Profinite semigroups and applications in Computer Science

Jorge Almeida

Departamento Matemática, CMUP
Faculdade de Ciências, Universidade do Porto
Porto, Portugal
http://www.fc.up.pt/cmup/jalmeida/

October, 2011
Ústav matematiky a statistiky
Přírodovědecká fakulta
Masarykova univerzita

Abstract

Finite semigroups appear naturally in Computer Science, namely as syntactic semigroups of regular languages, transition semigroups of finite automata, or as finite recognizing devices on their own. Eilenberg's correspondence theorem gives a general framework for the classification of regular languages through algebraic properties of their syntactic semigroups. Here is the resulting typical problem on the algebraic side: a recursively enumerable set R of finite semigroups is given and one wishes to decide whether a given finite semigroup is a homomorphic image of a subsemigroup of a finite product of members of R. Since such a problem is often undecidable, special techniques have been devised to handle special cases. Relatively free profinite semigroups turn out to be quite useful in this context. They play the role of free algebras in Universal Algebra, capturing in their algebraic-topological/metric structure combinatorial properties of the corresponding classes of languages.
The aim of this short course is to introduce reltively free profinite semigroups and to explore two topics in which there have been significant recent developments, namely the separation of a given word from a given regular language by a regular language of a special type (for instance, a group language), and connections with symbolic dynamics.
Tentative syllabus and preliminary references:

1. Relatively free profinite semigroups. (1 lecture)
   Reference:
   [1] J. Almeida, Profinite semigroups and applications, in "Structural Theory of Automata, Semigroups, and Universal Algebra", V. B. Kudryavtsev and I. G. Rosenberg (eds.), Proceedings of the NATO Advanced Study Institute on Structural Theory of Automata, Semigroups and Universal Algebra (Montréal, Québec, Canada, 7-18 July 2003), Springer, New York, 2005, pp. 1-45.
2. 2. Separating words and regular languages. (2 lectures)
   Reference:
   [2] S. Margolis, M. Sapir, and P. Weil, Closed subgroups in pro-V topologies and the extension problem for inverse automata, Int. J. Algebra and Comput. 11 (2001) 405-455.
3. Relatifely free profinite semigroups and Symbolic Dynamics. (2 lectures)
   Reference:
   [1] (see above).

Part I

Relatively free profinite semigroups

Outline

- Language recognition devices
- Eilenberg's correspondence
- Decidable pseudovarieties
- Metrics associated with pseudovarieties
- Pro-$\mathbf{V}$ semigroups
- Reiterman's Theorem

- A regular language is a subset of the free monoid A^* on an alphabet A admitting a regular expression, i.e., a formal expression describing it in terms of the empty set $\emptyset$ and the letters $a \in A$ using the following operations:
  - $(K, L) \mapsto K \cup L$ (union)
  - $(K, L) \mapsto KL$ (concatenation)
  - $L \mapsto L^*$ (Kleene star)
- The syntactic congruence of the language $L \subseteq A^*$ is the binary relation σ_L on A^* defined by:

$$u \, \sigma_L \, v \quad \text{if } \forall \, x, y \in A^* \; (xuy \in L \Leftrightarrow xvy \in L).$$

- The syntactic monoid $M(L)$ of the language $L \subseteq A^*$ is the quotient monoid A^*/σ_L.

Theorem 1.1

The following conditions are equivalent for a language L over a finite alphabet A:

(1) *L is regular;*

(2) *L is recognized by some finite automaton;*

(3) *L is recognized by some finite complete deterministic automaton;*

(4) *the syntactic monoid A^*/σ_L on A^* is finite;*

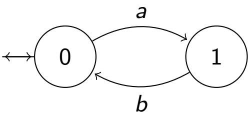
(5) *L is recognized by some homomorphism $\varphi : A^* \to M$ into a finite monoid, in the sense that $L = \varphi^{-1}\varphi L$.*

Corollary 1.2

The set $\mathrm{Reg}(A^*)$ of all regular languages over the alphabet A is a Boolean subalgebra of the Boolean algebra of all subsets of A^*.

Example: (restricted) Dyck languages

- Regular expression: $L_1 = (ab)^*$
- Minimal (incomplete) automaton:



- Transition monoid ($M(L_1)$):

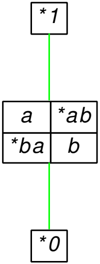
	0	1
a	1	-
b	-	0
ab	0	-
ba	-	1
0	-	-

	a	b	ab	ba	0
a	0	ab	0	a	0
b	ba	0	b	0	0
ab	a	0	ab	0	0
ba	0	b	0	ba	0
0	0	0	0	0	0

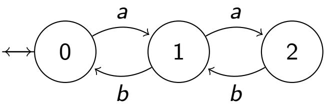
- Presentation: $\langle a, b;\ aba = a,\ bab = b,\ a^2 = b^2 = 0\rangle$.

One may then compute Green's relations, which are summarized in the following eggbox picture:

- same row: elements generate the same right ideal ($\mathcal{R}$)
- same column: elements generate the same left ideal ($\mathcal{L}$)
- elements above are factors of elements below ($\geq_{\mathcal{J}}$)
- $^*e$ marks an idempotent ($e^2 = e$)
- the "eggboxes" are the $\mathcal{J}$-classes ($\mathcal{J} = \geq_{\mathcal{J}} \cap \leq_{\mathcal{J}}$)
- $\mathcal{D} = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}$
- in a finite monoid, $\mathcal{D} = \mathcal{J}$



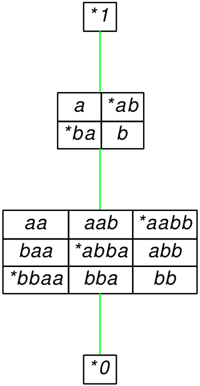
- Regular expression: $L_2 = (a(ab)^* b)^*$
- Minimal (incomplete) automaton:



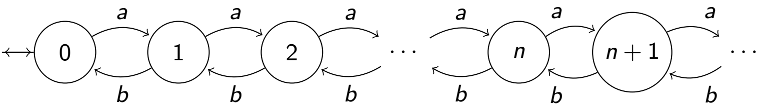
- Presentation of syntactic monoid $M(L_2)$:

$$\langle a, b;\ aba = a, bab = b, a^2b^2a^2 = a^2, b^2a^2b^2 = b^2, ab^2a = ba^2b, a^3 = b^3 = 0\rangle$$

- Eggbox picture:



- Dyck language: $L_\infty = \bigcup_{n \geq 0} L_n$, where $L_0 = \{1\}$, $L_{n+1} = (aL_nb)^*$.
- Recognition by infinite automaton:

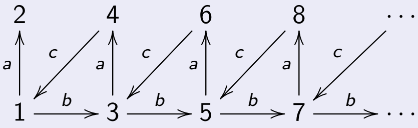


- Syntactic monoid: $M(L_\infty) = \langle a, b; ab = 1 \rangle$.
- Eggbox picture:

^{*1}	a	a^2	$\cdots$	a^n	a^{n+1}	$\cdots$
b	$^*ba$	ba^2	$\cdots$	ba^n	ba^{n+1}	$\cdots$
b^2	b^2a	$^*b^2a^2$	$\cdots$	b^2a^n	b^2a^{n+1}	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	
b^n	b^na	b^na^2	$\cdots$	$^*b^na^n$	b^na^{n+1}	$\cdots$
b^{n+1}	$b^{n+1}a$	$b^{n+1}a^2$	$\cdots$	$b^{n+1}a^n$	$^*b^{n+1}a^{n+1}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\ddots$

Exercise 1.3

Consider the transition semigroup S of the following infinite automaton:



1. Note that, in S, aca is a factor of a but a is not regular.
2. Verify that S admits the following presentation:

$$\langle a, b, c;\ baca = a, bacb = b^2ac = b, cbac = c,$$
$$a^2 = ab = bc = c^2 = 0\rangle.$$

3. Show that S has two $\mathcal{J}$-classes, one of which is reduced to zero.
4. Show that the non-trivial $\mathcal{J}$-class of S consists of two infinite $\mathcal{D}$-classes, one of which is regular and a bicyclic monoid, while the other is not regular and has only one $\mathcal{L}$-class. All $\mathcal{H}$-classes of S are trivial.

Outline

- A variety of languages is a correspondence $\mathcal{V}$ associating with each finitely generated free monoid A^* a set $\mathcal{V}(A^*)$ of languages over the finite alphabet A such that the following conditions hold:
  1. $\mathcal{V}(A^*)$ is a Boolean subalgebra of $\mathrm{Reg}(A^*)$;
  2. if $L \in \mathcal{V}(A^*)$ and $a \in A$, then the following languages also belong to $\mathcal{V}(A^*)$:

$$a^{-1}L = \{w \in A^* : aw \in L\}$$
$$La^{-1} = \{w \in A^* : wa \in L\};$$

  3. if $\varphi : A^* \to B^*$ is a homomorphism and $L \in \mathcal{V}(B^*)$, then $\varphi^{-1}(L) \in \mathcal{V}(A^*)$.
- A pseudovariety of monoids is a nonempty class $\mathbf{V}$ of finite monoids which is closed under taking homomorphic images, submonoids, and finite direct products.

Theorem 2.1 (Eilenberg [Eil76])

The complete lattices of varieties of languages and of pseudovarieties of monoids are isomorphic. More precisely, the following correspondences are mutually inverse isomorphisms between the two lattices:

- *to a variety $\mathcal{V}$ of languages, associate the pseudovariety $\mathbf{V}$ generated by all syntactic monoids $M(L)$ with $L \in \mathcal{V}(A^*)$ for some finite alphabet A;*
- *to a pseudovariety $\mathbf{V}$, associate the variety of languages $\mathcal{V}$ such that, for each finite alphabet A, $\mathcal{V}(A^*)$ consists of the languages $L \subseteq A^*$ such that $M(L) \in \mathbf{V}$.*

- Thus, problems about varieties of languages admit a translation into problems about pseudovarieties of monoids.
- For instance, to determine if a language $L \subseteq A^*$ belongs to smallest variety of languages containing two given varieties of languages $\mathcal{V}$ and $\mathcal{W}$ is equivalent to determine if $M(L)$ belongs to the pseudovariety join $\mathbf{V} \vee \mathbf{W}$.
- Typically, we are given a recursively enumerable set $\mathcal{R}$ of finite monoids and we want to determine an algorithm to decide whether a given finite monoid M belongs to the pseudovariety $\mathbf{V}(\mathcal{R})$ generated by $\mathcal{R}$.

Mutatis mutandis, we have

- languages $L \subseteq A^+$ without the empty word 1;
- syntactic congruence σ_L of L over A^+:

$$u \; \sigma_L \; v \quad \text{if } \forall\, x, y \in A^* \; (xuy \in L \Leftrightarrow xvy \in L).$$

- syntactic semigroup A^+/σ_L;
- varieties of languages without the empty word;
- pseudovarieties of semigroups;
- Eilenberg's correspondence in this setting.

Examples of pseudovarieties:

- **S**: all finite semigroups
- **I**: all singleton (trivial) semigroups
- **G**: all finite groups
- $\mathbf{G}_p$: all finite p-groups
- **A**: all finite aperiodic semigroups
- **Com**: all finite commutative semigroups
- **J**: all finite $\mathcal{J}$-trivial semigroups
- **R**: all finite $\mathcal{R}$-trivial semigroups
- **L**: all finite $\mathcal{L}$-trivial semigroups
- **Sl**: all finite semilattices
- **RZ**: all finite right-zero semigroups
- **B**: all finite bands
- **N**: all finite nilpotent semigroups
- **K**: all finite semigroups in which idempotents are left zeros
- **D**: all finite semigroups in which idempotents are right zeros

Important examples of instances of Eilenberg's correspondence

- A language $L \subseteq A^+$ is said to be star free if it admits an expression in terms of the languages $\{a\}$ $(a \in A)$ using only the operations: $\_ \cup \_$, $A^+ \setminus \_$, and concatenation.

Theorem 2.2 ([Sch65])

A language over a finite alphabet is star free if and only if its syntactic semigroup is finite and aperiodic.

- A language $L \subseteq A^*$ is piecewise testable if it is a Boolean combination of languages of the form $A^* a_1 A^* a_2 A^* \cdots a_n A^*$, with the $a_i \in A$.

Theorem 2.3 ([Sim75])

A language over a finite alphabet is piecewise testable if and only if its syntactic semigroup is finite and $\mathcal{J}$-trivial.

- A language $L \subseteq A^*$ is locally testable if it is a Boolean combination of languages of the forms A^*u, A^*vA^*, and wA^*, where $u, v, w \in A^+$.

Theorem 2.4 ([BS73, MP71])

A language L over a finite alphabet is locally testable if and only if its syntactic semigroup S is finite and a local semilattice (i.e., eSe is a semilattice for every idempotent $e \in S$).

Outline

Definition 3.1

We say that a pseudovariety $\mathbf{V}$ is decidable if there is an algorithm which, given a finite semigroup S as input, produces as output, in finite time, YES or NO according to whether or not $S \in \mathbf{V}$.

The semigroup S may be given in various ways:

- extensively, meaning the complete list of its elements together with its multiplication table;
- as the transformation semigroup on a finite set Q generated by a finite set A of transformations of Q ;
  $\rightarrow$ transition semigroup of a finite automaton (Q, A, δ, I, F);
- by means of a presentation.
- Different ways of describing S may lead to different complexity results, when such an algorithm exists.

Of course, not all pseudovarieties are decidable.

For instance, if P is a non-recursive set of primes, then the pseudovariety $\mathbf{Ab}_P$, generated by all groups $\mathbb{Z}/p\mathbb{Z}$ with $p \in P$, contains a group $\mathbb{Z}/q\mathbb{Z}$ of prime order q if and only if $q \in P$.

Since there are non-recursive sets of primes P, there are pseudovarieties of the form $\mathbf{Ab}_P$ which are not decidable.

Question 3.2 (Very imprecise!!)

Are all "natural" pseudovarieties decidable?

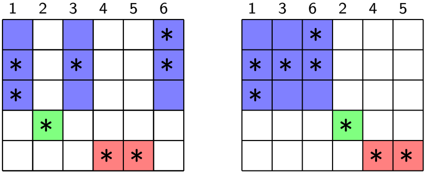
There are many ways to construct new pseudovarieties from known ones, that is by applying operators to pseudovarieties.
We proceed to introduce some natural operators.

Definition 3.3

Given a pseudovariety $\mathbf{V}$, consider the classes of all finite semigroups S such that, respectively:

- **LV**: $eSe \in \mathbf{V}$ for every idempotent $e \in S$;
- **EV**: $\langle E(S) \rangle \in \mathbf{V}$, where $\langle E(S) \rangle$ is the subsemigroup generated by the set $E(S)$ of all idempotents of S;
- **DV**: the regular $\mathcal{J}$-classes of S (are subsemigroups which) belong to $\mathbf{V}$;
- $\overline{\mathbf{V}}$: the subgroups of S belong to $\mathbf{V}$;

- Let S be a finite semigroup and let D be one of its regular $\mathcal{D}$-classes.
- Let $\sim$ be the equivalence relation on the set of group elements of D generated by the identification of elements which are either $\mathcal{R}$ or $\mathcal{L}$-equivalent.
- A block of D is the Rees quotient of the subsemigroup of S generated by a $\sim$-class modulo the ideal consisting of the elements which do not lie in D.



- The blocks of S are the blocks of its regular $\mathcal{D}$-classes.

Definition 3.4

For a pseudovariety **V**, let **BV** be the class of all finite semigroups whose blocks lie in **V**.

Proposition 3.5

For a pseudovariety $\mathbf{V}$*, the classes* $\mathbf{BV}$, $\mathbf{DV}$, $\mathbf{EV}$, $\mathbf{LV}$, $\overline{\mathbf{V}}$ *are pseudovarieties.*
Moreover, if $\mathbf{V}$ *is decidable then so are those pseudovarieties.*

Proof.

We consider only the case of $\mathbf{LV}$, leaving all other cases as exercises.

- If $\varphi : S \to T$ is an onto homomorphism, with $S \in \mathbf{S}$, and $f \in E(T)$, then $\exists e \in \varphi^{-1}(f) \cap E(S)$ and $\varphi|_{eSe} : eSe \to fTf$ is an onto homomorphism
  $\therefore$ $\mathbf{LV}$ is closed under taking homomorphic images.
- If $S \leq T$ and $e \in E(S)$, then $eSe \leq eTe$
  $\therefore$ $\mathbf{LV}$ is closed under taking subsemigroups.
- If S, T are semigroups, $e \in E(S)$, and $f \in E(T)$, then $(e, f)(S \times T)(e, f) \simeq eSe \times fTf$
  $\therefore$ $\mathbf{LV}$ is closed under taking finite direct products.

Given a finite semigroup, one can compute its set of idempotents $E(S)$ and, for each $e \in E(S)$, the monoid eSe.
Provided $\mathbf{V}$ is decidable, one can then effectively check whether $eSe \in \mathbf{V}$.
Hence one can effectively check whether $S \in \mathbf{LV}$. □

But, the most interesting operators are defined not in structural terms but rather by describing generators: the resulting pseudovariety is given as the smallest pseudovariety containing certain semigroups which are constructed from those in the argument pseudovarieties.

Definition 3.6

We say that a semigroup S divides a semigroup T, or that S is a divisor of T, and we write $S \prec T$, if S is a homomorphic image of a subsemigroup of T.

Proposition 3.7

Let $\mathcal{C}$ be a class of finite semigroups. Then the smallest pseudovariety $\mathbf{V}(\mathcal{C})$ containing $\mathcal{C}$ consists of all divisors of products of the form $S_1 \times \cdots \times S_n$ with $S_1, \ldots, S_n \in \mathcal{C}$.
In particular, if $\mathcal{C}$ is closed under finite direct product, then $\mathbf{V}(\mathcal{C})$ consists of all divisors of elements of $\mathcal{C}$.

Let S and T be semigroups and let $\varphi : T^1 \to \operatorname{End} S$ be a homomorphism of monoids, with endomorphisms acting on the left. For $s \in S$ and $t \in T^1$, let ${}^t s = \varphi(t)(s)$.
The semidirect product $S *_\varphi T$ is the set $S \times T$ under the multiplication

$$(s_1, t_1) \cdot (s_2, t_2) = (s_1 \, {}^{t_1} s_2, t_1 t_2).$$

Definition 3.8

The semidirect product $\mathbf{V} * \mathbf{W}$ of the pseudovarieties $\mathbf{V}$ and $\mathbf{W}$ is the smallest pseudovariety containing all semidirect products $S * T$ with $S \in \mathbf{V}$ and $T \in \mathbf{W}$.

Proposition 3.9

The pseudovariety $\mathbf{V} * \mathbf{W}$ consists of all divisors of semidirect products of the form $S * T$ with $S \in \mathbf{V}$ and $T \in \mathbf{W}$.

Proposition 3.10

The semidirect product of pseudovarieties is associative.

Definition 3.11

The Mal'cev product $\mathbf{V} ⓜ \mathbf{W}$ of two pseudovarieties $\mathbf{V}$ and $\mathbf{W}$ is the smallest pseudovariety containing all finite semigroups S for which there exists a homomorphism $\varphi : S \to T$ such that $T \in \mathbf{W}$ and $\varphi^{-1}(e) \in \mathbf{V}$ for all $e \in E(T)$.

Given two semigroups S and T, a relational morphism $S \to T$ is a relation $\mu : S \to T$ with domain S such that μ is a subsemigroup of $S \times T$.

Proposition 3.12

The pseudovariety $\mathbf{V} ⓜ \mathbf{W}$ consists of all finite semigroups S such that there is a relational morphism $\mu : S \to T$ such that $T \in \mathbf{W}$ and $\mu^{-1}(e) \in \mathbf{V}$ for all $e \in E(T)$.

For a semigroup S, denote by $\mathcal{P}(S)$ the semigroup of subsets of S under the product operation

$$X \cdot Y = \{xy : \ x \in X, \ y \in Y\}.$$

Note that the empty set $\emptyset$ is a zero and $\mathcal{P}'(S) = \mathcal{P}(S) \setminus \{\emptyset\}$ is a subsemigroup.

Definition 3.13

For a pseudovariety $\mathbf{V}$, denote by

- **PV**: the pseudovariety generated by all semigroups of the form $\mathcal{P}(S)$, with $S \in \mathbf{V}$;
- **P′V**: the pseudovariety generated by all semigroups of the form $\mathcal{P}'(S)$, with $S \in \mathbf{V}$.

Proposition 3.14

The pseudovariety $\mathbf{PV}$ consists of all divisors of semigroups of the form $\mathcal{P}(S)$ with $S \in \mathbf{V}$.
Similar statement for $\mathbf{P}'$.

Some examples of results on finite semigroups formulated in terms of these operators:

1. $\mathbf{J} = \mathbf{N} ⓜ \mathbf{SI}$
2. $\mathbf{DA} = \mathbf{LI} ⓜ \mathbf{SI}$, $\mathbf{DS} = \mathbf{LG} ⓜ \mathbf{SI}$
3. $\mathbf{R} = \mathbf{SI} * \mathbf{J}$ [Sti73]
4. $\mathbf{G} \vee \mathbf{Com} = \mathbf{ZE}$ (the pseudovariety of all finite semigroups in which idempotents are central) [Alm95]
5. $\mathbf{ESI} = \mathbf{SI} * \mathbf{G} = \mathbf{SI} ⓜ \mathbf{G} = \mathbf{Inv}$ (the pseudovariety generated by all finite inverse semigroups) [MP87, Ash87, Pin95],
   $\mathbf{ER} = \mathbf{R} * \mathbf{G}$ [Eil76], $\mathbf{EDS} = \mathbf{DS} * \mathbf{G}$ [AE03]
6. $\mathbf{PG} = \mathbf{J} * \mathbf{G} = \mathbf{J} ⓜ \mathbf{G} = \mathbf{EJ} = \mathbf{BG}$
   [MP84, HR91, Ash91, HMPR91, Pin95],
   $\mathbf{PJ} = \mathbf{PV}(Y)$ [PS85, Alm95] where $Y = \mathrm{Synt}(a^*bc^*)$
7. $\mathbf{S} = \bigcup_{n \geq 0} (\mathbf{A} * \mathbf{G})^n * \mathbf{A}$ [KR65]

$$\mathbf{S} = \bigcup_{n \geq 0} (\mathbf{A} * \mathbf{G})^n * \mathbf{A}$$

The (Krohn-Rhodes) hierarchy $\left((\mathbf{A} * \mathbf{G})^n * \mathbf{A}\right)_{n \geq 0}$ is strict.

The smallest n such that a given finite semigroup S belongs to $(\mathbf{A} * \mathbf{G})^n * \mathbf{A}$ is called the complexity of S, denoted $c(S)$.

Let T_n denote the full transformation semigroup of an n-element set It is known that $c(T_n) = n - 1$ [Eil76] and so certainly $c(S) \leq |S|$ (since $S \hookrightarrow T_{S^1}$).

NOTE 3.15

To know an algorithm to compute the complexity function is equivalent to know algorithms to decide the membership problem for each pseudovariety in the Krohn-Rhodes hierarchy.

This brings us to the following basic question:

QUESTION 3.16

For the operators which were defined above in terms of generators, do they preserve decidability?

THEOREM 3.17 (ALBERT, BALDINGER & RHODES'1992 [ABR92])

There exists a finite set Σ *of identities such that* $\mathbf{Com} \vee [\![\Sigma]\!]$ *is undecidable.*

Let $C_{2,1} = \langle a; a^2 = 0 \rangle^1$.

THEOREM 3.18 (AUINGER & STEINBERG'2003 [AS03])

There exists a decidable pseudovariety of groups $\mathbf{U}$ *such that the following pseudovarieties are all undecidable:*
$\mathbf{SI} * \mathbf{U}$ $(= \mathbf{SI} ⓜ \mathbf{U})$, $\mathbf{V}(C_{2,1}) \vee \mathbf{U}$, $\mathbf{PU}$ $(= \mathbf{P}'\mathbf{U})$.

The pseudovariety $\mathbf{U}$ is defined to be

$$\mathbf{U} = \bigvee_{p \in A} \mathbf{G}_p * (\mathbf{G}_{f(p)} \cap \mathbf{Com}) \vee \bigvee_{p \in D} (\mathbf{G}_p \cap \mathbf{Com})$$

where:

- A and B constitute a computable partition of the set of primes into two infinite sets;
- $f : A \to B$ is an injective recursive function whose range $C = f(A)$ is recursively enumerable but not recursive;
- $D = B \setminus C$ is not recursively enumerable.

Exercise 3.19

Show that $\mathbf{U}$ is decidable.

Outline

- Let $\mathbf{V}$ be a pseudovariety of semigroups.
- For two words $u, v \in A^+$, and $T \in \mathbf{V}$, let

$$T \models u = v \text{ if, for every homomorphism } \varphi : A^+ \to T,\ \varphi(u) = \varphi(v),$$
$$r_{\mathbf{V}}(u, v) = \min\{|S| : S \in \mathbf{V} \text{ and } S \not\models u = v\},$$
$$d_{\mathbf{V}}(u, v) = 2^{-r_{\mathbf{V}}(u,v)}$$

where we take $\min \emptyset = \infty$ and $2^{-\infty} = 0$.

Note 4.1

The following hold for $u, v, w, t \in A^+$ and a positive integer n:

(1) $r_{\mathbf{V}}(u, v) \geq n$ if and only if, for every $S \in \mathbf{V}$ with $|S| < n$, $S \models u = v$;

(2) $d_{\mathbf{V}}(u, v) \leq 2^{-n}$ if and only if, for every $S \in \mathbf{V}$ with $|S| < n$, $S \models u = v$;

(3) $d_{\mathbf{V}}(u, v) = 0$ if and only if, for every $S \in \mathbf{V}$, $S \models u = v$;

(4) $\min\{r_{\mathbf{V}}(u, v), r_{\mathbf{V}}(v, w)\} \leq r_{\mathbf{V}}(u, w)$;

(5) $\min\{r_{\mathbf{V}}(u, v), r_{\mathbf{V}}(w, z)\} \leq r_{\mathbf{V}}(uw, vz)$.

Definition 4.2

A function $d : X \times X \to \mathbb{R}_{\geq 0}$ is said to be a pseudo-ultrametric on the set X if the following properties hold for all $u, v, w \in X$:

1. $d(u, u) = 0$;
2. $d(u, v) = d(v, u)$;
3. $d(u, w) \leq \max\{d(u, v), d(v, w)\}$.

We then also say that X is a pseudo-ultrametric space.
If instead of Condition 3, the following weaker condition holds

4. $d(u, w) \leq d(u, v) + d(v, w)$ (triangle inequality).

then d is said to be a pseudo-metric on X, and X is said to be a pseudo-metric space. If the following condition holds

5. $d(u, v) = 0$ if and only if $u = v$,

then we drop the prefix "pseudo".

- A function $f : X \to Y$ between two pseudo-metric spaces is said to be uniformly continuous if the following condition holds:

$$\forall \epsilon > 0 \; \exists \delta > 0 \; \forall x_1, x_2 \in X \; \big(d(x_1, x_2) < \delta \Rightarrow d(f(u), f(v)) < \epsilon\big).$$

Proposition 4.3

1. *The function $d_{\mathbf{V}}$ is a pseudo-ultrametric on A^+.*
2. *The multiplication is contractive:*

$$d_{\mathbf{V}}(u_1 u_2, v_1 v_2) \leq \max\{d_{\mathbf{V}}(u_1, v_1), d_{\mathbf{V}}(u_2, v_2)\}.$$

In particular, the multiplication on A^+ is uniformly continuous.

- For a (pseudo-ultra)metric d, $u \in X$, and a positive real number ϵ, consider the open ball

$$B_\epsilon(u) = \{v \in X : d(u, v) < \epsilon\}.$$

The point u is the center and ϵ is the radius of the ball.

- A metric space that can be covered by a finite number of balls of any given positive radius is said to be totally bounded.

Proposition 4.4

The metric space $(A^+, d_{\mathbf{V}})$ is totally bounded.

Proof.

Let n be a positive integer such that $2^{-n} < \epsilon$. Note that, up to isomorphism, there are only finitely many semigroups of cardinality at most n in $\mathbf{V}$. For such a semigroup S_i consider all possible homomorphisms $\varphi_{i,j} : A^+ \to S_i$, let $S = \prod_{i,j} S_i$ and

$$\begin{aligned} \varphi : A^+ &\to S \\ u &\mapsto (\varphi_{i,j}(u))_{i,j}. \end{aligned}$$

Then $S \in \mathbf{V}$ and $d_{\mathbf{V}}(u, v) < 2^{-n}$ if and only if $\varphi(u) = \varphi(v)$.

For each $s \in S$, choose $u_s \in A^+$ such that $\varphi(u_s) = s$.

For $v \in A^+$ and $s = \varphi(v)$, we have $\varphi(v) = \varphi(u_s)$, and so $v \in B_\epsilon(u_s)$.

We have thus shown that $A^+ = \bigcup_{s \in S} B_\epsilon(u_s)$. □

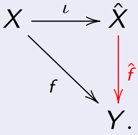
- A sequence $(u_n)_n$ in a (pseudo-ultra)metric space X is said to be a Cauchy sequence if

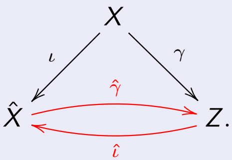
$$\forall \epsilon > 0 \, \exists N \, \big(m, n \geq N \Rightarrow d(u_m, u_n) < \epsilon\big).$$

- Note that every convergent sequence is a Cauchy sequence.
- The space X is complete if every Cauchy sequence in X converges in X.

Theorem 4.5

Let X be a pseudo-(ultra)metric space. Then there exists a complete metric space $\hat{X}$ and a uniformly continuous function $\iota : X \to \hat{X}$ with the following universal property: for every uniformly continuous function $f : X \to Y$ into a complete metric space Y, there exists a unique uniformly continuous function $\hat{f} : \hat{X} \to Y$ such that $\hat{f} \circ \iota = f$.





In particular, if $\gamma : X \to Z$ is another uniformly continuous function into another complete metric space with the above universal property then the induced unique uniformly continuous mappings $\hat{\iota} : \hat{X} \to Z$ and $\hat{\gamma} : Z \to \hat{X}$ are mutually inverse.

- The "unique" space $\hat{X}$ of Theorem 4.5 is called the Hausdorff completion of X.

- It may be constructed in the same way that the real numbers are obtained by completion of the rational numbers. Here is a sketch:
  (A) consider the set $C \subseteq X^{\mathbb{N}}$ of all Cauchy sequences of elements of X;
  (B) note that, for $s = (u_n)_n$ and $t(v_n)_n$ in C, the sequence of real numbers $\big(d(u_n, v_n)\big)_n$ is a Cauchy sequence and, therefore, it converges; its limit is denoted $d(s, t)$;

$$\begin{aligned}
&|d(u_n, v_n) - d(u_m, v_m)| \\
&\quad \leq |d(u_n, v_n) - d(u_n, v_m)| + |d(u_n, v_m) - d(u_m, v_m)| \\
&\quad \leq d(u_n, u_m) + d(v_n, v_m)
\end{aligned}$$

  (C) Step (B) defines a pseudo-(ultra)metric on C;
  (D) for $s = (u_n)_n$ and $t(v_n)_n$ in C, let $s \sim t \quad$ if $d(s, t) = 0$; this is an equivalence relation on C; the class of s is denoted $s/\sim$;
  (E) let $\hat{X} = C/\sim$ and put $d(s/\sim, t/\sim) = d(s, t)$, which can be easily checked to be defined;
  (F) finally, let $\iota : X \to \hat{X}$ map each $u \in X$ to the $\sim$-class of the constant sequence $(u)_n$, and check that this mapping is uniformly continuous and has the appropriate universal property.

- Note that $\iota(X)$ is dense in $\hat{X}$.
- In particular, we may consider the Hausdorff completion of the pseudo-ultrametric space $(A^+, d_{\mathbf{V}})$, which is denoted $\overline{\Omega}_A \mathbf{V}$.
- Since the multiplication of A^+ is uniformly continuous with respect to $d_{\mathbf{V}}$, it induces a uniformly continuous multiplication in $\overline{\Omega}_A \mathbf{V}$:

$$\begin{array}{ccc} A^+ \times A^+ & \xrightarrow[\text{(mult.)}]{\mu} & A^+ \\ \downarrow{\scriptstyle \iota \times \iota} & & \downarrow{\scriptstyle \iota} \\ \overline{\Omega}_A \mathbf{V} \times \overline{\Omega}_A \mathbf{V} & \xrightarrow{\hat{\mu}} & \overline{\Omega}_A \mathbf{V} \end{array}$$

- We endow each finite semigroup S with the discrete metric:

$$d(s,t) = \begin{cases} 0 & \text{if } s = t \\ 1 & \text{otherwise} \end{cases}$$

- Since $\iota(A^+)$ is dense in $\overline{\Omega}_A \mathbf{V}$, multiplication in $\overline{\Omega}_A \mathbf{V}$ is associative, and thus $\overline{\Omega}_A \mathbf{V}$ is naturally a semigroup.
- From hereon, we write d for $d_{\mathbf{V}}$. The context should leave clear which pseudovariety is involved.

- Note that, for $S \in \mathbf{V}$, every homomorphism $\varphi : A^+ \to S$ is uniformly continuous with respect to d.

$$d(u,v) < 2^{-|S|} \Rightarrow d(\varphi(u), \varphi(v)) = 0.$$

Thus, φ induces a unique uniformly continuous mapping $\hat{\varphi} : \overline{\Omega}_A\mathbf{V} \to S$ such that the following diagram commutes:

$$\begin{array}{ccc} A^+ & \xrightarrow{\ \iota\ } & \overline{\Omega}_A\mathbf{V} \\ & {\scriptstyle\varphi}\searrow & \downarrow{\scriptstyle\hat{\varphi}} \\ & & S. \end{array}$$

One can easily check that $\hat{\varphi}$ is a homomorphism:

$$\begin{aligned} \hat{\varphi}(uv) &= \lim \varphi(\iota(u_n v_n)) = \lim \varphi(\iota(u_n))\varphi(\iota(v_n)) \\ &= \lim \varphi(\iota(u_n)) \cdot \lim \varphi(\iota(v_n)) = \hat{\varphi}(u)\hat{\varphi}(v). \end{aligned}$$

- Given $u, v \in \overline{\Omega}_A \mathbf{V}$ and $S \in \mathbf{V}$, we write $S \models u = v$ if, for every homomorphism $\varphi : A^+ \to S$ (which is determined by $\varphi|_A$), the equality $\hat{\varphi}(u) = \hat{\varphi}(v)$ holds.
  We call the formal equality $u = v$ a **V**-pseudoidentity.
- Note that, if $u = \lim u_n$, $v = \lim v_n$, and $S \in \mathbf{V}$, then $S \models u = v$ if and only if $S \models u_n = v_n$ for all sufficiently large n.
- Given distinct elements $u, v \in \overline{\Omega}_A \mathbf{V}$, there exists a positive integer m such that $d(u, v) \geq 2^{-m}$.

  Consider sequences of words $(u_n)_n$ and $(v_n)_n$ such that $u = \lim \iota(u_n)$ and $v = \lim \iota(v_n)$.

  Then, for sufficiently large n, $d(u, \iota(u_n)) < 2^{-m}$ and $d(v, \iota(v_n)) < 2^{-m}$.

  Hence $d(u_n, v_n) = d(\iota(u_n), \iota(v_n)) \geq 2^{-m}$ for all sufficiently large n.

  It follows that every $S \in \mathbf{V}$ with $|S| < m$ fails the identity $u_n = v_n$ and, therefore, also the pseudoidentity $u = v$.

Proposition 4.6

For $u, v \in \overline{\Omega}_A \mathbf{V}$, *we have* $d(u,v) = 2^{-r(u,v)}$, *where*

$$r(u,v) = \min\{|S| : S \in \mathbf{V} \text{ and } S \not\models u = v\}.$$

Proof.

We have already shown that $d(u,v) \geq 2^{-m}$ implies $r(u,v) \leq m$. The converse, as well as how the equivalence gives the proposition are left as an exercise. □

- Recall that a metric space is compact if every sequence admits some convergent subsequence. Equivalently, every covering by open subsets contains a finite covering.

Proposition 4.7

1. *If X is a totally bounded pseudo-metric space, then $\hat{X}$ is also totally bounded.*
2. *If X is a totally bounded complete metric space, then X is compact.*

PROOF.

1. Given $\epsilon > 0$, let $u_1, \ldots, u_m \in X$ be such that $X = \bigcup_{i=1}^m B_{\epsilon/2}(u_i)$. Then $\hat{X} = \bigcup_{i=1}^m B_\epsilon(\iota(u_i))$ since every element of $\hat{X}$ is at distance at most $\epsilon/2$ of some element of $\iota(X)$.

2. For each positive integer m, let F_m be a finite subset of X such that $X = \bigcup_{x \in F_m} B_{2^{-m}}(x)$ and consider an arbitrary sequence $(u_n)_n$ in X.

For infinitely many indices n, the u_n belong to the same $B_{2^{-1}}(x_1)$. Let k_1 be the first of these indices. Similarly, among the remaining such indices, there are infinitely many n such that the u_n belong to the same $B_{2^{-2}}(x_2)$. We let k_2 be the first of them. And so on.

We thus construct a subsequence $(u_{k_n})_n$ with the property that $d(u_{k_m}, u_{k_n}) \leq 2^{-\min\{m,n\}+1}$,

if $p = \min\{m, n\}$, *then* $u_{k_m}, u_{k_n} \in B_{2^{-p}}(x_p)$, *which yields*

$$d(u_{k_m}, u_{k_n}) \leq d(u_{k_m}, x_p) + d(x_p, u_{k_n}) \leq 2^{-p} + 2^{-p}$$

whence a Cauchy sequence and, therefore, convergent. □

Outline

- By a pro-**V** semigroup we mean a semigroup S endowed with a metric such that the following properties hold:
  1. S is compact;
  2. the multiplication is uniformly continuous (metric semigroup);
  3. for every pair u, v of distinct elements of S, there is a uniform continuous homomorphism $\varphi : S \to T$ into a semigroup from **V** such that $\varphi(u) \neq \varphi(v)$ (S residually in **V**).
- By a profinite semigroup we mean a pro-**S** semigroup.

Proposition 5.1

Let S be a pro-$\mathbf{V}$ semigroup. Then there is a sequence $(S_n)_{n\in\mathbb{N}}$ of semigroups from $\mathbf{V}$ and an injective homomorphism $\varphi : S \to \prod_{n\in\mathbb{N}} S_n$ such that, for each component projection $\pi_m : \prod_{n\in\mathbb{N}} S_n \to S_m$, the homomorphism $\pi_m \circ \varphi$ is uniformly continuous.

We may define in $\prod_{n\in\mathbb{N}} S_n$ a metric structure by letting

$$d(u,v) = \sum_{n\in\mathbb{N}} 2^{-n} d_n(\pi_n(u), \pi_n(v))$$

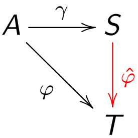
where d_n is the discrete metric on S_n. Then φ is uniformly continuous. In particular, the image T of φ is closed in $\prod_{n\in\mathbb{N}} S_n$, being a compact subset.

- Note that the sequence $(S_n)_{n\in\mathbb{N}}$ may be chosen so that there is a finite bound on the number of generators of the S_n if and only if S is finitely generated in the sense that there is a finite subset which generates a dense subsemigroup.
- On the other hand, if there is no such bound, one can show that S cannot have a countable dense subset, while it is easy to see that a compact metric space always admits a countable dense subset.

Proposition 5.2

Every pro-$\mathbf{V}$ metric semigroup is finitely generated.

- For a finite set A, we say that the pro-**V** semigroup S is freely generated by A if there is a mapping $\gamma : A \to S$ such that $\gamma(A)$ generates a dense subsemigroup of S and the following universal property is satisfied, where $\varphi : A \to T$ is an aribtrary mapping into a pro-**V** semigroup T, and $\hat{\varphi}$ is a unique continuous homomorphism:

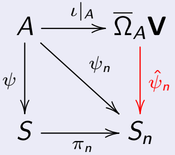


Theorem 5.3

For a pseudovariety of semigroups **V** *and a finite set A, the metric semigroup $\overline{\Omega}_A\mathbf{V}$ is a pro-***V** *semigroup freely generated by A via the mapping $\iota|_A$.*

Proof.

Let S be a pro-$\mathbf{V}$ semigroup and let $(S_n)_{n\in\mathbb{N}}$ be a countable family of semigroups from $\mathbf{V}$ as given by Proposition 5.1, so that there is an embedding $\varphi : S \to \prod_{n\in\mathbb{N}} S_n$ with each composite function $\pi_n \circ \varphi : S \to S_n$ uniformly continuous.
Given a mapping $\psi : A \to S$, let $\psi_n = \pi_n \circ \psi$.



The family $(\hat{\psi}_n)_{n\in\mathbb{N}}$ induces a homomorphism $\hat{\psi} : \overline{\Omega}_A\mathbf{V} \to \prod_{n\in\mathbb{N}} S_n$. Its image lies in the closed subsemigroup T, whence it lifts to the required continuous homomorphism $\overline{\Omega}_A\mathbf{V} \to S$. It is uniformly continuous because every continuous mapping from a compact metric space into another metric space is uniformly continuous. □

- A subset of a metric space is said to be clopen if it is both closed and open.
- A metric space is said to be zero-dimensional if every open set is a union of clopen subsets.

Proposition 5.4

*Every pro-**V** semigroup is zero-dimensional.*

Proof.

Let u be an element of the pro-**V** semigroup S. It suffices to show that the open ball $B_\epsilon(u)$ contains some clopen set which contains u.
For each $v \in S \setminus B_\epsilon(u)$, let $\varphi_v : S \to T_v$ be a uniformly continuous homomorphism into a semigroup from **V** such that $\varphi_v(u) \neq \varphi_v(v)$. Then $K_v = \varphi_v^{-1}\varphi_v(v)$ is a clopen set which contains v but not u. In particular, the K_v form a clopen covering of the closed set $S \setminus B_\epsilon(u)$, from which a finite covering $\mathcal{F}$ can be extracted.
The union of the clopen sets in $\mathcal{F}$ is itself a clopen set K. Note that $S \setminus K$ is also clopen, contains u, and is contained in $B_\epsilon(u)$. □

- For a mapping $\varphi : S \to T$, let $\ker \varphi = \{(u, v) : \varphi(u) = \varphi(v)\}$ be the kernel of φ.

Theorem 5.5

An A-generated profinite semigroup S is a continuous homomorphic image of $\overline{\Omega}_A \mathbf{V}$ if and only if it is a pro-$\mathbf{V}$ semigroup.

Corollary 5.6

Let S be a pro-$\mathbf{V}$ semigroup and suppose that $\varphi : S \to T$ is a continuous homomorphism onto a profinite semigroup. Then $T \in \mathbf{V}$. □

Proof of Theorem 5.5.

($\Leftarrow$) Apply Theorem 5.3.

($\Rightarrow$) Let $\varphi : \overline{\Omega}_A\mathbf{V} \to S$ be an onto continuous homomorphism. We need to show that S is residually in $\mathbf{V}$.
Given distinct points $s_1, s_2 \in S$, since S is residually in $\mathbf{S}$, there is an onto uniformly continuous homomorphism $\psi : S \to T$ such that $T \in \mathbf{S}$ and $\psi(s_1) \neq \psi(s_2)$. Note that T is a finite continuous homomorphic image of $\overline{\Omega}_A\mathbf{V}$. If we can show that $S \in \mathbf{V}$, we will be done. In other words, it suffices to consider the case where S is finite.
Since φ is continuous and $\overline{\Omega}_A\mathbf{V}$ is compact, φ is uniformly continuous. Hence, there is a positive integer n such that, for all $u, v \in \overline{\Omega}_A\mathbf{V}$,

$$d(u, v) < 2^{-n} \Rightarrow \varphi(u) = \varphi(v).$$

In view of Proposition 4.6, it follows that the intersection ρ of the kernels of the uniformly continuous homomorphisms $\overline{\Omega}_A\mathbf{V} \to V$ with $V \in \mathbf{V}$ and $|V| \leq n$ is contained in $\ker \varphi$. Hence, φ factors through the natural homomorphism $\overline{\Omega}_A\mathbf{V} \to \overline{\Omega}_A\mathbf{V}/\rho$. Since $\overline{\Omega}_A\mathbf{V}/\rho$ belongs to $\mathbf{V}$, so does S. □

Lemma 5.7 ([Num57, Hun88])

Let K be a clopen subset of a compact zero-dimensional metric semigroup S. Then there is a continuous homomorphism $\varphi : S \to T$ into a finite semigroup T such that $K = \varphi^{-1}\varphi(K)$.

Proof.

We may define on S a syntactic congruence of K by

$$u \; \sigma_K \; v \quad \text{if } \forall x, y \in S^1 \; (xuy \in K \Leftrightarrow xvy \in K).$$

It suffices to show that the classes of this congruence are open: then there are only finitely many of them, so that S/σ_K is a finite semigroup, and the natural mapping $S \to S/\sigma_K$ is a continuous homomorphism.
We show that, if $\lim u_n = u$, then all but finitely many terms in the sequence are σ_K-equivalent to u. Arguing by contradiction, otherwise, there is a subsequence consisting of terms which fail this property. We may as well assume that so does the original sequence.
For each n there are $x_n, y_n \in S^1$ such that one, but not both, of the products $x_n u_n y_n$ and $x_n u y_n$ lies in K. Again, by taking subsequences we may assume that $\lim x_n = x$, $\lim y_n = y$ (in S^1), and $x_n u y_n \notin K$. Then
$xuy = \lim x_n u_n y_n = \lim x_n u y_n$ must belong to both K and its complement. □

- A useful application of Lemma 5.7 is the following result, which completes that of Proposition 5.4.

Theorem 5.8

A compact metric semigroup is profinite if and only if it is zero-dimensional.

Proof.

($\Rightarrow$) This follows from Proposition 5.4.

($\Leftarrow$) Let S be a compact metric semigroup which is zero-dimensional. We need to show that it is residually in **S**, that is that, for every pair s, t of distinct points of S, there is a continuous homomorphism $\to T$ in to a finite semigroup T such that $\varphi(s) \neq \varphi(t)$.
Since S is a zero-dimensional metric space, there is some clopen subset K such that $s \in K$ and $t \notin K$. By Lemma 5.7, there is a continuous homomorphism $\varphi : S \to T$ into a finite semigroup T such that $K = \varphi^{-1}\varphi(K)$. In particular, we have $\varphi(s) \neq \varphi(t)$,as required. □

- A language $L \subseteq A^+$ is **V**-recognizable if its syntactic semigroup belongs to **V**.

Theorem 5.9

*A language $L \subseteq A^+$ is **V**-recognizable if and only if the closure $K = \overline{\iota(L)}$ is open in $\overline{\Omega}_A\mathbf{V}$ and $\iota^{-1}(K) = L$. The latter condition is superfluous if ι is injective and $\iota(A^+)$ is a discrete subset of $\overline{\Omega}_A\mathbf{V}$.*

Proof.

($\Rightarrow$) Use the universal property of $\overline{\Omega}_A\mathbf{V}$ (Theorem 5.3).

($\Leftarrow$) By Lemma 5.7, there is a continuous homomorphism $\varphi : \overline{\Omega}_A\mathbf{V} \to S$ such that $S \in \mathbf{V}$ and $K = \varphi^{-1}\varphi(K)$. Then $\psi = \varphi \circ \iota$ is a homomorphism $A^+ \to S$ such that $\psi^{-1}\varphi(K) = \iota^{-1}(K) = L$ and so L is **V**-recognizable. □

- Theorem 5.9 implies that, as a topological space, $\overline{\Omega}_A\mathbf{V}$ is the Stone dual of the Boolean algebra of **V**-recognizable languages of A^+.

Theorem 5.10

A set $\mathcal{S}$ of $\mathbf{V}$-recognizable languages over a finite alphabet A generates the Boolean algebra of all such languages if and only if the clopen sets of the form $\overline{\iota(L)}$ ($L \in \mathcal{S}$) suffice to separate points of $\overline{\Omega}_A\mathbf{V}$.

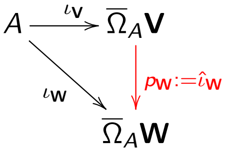
Proof.

($\Rightarrow$) Let $u, v \in \overline{\Omega}_A\mathbf{V}$ be distinct points. Then $\epsilon = d(u, v)$ is positive. Since $\overline{\Omega}_A\mathbf{V}$ is zero-dimensional (Proposition 5.4), there is a clopen subset K containing u and contained in $B_\epsilon(u)$, whence not containing v. By Theorem 5.9, $L = \iota^{-1}(K)$ is $\mathbf{V}$-recognizable. From the hypothesis, it follows that L is a Boolean combination $f(L_1, \dots, L_n)$ of languages L_i from $\mathcal{S}$. By Theorem 5.9 again, each set $\overline{\iota(L_i)}$ is clopen. Since $\overline{\iota(X_1 \cup X_2)} = \overline{\iota(X_1)} \cup \overline{\iota(X_2)}$ and $\overline{\Omega}_A\mathbf{V} \setminus \overline{\iota(X)} = \overline{\iota(A^+ \setminus X)}$ for $\mathbf{V}$-recognizable languages $X, X_1, X_2 \subseteq A^+$, we have $K = \overline{\iota(L)} = f(\overline{\iota(L_1)}, \dots, \overline{\iota(L_n)})$. Hence at least one of the sets $\overline{\iota(L_i)}$ must contain exactly one of the points u and v.

($\Leftarrow$) By Theorem 5.9, it suffices to show that the clopen sets of the form $\overline{\iota(L)}$, with $L \subseteq A^+$ $\mathbf{V}$-recognizable, generate the Boolean algebra of all clopen subsets of $\overline{\Omega}_A\mathbf{V}$. This is a nice exercise on compactness. □

Outline

- Recall that a $\mathbf{V}$-pseudoidentity is a formal equality $u = v$ with $u, v \in \overline{\Omega}_A\mathbf{V}$ for some finite set A.
- Recall also that, for $S \in \mathbf{V}$, we write $S \models u = v$ if $\varphi(u) = \varphi(v)$ for every continuous homomorphism $\varphi : \overline{\Omega}_A\mathbf{V} \to S$. In this case, we also say that $u = v$ **holds** in S.
- For a set Σ of $\mathbf{V}$-pseudoidentities, let $[\![\Sigma]\!]$ denote the class of all $S \in \mathbf{V}$ such that $S \models u = v$ for every pseudoidentity $u = v$ from Σ.
- For a subpseudovariety $\mathbf{W}$ of $\mathbf{V}$, let $p_{\mathbf{W}} : \overline{\Omega}_A\mathbf{V} \to \overline{\Omega}_A\mathbf{W}$ be the natural continuous homomorphism:



Lemma 6.1

A pseudoidentity $u = v$, with $u, v \in \overline{\Omega}_A \mathbf{V}$, holds in every member of a subpseudovariety $\mathbf{W}$ of $\mathbf{V}$ if and only if $p_{\mathbf{W}}(u) = p_{\mathbf{W}}(v)$.

Theorem 6.2 ([Rei82])

A subclass $\mathbf{W}$ of $\mathbf{V}$ is a subpseudovariety if and only if it is of the form $[\![\Sigma]\!]$ for some set Σ of $\mathbf{V}$-pseudoidentities.

- Usually, one takes $\mathbf{V} = \mathbf{S}$.

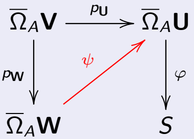
Proof of Theorem 6.2.

($\Leftarrow$) This amounts to verifying that the property $S \models u = v$ is preserved under taking homomorphic images, subsemigroups and finite direct products, which follows easily from the definitions.

($\Rightarrow$) Fix a countably infinite set X and let Σ be the set of all pseudoidentities $u = v$ such that $u, v \in \overline{\Omega}_A \mathbf{V}$ for some finite subset A of X and $S \models u = v$ for all $S \in \mathbf{W}$. Then $\mathbf{U} = [\![\Sigma]\!]$ is a subpseudovariety of $\mathbf{V}$ by the first part of the proof, and it clearly contains $\mathbf{W}$. We claim that $\mathbf{U} = \mathbf{W}$.

Let $S \in \mathbf{U}$ and choose an onto continuous homomorphism $\varphi : \overline{\Omega}_A \mathbf{U} \to S$ for some finite subset A of X (cf. Theorem 5.3).

Consider the natural continuous homomorphisms $p_{\mathbf{U}}$ and $p_{\mathbf{W}}$. By Lemma 6.1 and the choice of Σ, we have $\ker p_{\mathbf{W}} \subseteq \ker p_{\mathbf{U}}$ and so there is a factorization $p_{\mathbf{U}} = \psi \circ p_{\mathbf{W}}$ for some onto continuous homomorphism $\psi : \overline{\Omega}_A \mathbf{W} \to \overline{\Omega}_A \mathbf{U}$. Hence $\varphi \circ \psi : \overline{\Omega}_A \mathbf{W} \to S$ is an onto continuous homomorphism. Corollary 5.6 then implies that $S \in \mathbf{W}$ since $\overline{\Omega}_A \mathbf{W}$ is a pro-$\mathbf{W}$ semigroup by Theorem 5.3.



□

- To write pseudoidentities that are not identities, one needs to construct some elements of $\overline{\Omega}_A\mathbf{S} \setminus A^+$.

Lemma 6.3

Let S be a profinite semigroup, let s be an element of S, and let $k \in \mathbb{Z}$. Then the sequence of powers $(s^{n!+k})_{n\geq|k|}$ converges. For $k = 0$ the limit is an idempotent.

Proof.

Using Proposition 5.1, it suffices to consider the case where S is finite, which is left as an exercise. □

- The limit $\lim s^{n!+k}$ is denoted $s^{\omega+k}$.
- Note that $s^{\omega+k}s^{\omega+\ell} = s^{\omega+k+\ell}$.
  In particular, $s^{\omega} := s^{\omega+0}$ is an idempotent and $s^{\omega-k}$ and $s^{\omega+k}$ are mutual inverses in the maximal subgroup containing the idempotent s^{ω}.

Examples I

$$\mathbf{S} = [\![x = x]\!]$$
$$\mathbf{I} = [\![x = y]\!]$$
$$\mathbf{G} = [\![x^\omega = 1]\!]$$
$$\mathbf{G}_p = ?$$
$$\mathbf{A} = [\![x^{\omega+1} = x^\omega]\!]$$
$$\mathbf{Com} = [\![xy = yx]\!]$$
$$\mathbf{J} = [\![(xy)^\omega = (yx)^\omega, x^{\omega+1} = x^\omega]\!]$$
$$\mathbf{R} = [\![(xy)^\omega x = (xy)^\omega]\!]$$
$$\mathbf{L} = [\![y(xy)^\omega = (xy)^\omega]\!]$$

$$\mathbf{Sl} = [\![xy = yx, x^2 = x]\!]$$
$$\mathbf{RZ} = [\![xy = y]\!]$$
$$\mathbf{B} = [\![x^2 = x]\!]$$
$$\mathbf{N} = [\![x^\omega = 0]\!]$$
$$\mathbf{K} = [\![x^\omega y = x^\omega]\!]$$
$$\mathbf{D} = [\![yx^\omega = x^\omega]\!]$$

Examples II

- Since there are uncountably many pseudovarieties of the form $\mathbf{Ab}_P$, where P is a set of primes, and one can show that all of them admit a description of the form $[\![xy = yx, u = 1]\!]$ [Alm95, Corollary 3.7.8], for some $u \in \overline{\Omega}_{\{x\}}\mathbf{S}$, we conclude that $\overline{\Omega}_{\{x\}}\mathbf{S}$ is uncountable.
- Let P be an infinite set of primes and let $p_1, p_2, \ldots$ be an enumeration of its elements, without repetitions.
  Let u_P be an accumulation point in $\overline{\Omega}_{\{x\}}\mathbf{S}$ of the sequence $(x^{p_1 \cdots p_n})_n$.

$$\mathbf{Ab}_P = [\![xy = yx, u_P = 1]\!].$$

- *Does the sequence $(x^{p_1 \cdots p_n})_n$ converge?*

- To describe the pseudovariety $\mathbf{G}_p$ of all finite p-groups, we use the following result, whose proof is similar to that of Lemma 6.3.

Lemma 6.4

Let S be a profinite semigroup and $s \in S$. Then the sequence $(s^{p^{n!}})_n$ converges.

- We let $s^{p^\omega} = \lim s^{p^{n!}}$.

$$\mathbf{G}_p = [\![x^{p^\omega} = 1]\!].$$

Examples IV

Exercise 6.5 (For those that know some group theory)

Find, for each of the following pseudovarieties of groups, a single pseudoidentity defining them:

(1) the pseudovariety $\mathbf{G}_{p'}$ of all finite groups which have no lements of order p (p being a fixed prime number);
(2) the pseudovariety $\mathbf{G}_{\mathrm{nil}}$ of all finite nilpotent groups;
(3) the pseudovariety $\mathbf{G}_{\mathrm{sol}}$ of all finite solvable groups.

Section 7

References

[ABR92] D. Albert, R. Baldinger, and J. Rhodes, The identity problem for finite semigroups (the undecidability of), J. Symbolic Logic **57** (1992), 179–192.

[AE03] J. Almeida and A. Escada, Semidirect products with the pseudovariety of all finite groups, Proceedings of the International Conference Words, Languages and Combinatorics (Kyoto, March, 2000) (Singapore) (M. Ito and T. Imaoka, eds.), World Scientific, 2003, pp. 1–21.

[Alm95] J. Almeida, Finite semigroups and universal algebra, World Scientific, Singapore, 1995, English translation.

[AS03] K. Auinger and B. Steinberg, On the extension problem for partial permutations, Proc. Amer. Math. Soc. **131** (2003), 2693–2703.

[Ash87] C. J. Ash, Finite semigroups with commuting idempotents, J. Austral. Math. Soc., Ser. A **43** (1987), 81–90.

[Ash91] ______, Inevitable graphs: a proof of the type II conjecture and some related decision procedures, Int. J. Algebra Comput. **1** (1991), 127–146.

[BS73] J. A. Brzozowski and I. Simon, Characterizations of locally testable events, Discrete Math. **4** (1973), 243–271.

[Eil76] S. Eilenberg, Automata, languages and machines, vol. B, Academic Press, New York, 1976.

[HMPR91] K. Henckell, S. Margolis, J.-E. Pin, and J. Rhodes, Ash's type II theorem, profinite topology and Malcev products. Part I, Int. J. Algebra Comput. **1** (1991), 411–436.

[HR91] K. Henckell and J. Rhodes, The theorem of Knast, the PG=BG and Type II Conjectures, Monoids and Semigroups with Applications (Singapore) (J. Rhodes, ed.), World Scientific, 1991, pp. 453–463.

[Hun88] R. P. Hunter, Certain finitely generated compact zero-dimensional semigroups, J. Austral. Math. Soc., Ser. A **44** (1988), 265–270.

[KR65] K. Krohn and J. Rhodes, Algebraic theory of machines. I. Prime decomposition theorem for finite semigroups and machines, Trans. Amer. Math. Soc. **116** (1965), 450–464.

[MP71] R. McNaughton and S. Papert, Counter-free automata, MIT Press, Cambridge, MA, 1971.

[MP84] S. W. Margolis and J.-E. Pin, Varieties of finite monoids and topology for the free monoid, Proc. 1984 Marquette Semigroup Conference (Milwaukee), Marquette University, 1984, pp. 113–129.

[MP87] ______, Inverse semigroups and varieties of finite semigroups, J. Algebra **110** (1987), 306–323.

[Num57] K. Numakura, Theorems on compact totally disconnetced semigroups and lattices, Proc. Amer. Math. Soc. **8** (1957), 623–626.

[Pin95] J.-E. Pin, BG=PG: A success story, Semigroups, Formal Languages and Groups (Dordrecht) (J. Fountain, ed.), vol. 466, Kluwer, 1995, pp. 33–47.

[PS85] J.-E. Pin and H. Straubing, Monoids of upper triangular matrices, Semigroups: structure and universal algebraic problems (Amsterdam) (G. Pollák, ed.), North-Holland, 1985, pp. 259–272.

[Rei82] J. Reiterman, The Birkhoff theorem for finite algebras, Algebra Universalis **14** (1982), 1–10.

[Sch65] M. P. Schützenberger, On finite monoids having only trivial subgroups, Inform. and Control **8** (1965), 190–194.

[Sim75] I. Simon, Piecewise testable events, Proc. 2nd GI Conf. (Berlin), Lect. Notes in Comput. Sci., vol. 33, Springer, 1975, pp. 214–222.

[Sti73] P. Stiffler, Extension of the fundamental theorem of finite semigroups, Advances in Math. **11** (1973), 159–209.