 - 1) = 0, \quad y(0) = 2.$

$$yy'(1 - x^2) = y^2 - 1$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2 - 1}{y} \frac{1}{1 - x^2}$$

Přepíšeme derivaci jako podíl diferenciálů.

Řešte $y^2 - 1 + yy'(x^2 - 1) = 0, \quad y(0) = 2.$

$$yy'(1 - x^2) = y^2 - 1$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2 - 1}{y} \frac{1}{1 - x^2}$$

$$\frac{y}{y^2 - 1}\,\mathrm{d}y = \frac{1}{1 - x^2}\,\mathrm{d}x$$

Odseparujem proměnné. Přči tom násobíme rovnici výrazem $\frac{y}{y^2 - 1}$.

Toto lze provést pokud $y \neq \pm 1$, což je garantováno počáteční podmínkou.

Řešte $y^2 - 1 + yy'(x^2 - 1) = 0, \quad y(0) = 2.$

$$yy'(1 - x^2) = y^2 - 1$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2 - 1}{y} \frac{1}{1 - x^2}$$

$$\int \frac{y}{y^2 - 1}\,\mathrm{d}y = \int \frac{1}{1 - x^2}\,\mathrm{d}x$$

Připíšeme integrály...

Řešte $y^2 - 1 + yy'(x^2-1) = 0, \quad y(0) = 2.$

$$yy'(1-x^2) = y^2 - 1$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2-1}{y}\,\frac{1}{1-x^2}$$

$$\int \frac{y}{y^2-1}\,\mathrm{d}y = \int \frac{1}{1-x^2}\,\mathrm{d}x$$

$$\frac{1}{2}\ln|y^2-1| =$$

…a integrujeme. První integrál je (až na aditivní konstantu) typu $\int \frac{f'(x)}{f(x)}\,\mathrm{d}x$.

Řešte $y^2 - 1 + yy'(x^2 - 1) = 0, \quad y(0) = 2.$

$$yy'(1 - x^2) = y^2 - 1$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2 - 1}{y} \frac{1}{1 - x^2}$$

$$\int \frac{y}{y^2 - 1}\,\mathrm{d}y = \int \frac{1}{1 - x^2}\,\mathrm{d}x$$

$$\frac{1}{2} \ln |y^2 - 1| = \frac{1}{2} \ln\left|\frac{1 + x}{1 - x}\right| + c$$

Druhý integrál napíšeme pomocí vzorců.

Řešte $y^2 - 1 + yy'(x^2 - 1) = 0, \quad y(0) = 2.$

$$yy'(1 - x^2) = y^2 - 1$$

$$\frac{dy}{dx} = \frac{y^2 - 1}{y} \frac{1}{1 - x^2}$$

$$\int \frac{y}{y^2 - 1}\, dy = \int \frac{1}{1 - x^2}\, dx$$

$$\frac{1}{2} \ln |y^2 - 1| = \frac{1}{2} \ln \left| \frac{1 + x}{1 - x} \right| + c$$

$$\ln(y^2 - 1) = \ln \frac{1 + x}{1 - x} + 2c$$

vynásobíme rovnici dvěma. Vzhledem k počáteční podmínce vynecháme absolutní hodnoty.

Řešte $y^2 - 1 + yy'(x^2 - 1) = 0, \quad y(0) = 2.$

$$yy'(1 - x^2) = y^2 - 1$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2 - 1}{y} \frac{1}{1 - x^2}$$

$$\int \frac{y}{y^2 - 1}\,\mathrm{d}y = \int \frac{1}{1 - x^2}\,\mathrm{d}x$$

$$\frac{1}{2} \ln |y^2 - 1| = \frac{1}{2} \ln \left| \frac{1 + x}{1 - x} \right| + c$$

$$\ln(y^2 - 1) = \ln \frac{1 + x}{1 - x} + 2c$$

$$\ln(y^2 - 1) = \ln \frac{1 + x}{1 - x} + \ln e^{2c}$$

Napíšeme $2c$ ve tvaru logaritmu $\ln e^{2c}$...

Řešte $y^2 - 1 + yy'(x^2 - 1) = 0, \quad y(0) = 2.$

$$yy'(1 - x^2) = y^2 - 1$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2 - 1}{y} \frac{1}{1 - x^2}$$

$$\int \frac{y}{y^2 - 1}\,\mathrm{d}y = \int \frac{1}{1 - x^2}\,\mathrm{d}x$$

$$\frac{1}{2}\ln|y^2 - 1| = \frac{1}{2}\ln\left|\frac{1 + x}{1 - x}\right| + c$$

$$\ln(y^2 - 1) = \ln\frac{1 + x}{1 - x} + 2c$$

$$\ln(y^2 - 1) = \ln\frac{1 + x}{1 - x} + \ln e^{2c}$$

$$\ln(y^2 - 1) = \ln\left(\frac{1 + x}{1 - x}e^{2c}\right)$$

…a sečteme logaritmy.

Řešte $\quad y^2 - 1 + yy'(x^2-1) = 0, \quad y(0) = 2.$

$$yy'(1-x^2) = y^2 - 1$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2-1}{y}\,\frac{1}{1-x^2}$$

$$\int \frac{y}{y^2-1}\,\mathrm{d}y = \int \frac{1}{1-x^2}\,\mathrm{d}x$$

$$\frac{1}{2}\ln|y^2-1| = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + c$$

$$\ln(y^2-1) = \ln\frac{1+x}{1-x} + 2c$$

$$\ln(y^2-1) = \ln\frac{1+x}{1-x} + \ln e^{2c}$$

$$\cancel{\ln}(y^2-1) = \cancel{\ln}\left(\frac{1+x}{1-x}e^{2c}\right)$$

$$y^2 - 1 = \frac{1+x}{1-x}e^{2c}$$

Logaritmus je prostá funkce a můžeme jej na obou stranách rovnice vynechat.

Řešte $y^2 - 1 + yy'(x^2-1) = 0, \quad y(0) = 2.$

$$yy'(1-x^2) = y^2 - 1$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2-1}{y}\,\frac{1}{1-x^2}$$

$$\int \frac{y}{y^2-1}\,\mathrm{d}y = \int \frac{1}{1-x^2}\,\mathrm{d}x$$

$$\frac{1}{2}\ln|y^2-1| = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + c$$

$$\ln(y^2-1) = \ln\frac{1+x}{1-x} + 2c$$

$$\ln(y^2-1) = \ln\frac{1+x}{1-x} + \ln e^{2c}$$

$$\ln(y^2-1) = \ln\left(\frac{1+x}{1-x}e^{2c}\right)$$

$$y^2 - 1 = \frac{1+x}{1-x}e^{2c}$$

$$y^2 = 1 + C\cdot\frac{1+x}{1-x}$$

Obecné řešení. $C = e^{2c}$ je nová konstanta.

Řešte $y^2 - 1 + yy'(x^2 - 1) = 0, \quad y(0) = 2.$

$$yy'(1 - x^2) = y^2 - 1$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2 - 1}{y}\,\frac{1}{1 - x^2}$$

$$\int \frac{y}{y^2 - 1}\,\mathrm{d}y = \int \frac{1}{1 - x^2}\,\mathrm{d}x$$

$$\frac{1}{2}\ln|y^2 - 1| = \frac{1}{2}\ln\left|\frac{1 + x}{1 - x}\right| + c$$

$$\ln(y^2 - 1) = \ln\frac{1 + x}{1 - x} + 2c$$

$$\ln(y^2 - 1) = \ln\frac{1 + x}{1 - x} + \ln e^{2c}$$

$$\ln(y^2 - 1) = \ln\left(\frac{1 + x}{1 - x}e^{2c}\right)$$

$$y^2 - 1 = \frac{1 + x}{1 - x}e^{2c}$$

$$y^2 = 1 + C \cdot \frac{1 + x}{1 - x}$$

$x = 0$

$y = 2$

$$2^2 = 1 + C\frac{1 + 0}{1 - 0}$$

Dosadíme hodnoty z počáteční podmínky. . .

Řešte $y^2 - 1 + yy'(x^2 - 1) = 0, \quad y(0) = 2.$

$$yy'(1 - x^2) = y^2 - 1$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2 - 1}{y} \frac{1}{1 - x^2}$$
$$\int \frac{y}{y^2 - 1}\,\mathrm{d}y = \int \frac{1}{1 - x^2}\,\mathrm{d}x$$
$$\frac{1}{2} \ln |y^2 - 1| = \frac{1}{2} \ln\left|\frac{1 + x}{1 - x}\right| + c$$
$$\ln(y^2 - 1) = \ln \frac{1 + x}{1 - x} + 2c$$
$$\ln(y^2 - 1) = \ln \frac{1 + x}{1 - x} + \ln e^{2c}$$

$$\ln(y^2 - 1) = \ln\left(\frac{1 + x}{1 - x} e^{2c}\right)$$
$$y^2 - 1 = \frac{1 + x}{1 - x} e^{2c}$$
$$y^2 = 1 + C \cdot \frac{1 + x}{1 - x}$$
$$2^2 = 1 + C \frac{1 + 0}{1 - 0}$$
$$C = 3$$

…a nalezneme C.

Řešte $y^2 - 1 + yy'(x^2-1) = 0, \quad y(0) = 2.$

$$yy'(1-x^2) = y^2 - 1$$

$$\frac{dy}{dx} = \frac{y^2-1}{y}\,\frac{1}{1-x^2}$$

$$\int \frac{y}{y^2-1}\,dy = \int \frac{1}{1-x^2}\,dx$$

$$\frac{1}{2}\ln|y^2-1| = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + c$$

$$\ln(y^2-1) = \ln\frac{1+x}{1-x} + 2c$$

$$\ln(y^2-1) = \ln\frac{1+x}{1-x} + \ln e^{2c}$$

$$\ln(y^2-1) = \ln\left(\frac{1+x}{1-x}e^{2c}\right)$$

$$y^2 - 1 = \frac{1+x}{1-x}e^{2c}$$

$$y^2 = 1 + C\cdot\frac{1+x}{1-x}$$

$$2^2 = 1 + C\frac{1+0}{1-0}$$

$$C = 3$$

$$y^2 = 1 + 3\frac{1+x}{1-x}$$

Použijeme toto C v obecném řešení.

Řešte $y^2 - 1 + yy'(x^2-1) = 0, \quad y(0) = 2.$

$$yy'(1-x^2) = y^2 - 1$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2-1}{y}\,\frac{1}{1-x^2}$$

$$\int \frac{y}{y^2-1}\,\mathrm{d}y = \int \frac{1}{1-x^2}\,\mathrm{d}x$$

$$\frac{1}{2}\ln|y^2-1| = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + c$$

$$\ln(y^2-1) = \ln\frac{1+x}{1-x} + 2c$$

$$\ln(y^2-1) = \ln\frac{1+x}{1-x} + \ln e^{2c}$$

$$\ln(y^2-1) = \ln\left(\frac{1+x}{1-x}e^{2c}\right)$$

$$y^2 - 1 = \frac{1+x}{1-x}e^{2c}$$

$$y^2 = 1 + C\cdot\frac{1+x}{1-x}$$

$$2^2 = 1 + C\frac{1+0}{1-0}$$

$$C = 3$$

$$y^2 = 1 + 3\frac{1+x}{1-x}$$

$$y^2 = \frac{4+2x}{1-x}$$

Upravíme. Problém je vyřešen.

Solve the IVP $y' = \dfrac{2x+1}{2(y-1)}, \quad y(2) = 0$

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$$y' = \frac{2x+1}{2(y-1)}$$

We start with the equation.

Solve the IVP $\quad y' = \dfrac{2x+1}{2(y-1)}, \quad y(2) = 0$

$$y' = \frac{2x+1}{2(y-1)}$$

$$(2y-2)\,\mathrm{d}y = \quad (2x+1)\,\mathrm{d}x$$

The equation has separated variables and is meaningful for $y \neq 1$. To find the solution we multiply the equation by $2(y-1)$

Solve the IVP $y' = \dfrac{2x+1}{2(y-1)}, \quad y(2) = 0$

$$y' = \frac{2x+1}{2(y-1)}$$

$$\int (2y-2)\,\mathrm{d}y = \int (2x+1)\,\mathrm{d}x$$

We add integrals

Solve the IVP $\quad y' = \dfrac{2x+1}{2(y-1)}, \quad y(2) = 0$

$$y' = \frac{2x+1}{2(y-1)}$$

$$\int (2y-2)\,\mathrm{d}y = \int (2x+1)\,\mathrm{d}x$$

$$y^2 - 2y = x^2 + x + C$$

We integrate both sides of the equation. We have $\int 2y - 2\,\mathrm{d}y = y^2 - 2y$ and $\int 2x + 1\,\mathrm{d}x = x^2 + x$. We use the constant of integration on the right-hand side. We get the general solution of the equation.

Solve the IVP $y' = \dfrac{2x+1}{2(y-1)}, \quad y(2) = 0$

$$y' = \frac{2x+1}{2(y-1)}$$

$$\int (2y-2)\,\mathrm{d}y = \int (2x+1)\,\mathrm{d}x$$

$$y^2 - 2y = x^2 + x + C$$

$$(y-1)^2 - 1 = x^2 + x + C$$

We complete square on the left. . .

Solve the IVP $\quad y' = \dfrac{2x+1}{2(y-1)}, \quad y(2) = 0$

$$y' = \frac{2x+1}{2(y-1)}$$

$$\int (2y-2)\,\mathrm{d}y = \int (2x+1)\,\mathrm{d}x$$

$$y^2 - 2y = x^2 + x + C$$

$$(y-1)^2 - 1 = x^2 + x + C$$

$$(y-1)^2 = x^2 + x + C + 1$$

. . . and solve for y.

Solve the IVP $y' = \dfrac{2x+1}{2(y-1)}, \quad y(2) = 0$

$$y' = \frac{2x+1}{2(y-1)}$$

$$\int (2y-2)\,\mathrm{d}y = \int (2x+1)\,\mathrm{d}x$$

$$y^2 - 2y = x^2 + x + C$$

$$(y-1)^2 - 1 = x^2 + x + C$$

$$(y-1)^2 = x^2 + x + C + 1$$

$$y - 1 = \pm\sqrt{x^2 + x + K}$$

Let $K = C + 1$ be new constant. We take the second root of both sides of equation. . .

Solve the IVP $y' = \frac{2x+1}{2(y-1)}, \quad y(2) = 0$

$$y' = \frac{2x+1}{2(y-1)}$$

$$\int (2y-2)\,\mathrm{d}y = \int (2x+1)\,\mathrm{d}x$$

$$y^2 - 2y = x^2 + x + C$$

$$(y-1)^2 - 1 = x^2 + x + C$$

$$(y-1)^2 = x^2 + x + C + 1$$

$$y - 1 = \pm\sqrt{x^2 + x + K}$$

$$y = 1 \pm \sqrt{x^2 + x + K}$$

…and solve for y.

Solve the IVP $\quad y' = \dfrac{2x+1}{2(y-1)}, \quad y(2) = 0$

$$y' = \frac{2x+1}{2(y-1)}$$

$$\int (2y-2)\,\mathrm{d}y = \int (2x+1)\,\mathrm{d}x$$

$$y^2 - 2y = x^2 + x + C$$

$$(y-1)^2 - 1 = x^2 + x + C$$

$$(y-1)^2 = x^2 + x + C + 1$$

$$y - 1 = \pm\sqrt{x^2 + x + K}$$

$$y = 1 \pm \sqrt{x^2 + x + K}$$

$$y_1 = 1 + \sqrt{x^2 + x + K}$$

$$y_2 = 1 - \sqrt{x^2 + x + K}$$

This shows that there are two explicit formulas for general solution. Since $y_1(x) \geq 1$ and $y_2(x) \leq 1$ for all x, we consider the solution y_2 only (see the initial condition).

Solve the IVP $y' = \dfrac{2x+1}{2(y-1)}$, $y(2) = 0$

$$y' = \frac{2x+1}{2(y-1)}$$
$$\int (2y-2)\,\mathrm{d}y = \int (2x+1)\,\mathrm{d}x$$
$$y^2 - 2y = x^2 + x + C$$
$$(y-1)^2 - 1 = x^2 + x + C$$
$$(y-1)^2 = x^2 + x + C + 1$$
$$y - 1 = \pm\sqrt{x^2 + x + K}$$
$$y = 1 \pm \sqrt{x^2 + x + K}$$

$y = 0$

$x = 2$

$$y_1 = 1 + \sqrt{x^2 + x + K}$$
$$y_2 = 1 - \sqrt{x^2 + x + K}$$
$$0 = 1 - \sqrt{4 + 2 + K}$$

We substitute the initial condition into y_2.

Solve the IVP $\quad y' = \dfrac{2x+1}{2(y-1)}, \quad y(2) = 0$

$$y' = \frac{2x+1}{2(y-1)}$$

$$\int (2y-2)\,\mathrm{d}y = \int (2x+1)\,\mathrm{d}x$$

$$y^2 - 2y = x^2 + x + C$$

$$(y-1)^2 - 1 = x^2 + x + C$$

$$(y-1)^2 = x^2 + x + C + 1$$

$$y - 1 = \pm\sqrt{x^2 + x + K}$$

$$y = 1 \pm \sqrt{x^2 + x + K}$$

$$\sout{y_1 = 1 + \sqrt{x^2 + x + K}}$$

$$y_2 = 1 - \sqrt{x^2 + x + K}$$

$$0 = 1 - \sqrt{4 + 2 + K}$$

$$K = -5$$

The solution of $0 = 1 - \sqrt{4+2+K}$ is $K = -5$.

Solve the IVP $y' = \dfrac{2x+1}{2(y-1)}, \quad y(2) = 0$

$$y' = \frac{2x+1}{2(y-1)}$$

$$\int (2y-2)\,\mathrm{d}y = \int (2x+1)\,\mathrm{d}x$$

$$y^2 - 2y = x^2 + x + C$$

$$(y-1)^2 - 1 = x^2 + x + C$$

$$(y-1)^2 = x^2 + x + C + 1$$

$$y - 1 = \pm\sqrt{x^2 + x + K}$$

$$y = 1 \pm \sqrt{x^2 + x + K}$$

$$\cancel{y_1 = 1 + \sqrt{x^2 + x + K}}$$

$$y_2 = 1 - \sqrt{x^2 + x + K}$$

$$0 = 1 - \sqrt{4 + 2 + K}$$

$$K = -5$$

$$y_p = 1 - \sqrt{x^2 + x - 5}$$

We use the obtained value of K in the formula for y_2. The initial value problem is solved.

Solve DE $3xy^2y' = (y^3 - 1)(x^3 - 1)$.

Solve DE $3xy^2y' = (y^3 - 1)(x^3 - 1)$.

$$y' = \frac{y^3 - 1}{3y^2} \cdot \frac{x^3 - 1}{x}$$

- We solve the equation for y'.
- This shows that the equation has separated variables and is meaningful for $x \neq 0$ and $y \neq 0$.

Solve DE $3xy^2y' = (y^3-1)(x^3-1)$.

$$y' = \frac{y^3-1}{3y^2} \cdot \frac{x^3-1}{x}$$

The function $y \equiv 1$ is a solution.

The right-hand side equals zero for $y = 1$. Hence the constant function $y(x) = 1$ is a solution. This can be verified by direct substitution.

Solve DE $3xy^2y' = (y^3 - 1)(x^3 - 1)$.

$$y' = \frac{y^3 - 1}{3y^2} \cdot \frac{x^3 - 1}{x}$$

The function $y \equiv 1$ is a solution. From now suppose $y \not\equiv 1$.

Let us continue with the cases in which $y \not\equiv 1$. In this case we can multiply the equation by the factor $\frac{3y^2}{y^3 - 1}$. This separates the variables.

Solve DE $3xy^2y' = (y^3 - 1)(x^3 - 1)$.

$$\frac{\mathrm{d}x}{\mathrm{d}y} = y' = \frac{y^3 - 1}{3y^2} \cdot \frac{x^3 - 1}{x}$$

The function $y \equiv 1$ is a solution. From now suppose $y \not\equiv 1$.

$$\frac{3y^2}{y^3 - 1}\,\mathrm{d}y = \frac{x^3 - 1}{x}\,\mathrm{d}x$$

The variable y is on the left and x on the right.

Solve DE $3xy^2y' = (y^3-1)(x^3-1).$

The function $y \equiv 1$ is a solution. From now suppose $y \not\equiv 1$.

$$\int \frac{3y^2}{y^3-1}\,\mathrm{d}y = \int \frac{x^3-1}{x}\,\mathrm{d}x$$

We add integrals . . .

Solve DE $\quad 3xy^2y' = (y^3-1)(x^3-1)$.

The function $y \equiv 1$ is a solution. From now suppose $y \not\equiv 1$.

$$\int \frac{3y^2}{y^3-1}\,\mathrm{d}y = \int \frac{x^3-1}{x}\,\mathrm{d}x$$

$$\ln|y^3-1| = \frac{x^3}{3} - \ln|x| + c$$

…and evaluate. The integral on the left is of the type $\int \frac{f'(x)}{f(x)}\,\mathrm{d}x$ and the integral on the right can be written as the integral $\int \frac{x^3-1}{x}\,\mathrm{d}x = \int x^2 - \frac{1}{x}\,\mathrm{d}x = \frac{x^3}{3} - \ln|x|$.

Solve DE $\quad 3xy^2y' = (y^3-1)(x^3-1)$.

The function $y \equiv 1$ is a solution. From now suppose $y \not\equiv 1$.

$$\int \frac{3y^2}{y^3-1}\,\mathrm{d}y = \int \frac{x^3-1}{x}\,\mathrm{d}x$$

$$\ln|y^3-1| = \frac{x^3}{3} - \ln|x| + c$$

$$\ln|y^3-1| = \ln\left(e^{x^3/3}\,\frac{1}{|x|}\,e^c\right)$$

We write the expressions $\frac{x^3}{3}$ and c in logarithmic forms $\ln e^{x^3/3}$ and $\ln e^c$ and add (subtract) logarithms.

Solve DE $3xy^2y' = (y^3-1)(x^3-1)$.

The function $y \equiv 1$ is a solution. From now suppose $y \not\equiv 1$.

$$\int \frac{3y^2}{y^3-1}\,\mathrm{d}y = \int \frac{x^3-1}{x}\,\mathrm{d}x$$

$$\ln|y^3-1| = \frac{x^3}{3} - \ln|x| + c$$

$$\ln|y^3-1| = \ln\left(e^{x^3/3}\frac{1}{|x|}e^c\right)$$

$$|y^3-1| = e^{x^3/3}\frac{1}{|x|}e^c$$

Logarithm is one-to-one function and can be removed from both sides of equation.

Solve DE $3xy^2y' = (y^3-1)(x^3-1)$.

The function $y \equiv 1$ is a solution. From now suppose $y \not\equiv 1$.

$$\int \frac{3y^2}{y^3-1}\,\mathrm{d}y = \int \frac{x^3-1}{x}\,\mathrm{d}x$$

$$\ln|y^3-1| = \frac{x^3}{3} - \ln|x| + c$$

$$\ln|y^3-1| = \ln\left(e^{x^3/3}\,\frac{1}{|x|}\,e^c\right)$$

$$|y^3-1| = e^{x^3/3}\,\frac{1}{|x|}\,e^c$$

$$y^3-1 = (\pm e^c)\,e^{x^3/3}\,\frac{1}{x}$$

If we omit the absolute values, the right and left side can differ by the sign. We add this sign to the constant factor e^c...

Solve DE $\quad 3xy^2y' = (y^3-1)(x^3-1).$

The function $y \equiv 1$ is a solution. From now suppose $y \not\equiv 1$.

$$\int \frac{3y^2}{y^3-1}\,\mathrm{d}y = \int \frac{x^3-1}{x}\,\mathrm{d}x$$

$$\ln|y^3-1| = \frac{x^3}{3} - \ln|x| + c$$

$$\ln|y^3-1| = \ln\left(e^{x^3/3}\,\frac{1}{|x|}\,e^c\right)$$

$$|y^3-1| = e^{x^3/3}\,\frac{1}{|x|}\,e^c$$

$$y^3-1 = (\pm e^c)\,e^{x^3/3}\,\frac{1}{x} \qquad C = \pm e^c \in \mathbb{R}\setminus\{0\}$$

…and introduce new constant $C = \pm e^c$. Since c can take arbitrary real value, the expression e^c can take arbitrary positive value and $\pm e^c$ can take arbitrary real nonzero value.

Solve DE $3xy^2y' = (y^3-1)(x^3-1)$.

The function $y \equiv 1$ is a solution. From now suppose $y \not\equiv 1$.

$$\int \frac{3y^2}{y^3-1}\,\mathrm{d}y = \int \frac{x^3-1}{x}\,\mathrm{d}x$$

$$\ln|y^3-1| = \frac{x^3}{3} - \ln|x| + c$$

$$\ln|y^3-1| = \ln\left(e^{x^3/3}\,\frac{1}{|x|}\,e^{c}\right)$$

If we allow $C = 0$, the general solution gives $y = 1$ which is also a solution. Hence C can be arbitrary real value.

$$y^3 - 1 = \frac{C}{x}e^{x^3/3} \qquad C \in \mathbb{R}$$

Solve DE $3xy^2y' = (y^3-1)(x^3-1)$.

The function $y \equiv 1$ is a solution. From now suppose $y \not\equiv 1$.

$$\int \frac{3y^2}{y^3-1}\,\mathrm{d}y = \int \frac{x^3-1}{x}\,\mathrm{d}x$$

$$\ln|y^3-1| = \frac{x^3}{3} - \ln|x| + c$$

$$\ln|y^3-1| = \ln\left(e^{x^3/3}\,\frac{1}{|x|}\,e^c\right)$$

$$|y^3-1| = e^{x^3/3}\,\frac{1}{|x|}\,e^c$$

$$y^3-1 = (\pm e^c)\,e^{x^3/3}\,\frac{1}{x} \qquad C = \pm e^c \in \mathbb{R}$$

$$y^3-1 = \frac{C}{x}e^{x^3/3} \qquad C \in \mathbb{R}$$

The equation is solved.

Solve DE $(1 + e^x)y' + e^x y = 0$

Solve DE $(1+e^x)y' + e^x y = 0$

$$(1+e^x)y' = -e^x y$$

We start with the equation.

Solve DE $(1+e^x)y' + e^x y = 0$

$$(1+e^x)y' = -e^x y$$

$$y' = -\frac{e^x}{e^x+1}y$$

We solve the equation for y'.

Solve DE $(1+e^x)y' + e^x y = 0$

$$(1+e^x)y' = -e^x y$$
$$y' = -\frac{e^x}{e^x+1}y$$

The function $y \equiv 0$ is a solution. In the following suppose $y \neq 0$.

The right-hand side is zero for $y = 0$.

Solve DE $\quad (1+e^x)y' + e^x y = 0$

$$(1+e^x)y' = -e^x y$$
$$y' = -\frac{e^x}{e^x+1}y$$

The function $y \equiv 0$ is a solution. In the following suppose $y \neq 0$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{e^x}{e^x+1}y$$

We substitute $\frac{\mathrm{d}y}{\mathrm{d}x}$ for y'.

Solve DE $(1+e^x)y' + e^x y = 0$

$$(1+e^x)y' = -e^x y$$
$$y' = -\frac{e^x}{e^x+1}y$$

The function $y \equiv 0$ is a solution. In the following suppose $y \neq 0$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{e^x}{e^x+1}y$$
$$\frac{\mathrm{d}y}{y} = -\frac{e^x}{1+e^x}\,\mathrm{d}x$$

We multiply by $\mathrm{d}x$ and divide by y. Since $y \neq 0$, we can do the division.

Solve DE $(1+e^x)y' + e^x y = 0$

$$(1+e^x)y' = -e^x y$$
$$y' = -\frac{e^x}{e^x+1}y$$

The function $y \equiv 0$ is a solution. In the following suppose $y \neq 0$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{e^x}{e^x+1}y$$
$$\int \frac{\mathrm{d}y}{y} = -\int \frac{e^x}{1+e^x}\,\mathrm{d}x$$

We write integral signs.

Solve DE $(1+e^x)y' + e^x y = 0$

$$(1+e^x)y' = -e^x y$$
$$y' = -\frac{e^x}{e^x+1}y$$

The function $y \equiv 0$ is a solution. In the following suppose $y \neq 0$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{e^x}{e^x+1}y$$
$$\int \frac{\mathrm{d}y}{y} = -\int \frac{e^x}{1+e^x}\,\mathrm{d}x$$
$$\ln|y| = -\ln(1+e^x) + c$$

We evaluate the integrals. In the integral on the right we have the derivative of denominator in numerator.

Solve DE $(1+e^x)y' + e^x y = 0$

$$(1+e^x)y' = -e^x y$$
$$y' = -\frac{e^x}{e^x+1}y$$

The function $y \equiv 0$ is a solution. In the following suppose $y \neq 0$.

$$\frac{dy}{dx} = -\frac{e^x}{e^x+1}y$$
$$\int \frac{dy}{y} = -\int \frac{e^x}{1+e^x}\,dx$$
$$\ln|y| = -\ln(1+e^x) + c$$

$$\ln\left[|y|(1+e^x)\right] = \ln e^c$$

We convert logarithms to the left–hand side and add. Further we convert the number c into logarithmic form.

Solve DE $(1+e^x)y' + e^x y = 0$

$$(1+e^x)y' = -e^x y$$
$$y' = -\frac{e^x}{e^x+1}y$$

The function $y \equiv 0$ is a solution. In the following suppose $y \neq 0$.

$$\frac{dy}{dx} = -\frac{e^x}{e^x+1}y$$
$$\int \frac{dy}{y} = -\int \frac{e^x}{1+e^x}\,dx$$
$$\ln|y| = -\ln(1+e^x) + c$$

$$\ln\left[|y|(1+e^x)\right] = \ln e^c$$
$$|y|(1+e^x) = e^c$$

Logarithmic function is one-to-one and can be removed from both sides on equation.

Solve DE $(1+e^x)y' + e^x y = 0$

$$(1+e^x)y' = -e^x y$$
$$y' = -\frac{e^x}{e^x+1}y$$

The function $y \equiv 0$ is a solution. In the following suppose $y \neq 0$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{e^x}{e^x+1}y$$
$$\int \frac{\mathrm{d}y}{y} = -\int \frac{e^x}{1+e^x}\,\mathrm{d}x$$
$$\ln|y| = -\ln(1+e^x) + c$$

$$\ln\Big[|y|(1+e^x)\Big] = \ln e^c$$
$$|y|(1+e^x) = e^c$$
$$y(1+e^x) = K \qquad K = \pm e^c$$

We remove the absolute value. This yields $\pm$ sign on the right. We join this sign to the number e^c which gives a new constant K.

Solve DE $(1+e^x)y'+e^xy=0$

$$(1+e^x)y' = -e^xy$$
$$y' = -\frac{e^x}{e^x+1}y$$

The function $y\equiv 0$ is a solution. In the following suppose $y\neq 0$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{e^x}{e^x+1}y$$
$$\int\frac{\mathrm{d}y}{y} = -\int\frac{e^x}{1+e^x}\,\mathrm{d}x$$
$$\ln|y| = -\ln(1+e^x)+c$$

$$\ln\Big[|y|(1+e^x)\Big] = \ln e^c$$
$$|y|(1+e^x) = e^c$$
$$y(1+e^x) = K \qquad K=\pm e^c$$
$$y = \frac{K}{1+e^x} \qquad K\in\mathbb{R}\setminus\{0\}$$

We solve the obtained relation for y.

Solve DE $\quad (1+e^x)y' + e^x y = 0$

$$(1+e^x)y' = -e^x y$$
$$y' = -\frac{e^x}{e^x+1}y$$

The function $y \equiv 0$ is a solution. In the following suppose $y \neq 0$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{e^x}{e^x+1}y$$
$$\int \frac{\mathrm{d}y}{y} = -\int \frac{e^x}{1+e^x}\,\mathrm{d}x$$
$$\ln|y| = -\ln(1+e^x) + c$$

$$\ln\Big[|y|(1+e^x)\Big] = \ln e^c$$
$$|y|(1+e^x) = e^c$$
$$y(1+e^x) = K \qquad K = \pm e^c$$
$$y = \frac{K}{1+e^x} \qquad K \in \mathbb{R}\setminus\{0\}$$

The choice $K = 0$ gives $y \equiv 0$, which gives the constant solution.

Solve DE $(1+e^x)y' + e^x y = 0$

$$(1+e^x)y' = -e^x y$$

$$y' = -\frac{e^x}{e^x+1}y$$

The function $y \equiv 0$ is a solution. In the following suppose $y \neq 0$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{e^x}{e^x+1}y$$

$$\int \frac{\mathrm{d}y}{y} = -\int \frac{e^x}{1+e^x}\,\mathrm{d}x$$

$$\ln|y| = -\ln(1+e^x) + c$$

$$\ln\Big[|y|(1+e^x)\Big] = \ln e^c$$

$$|y|(1+e^x) = e^c$$

$$y(1+e^x) = K \qquad K = \pm e^c$$

$$y = \frac{K}{1+e^x} \qquad K \in \mathbb{R}$$

The problem is solved.

Solve DE $$y' e^{x^2+y} = -\frac{x}{y}$$

Solve DE $y'e^{x^2+y} = -\frac{x}{y}$

$$y'e^{x^2}e^{y} = -x\frac{1}{y}$$

We factor the exponential function e^{x^2+y}. This separates the variables in the exponent.

Solve DE $y'e^{x^2+y} = -\frac{x}{y}$

$$y'e^{x^2}e^y = -x\frac{1}{y}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}e^{x^2}e^y = -x\frac{1}{y}$$

We substitute $\frac{\mathrm{d}y}{\mathrm{d}x}$ for y'.

Solve DE $\quad y' e^{x^2+y} = -\frac{x}{y}$

$$y' e^{x^2} e^{y} = -x \frac{1}{y}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} e^{x^2} e^{y} = -x \frac{1}{y}$$

$$y e^{y}\,\mathrm{d}y = - \quad x e^{-x^2}\,\mathrm{d}x$$

We multiply by y and divide by e^{x^2}. The latter is equivalent to the multiplication by e^{-x^2}.

Solve DE $\quad y'e^{x^2+y} = -\frac{x}{y}$

$$y'e^{x^2}e^y = -x\frac{1}{y}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}e^{x^2}e^y = -x\frac{1}{y}$$

$$\int ye^y\,\mathrm{d}y = -\int xe^{-x^2}\,\mathrm{d}x$$

We write integral sings.

Solve DE $y'e^{x^2+y} = -\frac{x}{y}$

$$y'e^{x^2}e^y = -x\frac{1}{y}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}e^{x^2}e^y = -x\frac{1}{y}$$

$$\int ye^y\,\mathrm{d}y = -\int xe^{-x^2}\,\mathrm{d}x$$

$$ye^y - e^y =$$

On the left we integrate by parts:

$$\int ye^y\,\mathrm{d}y \quad \boxed{\begin{array}{ll} u = y & u' = 1 \\ v' = e^y & v = e^y \end{array}} = ye^y - \int e^y\,\mathrm{d}y = ye^y - e^y$$

Solve DE $\qquad y'e^{x^2+y} = -\frac{x}{y}$

$$y'e^{x^2}e^y = -x\frac{1}{y}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}e^{x^2}e^y = -x\frac{1}{y}$$

$$\int ye^y \,\mathrm{d}y = -\int xe^{-x^2}\,\mathrm{d}x$$

$$ye^y - e^y = \frac{1}{2}e^{-x^2} + C$$

On the right we use a substitution suggested by the inside function. Hence

$$-\int xe^{-x^2}\,\mathrm{d}x \quad \boxed{\begin{aligned} -x^2 &= t \\ -2x\,\mathrm{d}x &= \mathrm{d}t \\ -x\,\mathrm{d}x &= \frac{1}{2}\,\mathrm{d}t \end{aligned}} = \frac{1}{2}\int e^t\,\mathrm{d}t = \frac{1}{2}e^t = \frac{1}{2}e^{-x^2}$$

Solve DE $\quad y' e^{x^2+y} = -\frac{x}{y}$

$$y' e^{x^2} e^y = -x \frac{1}{y}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} e^{x^2} e^y = -x \frac{1}{y}$$

$$\int y e^y \,\mathrm{d}y = -\int x e^{-x^2} \,\mathrm{d}x$$

$$y e^y - e^y = \frac{1}{2} e^{-x^2} + C$$

$$2y e^y - 2e^y = e^{-x^2} + C \qquad C \in \mathbb{R}$$

We multiply the equation by the number 2. This gives the general solution in its implicit form. Unfortunately, we cannot solve explicitly this relation with respect to y. We keep the solution in its implicit form.

Solve DE $\quad y' e^{x^2+y} = -\frac{x}{y}$

$$y' e^{x^2} e^y = -x \frac{1}{y}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} e^{x^2} e^y = -x \frac{1}{y}$$

$$\int y e^y \, \mathrm{d}y = -\int x e^{-x^2} \, \mathrm{d}x$$

$$y e^y - e^y = \frac{1}{2} e^{-x^2} + C$$

$$2y e^y - 2e^y = e^{-x^2} + C \qquad C \in \mathbb{R}$$

The problem is solved.

2 Lineární diferenciální rovnice, variace konstanty

Definice (lineární DR). Nechť funkce a, b jsou spojité na intervalu I. Rovnice

$$y' + a(x)y = b(x) \tag{L}$$

se nazývá *obyčejná lineární diferenciální rovnice prvního řádu* (zkráceně píšeme LDR). Je-li navíc $b(x) \equiv 0$ na I, nazývá se rovnice (L) *homogenní*, v opačném případě *nehomogenní*.

Definice (homogenní rovnice). Buď dána rovnice (L). Homogenní rovnice, která vznikne z rovnice (L) nahrazením pravé strany nulovou funkcí, tj. rovnice

$$y' + a(x)y = 0 \tag{LH}$$

se nazývá *homogenní rovnice, příslušná nehomogenní rovnici* (L).

Homogenní LDR $y' + a(x)y = 0.$

$$y' = -a(x) \cdot y$$

U homogenní rovnice je derivace řešení rovna $-a(x)$ násobku tohoto řešení.

Homogenní LDR $y' + a(x)y = 0.$

$$y' = -a(x) \cdot y$$
$$y = e^{-\int a(x)\,\mathrm{d}x},$$

$$\left(e^{f(x)}\right)' = e^{f(x)} \cdot f'(x)$$

Porovnáme-li rovnici s derivací složené funkce s exponenciální vnější složkou vidíme okamžitě jedno řešení.

Homogenní LDR $y' + a(x)y = 0.$

$$y' = -a(x) \cdot y$$

$$y = C \cdot e^{-\int a(x)\,\mathrm{d}x}, \quad C \in \mathbb{R}$$

$$\left(e^{f(x)}\right)' = e^{f(x)} \cdot f'(x)$$

Všechna řešení jsou v souladu s principem superpozice násobky tohoto jednoho řešení.

Nehomogenní LDR $\quad y' + a(x) \cdot y = b(x).$

- Řešme nehomogenní LDR.
- Je-li $y_P(x)$ partikulární řešení a $y_{OH}(x)$ je obecné řešení odpovídající homogenní LDR, je funkce

$$y(x, C) = y_P(x) + y_{OH}(x)$$

obecným řešením nehomogenní rovnice.

Nehomogenní LDR $y' + a(x) \cdot y = \cancel{b(x)}.$

Asociovaná hom. LDR $y' + a(x)y = 0$

Obecné řešení hom. LDR $y_{OH}(x) = Ce^{-\int a(x)\,\mathrm{d}x} = C \cdot y_{PH(x)}$

- Uvažujme nejprve odpovídající homogenní rovnici.
- Obecné řešení této rovnice již známe.

Nehomogenní LDR $y' + a(x) \cdot y = b(x).$

Asociovaná hom. LDR $y' + a(x)y = 0$

Obecné řešení hom. LDR $y_{OH}(x) = Ce^{-\int a(x)\,\mathrm{d}x} = C \cdot y_{PH(x)}$

$y_P(x) = K(x) \cdot y_{PH}(x)$

variace konstanty

- Nyní stačí najít alespoň jedno řešení rovnice nehomogenní.
- Nahradíme konstantu C v obecném řešení homogenní LDR zatím neznámou funkcí $K(x)$ a budeme hledat, za jakých podmínek je výsledná funkce řešením nehomogenní LDR.

Nehomogenní LDR $y' + a(x) \cdot y = b(x).$

Asociovaná hom. LDR $y' + a(x)y = 0$

Obecné řešení hom. LDR $y_{OH}(x) = Ce^{-\int a(x)\,\mathrm{d}x} = C \cdot y_{PH(x)}$

$$y_P(x) = K(x) \cdot y_{PH}(x)$$

$$y_P'(x) = K'(x) \cdot y_{PH}(x) + K(x) \cdot y_{PH}'(x)$$

- Musíme najít funkci $K(x)$.
- Pro dosazení do rovnice je nutné znát derivaci y'.
- Derivujeme jako součin podle vzorce $(uv)' = u' \cdot v + u \cdot v'$

Nehomogenní LDR $y' + a(x) \cdot y = b(x).$

Asociovaná hom. LDR $y' + a(x)y = 0$

Obecné řešení hom. LDR $y_{OH}(x) = Ce^{-\int a(x)\,\mathrm{d}x} = C \cdot y_{PH(x)}$

$$y_P(x) = K(x) \cdot y_{PH}(x)$$

$$y'_P(x) = K'(x) \cdot y_{PH}(x) + K(x) \cdot y'_{PH}(x)$$

$$\overbrace{K'(x)y_{PH}(x) + K(x)y'_{PH}(x)}^{y'}$$

Dosadíme do rovnice.

Nehomogenní LDR $y' + a(x) \cdot y = b(x)$.

Asociovaná hom. LDR $y' + a(x)y = 0$

Obecné řešení hom. LDR $y_{OH}(x) = Ce^{-\int a(x)\,\mathrm{d}x} = C \cdot y_{PH(x)}$

$$y_P(x) = K(x) \cdot y_{PH}(x)$$

$$y'_P(x) = K'(x) \cdot y_{PH}(x) + K(x) \cdot y'_{PH}(x)$$

$$\overbrace{K'(x)y_{PH}(x) + K(x)y'_{PH}(x)}^{y'} + a(x) \cdot \overbrace{K(x)y_{PH}(x)}^{y} = b(x)$$

Dosadíme do rovnice.

Nehomogenní LDR $y' + a(x) \cdot y = b(x).$

Asociovaná hom. LDR $y' + a(x)y = 0$

Obecné řešení hom. LDR $y_{OH}(x) = Ce^{-\int a(x)\,\mathrm{d}x} = C \cdot y_{PH(x)}$

$$y_P(x) = K(x) \cdot y_{PH}(x)$$

$$y_P'(x) = K'(x) \cdot y_{PH}(x) + K(x) \cdot y_{PH}'(x)$$

$$\overbrace{K'(x)y_{PH}(x) + K(x)y_{PH}'(x)}^{y'} + a(x) \cdot \overbrace{K(x)y_{PH}(x)}^{y} = b(x)$$

$$K'(x)y_{PH}(x) + K(x)\left[y_{PH}'(x) + a(x)y_{PH}(x)\right] = b(x)$$

Vytkneme na levé straně $K(x)$.

Nehomogenní LDR $y' + a(x)\cdot y = b(x).$

Asociovaná hom. LDR $y' + a(x)y = 0$

Obecné řešení hom. LDR $y_{OH}(x) = Ce^{-\int a(x)\,\mathrm{d}x} = C\cdot y_{PH(x)}$

$$y_P(x) = K(x)\cdot y_{PH}(x)$$

$$y_P'(x) = K'(x)\cdot y_{PH}(x) + K(x)\cdot y_{PH}'(x)$$

$$\overbrace{K'(x)y_{PH}(x) + K(x)y_{PH}'(x)}^{y'} + a(x)\cdot\overbrace{K(x)y_{PH}(x)}^{y} = b(x)$$

$$K'(x)y_{PH}(x) + K(x)\left[y_{PH}'(x) + a(x)y_{PH}(x)\right] = b(x)$$

$$K'(x)y_{PH}(x) = b(x)$$

Vyznačený výraz je roven nule.

Nehomogenní LDR $y' + a(x) \cdot y = b(x).$

Asociovaná hom. LDR $y' + a(x)y = 0$

Obecné řešení hom. LDR $y_{OH}(x) = Ce^{-\int a(x)\,\mathrm{d}x} = C \cdot y_{PH(x)}$

$$y_P(x) = K(x) \cdot y_{PH}(x)$$

$$y_P'(x) = K'(x) \cdot y_{PH}(x) + K(x) \cdot y_{PH}'(x)$$

$$\overbrace{K'(x)y_{PH}(x) + K(x)y_{PH}'(x)}^{y'} + a(x) \cdot \overbrace{K(x)y_{PH}(x)}^{y} = b(x)$$

$$K'(x)y_{PH}(x) + K(x)\left[y_{PH}'(x) + a(x)y_{PH}(x)\right] = b(x)$$

$$K'(x)y_{PH}(x) = b(x)$$

$$K'(x) = \frac{b(x)}{y_{PH}(x)}$$

Dostali jsme rovnici, která neobsahuje funkci $K(x)$, ale jenom její derivaci $K'(x)$. Vyjádříme $K'(x)$.

Nehomogenní LDR $y' + a(x) \cdot y = b(x)$.

Asociovaná hom. LDR $y' + a(x)y = 0$

Obecné řešení hom. LDR $y_{OH}(x) = Ce^{-\int a(x)\,\mathrm{d}x} = C \cdot y_{PH(x)}$

$$y_P(x) = K(x) \cdot y_{PH}(x)$$

$$y'_P(x) = K'(x) \cdot y_{PH}(x) + K(x) \cdot y'_{PH}(x)$$

$$\overbrace{K'(x)y_{PH}(x) + K(x)y'_{PH}(x)}^{y'} + a(x) \cdot \overbrace{K(x)y_{PH}(x)}^{y} = b(x)$$

$$K'(x)y_{PH}(x) + K(x)\left[y'_{PH}(x) + a(x)y_{PH}(x)\right] = b(x)$$

$$K'(x)y_{PH}(x) = b(x)$$

$$K(x) = \int \frac{b(x)}{y_{PH}(x)}\,\mathrm{d}x$$

Integrací nalezneme $K(x)$. Integrační konstantu volíme libovolnou.

Nehomogenní LDR $y' + a(x) \cdot y = b(x).$

Asociovaná hom. LDR $y' + a(x)y = 0$

Obecné řešení hom. LDR $y_{OH}(x) = Ce^{-\int a(x)\,\mathrm{d}x} = C \cdot y_{PH(x)}$

$$y_P(x) = K(x) \cdot y_{PH}(x)$$

$$y'_P(x) = K'(x) \cdot y_{PH}(x) + K(x) \cdot y'_{PH}(x)$$

$$\overbrace{K'(x)y_{PH}(x) + K(x)y'_{PH}(x)}^{y'} + a(x) \cdot \overbrace{K(x)y_{PH}(x)}^{y} = b(x)$$

$$K'(x)y_{PH}(x) + K(x)\left[y'_{PH}(x) + a(x)y_{PH}(x)\right] = b(x)$$

$$K'(x)y_{PH}(x) = b(x)$$

$$K(x) = \int \frac{b(x)}{y_{PH}(x)}\,\mathrm{d}x$$

Zapomeneme nyní již nepodstatné informace.

Nehomogenní LDR $y' + a(x) \cdot y = b(x)$.

Asociovaná hom. LDR $y' + a(x)y = 0$

Obecné řešení hom. LDR $y_{OH}(x) = Ce^{-\int a(x)\,\mathrm{d}x} = C \cdot y_{PH(x)}$

$$y_P(x) = K(x) \cdot y_{PH}(x)$$

$$K(x) = \int \frac{b(x)}{y_{PH}(x)}\,\mathrm{d}x$$

Zapomeneme nyní již nepodstatné informace.

Nehomogenní LDR $y' + a(x) \cdot y = b(x).$

Asociovaná hom. LDR $y' + a(x)y = 0$

Obecné řešení hom. LDR $y_{OH}(x) = Ce^{-\int a(x)\,\mathrm{d}x} = C \cdot y_{PH(x)}$

$y_P(x) = K(x) \cdot y_{PH}(x)$

$$K(x) = \int \frac{b(x)}{y_{PH}(x)}\,\mathrm{d}x$$

$$y_P(x) = y_{PH}(x) \cdot \int \frac{b(x)}{y_{PH}(x)}\,\mathrm{d}x$$

Použijeme funkci $K(x)$ pro obdržení partikulárního řešení rovnice.

Nehomogenní LDR $y' + a(x) \cdot y = b(x)$.

Asociovaná hom. LDR $y' + a(x)y = 0$

Obecné řešení hom. $y_{OH}(x) = Ce^{-\int a(x)\,\mathrm{d}x} = C \cdot y_{PH(x)}$

$$y_P(x) = K(x) \cdot y_{PH}(x)$$

$$K(x) = \int \frac{b(x)}{y_{PH}(x)}\,\mathrm{d}x$$

$$y_P(x) = y_{PH}(x) \cdot \int \frac{b(x)}{y_{PH}(x)}\,\mathrm{d}x$$

$$y(x) = y_P(x) + y_{OH}(x)$$

Sečteme partikulární řešení nehomogenní a obecné řešení homogenní rovnice a rovnice je vyřešena.

Řešte DR $y' + \frac{2}{x} y = \frac{1}{x+1}$

Rovnice je lineární.

Řešte DR $y' + \frac{2}{x} y = \frac{1}{x+1}$

$$y' + \frac{2}{x}y = 0$$

- Uvažujme nejprve homogenní rovnici.
- Nahradíme pravou stranu rovnice nulou.

Řešte DR $y' + \frac{2}{x} y = \frac{1}{x+1}$

$$y' + \frac{2}{x} y = 0$$

$$y_{OH}(x) = Ke^{-\int \frac{2}{x}\,\mathrm{d}x}$$

- Obecné řešení rovnice $y + a(x)y = 0$ je dáno formulkou $y = Ke^{-\int a(x)\,\mathrm{d}x}$.
- V našem případě $a(x) = \frac{2}{x}$.

Řešte DR $y' + \frac{2}{x} y = \frac{1}{x+1}$

$$y' + \frac{2}{x}y = 0$$

$$y_{OH}(x) = Ke^{-\int \frac{2}{x}\,\mathrm{d}x} = Ke^{-2\ln|x|}$$

Integrujeme...

Řešte DR $y' + \frac{2}{x} y = \frac{1}{x+1}$

$$y' + \frac{2}{x}y = 0$$

$$y_{OH}(x) = Ke^{-\int \frac{2}{x}\,\mathrm{d}x} = Ke^{-2\ln|x|} = Ke^{\ln x^{-2}}$$

…upravujeme…

Řešte DR $y' + \frac{2}{x} y = \frac{1}{x+1}$

$y_{OH}(x) = Kx^{-2}$

$$y' + \frac{2}{x}y = 0$$

$$y_{OH}(x) = Ke^{-\int \frac{2}{x}\,\mathrm{d}x} = Ke^{-2\ln|x|} = Ke^{\ln x^{-2}} = Kx^{-2}$$

a upravujeme ještě více. Nezapomeňme že exponenciální a logaritmická funkce jsou navzájem inverzní a jejich složením dostaneme identitu.

Řešte DR $y' + \frac{2}{x} y = \frac{1}{x+1}$

$y_{OH}(x) = Kx^{-2}$

$y_{PN}(x) = K(x) \cdot x^{-2}$

- Nyní budeme hledat partikulární řešení nehomogenní rovnice.
- Nahradíme tedy konstantu v $y_{OH}(x)$ funkcí.

Řešte DR $y' + \frac{2}{x} y = \frac{1}{x+1}$

$y_{OH}(x) = Kx^{-2}$

$y_{PN}(x) = K(x) \cdot x^{-2}$

$y'_{PN} = K'(x)x^{-2} + (-2)K(x)x^{-3}$

- Najdeme derivaci $y'_{PN}(x)$.
- K tomu využijeme pravidlo pro derivaci součinu: $(uv)' = u'v + uv'$.

Řešte DR $y' + \frac{2}{x} y = \frac{1}{x+1}$

$y_{OH}(x) = Kx^{-2}$

$y_{PN}(x) = K(x) \cdot x^{-2}$

$y'_{PN} = K'(x)x^{-2} + (-2)K(x)x^{-3}$

$$\overbrace{K'(x)x^{-2} + (-2)K(x)x^{-3}}^{y'} + \frac{2}{x}\overbrace{K(x)x^{-2}}^{y} = \frac{1}{x+1}$$

Dosadíme do rovnice.

Řešte DR $\quad y' + \frac{2}{x} y = \frac{1}{x+1}$

$$y_{OH}(x) = Kx^{-2}$$

$$y_{PN}(x) = K(x) \cdot x^{-2}$$

$$y'_{PN} = K'(x)x^{-2} + (-2)K(x)x^{-3}$$

$$\overbrace{K'(x)x^{-2} + (-2)K(x)x^{-3}}^{y'} + \frac{2}{x}\overbrace{K(x)x^{-2}}^{y} = \frac{1}{x+1}$$

$$K'(x) = \frac{x^2}{x+1}$$

- Nalezneme rovnici pro K'.
- Výrazy s K se podle očekávání odečtou. Skutečně: $(-2)Kx^{-3} + \frac{2}{x}Kx^{-2} = 0$.

Řešte DR $y' + \frac{2}{x} y = \frac{1}{x+1}$

$$y_{OH}(x) = Kx^{-2}$$

$$y_{PN}(x) = K(x) \cdot x^{-2}$$

$$y'_{PN} = K'(x)x^{-2} + (-2)K(x)x^{-3}$$

$$\overbrace{K'(x)x^{-2} + (-2)K(x)x^{-3}}^{y'} + \frac{2}{x}\overbrace{K(x)x^{-2}}^{y} = \frac{1}{x+1}$$

$$K'(x) = \frac{x^2}{x+1}$$

$$K'(x) = x - 1 + \frac{1}{x+1}$$

- Funkci K získáme jako libovolný integrál z K'.
- Před výpočtem integrálu musíme vydělit polynom v čitateli polynomem ve jmenovateli.

Řešte DR $y' + \frac{2}{x} y = \frac{1}{x+1}$

$$y_{OH}(x) = Kx^{-2}$$

$$y_{PN}(x) = K(x) \cdot x^{-2}$$

$$y'_{PN} = K'(x)x^{-2} + (-2)K(x)x^{-3}$$

$$\overbrace{K'(x)x^{-2} + (-2)K(x)x^{-3}}^{y'} + \frac{2}{x}\overbrace{K(x)x^{-2}}^{y} = \frac{1}{x+1}$$

$$K'(x) = \frac{x^2}{x+1}$$

$$K'(x) = x - 1 + \frac{1}{x+1}$$

$$K(x) = \int x - 1 + \frac{1}{x+1}\,\mathrm{d}x$$

Integrujeme. . .

Řešte DR $y' + \frac{2}{x} y = \frac{1}{x+1}$

$$y_{OH}(x) = Kx^{-2}$$

$$y_{PN}(x) = K(x) \cdot x^{-2}$$

$$y'_{PN} = K'(x)x^{-2} + (-2)K(x)x^{-3}$$

$$\overbrace{K'(x)x^{-2} + (-2)K(x)x^{-3}}^{y'} + \frac{2}{x}\overbrace{K(x)x^{-2}}^{y} = \frac{1}{x+1}$$

$$K'(x) = \frac{x^2}{x+1}$$

$$K'(x) = x - 1 + \frac{1}{x+1}$$

$$K(x) = \int x - 1 + \frac{1}{x+1}\,\mathrm{d}x$$

$$= \frac{x^2}{2} - x + \ln|x+1|$$

... a dostáváme $K(x) = \frac{x^2}{2} + x + \ln|x+1|$.

Řešte DR $y' + \frac{2}{x} y = \frac{1}{x+1}$

$$y_{OH}(x) = Kx^{-2}$$

$$y_{PN}(x) = K(x) \cdot x^{-2}$$

$$y'_{PN} = K'(x)x^{-2} + (-2)K(x)x^{-3}$$

$$\overbrace{K'(x)x^{-2} + (-2)K(x)x^{-3}}^{y'} + \frac{2}{x}\overbrace{K(x)x^{-2}}^{y} = \frac{1}{x+1}$$

$$K'(x) = \frac{x^2}{x+1}$$

$$K'(x) = x - 1 + \frac{1}{x+1}$$

$$K(x) = \int x - 1 + \frac{1}{x+1}\,\mathrm{d}x$$

$$= \frac{x^2}{2} - x + \ln|x+1|$$

Odstraníme nyní již nepotřebné výpočty.

Řešte DR $y' + \frac{2}{x} y = \frac{1}{x+1}$

$$y_{OH}(x) = Kx^{-2}$$

$$y_{PN}(x) = K(x) \cdot x^{-2}$$

$$K(x) = \frac{x^2}{2} - x + \ln|x+1|$$

Známe funkci $K(x)$ a hledáme $y_{PN}(x)$.

Řešte DR $y' + \frac{2}{x} y = \frac{1}{x+1}$

$y_{OH}(x) = Kx^{-2}$

$y_{PN}(x) = K(x) \cdot x^{-2}$

$K(x) = \frac{x^2}{2} - x + \ln|x+1|$

$y_{PN}(x) = \left(\frac{x^2}{2} - x + \ln(x+1)\right) \cdot x^{-2}$

Dosadíme $K(x)$...

Řešte DR $y' + \frac{2}{x} y = \frac{1}{x+1}$

$$y_{OH}(x) = Kx^{-2}$$

$$y_{PN}(x) = K(x) \cdot x^{-2}$$

$$K(x) = \frac{x^2}{2} - x + \ln|x+1|$$

$$y_{PN}(x) = \left(\frac{x^2}{2} - x + \ln(x+1)\right) \cdot x^{-2} = \frac{1}{2} - \frac{1}{x} + \frac{\ln(x+1)}{x^2}$$

... a upravíme

Řešte DR $y' + \frac{2}{x} y = \frac{1}{x+1}$

$$y_{OH}(x) = Kx^{-2}$$

$$y_{PN}(x) = K(x) \cdot x^{-2}$$

$$K(x) = \frac{x^2}{2} - x + \ln|x+1|$$

$$y_{PN}(x) = \left(\frac{x^2}{2} - x + \ln(x+1)\right) \cdot x^{-2} = \frac{1}{2} - \frac{1}{x} + \frac{\ln(x+1)}{x^2}$$

$$y(x) = y_{OH}(x) + y_{PN}(x)$$

Obecné řešení $y(x)$ je součtem $y_{OH}(x)$ a $y_{PN}(x)$.

Řešte DR $y' + \frac{2}{x} y = \frac{1}{x+1}$

$y_{OH}(x) = Kx^{-2}$

$y_{PN}(x) = K(x) \cdot x^{-2}$

$K(x) = \frac{x^2}{2} - x + \ln|x+1|$

$y_{PN}(x) = \left(\frac{x^2}{2} - x + \ln(x+1)\right) \cdot x^{-2} = \frac{1}{2} - \frac{1}{x} + \frac{\ln(x+1)}{x^2}$

$y(x) = y_{OH}(x) + y_{PN}(x) = \frac{K}{x^2} + \frac{1}{2} - \frac{1}{x} + \frac{\ln(x+1)}{x^2}$

Dosadíme za y_{PN} a y_{OH}.

Řešte DR $y' + \frac{2}{x} y = \frac{1}{x+1}$

$$y_{OH}(x) = Kx^{-2}$$

$$y_{PN}(x) = K(x) \cdot x^{-2}$$

$$K(x) = \frac{x^2}{2} - x + \ln|x+1|$$

$$y_{PN}(x) = \left(\frac{x^2}{2} - x + \ln(x+1)\right) \cdot x^{-2} = \frac{1}{2} - \frac{1}{x} + \frac{\ln(x+1)}{x^2}$$

$$y(x) = y_{OH}(x) + y_{PN}(x) = \frac{K}{x^2} + \frac{1}{2} - \frac{1}{x} + \frac{\ln(x+1)}{x^2}, \quad K \in \mathbb{R}$$

Problém je vyřešen.

Solve DE $\quad y' = 1 + 3y \operatorname{tg} x.$

Solve DE $y' = 1 + 3y \operatorname{tg} x$.

$y' - 3y \operatorname{tg} x = 1 \ldots$ original equation

We convert the linear equation into the form

$$y' - a(x)y = b(x).$$

Hence $a(x) = -3 \operatorname{tg} x$ and $b(x) = 1$.

Solve DE $y' = 1 + 3y \operatorname{tg} x$.

$y' - 3y \operatorname{tg} x = \cancel{1}$ . . . original equation

$y' - 3y \operatorname{tg} x = 0$ . . . associated homogeneous equation

We write the corresponding homogeneous equation. We replace the right-hand side by zero.

Solve DE $y' = 1 + 3y \operatorname{tg} x$.

$$y' - 3y \operatorname{tg} x = 1 \ldots \text{original equation}$$
$$y' - 3y \operatorname{tg} x = 0 \ldots \text{associated homogeneous equation}$$

$$y_{GH}(x) = Ce^{-\int -3 \operatorname{tg} x \, \mathrm{d}x}$$

The general solution of
$$y' + a(x)y = 0$$
is given by the formula $y = Ce^{-\int a(x)\, \mathrm{d}x}$.

Solve DE $y' = 1 + 3y \operatorname{tg} x.$

$$y' - 3y \operatorname{tg} x = 1 \ldots \text{original equation}$$
$$y' - 3y \operatorname{tg} x = 0 \ldots \text{associated homogeneous equation}$$

$$y_{GH}(x) = Ce^{-\int -3 \operatorname{tg} x \, \mathrm{d}x} = Ce^{-3 \ln \cos x}$$

We evaluate the integral as follows:

$$\int -3 \operatorname{tg} x \, \mathrm{d}x = \int 3 \frac{-\sin x}{\cos x} \, \mathrm{d}x = 3 \ln |\cos x|.$$

Here we used the formula $\int \frac{f'(x)}{f(x)} \, \mathrm{d}x = \ln |f(x)|$. In the following we will suppose that we work on the interval, where $\cos x > 0$. In this case we omit the absolute value.

Solve DE $y' = 1 + 3y \operatorname{tg} x.$

$$y' - 3y \operatorname{tg} x = 1 \ldots \text{original equation}$$
$$y' - 3y \operatorname{tg} x = 0 \ldots \text{associated homogeneous equation}$$

$$y_{GH}(x) = Ce^{-\int -3 \operatorname{tg} x \, \mathrm{d}x} = Ce^{-3 \ln \cos x} = Ce^{\ln \cos^{-3} x}$$

We convert the function into the form in which the exponential function follows the logarithmic function.

Solve DE $y' = 1 + 3y \operatorname{tg} x$.

$$y' - 3y \operatorname{tg} x = 1 \dots \text{original equation}$$
$$y' - 3y \operatorname{tg} x = 0 \dots \text{associated homogeneous equation}$$

$$y_{GH}(x) = Ce^{-\int -3 \operatorname{tg} x \, dx} = Ce^{-3 \ln \cos x} = Ce^{\ln \cos^{-3} x} = C \cos^{-3} x$$

The functions $\ln(x)$ and e^x are mutually inverse function and the composition $e^{\ln x}$ is identity.

Solve DE $y' = 1 + 3y \operatorname{tg} x$.

$$y' - 3y \operatorname{tg} x = 1 \ldots \text{original equation}$$
$$y' - 3y \operatorname{tg} x = 0 \ldots \text{associated homogeneous equation}$$

$$y_{GH}(x) = Ce^{-\int -3 \operatorname{tg} x \, dx} = Ce^{-3 \ln \cos x} = Ce^{\ln \cos^{-3} x} = C \cos^{-3} x$$

$$y_{PN}(x) = K(x) \cos^{-3} x$$

- Now we have the general solution of homogeneous equation.
- We look for the particular solution of nonhomogeneous equation in the form, in which the constant from y_{GH} is replaced by the function $K(x)$.

Solve DE $y' = 1 + 3y \operatorname{tg} x$.

$$y' - 3y \operatorname{tg} x = 1 \ldots \text{original equation}$$
$$y' - 3y \operatorname{tg} x = 0 \ldots \text{associated homogeneous equation}$$

$$y_{GH}(x) = Ce^{-\int -3 \operatorname{tg} x \, dx} = Ce^{-3 \ln \cos x} = Ce^{\ln \cos^{-3} x} = C \cos^{-3} x$$

$$y_{PN}(x) = K(x) \cos^{-3} x$$
$$y'_{PN}(x) = K'(x) \cos^{-3} x + K(x)(-3) \cos^{-4} x(-\sin x)$$

- When evaluating the derivative of $y'_{PN}(x)$ we use the product rule $(uv)' = u'v + uv'$.
- The derivative of $\cos^{-3} x$ is evaluated by chain rule.

Solve DE $y' = 1 + 3y \operatorname{tg} x$.

$$y' - 3y \operatorname{tg} x = 1 \dots \text{original equation}$$
$$y' - 3y \operatorname{tg} x = 0 \dots \text{associated homogeneous equation}$$

$$y_{GH}(x) = Ce^{-\int -3 \operatorname{tg} x \, dx} = Ce^{-3 \ln \cos x} = Ce^{\ln \cos^{-3} x} = C \cos^{-3} x$$

$$y_{PN}(x) = K(x) \cos^{-3} x$$
$$y'_{PN}(x) = K'(x) \cos^{-3} x + K(x)(-3) \cos^{-4} x(-\sin x)$$

$$\overbrace{K'(x) \cos^{-3} x + K(x)(-3) \cos^{-4} x(-\sin x)}^{y'} - 3\overbrace{K(x) \cos^{-3} x}^{y} \operatorname{tg} x = 1$$

We substitute for y and y' into the original equation.

Solve DE $y' = 1 + 3y \operatorname{tg} x$.

$y' - 3y \operatorname{tg} x = 1$ … original equation

$y' - 3y \operatorname{tg} x = 0$ … associated homogeneous equation

$$y_{GH}(x) = Ce^{-\int -3 \operatorname{tg} x \, dx} = Ce^{-3 \ln \cos x} = Ce^{\ln \cos^{-3} x} = C \cos^{-3} x$$

$$y_{PN}(x) = K(x) \cos^{-3} x$$

$$y'_{PN}(x) = K'(x) \cos^{-3} x + K(x)(-3) \cos^{-4} x(-\sin x)$$

$$\overbrace{K'(x) \cos^{-3} x + K(x)(-3) \cos^{-4} x(-\sin x)}^{y'} - 3\overbrace{K(x) \cos^{-3} x}^{y} \operatorname{tg} x = 1$$

We clean the informations which are no more important.

Solve DE $y' = 1 + 3y \operatorname{tg} x.$

Summary: $y_{GH}(x) = C\cos^{-3} x$ $\qquad$ $y_{PN}(x) = K(x) \cdot \cos^{-3}(x)$

$$\overbrace{K'(x)\cos^{-3} x + K(x)(-3)\cos^{-4} x(-\sin x)}^{y'} \overbrace{-\,3K(x)\cos^{-3} x}^{y} \operatorname{tg} x = 1$$

$$K'(x)\cos^{-3} x = 1$$

The term with $K(x)$ disappear, since

$$K(x)(-3)\cos^{-4} x(-\sin x) - 3K(x)\cos^{-3} x \operatorname{tg} x = 0.$$

We obtain the equation for $K'(x)$.

Solve DE $y' = 1 + 3y \operatorname{tg} x.$

Summary: $y_{GH}(x) = C\cos^{-3} x \qquad y_{PN}(x) = K(x)\cdot\cos^{-3}(x)$

$$\overbrace{K'(x)\cos^{-3} x + K(x)(-3)\cos^{-4} x(-\sin x)}^{y'} - 3\overbrace{K(x)\cos^{-3} x}^{y} \operatorname{tg} x = 1$$

$$K'(x)\cos^{-3} x = 1$$

$$K'(x) = \cos^{3} x$$

We solve that equation for $K'(x)$...

Solve DE $y' = 1 + 3y \operatorname{tg} x.$

Summary: $y_{GH}(x) = C\cos^{-3} x \qquad y_{PN}(x) = K(x)\cdot\cos^{-3}(x)$

$$\overbrace{K'(x)\cos^{-3} x + K(x)(-3)\cos^{-4} x(-\sin x)}^{y'} - 3\overbrace{K(x)\cos^{-3} x}^{y} \operatorname{tg} x = 1$$

$$K'(x)\cos^{-3} x = 1$$

$$K'(x) = \cos^3 x$$

$$K(x) = \int \cos^3 x \, dx$$

…and integrate. This gives $K(x)$.

Solve DE $y' = 1 + 3y \operatorname{tg} x.$

Summary: $y_{GH}(x) = C \cos^{-3} x \qquad y_{PN}(x) = K(x) \cdot \cos^{-3}(x)$

$$\overbrace{K'(x)\cos^{-3} x + K(x)(-3)\cos^{-4} x(-\sin x)}^{y'} - 3\overbrace{K(x)\cos^{-3} x}^{y} \operatorname{tg} x = 1$$

$$K'(x)\cos^{-3} x = 1$$

$$K'(x) = \cos^3 x$$

$$K(x) = \int \cos^3 x \, dx = \int (1 - \sin^2 x) \cos x \, dx$$

We write $\cos^3 x$ in the form

$$\cos^3 x = \cos^2 x \cos x = (1 - \sin^2 x) \cos x.$$

Solve DE $y' = 1 + 3y \operatorname{tg} x.$

Summary: $y_{GH}(x) = C\cos^{-3} x \qquad y_{PN}(x) = K(x)\cdot\cos^{-3}(x)$

$$\overbrace{K'(x)\cos^{-3} x + K(x)(-3)\cos^{-4} x(-\sin x)}^{y'} - 3\overbrace{K(x)\cos^{-3} x}^{y} \operatorname{tg} x = 1$$

$$K'(x)\cos^{-3} x = 1$$

$$K'(x) = \cos^3 x$$

$$K(x) = \int \cos^3 x \, dx = \int (1-\sin^2 x)\cos x \, dx$$

$$= \sin x - \frac{\sin^3 x}{3}$$

The integral is ready for substitution $\sin x = t$, $\cos x \, dx = dt$. This converts the integral into

$$\int (1-t^2)\, dt = t - \frac{t^3}{3} = \sin x - \frac{\sin^3 x}{3}.$$

Solve DE $y' = 1 + 3y \operatorname{tg} x.$

Summary: $y_{GH}(x) = C\cos^{-3} x$ $y_{PN}(x) = K(x) \cdot \cos^{-3}(x)$

$$\overbrace{K'(x)\cos^{-3} x + K(x)(-3)\cos^{-4} x(-\sin x)}^{y'} - 3\overbrace{K(x)\cos^{-3} x}^{y} \operatorname{tg} x = 1$$

$$K'(x)\cos^{-3} x = 1$$

$$K'(x) = \cos^3 x$$

$$K(x) = \int \cos^3 x \, dx = \int (1 - \sin^2 x)\cos x \, dx$$

$$= \sin x - \frac{\sin^3 x}{3}$$

$$y_{PN}(x) = \left(\sin x - \frac{\sin^3 x}{3}\right) \cdot \cos^{-3} x = \frac{\sin x}{\cos^3 x} - \frac{\sin^3 x}{3\cos^3 x}$$

We use the function K in the formula for y_{PN}.

Solve DE $y' = 1 + 3y \operatorname{tg} x$.

Summary: $y_{GH}(x) = C\cos^{-3} x$ $\qquad y_{PN}(x) = K(x)\cdot\cos^{-3}(x)$

$$\overbrace{K'(x)\cos^{-3} x + K(x)(-3)\cos^{-4} x(-\sin x)}^{y'} - 3\overbrace{K(x)\cos^{-3} x}^{y}\operatorname{tg} x = 1$$

$$K'(x)\cos^{-3} x = 1$$

$$K'(x) = \cos^3 x$$

$$K(x) = \int \cos^3 x \,\mathrm{d}x = \int (1-\sin^2 x)\cos x \,\mathrm{d}x$$

The general solution of nonhomogeneous equation is a sum of particular solution of that equation and general solution of homogeneous equation.

$$y_{PN}(x) = \left(\sin x - \frac{\sin^3 x}{3}\right)\cdot\cos^{-3} x = \frac{\sin x}{\cos^3 x} - \frac{\sin^3 x}{3\cos^3 x}$$

$$y_{GN}(x) = y_{GH}(x) + y_{PN}(x)$$

Solve DE $y' = 1 + 3y \operatorname{tg} x$.

Summary: $y_{GH}(x) = C\cos^{-3} x$ $y_{PN}(x) = K(x)\cdot\cos^{-3}(x)$

$$\overbrace{K'(x)\cos^{-3} x + K(x)(-3)\cos^{-4} x(-\sin x)}^{y'} - 3\overbrace{K(x)\cos^{-3} x}^{y}\operatorname{tg} x = 1$$

$$K'(x)\cos^{-3} x = 1$$

$$K'(x) = \cos^3 x$$

$$K(x) = \int \cos^3 x\,\mathrm{d}x = \int (1-\sin^2 x)\cos x\,\mathrm{d}x$$

Both y_{GH} and y_{PN} are known and we can substitute.

$$y_{PN}(x) = \left(\sin x - \frac{\sin^3 x}{3}\right)\cdot\cos^{-3} x = \frac{\sin x}{\cos^3 x} - \frac{\sin^3 x}{3\cos^3 x}$$

$$y_{GN}(x) = y_{GH}(x) + y_{PN}(x) = \frac{C}{\cos^3 x} + \frac{\sin x}{\cos^3 x} - \frac{\sin^3 x}{3\cos^3 x}, \quad C \in \mathbb{R}$$

Solve DE $y' = 1 + 3y \operatorname{tg} x.$

Summary: $y_{GH}(x) = C\cos^{-3} x \qquad y_{PN}(x) = K(x)\cdot\cos^{-3}(x)$

$$\overbrace{K'(x)\cos^{-3}x + K(x)(-3)\cos^{-4}x(-\sin x)}^{y'} - 3\overbrace{K(x)\cos^{-3}x}^{y}\operatorname{tg} x = 1$$

$$K'(x)\cos^{-3}x = 1$$

$$K'(x) = \cos^3 x$$

$$K(x) = \int \cos^3 x\,\mathrm{d}x = \int (1-\sin^2 x)\cos x\,\mathrm{d}x$$

The problem is solved.

$$y_{PN}(x) = \left(\sin x - \frac{\sin^3 x}{3}\right)\cdot\cos^{-3}x = \frac{\sin x}{\cos^3 x} - \frac{\sin^3 x}{3\cos^3 x}$$

$$y_{GN}(x) = y_{GH}(x) + y_{PN}(x) = \frac{C}{\cos^3 x} + \frac{\sin x}{\cos^3 x} - \frac{\sin^3 x}{3\cos^3 x}, \qquad C\in\mathbb{R}$$

Solve DE $xy' + y = x\ln(x+1)$

Solve DE $xy' + y = x\ln(x+1)$

$$y' + \frac{1}{x}y = \ln(x+1)$$

- We write the equation in its normal form $y' + a(x)y = b$.
- We divide by x. Hence we lok for the solution either on $(-1, 0)$ (see the logarithmic function) or on $(0, \infty)$.

Solve DE $xy' + y = x\ln(x+1)$

$$y' + \frac{1}{x}y = \ln(x+1) \qquad\qquad y' + \frac{1}{x}y = 0$$

We write the corresponding homogeneous equation.

Solve DE $xy' + y = x\ln(x+1)$

$$y' + \frac{1}{x}y = \ln(x+1) \qquad y' + \frac{1}{x}y = 0$$

$$y_{GH} = Ce^{-\int \frac{1}{x}\,\mathrm{d}x}$$

The general solution of the homogeneous equation

$$y' + a(x)y = 0$$

is

$$y_{GH} = Ce^{-\int a(x)\,\mathrm{d}x}.$$

In our case we have $a(x) = \frac{1}{x}$.

Solve DE $xy' + y = x\ln(x+1)$

$$y' + \frac{1}{x}y = \ln(x+1) \qquad y' + \frac{1}{x}y = 0$$

$$y_{GH} = Ce^{-\int \frac{1}{x}\,\mathrm{d}x} = Ce^{-\ln|x|}$$

We evaluate the integral. . .

Solve DE $xy' + y = x\ln(x+1)$

$$y' + \frac{1}{x}y = \ln(x+1) \qquad y' + \frac{1}{x}y = 0$$

$$y_{GH} = Ce^{-\int \frac{1}{x}\,\mathrm{d}x} = Ce^{-\ln|x|} = Ce^{\ln|x|^{-1}}$$

...and simplify.

Solve DE $\quad xy' + y = x\ln(x+1)$

$$y' + \frac{1}{x}y = \ln(x+1) \qquad y' + \frac{1}{x}y = 0$$

$$y_{GH} = Ce^{-\int \frac{1}{x}\,\mathrm{d}x} = Ce^{-\ln|x|} = Ce^{\ln|x|^{-1}} = C|x|^{-1} = \frac{C}{|x|}$$

The composition $e^{\ln x}$ is identity.

Solve DE $xy' + y = x\ln(x+1)$

$$y' + \frac{1}{x}y = \ln(x+1) \qquad y' + \frac{1}{x}y = 0$$

$$y_{GH} = Ce^{-\int \frac{1}{x}\,\mathrm{d}x} = Ce^{-\ln|x|} = Ce^{\ln|x|^{-1}} = C|x|^{-1} = \frac{C}{|x|} = \frac{K}{x}$$

If we introduce the new constant $C = \pm K$, we can write the general solution of homogeneous equation in the form $y_{GH} = \dfrac{K}{x}$

Solve DE $xy' + y = x\ln(x+1)$

$$y' + \frac{1}{x}y = \ln(x+1) \qquad y' + \frac{1}{x}y = 0 \qquad y_{GH} = \frac{K}{x} = K \cdot \frac{1}{x}$$

$$y_{PN} = K(x)\frac{1}{x}$$

- Now let us look for the solution of nonhomogeneous equation.
- We replace the constant K in the formula for y_{GH} by the function $K(x)$.

Solve DE $xy' + y = x\ln(x+1)$

$$y' + \frac{1}{x}y = \ln(x+1) \qquad y' + \frac{1}{x}y = 0 \qquad y_{GH} = \frac{K}{x} = K \cdot \frac{1}{x}$$

$$y_{PN} = K(x)\frac{1}{x} \qquad y'_{PN} = K'(x)\frac{1}{x} + K(x)(-1)x^{-2}$$

We evaluate the derivative of the function y_{PN} by the product rule

$$(uv)' = u'v + uv'.$$

We differentiate the function $\frac{1}{x}$ as a power function x^{-1}. Hence

$$\left(\frac{1}{x}\right)' = (x^{-1})' = (-1)x^{-2}.$$

Solve DE $xy' + y = x\ln(x+1)$

$$y' + \frac{1}{x}y = \ln(x+1) \qquad y' + \frac{1}{x}y = 0 \qquad y_{GH} = \frac{K}{x} = K \cdot \frac{1}{x}$$

$$y_{PN} = K(x)\frac{1}{x} \qquad y'_{PN} = K'(x)\frac{1}{x} + K(x)(-1)x^{-2}$$

$$\overbrace{K'(x)\frac{1}{x} + K(x)(-1)x^{-2}}^{y'} + \frac{1}{x}\overbrace{K(x)\frac{1}{x}}^{y} = \ln(x+1)$$

We substitute for y' and y into original equation

$$y' + \frac{1}{x}y = \ln(x+1).$$

Solve DE $\quad xy' + y = x\ln(x+1)$

$$y' + \frac{1}{x}y = \ln(x+1) \qquad y' + \frac{1}{x}y = 0 \qquad y_{GH} = \frac{K}{x} = K\cdot\frac{1}{x}$$

$$y_{PN} = K(x)\frac{1}{x} \qquad y'_{PN} = K'(x)\frac{1}{x} + K(x)(-1)x^{-2}$$

$$\overbrace{K'(x)\frac{1}{x} + K(x)(-1)x^{-2}}^{y'} + \frac{1}{x}\overbrace{K(x)\frac{1}{x}}^{y} = \ln(x+1)$$

$$K'(x)\frac{1}{x} = \ln(x+1)$$

The terms with $K(x)$ cancel and only $K'(x)$ remains.

Solve DE $xy' + y = x\ln(x+1)$

$$y' + \frac{1}{x}y = \ln(x+1) \qquad y' + \frac{1}{x}y = 0 \qquad y_{GH} = \frac{K}{x} = K\cdot\frac{1}{x}$$

$$y_{PN} = K(x)\frac{1}{x} \qquad y'_{PN} = K'(x)\frac{1}{x} + K(x)(-1)x^{-2}$$

$$\overbrace{K'(x)\frac{1}{x} + K(x)(-1)x^{-2}}^{y'} + \frac{1}{x}\overbrace{K(x)\frac{1}{x}}^{y} = \ln(x+1)$$

$$K'(x)\frac{1}{x} = \ln(x+1) \qquad K'(x) = x\ln(x+1)$$

We solve the equation for $K'(x)$ ...

Solve DE $xy' + y = x\ln(x+1)$

$$y' + \frac{1}{x}y = \ln(x+1) \qquad y' + \frac{1}{x}y = 0 \qquad y_{GH} = \frac{K}{x} = K\cdot\frac{1}{x}$$

$$y_{PN} = K(x)\frac{1}{x} \qquad y'_{PN} = K'(x)\frac{1}{x} + K(x)(-1)x^{-2}$$

$$K(x) = \int x\ln(x+1)\,\mathrm{d}x$$

…and integrate.

Solve DE $xy' + y = x\ln(x+1)$

$$y' + \frac{1}{x}y = \ln(x+1) \qquad y' + \frac{1}{x}y = 0 \qquad y_{GH} = \frac{K}{x} = K \cdot \frac{1}{x}$$

$$y_{PN} = K(x)\frac{1}{x} \qquad y'_{PN} = K'(x)\frac{1}{x} + K(x)(-1)x^{-2}$$

$$K(x) = \int x\ln(x+1)\,\mathrm{d}x = \frac{x^2}{2}\ln(x+1) - \frac{x^2}{4} + \frac{x}{2} - \frac{1}{2}\ln(x+1)$$

We use integration by parts with $\boxed{\begin{array}{ll} u = \ln(x+1) & u' = \frac{1}{x+1} \\ v' = x & v = \frac{x^2}{2} \end{array}}$. This gives

$$\begin{aligned}\int x\ln(x+1)\,\mathrm{d}x &= \frac{x^2}{2}\ln(x+1) - \frac{1}{2}\int \frac{x^2}{x+1}\,\mathrm{d}x \\ &= \frac{x^2}{2}\ln(x+1) - \frac{1}{2}\int x - 1 + \frac{1}{x+1}\,\mathrm{d}x\end{aligned}$$

Solve DE $xy' + y = x\ln(x+1)$

$$y' + \frac{1}{x}y = \ln(x+1) \qquad y' + \frac{1}{x}y = 0 \qquad y_{GH} = \frac{K}{x} = K \cdot \frac{1}{x}$$

$$y_{PN} = K(x)\frac{1}{x} \qquad y'_{PN} = K'(x)\frac{1}{x} + K(x)(-1)x^{-2}$$

$$K(x) = \int x\ln(x+1)\,\mathrm{d}x = \frac{x^2}{2}\ln(x+1) - \frac{x^2}{4} + \frac{x}{2} - \frac{1}{2}\ln(x+1)$$

$$y_{PN}(x) = K(x)\frac{1}{x} = \frac{x}{2}\ln(x+1) - \frac{x}{4} + \frac{1}{2} - \frac{1}{2x}\ln(x+1)$$

We substitute for $K(x)$ into the relation for $y_{PN}(x)$

Solve DE $xy' + y = x\ln(x+1)$

$$y' + \frac{1}{x}y = \ln(x+1) \qquad y' + \frac{1}{x}y = 0 \qquad y_{GH} = \frac{K}{x} = K\cdot\frac{1}{x}$$

$$y_{PN} = K(x)\frac{1}{x} \qquad y'_{PN} = K'(x)\frac{1}{x} + K(x)(-1)x^{-2}$$

$$K(x) = \int x\ln(x+1)\,\mathrm{d}x = \frac{x^2}{2}\ln(x+1) - \frac{x^2}{4} + \frac{x}{2} - \frac{1}{2}\ln(x+1)$$

$$y_{PN}(x) = K(x)\frac{1}{x} = \frac{x}{2}\ln(x+1) - \frac{x}{4} + \frac{1}{2} - \frac{1}{2x}\ln(x+1)$$

$$y_{GN} = y_{GH} + y_{PN}$$

The general solution of nonhomogeneous equation is a sum of general solution of homogeneous equation and the particular solution of nonhomogeneous equation.

Solve DE $xy' + y = x\ln(x+1)$

$$y' + \frac{1}{x}y = \ln(x+1) \qquad y' + \frac{1}{x}y = 0 \qquad y_{GH} = \frac{K}{x} = K\cdot\frac{1}{x}$$

$$y_{PN} = K(x)\frac{1}{x} \qquad y'_{PN} = K'(x)\frac{1}{x} + K(x)(-1)x^{-2}$$

$$K(x) = \int x\ln(x+1)\,\mathrm{d}x = \frac{x^2}{2}\ln(x+1) - \frac{x^2}{4} + \frac{x}{2} - \frac{1}{2}\ln(x+1)$$

$$y_{PN}(x) = K(x)\frac{1}{x} \quad \frac{x}{2}\ln(x+1) - \frac{x}{4} + \frac{1}{2} - \frac{1}{2x}\ln(x+1)$$

$$y_{GN} = y_{GH} + y_{PN} = \frac{K}{x} + \frac{x}{2}\ln(x+1) - \frac{x}{4} + \frac{1}{2} - \frac{1}{2x}\ln(x+1)\,, \qquad K\in\mathbb{R}$$

We use that solutions. . .

Solve DE $xy' + y = x\ln(x+1)$

$$y' + \frac{1}{x}y = \ln(x+1) \qquad y' + \frac{1}{x}y = 0 \qquad y_{GH} = \frac{K}{x} = K\cdot\frac{1}{x}$$

$$y_{PN} = K(x)\frac{1}{x} \qquad y'_{PN} = K'(x)\frac{1}{x} + K(x)(-1)x^{-2}$$

$$K(x) = \int x\ln(x+1)\,\mathrm{d}x = \frac{x^2}{2}\ln(x+1) - \frac{x^2}{4} + \frac{x}{2} - \frac{1}{2}\ln(x+1)$$

$$y_{PN}(x) = K(x)\frac{1}{x} = \frac{x}{2}\ln(x+1) - \frac{x}{4} + \frac{1}{2} - \frac{1}{2x}\ln(x+1)$$

$$y_{GN} = y_{GH} + y_{PN} = \frac{K}{x} + \frac{x}{2}\ln(x+1) - \frac{x}{4} + \frac{1}{2} - \frac{1}{2x}\ln(x+1), \qquad K \in \mathbb{R}$$

…and the problem is solved.

Část II

DR druhého řádu

Definice (lineární diferenciální rovnice druhého řádu). Buďte p, q a f funkce definované a spojité na intervalu I. Diferenciální rovnice

$$y'' + p(x)y' + q(x)y = f(x) \tag{1}$$

se nazývá *lineární diferenciální rovnice druhého řádu* (zkráceně LDR druhého řádu). *Řešením rovnice* (nebo též *integrálem rovnice*) na intervalu I rozumíme funkci, která má spojité derivace do řádu 2 na intervalu I a po dosazení identicky splňuje rovnost (1) na I. Úloha nalézt řešení rovnice, které splňuje v bodě $x_0 \in I$ *počáteční podmínky*

$$\begin{cases} y(x_0) = y_0, \\ y'(x_0) = y_0', \end{cases} \tag{2}$$

kde y_0 a y_0' jsou reálná čísla, se nazývá *počáteční úloha* (*Cauchyova úloha*). Řešení počáteční úlohy se nazývá *partikulární řešení rovnice* (1).

Věta 1 (fundamentální systém řešení LDR s konstantními koeficienty). Uvažujme LDR druhého řádu s konst. koef. a její charakteristickou rovnici

- Jsou-li $z_1, z_2 \in \mathbb{R}$ dva různé reálné kořeny charakteristické rovnice, definujme
$$y_1 = e^{z_1 x} \qquad \text{a} \qquad y_2 = e^{z_2 x}.$$
- Je-li $z_1 \in \mathbb{R}$ dvojnásobným kořenem charakteristické rovnice, definujme
$$y_1 = e^{z_1 x} \qquad \text{a} \qquad y_2 = x e^{z_1 x}.$$
- Jsou-li $z_{1,2} = \alpha \pm i\beta \notin \mathbb{R}$ dva komplexně sdružené kořeny charakteristické rovnice, definujme
$$y_1(x) = e^{\alpha x} \cos(\beta x) \qquad \text{a} \qquad y_2(x) = e^{\alpha x} \sin(\beta x).$$

Potom funkce y_1 a y_2 tvoří fundamentální systém řešení rovnice na množině $\mathbb{R}$. Obecné řešení rovnice je tedy

$$y(x, C_1, C_2) = C_1 y_1(x) + C_2 y_2(x), \qquad C_1 \in \mathbb{R},\ C_2 \in \mathbb{R}.$$

Řešte poč. úlohu $y'' + y = 0 \quad y(0) = 1, y'(0) = -1.$

Řešte poč. úlohu $y'' + y = 0 \quad y(0) = 1, y'(0) = -1.$

$z^2 + 1 = 0 \qquad \Rightarrow$

Sestavíme charakteristickou rovnici. . .

Řešte poč. úlohu $y'' + y = 0 \quad y(0) = 1, y'(0) = -1.$

$z^2 + 1 = 0 \quad \Rightarrow \quad z^2 = -1 \quad \Rightarrow$

…a vyřešíme ji.

Řešte poč. úlohu $y'' + y = 0 \quad y(0) = 1, y'(0) = -1.$

$z^2 + 1 = 0 \quad \Rightarrow \quad z^2 = -1 \quad \Rightarrow \quad z^2 = \pm\sqrt{-1} = \pm i$

Řešením jsou dvě komplexně sdružená čísla.

Řešte poč. úlohu $y'' + y = 0 \quad y(0) = 1, y'(0) = -1.$

$$z^2 + 1 = 0 \quad \Rightarrow \quad z^2 = -1 \quad \Rightarrow \quad z^2 = \pm\sqrt{-1} = \pm i$$

$$y_1(x) = \sin x$$

$$y_2(x) = \cos x$$

Reálná část kořenů charakteristické rovnice je $\alpha = 0$, imaginární část je $\beta = 1$. Fundamentální systém řešení je

$$y_1(x) = e^{\alpha x}\cos(\beta x)$$

a

$$y_2(x) = e^{\alpha x}\sin(\beta x)$$

.

Řešte poč. úlohu $y'' + y = 0 \quad y(0) = 1, y'(0) = -1.$

$$z^2 + 1 = 0 \quad \Rightarrow \quad z^2 = -1 \quad \Rightarrow \quad z^2 = \pm\sqrt{-1} = \pm i$$

Fundamentální systém: $\begin{cases} y_1(x) = \sin x \\ y_2(x) = \cos x \end{cases}$

Získali jsme fundamentální systém. . .

Řešte poč. úlohu $y'' + y = 0 \quad y(0) = 1, y'(0) = -1.$

$z^2 + 1 = 0 \quad \Rightarrow \quad z^2 = -1 \quad \Rightarrow \quad z^2 = \pm\sqrt{-1} = \pm i$

Fundamentální systém: $\begin{cases} y_1(x) = \sin x \\ y_2(x) = \cos x \end{cases}$

Obecné řešení: $y(x) = C_1 \sin x + C_2 \cos x, \qquad C_1, C_2 \in \mathbb{R}$

... a můžeme napsat obecné řešení. Obecným řešením je obecná lineární kombinace funkcí tvořících fundamentální systém.

Řešte poč. úlohu $y'' + y = 0 \quad y(0) = 1, y'(0) = -1.$

$z^2 + 1 = 0 \quad \Rightarrow \quad z^2 = -1 \quad \Rightarrow \quad z^2 = \pm\sqrt{-1} = \pm i$

Fundamentální systém: $\begin{cases} y_1(x) = \sin x \\ y_2(x) = \cos x \end{cases}$

Obecné řešení: $y(x) = C_1 \sin x + C_2 \cos x, \qquad C_1, C_2 \in \mathbb{R}$

$y'(x) = C_1 \cos x - C_2 \sin x$

nyní budeme pracovat s počáteční podmínkou. Nalezneme y'...

Řešte poč. úlohu $y'' + y = 0 \quad y(0) = 1, y'(0) = -1.$

$z^2 + 1 = 0 \quad \Rightarrow \quad z^2 = -1 \quad \Rightarrow \quad z^2 = \pm\sqrt{-1} = \pm i$

Fundamentální systém: $\begin{cases} y_1(x) = \sin x \\ y_2(x) = \cos x \end{cases}$

Obecné řešení: $y(x) = C_1 \sin x + C_2 \cos x, \qquad C_1, C_2 \in \mathbb{R}$

$y'(x) = C_1 \cos x - C_2 \sin x$

$1 = C_1 \sin 0 + C_2 \cos 0$

. . . a dosadíme za y. . .

Řešte poč. úlohu $y'' + y = 0 \quad y(0) = 1, y'(0) = -1.$

$$z^2 + 1 = 0 \quad \Rightarrow \quad z^2 = -1 \quad \Rightarrow \quad z^2 = \pm\sqrt{-1} = \pm i$$

Fundamentální systém: $\begin{cases} y_1(x) = \sin x \\ y_2(x) = \cos x \end{cases}$

Obecné řešení: $y(x) = C_1 \sin x + C_2 \cos x, \qquad C_1, C_2 \in \mathbb{R}$

$$y'(x) = C_1 \cos x - C_2 \sin x$$

$$1 = C_1 \sin 0 + C_2 \cos 0$$

$$-1 = C_1 \cos 0 - C_2 \sin 0$$

. . . a za y'.

Řešte poč. úlohu $y'' + y = 0 \quad y(0) = 1, y'(0) = -1.$

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$$\left.\begin{aligned} 1 &= C_1 \sin 0 + C_2 \cos 0 \\ -1 &= C_1 \cos 0 - C_2 \sin 0 \end{aligned}\right\} \Rightarrow C_1 = -1, \quad C_2 = 1$$

Obdrželi jsme soustavu lineárních rovnic, kterou vyřešíme.

Řešte poč. úlohu $y'' + y = 0 \quad y(0) = 1, y'(0) = -1.$

$$z^2 + 1 = 0 \quad \Rightarrow \quad z^2 = -1 \quad \Rightarrow \quad z^2 = \pm\sqrt{-1} = \pm i$$

Fundamentální systém: $\begin{cases} y_1(x) = \sin x \\ y_2(x) = \cos x \end{cases}$

Obecné řešení: $y(x) = C_1 \sin x + C_2 \cos x, \qquad C_1, C_2 \in \mathbb{R}$

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$$\left.\begin{aligned} 1 &= C_1 \sin 0 + C_2 \cos 0 \\ -1 &= C_1 \cos 0 - C_2 \sin 0 \end{aligned}\right\} \Rightarrow C_1 = -1, \quad C_2 = 1$$

Řešení PÚ: $y(x) = -\sin x + \cos x$

A konečně použijeme vypočtené hodnoty C_1 a C_2 v obecném řešení. Tím získáme obecné řešení počáteční úlohy.

Řešte poč. úlohu $y'' + y = 0 \quad y(0) = 1, y'(0) = -1.$

$z^2 + 1 = 0 \quad \Rightarrow \quad z^2 = -1 \quad \Rightarrow \quad z^2 = \pm\sqrt{-1} = \pm i$

Fundamentální systém: $\begin{cases} y_1(x) = \sin x \\ y_2(x) = \cos x \end{cases}$

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$y'(x) = C_1 \cos x - C_2 \sin x$

$$\left.\begin{aligned} 1 &= C_1 \sin 0 + C_2 \cos 0 \\ -1 &= C_1 \cos 0 - C_2 \sin 0 \end{aligned}\right\} \Rightarrow C_1 = -1, \quad C_2 = 1$$

Řešení PÚ: $y(x) = -\sin x + \cos x$

Hotovo!

Řešte DR $4y'' + 4y' + y = 0.$

Řešte DR $4y'' + 4y' + y = 0.$

$4z^2 + 4z + 1 = 0$

Sestavíme charakteristickou rovnici. . .

Řešte DR $4y'' + 4y' + y = 0.$

$$4z^2 + 4z + 1 = 0$$

$$z_{1,2} = \frac{-4 \pm \sqrt{4^2 - 4.4.1}}{2.4}$$

... a vyřešíme ji. Pro řešení kvadratické rovnice

$$az^2 + bz + c = 0$$

používáme vzorec

$$z_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Řešte DR $4y'' + 4y' + y = 0.$

$$4z^2 + 4z + 1 = 0$$

$$z_{1,2} = \frac{-4 \pm \sqrt{4^2 - 4.4.1}}{2.4} = \frac{-4 \pm 0}{8}$$

Upravíme.

Řešte DR $4y'' + 4y' + y = 0.$

$$4z^2 + 4z + 1 = 0$$

$$z_{1,2} = \frac{-4 \pm \sqrt{4^2 - 4.4.1}}{2.4} = \frac{-4 \pm 0}{8} = -\frac{1}{2} \dots \text{dvojnásobný kořen}$$

Charakteristická rovnice má dvojnásobný kořen $z_{1,2} = -\frac{1}{2}$.

Řešte DR $4y'' + 4y' + y = 0.$

$$4z^2 + 4z + 1 = 0$$

$$z_{1,2} = \frac{-4 \pm \sqrt{4^2 - 4.4.1}}{2.4} = \frac{-4 \pm 0}{8} = -\frac{1}{2} \ldots \text{dvojnásobný kořen}$$

Fundamentální systém: $\begin{cases} y_1 = e^{-\frac{x}{2}} \\ y_2 = xe^{-\frac{x}{2}} \end{cases}$

V případě dvojnásobného kořene z charakteristické rovnice je fundamentální systém tvořen funkcemi

$$y_1(x) = e^{zx}, \qquad y_2(x) = xe^{zx}.$$

Řešte DR $4y'' + 4y' + y = 0.$

$$4z^2 + 4z + 1 = 0$$

$$z_{1,2} = \frac{-4 \pm \sqrt{4^2 - 4.4.1}}{2.4} = \frac{-4 \pm 0}{8} = -\frac{1}{2} \text{ \ldots dvojnásobný kořen}$$

Fundamentální systém: $\begin{cases} y_1 = e^{-\frac{x}{2}} \\ y_2 = xe^{-\frac{x}{2}} \end{cases}$

Obecné řešení: $y(x) = C_1 e^{-\frac{x}{2}} + C_2 x e^{-\frac{x}{2}}$, $C_1, C_2 \in \mathbb{R}$

Obecné řešení je lineární kombinací funcí z fundamentálního systému řešení.

Řešte DR $4y'' + 4y' + y = 0.$

$4z^2 + 4z + 1 = 0$

$$z_{1,2} = \frac{-4 \pm \sqrt{4^2 - 4.4.1}}{2.4} = \frac{-4 \pm 0}{8} = -\frac{1}{2}$$ ... dvojnásobný kořen

Fundamentální systém: $\begin{cases} y_1 = e^{-\frac{x}{2}} \\ y_2 = xe^{-\frac{x}{2}} \end{cases}$

Obecné řešení: $y(x) = C_1 e^{-\frac{x}{2}} + C_2 x e^{-\frac{x}{2}} = e^{-\frac{x}{2}}(C_1 + C_2 x)$, $C_1, C_2 \in \mathbb{R}$

Upravíme obecné řešení. Hotovo!

Řešte DR $y'' + 4y' + 29y = 0, y(0) = 0, y'(0) = 10.$

Řešte DR $y'' + 4y' + 29y = 0, y(0) = 0, y'(0) = 10.$

$z^2 + 4z + 29 = 0$

Rovnice je lineární homogenní druhého řádu. Sestavíme nejprve charakteristickou rovnici.

Řešte DR $y'' + 4y' + 29y = 0, y(0) = 0, y'(0) = 10.$

$$z^2 + 4z + 29 = 0$$

$$z_{1,2} = \frac{-4 \pm \sqrt{16 - 4.1.29}}{2.1}$$

Řešením rovnice

$$az^2 + bz + c = 0$$

jsou čísla která obdržíme ze vzorce

$$z_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Řešte DR $y'' + 4y' + 29y = 0, y(0) = 0, y'(0) = 10.$

$$z^2 + 4z + 29 = 0$$

$$z_{1,2} = \frac{-4 \pm \sqrt{16 - 4.1.29}}{2.1} = \frac{-4 \pm \sqrt{-100}}{2}$$

Upravíme ...

Řešte DR $y'' + 4y' + 29y = 0, y(0) = 0, y'(0) = 10.$

$$z^2 + 4z + 29 = 0$$

$$z_{1,2} = \frac{-4 \pm \sqrt{16 - 4.1.29}}{2.1} = \frac{-4 \pm \sqrt{-100}}{2} = -2 \pm 5i$$

… a najdeme řešení charakteristické rovnice. použijeme skutečnost, že

$$\sqrt{-100} = \sqrt{100}\sqrt{-1} = 10\sqrt{-1} = 10i.$$

Řešte DR $y'' + 4y' + 29y = 0, y(0) = 0, y'(0) = 10.$

$$z^2 + 4z + 29 = 0$$

$$z_{1,2} = \frac{-4 \pm \sqrt{16 - 4.1.29}}{2.1} = \frac{-4 \pm \sqrt{-100}}{2} = -2 \pm 5i$$

$y_1(x) = e^{-2x}\cos(5x)$ $y_2(x) = e^{-2x}\sin(5x)$

Z kořenů charakteristické rovnice sestavíme fundamentální systém řešení. Reálná část kořenů je $\alpha = -2$, imaginární je $\beta = 5$. Fundamentální systém je tvořen funkcemi

$$y_1(x) = e^{\alpha x}\cos(\beta x) \quad \text{a} \quad y_2(x) = e^{\alpha x}\sin(\beta x).$$

Řešte DR $y'' + 4y' + 29y = 0, y(0) = 0, y'(0) = 10.$

$$z^2 + 4z + 29 = 0$$

$$z_{1,2} = \frac{-4 \pm \sqrt{16 - 4.1.29}}{2.1} = \frac{-4 \pm \sqrt{-100}}{2} = -2 \pm 5i$$

$$y_1(x) = e^{-2x} \cos(5x) \qquad y_2(x) = e^{-2x} \sin(5x)$$

$$y(x) = C_1 e^{-2x} \cos(5x) + C_2 e^{-2x} \sin(5x)$$

Obecné řešení je lineární kombinací funkcí z fundamentálního systému řešení.

Řešte DR $y'' + 4y' + 29y = 0, y(0) = 0, y'(0) = 10.$

$z^2 + 4z + 29 = 0$

$$z_{1,2} = \frac{-4 \pm \sqrt{16 - 4.1.29}}{2.1} = \frac{-4 \pm \sqrt{-100}}{2} = -2 \pm 5i$$

$y_1(x) = e^{-2x}\cos(5x)$ $\qquad$ $y_2(x) = e^{-2x}\sin(5x)$

$$\begin{aligned} y(x) =& C_1 e^{-2x}\cos(5x) + C_2 e^{-2x}\sin(5x) \\ y'(x) =& C_1\left[-2e^{-2x}\cos(5x) - 5e^{-2x}\sin(5x)\right] \\ &+ C_2\left[-2e^{-2x}\sin(5x) + 5e^{-2x}\cos(5x)\right] \end{aligned}$$

Vypočteme derivaci y'. musíme použít pravidlo pro derivaci součinu

$$(uv)' = u'v + uv'.$$

Při derivování e^{-2x} a $\sin(5x)$ použijeme pravidlo pro derivaci složené funkce.

Řešte DR $y'' + 4y' + 29y = 0, y(0) = 0, y'(0) = 10.$

$$z^2 + 4z + 29 = 0$$

$$z_{1,2} = \frac{-4 \pm \sqrt{16 - 4.1.29}}{2.1} = \frac{-4 \pm \sqrt{-100}}{2} = -2 \pm 5i$$

$y_1(x) = e^{-2x}\cos(5x)$ $\qquad$ $y_2(x) = e^{-2x}\sin(5x)$

$$y(x) = C_1 e^{-2x}\cos(5x) + C_2 e^{-2x}\sin(5x)$$

$$y'(x) = C_1\left[-2e^{-2x}\cos(5x) - 5e^{-2x}\sin(5x)\right] + C_2\left[-2e^{-2x}\sin(5x) + 5e^{-2x}\cos(5x)\right]$$

$$0 = C_1 + 0C_2$$

Dosadíme za y...

Řešte DR $y'' + 4y' + 29y = 0,\ y(0) = 0,\ y'(0) = 10.$

$$z^2 + 4z + 29 = 0$$

$$z_{1,2} = \frac{-4 \pm \sqrt{16 - 4.1.29}}{2.1} = \frac{-4 \pm \sqrt{-100}}{2} = -2 \pm 5i$$

$$y_1(x) = e^{-2x}\cos(5x) \qquad y_2(x) = e^{-2x}\sin(5x)$$

$$\begin{aligned} y(x) &= C_1 e^{-2x}\cos(5x) + C_2 e^{-2x}\sin(5x) \\ y'(x) &= C_1\left[-2e^{-2x}\cos(5x) - 5e^{-2x}\sin(5x)\right] \\ &\quad + C_2\left[-2e^{-2x}\sin(5x) + 5e^{-2x}\cos(5x)\right] \end{aligned}$$

$$\begin{aligned} 0 &= C_1 + 0C_2 \\ 10 &= -2C_1 + 5C_2 \end{aligned}$$

…a za y'.

Řešte DR $y'' + 4y' + 29y = 0, y(0) = 0, y'(0) = 10.$

$$z^2 + 4z + 29 = 0$$

$$z_{1,2} = \frac{-4 \pm \sqrt{16 - 4.1.29}}{2.1} = \frac{-4 \pm \sqrt{-100}}{2} = -2 \pm 5i$$

$y_1(x) = e^{-2x}\cos(5x)$ $\qquad$ $y_2(x) = e^{-2x}\sin(5x)$

$$\begin{aligned} y(x) =& C_1 e^{-2x}\cos(5x) + C_2 e^{-2x}\sin(5x) \\ y'(x) =& C_1\left[-2e^{-2x}\cos(5x) - 5e^{-2x}\sin(5x)\right] \\ &+ C_2\left[-2e^{-2x}\sin(5x) + 5e^{-2x}\cos(5x)\right] \end{aligned}$$

$$\begin{aligned} 0 &= C_1 + 0C_2 \\ 10 &= -2C_1 + 5C_2 \end{aligned} \Rightarrow C_1 = 0,\ C_2 = 2$$

Vyřešeíme soustavu rovnic pro C_1 a C_2.

Řešte DR $y'' + 4y' + 29y = 0, y(0) = 0, y'(0) = 10.$

$$z^2 + 4z + 29 = 0$$

$$z_{1,2} = \frac{-4 \pm \sqrt{16 - 4.1.29}}{2.1} = \frac{-4 \pm \sqrt{-100}}{2} = -2 \pm 5i$$

$$y_1(x) = e^{-2x}\cos(5x) \qquad y_2(x) = e^{-2x}\sin(5x)$$

$$\begin{aligned} y(x) =& C_1 e^{-2x}\cos(5x) + C_2 e^{-2x}\sin(5x) \\ y'(x) =& C_1\left[-2e^{-2x}\cos(5x) - 5e^{-2x}\sin(5x)\right] \\ &+ C_2\left[-2e^{-2x}\sin(5x) + 5e^{-2x}\cos(5x)\right] \end{aligned}$$

$$\begin{aligned} 0 &= C_1 + 0C_2 \\ 10 &= -2C_1 + 5C_2 \end{aligned} \Rightarrow C_1 = 0,\ C_2 = 2$$

$$y_p(x) = 2e^{-2x}\sin(5x)$$

Dosadíme vypočtené hodnoty koeficientů C_1 a C_2. hotovo!

Řešte DR $y'' - 4y = x^2 - 1.$
Návod: partikulární řešení hledejte jako kvadratickou funkci.

Máme za úkol řešit lineární nehomogenní rovnici druhého řádu.

Řešte DR $y'' - 4y = x^2 - 1$.
Návod: partikulární řešení hledejte jako kvadratickou funkci.

$y'' - 4y = 0$

Budeme uvažovat nejprve odpovídající homogenní rovnici.

Řešte DR $y'' - 4y = x^2 - 1$.
Návod: partikulární řešení hledejte jako kvadratickou funkci.

$y'' - 4y = 0$
$z^2 - 4 = 0 \Rightarrow z_{1,2} = \pm 2$

Sestavíme charakteristickou rovnici a vyřešíme ji.

Řešte DR $y'' - 4y = x^2 - 1$.
Návod: partikulární řešení hledejte jako kvadratickou funkci.

$y'' - 4y = 0 \Rightarrow y_{OH} = C_1 e^{2x} + C_2 e^{-2x}$
$z^2 - 4 = 0 \Rightarrow z_{1,2} = \pm 2$

Z kořenů charakteristické rovnice určíme fundamentální systém řešení a obecné řešení *homogenní* rovnice.

Řešte DR $y'' - 4y = x^2 - 1$.
Návod: partikulární řešení hledejte jako kvadratickou funkci.

$y'' - 4y = 0 \Rightarrow y_{OH} = C_1 e^{2x} + C_2 e^{-2x}$

$z^2 - 4 = 0 \Rightarrow z_{1,2} = \pm 2$

$y_p = ax^2 + bx + c$

Budeme postupovat podle návodu a hledat partikulární řešení, které je kvadratickou funkcí. Nejobecnější možná kvadratická funkce je $y = ax^2 + bx + c$.

Řešte DR $y'' - 4y = x^2 - 1$.
Návod: partikulární řešení hledejte jako kvadratickou funkci.

$y'' - 4y = 0 \Rightarrow y_{OH} = C_1 e^{2x} + C_2 e^{-2x}$
$z^2 - 4 = 0 \Rightarrow z_{1,2} = \pm 2$
$y_p = ax^2 + bx + c \quad \Rightarrow \quad y_p' = 2ax + b \quad \Rightarrow \quad y_p'' = 2a$

Hledáme hodnoty parametrů a, b a c tak, aby tato funkce byl řešením zadané rovnice. Abychom mohli do rovnice dosadit, je nutno vypočítat druhou derivaci.

Řešte DR $y'' - 4y = x^2 - 1.$

Návod: partikulární řešení hledejte jako kvadratickou funkci.

$y'' - 4y = 0 \Rightarrow y_{OH} = C_1 e^{2x} + C_2 e^{-2x}$

$z^2 - 4 = 0 \Rightarrow z_{1,2} = \pm 2$

$y_p = ax^2 + bx + c \quad \Rightarrow \quad y_p' = 2ax + b \quad \Rightarrow \quad y_p'' = 2a$

$y'' - 4y = x^2 - 1$

Vrátíme se k zadané rovnici.

Řešte DR $y'' - 4y = x^2 - 1$.
Návod: partikulární řešení hledejte jako kvadratickou funkci.

$y'' - 4y = 0 \Rightarrow y_{OH} = C_1 e^{2x} + C_2 e^{-2x}$

$z^2 - 4 = 0 \Rightarrow z_{1,2} = \pm 2$

$y_p = ax^2 + bx + c \quad \Rightarrow \quad y_p' = 2ax + b \quad \Rightarrow \quad y_p'' = 2a$

$$y'' - 4y = x^2 - 1$$

$$2a - 4 \cdot (ax^2 + bx + c) = x^2 - 1$$

Dosadíme.

Řešte DR $y'' - 4y = x^2 - 1$.
Návod: partikulární řešení hledejte jako kvadratickou funkci.

$y'' - 4y = 0 \Rightarrow y_{OH} = C_1 e^{2x} + C_2 e^{-2x}$
$z^2 - 4 = 0 \Rightarrow z_{1,2} = \pm 2$
$y_p = ax^2 + bx + c \quad \Rightarrow \quad y_p' = 2ax + b \quad \Rightarrow \quad y_p'' = 2a$

$$y'' - 4y = x^2 - 1$$
$$2a - 4 \cdot (ax^2 + bx + c) = x^2 - 1$$
$$-4a \cdot x^2 - 4b \cdot x + 2a - 4c = 1 \cdot x^2 + 0 \cdot x - 1$$

Roznásobíme závorku a přeskupíme členy polynomu tak, abychom viděli koeficienty u jednotlivých mocnin.

Řešte DR $y'' - 4y = x^2 - 1.$
Návod: partikulární řešení hledejte jako kvadratickou funkci.

$y'' - 4y = 0 \Rightarrow y_{OH} = C_1 e^{2x} + C_2 e^{-2x}$
$z^2 - 4 = 0 \Rightarrow z_{1,2} = \pm 2$
$y_p = ax^2 + bx + c \quad \Rightarrow \quad y_p' = 2ax + b \quad \Rightarrow \quad y_p'' = 2a$

$$y'' - 4y = x^2 - 1$$
$$2a - 4 \cdot (ax^2 + bx + c) = x^2 - 1$$
$$-4a \cdot x^2 - 4b \cdot x + 2a - 4c = 1 \cdot x^2 + 0 \cdot x - 1$$

$$-4a = 1$$
$$-4b = 0$$
$$2a - 4c = -1$$

Polynom na levé straně se bude rovnat polynomu na straně pravé právě tehdy, když koeficienty u odpovídajících si mocnin budou totožné.

Řešte DR $y'' - 4y = x^2 - 1$.
Návod: partikulární řešení hledejte jako kvadratickou funkci.

$y'' - 4y = 0 \Rightarrow y_{OH} = C_1e^{2x} + C_2e^{-2x}$
$z^2 - 4 = 0 \Rightarrow z_{1,2} = \pm 2$
$y_p = ax^2 + bx + c \quad \Rightarrow \quad y_p' = 2ax + b \quad \Rightarrow \quad y_p'' = 2a$

$$y'' - 4y = x^2 - 1$$
$$2a - 4 \cdot (ax^2 + bx + c) = x^2 - 1$$
$$-4a \cdot x^2 - 4b \cdot x + 2a - 4c = 1 \cdot x^2 + 0 \cdot x - 1$$

$$\begin{aligned} -4a &= 1 \\ -4b &= 0 \\ 2a - 4c &= -1 \end{aligned} \Rightarrow \begin{aligned} a &= -\frac{1}{4} \\ b &= 0 \\ c &= \frac{1}{8} \end{aligned}$$

Vyřešíme soustavu rovnic.

Řešte DR $y'' - 4y = x^2 - 1$.
Návod: partikulární řešení hledejte jako kvadratickou funkci.

$y'' - 4y = 0$ $\quad y_{OH} = C_1 e^{2x} + C_2 e^{-2x}$

$z^2 - 4 = 0 \Rightarrow z_{1,2} = \pm 2$

$y_p = ax^2 + bx + c \quad \Rightarrow \quad y_p' = 2ax + b \quad \Rightarrow \quad y_p'' = 2a$

$$y'' - 4y = x^2 - 1$$
$$2a - 4 \cdot (ax^2 + bx + c) = x^2 - 1$$
$$-4a \cdot x^2 - 4b \cdot x + 2a - 4c = 1 \cdot x^2 + 0 \cdot x - 1$$

$$\begin{aligned} -4a &= 1 \\ -4b &= 0 \\ 2a - 4c &= -1 \end{aligned} \Rightarrow \begin{aligned} a &= -\frac{1}{4} \\ b &= 0 \\ c &= \frac{1}{8} \end{aligned} \Rightarrow y = C_1 e^{2x} + C_2 e^{-2x} - \frac{1}{4}x^2 + \frac{1}{8}$$

Sestrojíme obecné řešení. Hotovo!

Solve DE $y'' - 4y' + 4y = e^{-x}$.

Solve DE $y'' - 4y' + 4y = e^{-x}$.

$y'' - 4y' + 4y = 0$

The equation is not homogeneous. We start with the corresponding homogeneous equation.

Solve DE $y'' - 4y' + 4y = e^{-x}$.

$$y'' - 4y' + 4y = 0 \quad \Rightarrow \quad z^2 - 4z + 4 = 0$$

We write the characteristic equation for the homogeneous equation...

Solve DE $y'' - 4y' + 4y = e^{-x}$.

$$y'' - 4y' + 4y = 0 \quad \Rightarrow \quad z^2 - 4z + 4 = 0 \quad \Rightarrow \quad z_{1,2} = \frac{4 - \sqrt{16 - 4.1.4}}{2.1}$$

... and solve it.

Solve DE $y'' - 4y' + 4y = e^{-x}$.

$$y'' - 4y' + 4y = 0 \quad \Rightarrow \quad z^2 - 4z + 4 = 0 \quad \Rightarrow \quad z_{1,2} = \frac{4 - \sqrt{16 - 4.1.4}}{2.1} = 2$$

The characteristic equation has a double root $z_{1,2} = 2$.

Solve DE $y'' - 4y' + 4y = e^{-x}$.

$$y'' - 4y' + 4y = 0 \quad \Rightarrow \quad z^2 - 4z + 4 = 0 \quad \Rightarrow \quad z_{1,2} = \frac{4 - \sqrt{16 - 4.1.4}}{2.1} = 2$$

$y_1(x) = e^{2x}, y_2(x) = xe^{2x}$

The fundamental system is in the case of double root z given by the functions

$$y_1 = e^{zx}, \quad y_2 = xe^{zx}.$$

Solve DE $y'' - 4y' + 4y = e^{-x}$.

$y_1(x) = e^{2x}, y_2(x) = xe^{2x}$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

We look for the particular solution in this form.

Solve DE $y'' - 4y' + 4y = e^{-x}$.

$y_1(x) = e^{2x}, y_2(x) = xe^{2x}$ $\qquad$ $y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$y_1'(x) = 2e^{2x}, y_2'(x) = e^{2x}(1+2x)$

We find the derivatives $y_1'(x)$ and $y_2'(x)$.

Solve DE $y'' - 4y' + 4y = e^{-x}$.

$y_1(x) = e^{2x}, y_2(x) = xe^{2x}$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$y_1'(x) = 2e^{2x}, y_2'(x) = e^{2x}(1+2x)$

$$\begin{aligned} A'e^{2x} + B'xe^{2x} &= 0 \\ 2A'e^{2x} + B'(1+2x)e^{2x} &= e^{-x} \end{aligned}$$

We write the system for the coefficients $A'(x)$ and $B'(x)$.

Solve DE $y'' - 4y' + 4y = e^{-x}$.

$y_1(x) = e^{2x}, y_2(x) = xe^{2x}$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$y_1'(x) = 2e^{2x}, y_2'(x) = e^{2x}(1+2x)$

$$\begin{aligned} A'e^{2x} + B'xe^{2x} &= 0 \\ 2A'e^{2x} + B'(1+2x)e^{2x} &= e^{-x} \end{aligned} \quad \Rightarrow \quad \begin{aligned} A' + B'x &= 0 \\ 2A' + B'(1+2x) &= e^{-3x} \end{aligned}$$

We divide both equations by the factor e^{2x}.

Solve DE $y'' - 4y' + 4y = e^{-x}$.

$y_1(x) = e^{2x}, y_2(x) = xe^{2x}$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$y_1'(x) = 2e^{2x}, y_2'(x) = e^{2x}(1+2x)$

$$\begin{aligned} A'e^{2x} + B'xe^{2x} &= 0 \\ 2A'e^{2x} + B'(1+2x)e^{2x} &= e^{-x} \end{aligned} \quad \Rightarrow \quad \begin{aligned} A' + B'x &= 0 \\ 2A' + B'(1+2x) &= e^{-3x} \end{aligned}$$

$B' = e^{-3x}$

We multiply the first equation by (-2) and add to the second equation. We obtain

$$B' = e^{-3x}.$$

Solve DE $y'' - 4y' + 4y = e^{-x}$.

$y_1(x) = e^{2x}, y_2(x) = xe^{2x}$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$y_1'(x) = 2e^{2x}, y_2'(x) = e^{2x}(1 + 2x)$

$$\begin{aligned} A'e^{2x} + B'xe^{2x} &= 0 \\ 2A'e^{2x} + B'(1 + 2x)e^{2x} &= e^{-x} \end{aligned} \quad \Rightarrow \quad \begin{aligned} A' + B'x &= 0 \\ 2A' + B'(1 + 2x) &= e^{-3x} \end{aligned}$$

$B' = e^{-3x}$

$A' = -xe^{-3x}$

We put $B' = e^{-3x}$ to the first equation and obtain

$$A' + xe^{-3x} = 0.$$

We solve this equation for A'.

Solve DE $y'' - 4y' + 4y = e^{-x}$.

$y_1(x) = e^{2x}, y_2(x) = xe^{2x}$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$y_1'(x) = 2e^{2x}, y_2'(x) = e^{2x}(1+2x)$

$B' = e^{-3x}$

$A' = -xe^{-3x}$

$$A(x) = -\int xe^{-3x}\,\mathrm{d}x = \frac{1}{3}xe^{-3x} - \frac{1}{3}\int e^{-3x}\,\mathrm{d}x = \frac{1}{3}xe^{-3x} + \frac{1}{9}e^{-3x}$$

We integrate by parts:

$$\int xe^{-3x}\,\mathrm{d}x \quad \begin{array}{|ll|} u = x & u' = 1 \\ v' = e^{-3x} & v = -\frac{1}{3}e^{-3x} \end{array} = -\frac{1}{3}e^{-3x}x - \int -\frac{1}{3}e^{-3x}\,\mathrm{d}x$$

$$= -\frac{1}{3}e^{-3x}x - \left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)e^{-3x}$$

Solve DE $y'' - 4y' + 4y = e^{-x}$.

$y_1(x) = e^{2x}, y_2(x) = xe^{2x}$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$y_1'(x) = 2e^{2x}, y_2'(x) = e^{2x}(1 + 2x)$

$$B' = e^{-3x} \qquad A' = -xe^{-3x}$$

$$A(x) = -\int xe^{-3x}\,\mathrm{d}x = \frac{1}{3}xe^{-3x} - \frac{1}{3}\int e^{-3x}\,\mathrm{d}x = \frac{1}{3}xe^{-3x} + \frac{1}{9}e^{-3x}$$

$$B(x) = \int e^{-3x}\,\mathrm{d}x = -\frac{1}{3}e^{-3x}$$

The integral for B is easy.

Solve DE $y'' - 4y' + 4y = e^{-x}$.

$y_1(x) = e^{2x}, y_2(x) = xe^{2x}$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$y_1'(x) = 2e^{2x}, y_2'(x) = e^{2x}(1+2x)$

$B' = e^{-3x}$ $\qquad$ $A' = -xe^{-3x}$

$$A(x) = -\int xe^{-3x}\,\mathrm{d}x = \frac{1}{3}xe^{-3x} - \frac{1}{3}\int e^{-3x}\,\mathrm{d}x = \frac{1}{3}xe^{-3x} + \frac{1}{9}e^{-3x}$$

$$B(x) = \int e^{-3x}\,\mathrm{d}x = -\frac{1}{3}e^{-3x}$$

$y_p = Ay_1 + By_2$

We return to the formula for the particular solution.

Solve DE $y'' - 4y' + 4y = e^{-x}$.

$y_1(x) = e^{2x}, y_2(x) = xe^{2x}$ $\qquad$ $y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$y_1'(x) = 2e^{2x}, y_2'(x) = e^{2x}(1+2x)$

$$B' = e^{-3x} \qquad A' = -xe^{-3x}$$

$$A(x) = -\int xe^{-3x}\,\mathrm{d}x = \frac{1}{3}xe^{-3x} - \frac{1}{3}\int e^{-3x}\,\mathrm{d}x = \frac{1}{3}xe^{-3x} + \frac{1}{9}e^{-3x}$$

$$B(x) = \int e^{-3x}\,\mathrm{d}x = -\frac{1}{3}e^{-3x}$$

$$y_p = Ay_1 + By_2 = \underbrace{\left(\frac{1}{3}xe^{-3x} + \frac{1}{9}e^{-3x}\right)}_{A} \cdot \underbrace{e^{2x}}_{y_1} \underbrace{-\frac{1}{3}e^{-3x}}_{B} \cdot \underbrace{xe^{2x}}_{y_2}$$

We know all the function here: A, B, y_1, y_2. We substitute...

Solve DE $y'' - 4y' + 4y = e^{-x}$.

$y_1(x) = e^{2x}, y_2(x) = xe^{2x}$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$y_1'(x) = 2e^{2x}, y_2'(x) = e^{2x}(1 + 2x)$

$B' = e^{-3x}$

$A' = -xe^{-3x}$

$$A(x) = -\int xe^{-3x}\,\mathrm{d}x = \frac{1}{3}xe^{-3x} - \frac{1}{3}\int e^{-3x}\,\mathrm{d}x = \frac{1}{3}xe^{-3x} + \frac{1}{9}e^{-3x}$$

$$B(x) = \int e^{-3x}\,\mathrm{d}x = -\frac{1}{3}e^{-3x}$$

$$y_p = Ay_1 + By_2 = \underbrace{\left(\frac{1}{3}xe^{-3x} + \frac{1}{9}e^{-3x}\right)}_{A} \cdot \underbrace{e^{2x}}_{y_1} \underbrace{-\frac{1}{3}e^{-3x}}_{B} \cdot \underbrace{xe^{2x}}_{y_2} = \frac{1}{9}e^{-x}$$

...and simplify.

$$\left(\frac{1}{3}xe^{-3x} + \frac{1}{9}e^{-3x}\right)e^{2x} - \frac{1}{3}e^{-3x}xe^{2x} = \frac{1}{3}xe^{-x} + \frac{1}{9}e^{-x} - \frac{1}{3}xe^{-x}.$$

Solve DE $y'' - 4y' + 4y = e^{-x}$.

$$y_1(x) = e^{2x},\ y_2(x) = xe^{2x} \qquad y_p(x) = A(x)y_1(x) + B(x)y_2(x)$$

$$y_1'(x) = 2e^{2x},\ y_2'(x) = e^{2x}(1+2x)$$

We write the solution as a sum of particular solution and linear combination of functions from fundamental system.

$$B(x) = \int e^{-3x}\,\mathrm{d}x = -\frac{1}{3}e^{-3x}$$

$$y_p = Ay_1 + By_2 = \underbrace{\left(\frac{1}{3}xe^{-3x} + \frac{1}{9}e^{-3x}\right)}_{A} \cdot \underbrace{e^{2x}}_{y_1} \underbrace{-\frac{1}{3}e^{-3x}}_{B} \cdot \underbrace{xe^{2x}}_{y_2} = \frac{1}{9}e^{-x}$$

$$y = C_1e^{2x} + C_2xe^{2x} + \frac{1}{9}e^{-x}, \qquad C_1, C_2 \in \mathbb{R}$$

Solve DE $y'' - 4y' + 4y = e^{-x}$.

$y_1(x) = e^{2x}, y_2(x) = xe^{2x}$ $\qquad$ $y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$y_1'(x) = 2e^{2x}, y_2'(x) = e^{2x}(1+2x)$

$B' = e^{-3x}$ $\qquad$ $A' = -xe^{-3x}$

$$A(x) = -\int xe^{-3x}\,\mathrm{d}x = \frac{1}{3}xe^{-3x} - \frac{1}{3}\int e^{-3x}\,\mathrm{d}x = \frac{1}{3}xe^{-3x} + \frac{1}{9}e^{-3x}$$

$$B(x) = \int e^{-3x}\,\mathrm{d}x = -\frac{1}{3}e^{-3x}$$

$$y_p = Ay_1 + By_2 = \underbrace{\left(\frac{1}{3}xe^{-3x} + \frac{1}{9}e^{-3x}\right)}_{A} \cdot \underbrace{e^{2x}}_{y_1} \underbrace{-\frac{1}{3}e^{-3x}}_{B} \cdot \underbrace{xe^{2x}}_{y_2} = \frac{1}{9}e^{-x}$$

$$y = C_1e^{2x} + C_2xe^{2x} + \frac{1}{9}e^{-x}, \qquad C_1, C_2 \in \mathbb{R}$$

The problem is solved.

Solve DE $y'' - 5y' + 6y = xe^x$.

Solve DE $y'' - 5y' + 6y = xe^x$.

$y'' - 5y' + 6y = 0$

We start with the homogeneous equation...

Solve DE $y'' - 5y' + 6y = xe^x$.

$y'' - 5y' + 6y = 0 \qquad \Rightarrow \qquad z^2 - 5z + 6 = 0$

…and its characteristic equation.

Solve DE $y'' - 5y' + 6y = xe^x$.

$y'' - 5y' + 6y = 0 \quad \Rightarrow \quad z^2 - 5z + 6 = 0 \quad \Rightarrow \quad z_1 = 2, z_2 = 3$

The roots of the equation are real and distinct.

Solve DE $y'' - 5y' + 6y = xe^x$.

$y'' - 5y' + 6y = 0 \quad \Rightarrow \quad z^2 - 5z + 6 = 0 \quad \Rightarrow \quad z_1 = 2, z_2 = 3$

$y_1(x) = e^{2x}$ $\qquad$ $y_2(x) = e^{3x}$

We write the fundamental system...

Solve DE $y'' - 5y' + 6y = xe^x.$

$y'' - 5y' + 6y = 0 \quad \Rightarrow \quad z^2 - 5z + 6 = 0 \quad \Rightarrow \quad z_1 = 2, z_2 = 3$

$y_1(x) = e^{2x} \qquad y_2(x) = e^{3x}$

$y_1'(x) = 2e^{2x} \qquad y_2'(x) = 3e^{3x}$

…and the derivatives of the functions from that fundamental system.

Solve DE $y'' - 5y' + 6y = xe^x$.

$$y'' - 5y' + 6y = 0 \quad \Rightarrow \quad z^2 - 5z + 6 = 0 \quad \Rightarrow \quad z_1 = 2, z_2 = 3$$

$y_1(x) = e^{2x}$ $\qquad$ $y_2(x) = e^{3x}$

$y_1'(x) = 2e^{2x}$ $\qquad$ $y_2'(x) = 3e^{3x}$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

We will use the variation of parameters.

Solve DE $y'' - 5y' + 6y = xe^x$.

$$y'' - 5y' + 6y = 0 \quad \Rightarrow \quad z^2 - 5z + 6 = 0 \quad \Rightarrow \quad z_1 = 2, z_2 = 3$$

$y_1(x) = e^{2x}$ $\qquad$ $y_2(x) = e^{3x}$

$y_1'(x) = 2e^{2x}$ $\qquad$ $y_2'(x) = 3e^{3x}$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$$A'e^{2x} + B'e^{3x} = 0$$
$$2A'e^{2x} + 3B'e^{3x} = xe^x$$

The functions A and B satisfy the following relations…

Solve DE $y'' - 5y' + 6y = xe^x$.

$y'' - 5y' + 6y = 0 \quad \Rightarrow \quad z^2 - 5z + 6 = 0 \quad \Rightarrow \quad z_1 = 2, z_2 = 3$

$y_1(x) = e^{2x}$ $\qquad y_2(x) = e^{3x}$

$y_1'(x) = 2e^{2x}$ $\qquad y_2'(x) = 3e^{3x}$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$$\begin{aligned} A'e^{2x} + B'e^{3x} &= 0 \\ 2A'e^{2x} + 3B'e^{3x} &= xe^x \end{aligned} \quad \Rightarrow \quad \begin{aligned} A' + B'e^x &= 0 \\ 2A' + 3B'e^x &= xe^{-x} \end{aligned}$$

…which are equivalent to this linear system.

Solve DE $y'' - 5y' + 6y = xe^x$.

$y'' - 5y' + 6y = 0 \quad \Rightarrow \quad z^2 - 5z + 6 = 0 \quad \Rightarrow \quad z_1 = 2, z_2 = 3$

$y_1(x) = e^{2x} \qquad y_2(x) = e^{3x}$

$y_1'(x) = 2e^{2x} \qquad y_2'(x) = 3e^{3x}$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$$\begin{aligned} A'e^{2x} + B'e^{3x} &= 0 \\ 2A'e^{2x} + 3B'e^{3x} &= xe^x \end{aligned} \quad \Rightarrow \quad \begin{aligned} A' + B'e^x &= 0 \\ 2A' + 3B'e^x &= xe^{-x} \end{aligned}$$

$$A' = -B'e^x$$

We solve the first equation with respect to A'...

Solve DE $y'' - 5y' + 6y = xe^x$.

$y'' - 5y' + 6y = 0 \quad \Rightarrow \quad z^2 - 5z + 6 = 0 \quad \Rightarrow \quad z_1 = 2, z_2 = 3$

$y_1(x) = e^{2x}$ $\qquad y_2(x) = e^{3x}$

$y_1'(x) = 2e^{2x}$ $\qquad y_2'(x) = 3e^{3x}$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$$\begin{aligned} A'e^{2x} + B'e^{3x} &= 0 \\ 2A'e^{2x} + 3B'e^{3x} &= xe^x \end{aligned} \quad \Rightarrow \quad \begin{aligned} A' + B'e^{x} &= 0 \\ 2A' + 3B'e^{x} &= xe^{-x} \end{aligned}$$

$$A' = -B'e^x$$

$$B'e^x = xe^{-x}$$

… and substitute into the second equation. We obtain

$$2(-B'e^{-x}) + 3B'e^x = xe^{-x}$$

which is equivalent to the blue expression.

Solve DE $y'' - 5y' + 6y = xe^x$.

$y'' - 5y' + 6y = 0 \quad \Rightarrow \quad z^2 - 5z + 6 = 0 \quad \Rightarrow \quad z_1 = 2, z_2 = 3$

$y_1(x) = e^{2x}$ $\qquad y_2(x) = e^{3x}$

$y_1'(x) = 2e^{2x}$ $\qquad y_2'(x) = 3e^{3x}$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$$\begin{aligned} A'e^{2x} + B'e^{3x} &= 0 \\ 2A'e^{2x} + 3B'e^{3x} &= xe^x \end{aligned} \quad \Rightarrow \quad \begin{aligned} A' + B'e^x &= 0 \\ 2A' + 3B'e^x &= xe^{-x} \end{aligned}$$

$$A' = -B'e^x$$

$$B'e^x = xe^{-x}$$

$$B' = xe^{-2x}$$

We can find B' ...

Solve DE $y'' - 5y' + 6y = xe^x$.

$$y'' - 5y' + 6y = 0 \quad \Rightarrow \quad z^2 - 5z + 6 = 0 \quad \Rightarrow \quad z_1 = 2, z_2 = 3$$

$y_1(x) = e^{2x}$ $\qquad$ $y_2(x) = e^{3x}$

$y_1'(x) = 2e^{2x}$ $\qquad$ $y_2'(x) = 3e^{3x}$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$$\begin{aligned} A'e^{2x} + B'e^{3x} &= 0 \\ 2A'e^{2x} + 3B'e^{3x} &= xe^x \end{aligned} \quad \Rightarrow \quad \begin{aligned} A' + B'e^x &= 0 \\ 2A' + 3B'e^x &= xe^{-x} \end{aligned}$$

$$A' = -B'e^x$$

$$B'e^x = xe^{-x}$$

$$B' = xe^{-2x}$$

$$A' = -B'e^x = -xe^{-x}$$

... and A'.

Solve DE $y'' - 5y' + 6y = xe^x$.

$y'' - 5y' + 6y = 0 \quad \Rightarrow \quad z^2 - 5z + 6 = 0 \quad \Rightarrow \quad z_1 = 2, z_2 = 3$

$y_1(x) = e^{2x}$ $\qquad$ $y_2(x) = e^{3x}$

$y_1'(x) = 2e^{2x}$ $\qquad$ $y_2'(x) = 3e^{3x}$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$$\begin{aligned} A'e^{2x} + B'e^{3x} &= 0 \\ 2A'e^{2x} + 3B'e^{3x} &= xe^x \end{aligned} \quad \Rightarrow \quad \begin{aligned} A' + B'e^{x} &= 0 \\ 2A' + 3B'e^{x} &= xe^{-x} \end{aligned}$$

$A' = -xe^{-x}, B' = xe^{-2x}$

We will look for $A(x)$ and $B(x)$ from A' and B'.

Solve DE $y'' - 5y' + 6y = xe^x$.

$y'' - 5y' + 6y = 0 \quad \Rightarrow \quad z^2 - 5z + 6 = 0 \quad \Rightarrow \quad z_1 = 2, z_2 = 3$

$y_1(x) = e^{2x} \qquad y_2(x) = e^{3x}$

$y_1'(x) = 2e^{2x} \qquad y_2'(x) = 3e^{3x}$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$$\begin{aligned} A'e^{2x} + B'e^{3x} &= 0 \\ 2A'e^{2x} + 3B'e^{3x} &= xe^x \end{aligned} \quad \Rightarrow \quad \begin{aligned} A' + B'e^x &= 0 \\ 2A' + 3B'e^x &= xe^{-x} \end{aligned}$$

$A' = -xe^{-x}, B' = xe^{-2x}$

$A(x) = (x+1)e^{-x},$

We integrate by parts

$$A = -\int xe^{-x}\,\mathrm{d}x \quad \begin{array}{|ll|} u = x & u' = 1 \\ v' = e^{-x} & v = -e^{-x} \end{array} = -\left(-xe^{-x} + \int e^{-x}\,\mathrm{d}x\right)$$

$$= -\left(-xe^{-x} - e^{-x}\right) = (x+1)e^{-x}$$

Solve DE $y'' - 5y' + 6y = xe^x$.

$y'' - 5y' + 6y = 0 \quad \Rightarrow \quad z^2 - 5z + 6 = 0 \quad \Rightarrow \quad z_1 = 2, z_2 = 3$

$y_1(x) = e^{2x} \qquad y_2(x) = e^{3x}$

$y_1'(x) = 2e^{2x} \qquad y_2'(x) = 3e^{3x}$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$$\begin{aligned} A'e^{2x} + B'e^{3x} &= 0 \\ 2A'e^{2x} + 3B'e^{3x} &= xe^x \end{aligned} \quad \Rightarrow \quad \begin{aligned} A' + B'e^x &= 0 \\ 2A' + 3B'e^x &= xe^{-x} \end{aligned}$$

$A' = -xe^{-x}$, $B' = xe^{-2x}$

$A(x) = (x+1)e^{-x}$, $\qquad B(x) = -\left(\frac{1}{2}x + \frac{1}{4}\right)e^{-2x}$

$$B(x) = \int xe^{-2x}\,\mathrm{d}x \quad \begin{array}{|ll|} u = x & u' = 1 \\ v' = e^{-2x} & v = -\frac{1}{2}e^{-2x} \end{array}$$

$$= -\frac{1}{2}xe^{-2x} + \frac{1}{2}\int e^{-2x}\,\mathrm{d}x = -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}$$

Solve DE $y'' - 5y' + 6y = xe^x$.

$y'' - 5y' + 6y = 0 \quad \Rightarrow \quad z^2 - 5z + 6 = 0 \quad \Rightarrow \quad z_1 = 2, z_2 = 3$

$y_1(x) = e^{2x}$ $\qquad y_2(x) = e^{3x}$

$y_1'(x) = 2e^{2x}$ $\qquad y_2'(x) = 3e^{3x}$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$$\begin{aligned} A'e^{2x} + B'e^{3x} &= 0 \\ 2A'e^{2x} + 3B'e^{3x} &= xe^x \end{aligned} \quad \Rightarrow \quad \begin{aligned} A' + B'e^{x} &= 0 \\ 2A' + 3B'e^{x} &= xe^{-x} \end{aligned}$$

$A' = -xe^{-x}$, $B' = xe^{-2x}$

$A(x) = (x+1)e^{-x}$, $\qquad B(x) = -\left(\frac{1}{2}x + \frac{1}{4}\right)e^{-2x}$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

We return to the formula for the particular equation, ...

Solve DE $y'' - 5y' + 6y = xe^x$.

$$y'' - 5y' + 6y = 0 \quad \Rightarrow \quad z^2 - 5z + 6 = 0 \quad \Rightarrow \quad z_1 = 2, z_2 = 3$$

$y_1(x) = e^{2x}$ $\qquad$ $y_2(x) = e^{3x}$

$y_1'(x) = 2e^{2x}$ $\qquad$ $y_2'(x) = 3e^{3x}$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$$\begin{aligned} A'e^{2x} + B'e^{3x} &= 0 \\ 2A'e^{2x} + 3B'e^{3x} &= xe^x \end{aligned} \quad \Rightarrow \quad \begin{aligned} A' + B'e^x &= 0 \\ 2A' + 3B'e^x &= xe^{-x} \end{aligned}$$

$A' = -xe^{-x}$, $B' = xe^{-2x}$

$A(x) = (x+1)e^{-x}$, $\qquad$ $B(x) = -\left(\frac{1}{2}x + \frac{1}{4}\right)e^{-2x}$

$$y_p(x) = A(x)y_1(x) + B(x)y_2(x) = \underbrace{(x+1)e^{-x}}_{A}\underbrace{e^{2x}}_{y_1}\underbrace{-\left(\frac{1}{2}x + \frac{1}{4}\right)e^{-2x}}_{B}\underbrace{e^{3x}}_{y_2}$$

…substitute…

Solve DE $y'' - 5y' + 6y = xe^x.$

$y'' - 5y' + 6y = 0 \quad \Rightarrow \quad z^2 - 5z + 6 = 0 \quad \Rightarrow \quad z_1 = 2, z_2 = 3$

$y_1(x) = e^{2x} \qquad y_2(x) = e^{3x}$

$y_1'(x) = 2e^{2x} \qquad y_2'(x) = 3e^{3x}$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$$\begin{aligned} A'e^{2x} + B'e^{3x} &= 0 \\ 2A'e^{2x} + 3B'e^{3x} &= xe^x \end{aligned} \quad \Rightarrow \quad \begin{aligned} A' + B'e^x &= 0 \\ 2A' + 3B'e^x &= xe^{-x} \end{aligned}$$

$A' = -xe^{-x}, B' = xe^{-2x}$

$A(x) = (x+1)e^{-x}, \qquad B(x) = -\left(\frac{1}{2}x + \frac{1}{4}\right)e^{-2x}$

$$y_p(x) = A(x)y_1(x) + B(x)y_2(x) = \underbrace{(x+1)e^{-x}}_{A}\underbrace{e^{2x}}_{y_1} \underbrace{-\left(\frac{1}{2}x + \frac{1}{4}\right)e^{-2x}}_{B}\underbrace{e^{3x}}_{y_2}$$

$$= \frac{1}{4}e^x(2x+3)$$

. . . and simplify.

Solve DE $y'' - 5y' + 6y = xe^x.$

$y'' - 5y' + 6y = 0 \quad \Rightarrow \quad z^2 - 5z + 6 = 0 \quad \Rightarrow \quad z_1 = 2, z_2 = 3$

$y_1(x) = e^{2x} \qquad y_2(x) = e^{3x}$

$y_1'(x) = 2e^{2x} \qquad y_2'(x) = 3e^{3x}$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

The general solution is the sum of the particular solution and the general solution of the corresponding homogeneous equation. The general solution of the corresponding homogeneous equation is a linear combination of the functions from the fundamental system.

$$y_p(x) = A(x)y_1(x) + B(x)y_2(x) = \underbrace{(x+1)e^{-x}}_{A}\underbrace{e^{2x}}_{y_1} \underbrace{-\left(\frac{1}{2}x + \frac{1}{4}\right)e^{-2x}}_{B}\underbrace{e^{3x}}_{y_2}$$

$$= \frac{1}{4}e^x(2x+3)$$

$$y = C_1e^{2x} + C_2e^{3x} + \frac{1}{4}e^x(2x+3), \quad C_1, C_2 \in \mathbb{R}$$

Solve DE $y'' - 5y' + 6y = xe^x$.

$y'' - 5y' + 6y = 0 \quad \Rightarrow \quad z^2 - 5z + 6 = 0 \quad \Rightarrow \quad z_1 = 2, z_2 = 3$

$y_1(x) = e^{2x}$ $\qquad y_2(x) = e^{3x}$

$y_1'(x) = 2e^{2x}$ $\qquad y_2'(x) = 3e^{3x}$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$$\begin{aligned} A'e^{2x} + B'e^{3x} &= 0 \\ 2A'e^{2x} + 3B'e^{3x} &= xe^x \end{aligned} \quad \Rightarrow \quad \begin{aligned} A' + B'e^x &= 0 \\ 2A' + 3B'e^x &= xe^{-x} \end{aligned}$$

$A' = -xe^{-x}, B' = xe^{-2x}$

$A(x) = (x+1)e^{-x}, \qquad B(x) = -\left(\frac{1}{2}x + \frac{1}{4}\right)e^{-2x}$

$$\begin{aligned} y_p(x) &= A(x)y_1(x) + B(x)y_2(x) = \underbrace{(x+1)e^{-x}}_{A}\underbrace{e^{2x}}_{y_1} \underbrace{-\left(\frac{1}{2}x + \frac{1}{4}\right)e^{-2x}}_{B}\underbrace{e^{3x}}_{y_2} \\ &= \frac{1}{4}e^x(2x+3) \end{aligned}$$

$y = C_1e^{2x} + C_2e^{3x} + \frac{1}{4}e^x(2x+3), \quad C_1, C_2 \in \mathbb{R}$

The problem is solved.

Solve DE $\quad y'' + y = \dfrac{\cos x}{\sin x}$. Work on the interval where $\sin(x) > 0$.

Solve DE $y'' + y = \frac{\cos x}{\sin x}$. Work on the interval where $\sin(x) > 0$.

$y'' + y = 0$

We start with the corresponding homogeneous equation...

Solve DE $y'' + y = \frac{\cos x}{\sin x}$. Work on the interval where $\sin(x) > 0$.

$y'' + y = 0 \qquad \Rightarrow \qquad z^2 + 1 = 0$

. . . and its characteristic equation.

Solve DE $y'' + y = \dfrac{\cos x}{\sin x}$. Work on the interval where $\sin(x) > 0$.

$y'' + y = 0 \quad \Rightarrow \quad z^2 + 1 = 0 \quad \Rightarrow \quad z_1 = i, z_2 = -i$

The characteristic equation has complex roots.

Solve DE $y'' + y = \dfrac{\cos x}{\sin x}$. Work on the interval where $\sin(x) > 0$.

$y'' + y = 0 \quad \Rightarrow \quad z^2 + 1 = 0 \quad \Rightarrow \quad z_1 = i, z_2 = -i$

$y_1(x) = \cos x,$ $\quad y_2(x) = \sin x$

We write the fundamental system.

Solve DE $y'' + y = \frac{\cos x}{\sin x}$. Work on the interval where $\sin(x) > 0$.

$y_1(x) = \cos x,$ $y_2(x) = \sin x$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

We look for the particular solution in this form.

Solve DE $y'' + y = \dfrac{\cos x}{\sin x}$. Work on the interval where $\sin(x) > 0$.

$y_1(x) = \cos x,$ $y_2(x) = \sin x$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$$\begin{aligned} A' \cos x + B' \sin x &= 0 \\ -A' \sin x + B' \cos x &= \frac{\cos x}{\sin x} \end{aligned}$$

The functions A and B have to satisfy this linear system.

Solve DE $y'' + y = \dfrac{\cos x}{\sin x}$. Work on the interval where $\sin(x) > 0$.

$y_1(x) = \cos x,$ $y_2(x) = \sin x$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$$\begin{aligned} A' \cos x + B' \sin x &= 0 \\ -A' \sin x + B' \cos x &= \frac{\cos x}{\sin x} \end{aligned}$$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1;$$

We will solve this system by Cramer's rule. We find the determinant of the coefficients matrix. This determinant is called *wronskian*.

Solve DE $y'' + y = \dfrac{\cos x}{\sin x}$. Work on the interval where $\sin(x) > 0$.

$y_1(x) = \cos x,$ $\qquad y_2(x) = \sin x$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$$A' \cos x + B' \sin x = 0$$
$$-A' \sin x + B' \cos x = \frac{\cos x}{\sin x}$$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1;$$

$$W_1 = \begin{vmatrix} 0 & \sin x \\ \dfrac{\cos x}{\sin x} & \cos x \end{vmatrix} = -\cos x;$$

We evaluate the auxiliary determinants. . .

Solve DE $y'' + y = \frac{\cos x}{\sin x}$. Work on the interval where $\sin(x) > 0$.

$y_1(x) = \cos x, \qquad y_2(x) = \sin x$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$$\begin{aligned} A' \cos x + B' \sin x &= 0 \\ -A' \sin x + B' \cos x &= \frac{\cos x}{\sin x} \end{aligned}$$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1;$$

$$W_1 = \begin{vmatrix} 0 & \sin x \\ \frac{\cos x}{\sin x} & \cos x \end{vmatrix} = -\cos x; \qquad W_2 = \begin{vmatrix} \cos x & 0 \\ -\sin x & \frac{\cos x}{\sin x} \end{vmatrix} = \frac{\cos^2 x}{\sin x}$$

We evaluate the auxiliary determinants. . .

Solve DE $y'' + y = \dfrac{\cos x}{\sin x}$. Work on the interval where $\sin(x) > 0$.

$y_1(x) = \cos x,\qquad y_2(x) = \sin x$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$$\begin{aligned} A'\cos x + B'\sin x &= 0 \\ -A'\sin x + B'\cos x &= \frac{\cos x}{\sin x} \end{aligned}$$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1;$$

$$W_1 = \begin{vmatrix} 0 & \sin x \\ \frac{\cos x}{\sin x} & \cos x \end{vmatrix} = -\cos x; \qquad W_2 = \begin{vmatrix} \cos x & 0 \\ -\sin x & \frac{\cos x}{\sin x} \end{vmatrix} = \frac{\cos^2 x}{\sin x}$$

$$A' = \frac{W_1}{W} = -\cos x$$

…and use the formula of Cramer for A'…

Solve DE $y'' + y = \frac{\cos x}{\sin x}$. Work on the interval where $\sin(x) > 0$.

$y_1(x) = \cos x, \qquad y_2(x) = \sin x$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$$A'\cos x + B'\sin x = 0$$
$$-A'\sin x + B'\cos x = \frac{\cos x}{\sin x}$$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1;$$

$$W_1 = \begin{vmatrix} 0 & \sin x \\ \frac{\cos x}{\sin x} & \cos x \end{vmatrix} = -\cos x; \qquad W_2 = \begin{vmatrix} \cos x & 0 \\ -\sin x & \frac{\cos x}{\sin x} \end{vmatrix} = \frac{\cos^2 x}{\sin x}$$

$$A' = \frac{W_1}{W} = -\cos x \qquad B' = \frac{W_2}{W} = \frac{\cos^2 x}{\sin x}$$

…and for B'.

Solve DE $y'' + y = \dfrac{\cos x}{\sin x}$. Work on the interval where $\sin(x) > 0$.

$y_1(x) = \cos x,$ $y_2(x) = \sin x$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$A' = \dfrac{W_1}{W} = -\cos x$ $B' = \dfrac{W_2}{W} = \dfrac{\cos^2 x}{\sin x}$

$A = -\sin x$

We integrate. The integral for A is easy.

Solve DE $y'' + y = \frac{\cos x}{\sin x}$. Work on the interval where $\sin(x) > 0$.

$y_1(x) = \cos x,\qquad y_2(x) = \sin x$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$A' = \frac{W_1}{W} = -\cos x \qquad B' = \frac{W_2}{W} = \frac{\cos^2 x}{\sin x}$

$A = -\sin x$

$$B = \int \frac{\cos^2 x}{\sin x}\,\mathrm{d}x = \int \frac{\cos^2 x \sin x}{1-\cos^2 x}\,\mathrm{d}x$$

The integral for B is more complicated. The odd power of the goniometric function is in the denominator. We have to multiply and divide by $\sin x$ and use the formula $\cos^2 + \sin^2 x = 1$. This gives

$$B(x) = \int \frac{\cos^2 x}{\sin x}\,\mathrm{d}x = \int \frac{\cos^2 x}{\sin^2 x}\sin x\,\mathrm{d}x = \int \frac{\cos^2 x}{1-\cos^2 x}\sin x\,\mathrm{d}x$$

Solve DE $y'' + y = \dfrac{\cos x}{\sin x}$. Work on the interval where $\sin(x) > 0$.

$y_1(x) = \cos x,$ $\qquad y_2(x) = \sin x$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$A' = \dfrac{W_1}{W} = -\cos x$ $\qquad B' = \dfrac{W_2}{W} = \dfrac{\cos^2 x}{\sin x}$

$A = -\sin x$

$$B = \int \frac{\cos^2 x}{\sin x}\,\mathrm{d}x = \int \frac{\cos^2 x \sin x}{1-\cos^2 x}\,\mathrm{d}x = \int \frac{t^2}{t^2-1}\,\mathrm{d}t = \int 1 + \frac{1}{t^2-1}\,\mathrm{d}t$$

Now we use the substitution $\begin{array}{c} \cos x = t \\ \sin x\,\mathrm{d}x = -\,\mathrm{d}t \end{array}$. This gives

$$B(x) = \int \frac{t^2}{1-t^2}(-1)\,\mathrm{d}t = \int \frac{t^2}{t^2-1}\,\mathrm{d}t.$$

Further we divide the numerator t^2 by the denominator (t^2-1).

Solve DE $y'' + y = \dfrac{\cos x}{\sin x}$. Work on the interval where $\sin(x) > 0$.

$y_1(x) = \cos x, \qquad y_2(x) = \sin x$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$A' = \dfrac{W_1}{W} = -\cos x \qquad B' = \dfrac{W_2}{W} = \dfrac{\cos^2 x}{\sin x}$

$A = -\sin x$

$$B = \int \frac{\cos^2 x}{\sin x}\,\mathrm{d}x = \int \frac{\cos^2 x \sin x}{1-\cos^2 x}\,\mathrm{d}x = \int \frac{t^2}{t^2-1}\,\mathrm{d}t = \int 1 + \frac{1}{t^2-1}\,\mathrm{d}t$$

$$= t + \frac{1}{2}\ln\frac{1-t}{1+t}$$

We expand the fraction $\dfrac{1}{t^2-1}$ into partial fractions, integrate and add logarithms. This gives

$$B(x) = t + \int \frac{1}{2}\frac{1}{t-1} - \frac{1}{2}\frac{1}{t+1}\,\mathrm{d}t = t + \frac{1}{2}\ln|t-1| - \frac{1}{2}\ln|t+1|$$

$$= t + \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| = t + \frac{1}{2}\ln\frac{1-t}{1+t}$$

Solve DE $y'' + y = \dfrac{\cos x}{\sin x}$. Work on the interval where $\sin(x) > 0$.

$y_1(x) = \cos x, \qquad y_2(x) = \sin x$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$A' = \dfrac{W_1}{W} = -\cos x \qquad B' = \dfrac{W_2}{W} = \dfrac{\cos^2 x}{\sin x}$

$A = -\sin x$

$$B = \int \frac{\cos^2 x}{\sin x}\,\mathrm{d}x = \int \frac{\cos^2 x \sin x}{1-\cos^2 x}\,\mathrm{d}x = \int \frac{t^2}{t^2-1}\,\mathrm{d}t = \int 1 + \frac{1}{t^2-1}\,\mathrm{d}t$$

$$= t + \frac{1}{2}\ln\frac{1-t}{1+t} = \cos x + \frac{1}{2}\ln\frac{1-\cos x}{1+\cos x}$$

We use the back substitution $\cos x = t$.

Solve DE $y'' + y = \dfrac{\cos x}{\sin x}$. Work on the interval where $\sin(x) > 0$.

$y_1(x) = \cos x,$ $y_2(x) = \sin x$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$A' = \dfrac{W_1}{W} = -\cos x$ $B' = \dfrac{W_2}{W} = \dfrac{\cos^2 x}{\sin x}$

$A = -\sin x$

$B = \cos x + \dfrac{1}{2} \ln \dfrac{1 - \cos x}{1 + \cos x}$

Now both A and B are known.

Solve DE $y'' + y = \dfrac{\cos x}{\sin x}$. Work on the interval where $\sin(x) > 0$.

$y_1(x) = \cos x,$ $\quad y_2(x) = \sin x$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$$A' = \frac{W_1}{W} = -\cos x \qquad B' = \frac{W_2}{W} = \frac{\cos^2 x}{\sin x}$$

$A = -\sin x$

$$B = \cos x + \frac{1}{2}\ln\frac{1-\cos x}{1+\cos x}$$

$y_p(x) = Ay_1 + By_2$

The particular solution we looked in this form.

Solve DE $y'' + y = \dfrac{\cos x}{\sin x}$. Work on the interval where $\sin(x) > 0$.

$y_1(x) = \cos x, \qquad y_2(x) = \sin x$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$A' = \dfrac{W_1}{W} = -\cos x \qquad B' = \dfrac{W_2}{W} = \dfrac{\cos^2 x}{\sin x}$

$A = -\sin x$

$B = \cos x + \dfrac{1}{2}\ln\dfrac{1-\cos x}{1+\cos x}$

$$y_p(x) = Ay_1 + By_2 = \underbrace{-\sin x}_{A}\underbrace{\cos x}_{y_1} + \underbrace{\left[\cos x + \frac{1}{2}\ln\frac{1-\cos x}{1+\cos x}\right]}_{B}\underbrace{\sin x}_{y_2}$$

We can substitute. . .

Solve DE $y'' + y = \dfrac{\cos x}{\sin x}$. Work on the interval where $\sin(x) > 0$.

$y_1(x) = \cos x, \qquad y_2(x) = \sin x$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$A' = \dfrac{W_1}{W} = -\cos x \qquad B' = \dfrac{W_2}{W} = \dfrac{\cos^2 x}{\sin x}$

$A = -\sin x$

$B = \cos x + \dfrac{1}{2}\ln\dfrac{1-\cos x}{1+\cos x}$

$$y_p(x) = Ay_1 + By_2 = \underbrace{-\sin x}_{A}\underbrace{\cos x}_{y_1} + \underbrace{\left[\cos x + \frac{1}{2}\ln\frac{1-\cos x}{1+\cos x}\right]}_{B}\underbrace{\sin x}_{y_2}$$

$$= \frac{1}{2}\sin x \ln\frac{1-\cos x}{1+\cos x}$$

... and simplify

Solve DE $\quad y'' + y = \dfrac{\cos x}{\sin x}$. Work on the interval where $\sin(x) > 0$.

$y_1(x) = \cos x, \qquad y_2(x) = \sin x$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$$A' = \frac{W_1}{W} = -\cos x \qquad B' = \frac{W_2}{W} = \frac{\cos^2 x}{\sin x}$$

$A = -\sin x$

$$B = \cos x + \frac{1}{2}\ln\frac{1-\cos x}{1+\cos x}$$

$$y_p(x) = Ay_1 + By_2 = \underbrace{-\sin x}_{A}\underbrace{\cos x}_{y_1} + \underbrace{\left[\cos x + \frac{1}{2}\ln\frac{1-\cos x}{1+\cos x}\right]}_{B}\underbrace{\sin x}_{y_2}$$

$$= \frac{1}{2}\sin x \ln\frac{1-\cos x}{1+\cos x}$$

$$y(x) = C_1\cos x + C_2\sin x + \frac{\sin x}{2}\ln\frac{1-\cos x}{1+\cos x}$$

The general solution is a sum of the general solution of the homogeneous system and the particular solution of nonhomogeneous system.

Solve DE $y'' + y = \dfrac{\cos x}{\sin x}$. Work on the interval where $\sin(x) > 0$.

$y_1(x) = \cos x,$ $\qquad y_2(x) = \sin x$

$y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$A' = \dfrac{W_1}{W} = -\cos x$ $\qquad B' = \dfrac{W_2}{W} = \dfrac{\cos^2 x}{\sin x}$

$A = -\sin x$

$B = \cos x + \dfrac{1}{2} \ln \dfrac{1-\cos x}{1+\cos x}$

$$y_p(x) = Ay_1 + By_2 = \underbrace{-\sin x}_{A}\underbrace{\cos x}_{y_1} + \underbrace{\left[\cos x + \frac{1}{2}\ln\frac{1-\cos x}{1+\cos x}\right]}_{B}\underbrace{\sin x}_{y_2}$$

$$= \frac{1}{2}\sin x \ln \frac{1-\cos x}{1+\cos x}$$

$$y(x) = C_1 \cos x + C_2 \sin x + \frac{\sin x}{2} \ln \frac{1-\cos x}{1+\cos x}$$

The problem is solved

FINISHED.