Problems for the Course F5170 – Introduction to Plasma Physics

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Preface

This document contains exercise problems for the course F5170 – Introduction to plasma physics. This work was supported by the Ministry of Research and Education of the Czech Republic, project no. FRVŠ 12/2013/G6. The most valuable source of information for this document was the book Fundamentals of Plasma Physics by J. A. Bittencourt [4]. The authors would be grateful for any notification about eventual errors.

The complete and up-to-date version of this document can be found at http://physics.muni.cz/~sperka/exercises.html.

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Physical constants

Proton rest mass	m_p	$1,67 \cdot 10^{-27}\,\mathrm{kg}$
Electron rest mass	m_e	$9.109 \cdot 10^{-31}\,\mathrm{kg}$
Elementary charge	e	$1.602 \cdot 10^{-19}\,\mathrm{C}$
Boltzmann's constant	k	$1.38 \cdot 10^{-23}\,\mathrm{J\,K^{-1}}$
Vacuum permittivity	ε_0	$8.854 \cdot 10^{-12}\,\mathrm{A^2\,s^4\,kg^{-1}\,m^{-3}}$

Used symbols

Vector quantities are typed in bold face ($\mathbf{v}$), scalar quantities, including magnitudes of vectors are in italic (v). Tensors are usually in upper-case calligraphic typeface ($\mathcal{P}$).

Operators

scalar product	$\mathbf{a} \cdot \mathbf{b}$
vector product	$\mathbf{a} \times \mathbf{b}$
i^{th} derivative with respect to x	$\frac{\mathrm{d}^{\mathrm{i}}}{\mathrm{d}x^{i}}$
partial derivative	$\frac{\partial}{\partial x}$
nabla operator	$\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$
Laplace operator	$\Delta = \nabla^2$
total time derivative	$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla$

Physical quantities

electron concentration	n_{e}
electron temperature	T_{e}
electron plasma frequency	ω_{pe}
Debye length	λ_{D}
Larmor radius	r_c
Larmor frequency	Ω_c
magnetic moment	$\mathbf{m}$
force	$\mathbf{F}$
electric field intensity	$\mathbf{E}$
magnetic field induction	$\mathbf{B}$
arb. quantity for one type of particles	χ_α
distribution function	$f(\chi_\alpha)$
mean velocity	$\mathbf{u}$
charge denisty	ρ
mass density	ρ_m
collision frequency	ν
source term due to collisions	S_α
scalar pressure	p
tensor of kinetic pressure	$\mathcal{P}$
mobility of particles	$\mathcal{M}_\alpha$

Chapter 1

Introduction

1.1 Theory

Electron plasma frequency

$$\omega_{\mathrm{p}} = \sqrt{\frac{n_{\mathrm{e}}\, e^2}{\varepsilon_0\, m_{\mathrm{e}}}} = \mathrm{const}\,\sqrt{n_{\mathrm{e}}} \tag{1.1}$$

describes the typical electrostatic collective electron oscillations due to little separation of electric charge. Plasma frequencies of other particles can be defined in a similar way. However, the electron plasma frequency is the most important because of high mobility of electrons (the proton/electron mass ratio $m_{\mathrm{p}}/m_{\mathrm{e}}$ is 1.8×10^3).

Note that plasma oscillations will only be observed if the plasma system is studied over time periods longer than the plasma period ω_{p}^{-1} and if external actions change the system at a rate no faster than ω_{p}. Observations over length-scales shorter than the distance traveled by a typical plasma particle during a plasma period will also not detect plasma behaviour. This distance, which is the spatial equivalent to the plasma period , is called the Debye length, and takes the form

$$\lambda_{\mathrm{D}} = \sqrt{\frac{T_{\mathrm{e}}}{m_{\mathrm{e}}}}\,\omega_{\mathrm{p}}^{-1} = \sqrt{\frac{\varepsilon_0\, T_{\mathrm{e}}}{n_{\mathrm{e}}\, e^2}} = \mathrm{const}\,\sqrt{T_{\mathrm{e}}/n_{\mathrm{e}}}. \tag{1.2}$$

The Debye length is independent of mass and is therefore comparable for different species.

	n_e [cm^{-3}]	T_e [eV]	tlak [Pa]	Ref.
Plasma Displays	(2.5–3.7) $\times 10^{11}$	0.8–1.8	(20–50) $\times 10^3$	[8]
	max 3 $\times 10^{12}$		(40–67) $\times 10^3$	[22]
	(0.2–3)$\times 10^{13}$	1.6–3.4		[19]
Earth's ionosphere	max 10^6	max 0.26		[6]
			10^{-5}	[2]
RF Magnetrons			0.5-10	[15]
	1–8$\times 10^9$	2–9	0.3–2.6	[20]
DC Magnetrons	10^{18}	1-5	0.5–2.5	[23]
RF Atmospheric plasma	10^{13}–10^{14}		10^5	[10]
		0.2–6	10^5	[12]
MW Atmospheric plasma		1.2–1.9	10^5	[14]
	3×10^{14}			[11]
Welding arc		1.5	10^5	[3]
	1.5×10^{17}		10^5	[21]
	1.6×10^{17}	1.3	10^5	[18]
Low-pressure CCP	6×10^8		6–7	[24]
	(0.5–4.5) $\times 10^{10}$	1.4–1.6	4.7	[5]
Fluorescent lamps	10^{10}–10^{11}	1	8×10^3	[7]

Table 1.1: An overview of typical values of the most important parameters for various plasmas.

1.2 Problems

1.2.1 Derivation of the plasma frequency

Consider a steady initial state with a uniform number density of electrons and an equal number of ions such that the total electrical charge is neutral. Neglect the thermal motion of the particles and assume that the ions are stationary. Show that a small displacement of a group of electrons leads to oscillations with the plasma frequency according to the equation (1.1).

Solution The situation is sketched in the figure 1.1. Assume that the electric field in the plane perpendicular to the x-axis is zero (just like in the case of an infinitely large charged plane or capacitor). Let us apply the Gauss's law to a closed cylindrical surface (only contour of which is sketched in the figure):

$$\oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \frac{Q}{\epsilon_0} = \left(\frac{S n_e e}{\epsilon_0}\right) x, \tag{1.3}$$

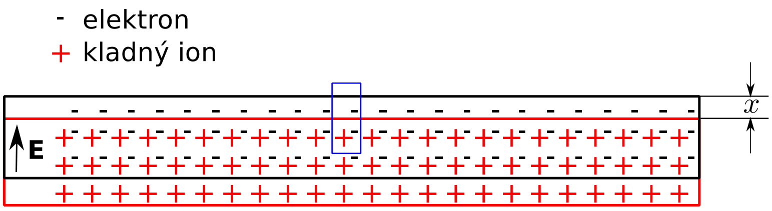

Figure 1.1: Illustration of the problem no. 1.2.1.

where S is the area of the base of the cylinder. The resulting electric field is

$$E_x = \left(\frac{n_0 e}{\epsilon_0}\right) x. \tag{1.4}$$

Inserting this electric field into the equation of motion of a single electron yields

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \left(\frac{n_0 e^2}{m_\mathrm{e}\epsilon_0}\right) x = 0, \tag{1.5}$$

Which is an equation of a harmonic oscillator with the frequency

$$\omega_\mathrm{pe} = \left(\frac{n_0 e^2}{m_\mathrm{e}\epsilon_0}\right)^{1/2}. \tag{1.6}$$

1.2.2 Plasma frequency and Debye length

Compute the plasma frequency and the Debye length for the following plasmas

(a) Earth's ionosphere with electron concentration $n_e = 10^6\,\mathrm{cm}^{-3}$ and electron temperature $k\,T_\mathrm{e} = 0.2\,\mathrm{eV}$.
[$\omega_\mathrm{p} = 5,6 \times 10^7\,\mathrm{rad\cdot s^{-1}} = 3,5 \times 10^8\,\mathrm{Hz},\ \lambda_\mathrm{D} = 3,3\,\mathrm{mm}$]

(b) A cell of a typical plasma display with electron concentration of $10^{13}\,\mathrm{cm}^{-3}$ and electron temperature of $1\,\mathrm{eV}$. The cell dimension is about $100\,\mu\mathrm{m}$. Is the condition that the system dimension should be much greater than the Debye length fulfilled?
[$\omega_\mathrm{pe} = 2,3 \times 10^{13}\,\mathrm{rad\cdot s^{-1}} = 3,6\,\mathrm{THz},\ \lambda_\mathrm{D} = 21\,\mathrm{nm}$]

(c) A welding arc with electron concentration of $1,6\times10^{17}\,\mathrm{cm}^{-3}$ and electron temperature of $1,3\,\mathrm{eV}$
[$\omega_\mathrm{pe} = 2,3 \times 10^{13}\,\mathrm{rad\cdot s^{-1}} = 3,6\,\mathrm{THz},\ \lambda_\mathrm{D} = 21\,\mathrm{nm}$]

(d) A fluorescent lamp with electron concentration of $10^{10}\,\mathrm{cm}^{-3}$ and electron temperature of $1\,\mathrm{eV}$
[$\omega_\mathrm{pe} = 5,6 \times 10^9\,\mathrm{rad\cdot s^{-1}} = 0,90\,\mathrm{GHz},\ \lambda_\mathrm{D} = 74\,\mu\mathrm{m}$]

1.2.3 Debye-Hückel potential

Show that Debye–Hückel potential

$$\varphi(\mathbf{r}) = \frac{e}{4\,\pi\,\varepsilon_0}\frac{\exp\left(-\frac{r}{\lambda_{\mathrm{D}}}\right)}{r} \tag{1.7}$$

is solution of equation

$$\nabla^2\varphi(\mathbf{r}) = \frac{\varphi(\mathbf{r})}{r_{\mathrm{D}}^2} = \frac{n_{\mathrm{e}}\,e^2}{\varepsilon_0 k T_{\mathrm{e}}}\varphi(\mathbf{r}) \tag{1.8}$$

where r_{D} is Debye–Hückel radius.

Remark: Debye–Hückel potential which is called after Pieter Debye (1884-1966) and Erich Hückel (1896-1980) who studied polarisation effects in electrolytes [9].

Solution Put simply the Debye-Hückel potential into the equation (1.8) and calculate Laplace operator in spherical coordinates

$$\Delta f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2} \tag{1.9}$$

Chapter 2

Motion of particles in electromagnetic fields

2.1 Theory

The Lorentz force is the combination of electric and magnetic force on a point charge due to electromagnetic fields. If a particle of charge q moves with velocity v in the presence of an electric field $\mathbf{E}$ and a magnetic field $\mathbf{B}$, then it will experience the Lorentz force

$$\mathbf{F} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right). \tag{2.1}$$

The gyroradius (also known as Larmor radius or cyclotron radius) is the radius of the circular motion of a charged particle in the presence of a uniform magnetic field:

$$r_g = \frac{mv_\perp}{|q|B}, \tag{2.2}$$

where r_g is the gyroradius, m is the mass of the charged particle, $v_\perp$ is the velocity component perpendicular to the direction of the magnetic field, q is the charge of the particle, and B is magnitude of the constant magnetic field.

Similarly, the frequency of this circular motion is known as the gyrofrequency or cyclotron frequency, and is given by:

$$\omega_g = \frac{|q|B}{m}. \tag{2.3}$$

Note: A cyclotron is a type of particle accelerator in which charged particles accelerate due to high-frequency electric field. The cyclotron was invented and patented by Ernest Lawrence of the University of California, Berkeley, where it was first operated in 1932.

2.2 Problems

2.2.1 Magnetic mirror

Magnetic mirrors are used to confine charged particles in a limited volume. The gradient of magnetic field induction can result in reversing the direction of drift of a charged particle.

Suppose we have an electron located at $z = 0$ with initial velocity v_0 and an initial pitch angle ϑ. The magnetic field induction is given by

$$B(z) = B_0 \left(1 + (\gamma\, z)^2\right). \tag{2.4}$$

Calculate the turning point z_t [13].

Solution We start with the conservation of kinetic energy and the magnetic moment. The kinetic energy conservation condition yields

$$v_0^2 = v_\mathrm{t}^2. \tag{2.5}$$

The z-component of the velocity at the turning point must be zero, which we immediately use in the equation describing the conservation of the magnetic moment

$$\begin{aligned}
\frac{m_\mathrm{e}\, v_0^2 \sin^2\vartheta}{2\, B_0} &= \frac{m_\mathrm{e}\, v_\mathrm{t}^2}{2\, B_0\, (1 + (\gamma\, z_\mathrm{t})^2)} \\
v_0^2 \sin^2\vartheta \left(1 + (\gamma\, z_\mathrm{t})^2\right) &= v_\mathrm{t}^2 \qquad \left(= v_0^2\right) \\
\gamma^2\, z_\mathrm{t}^2 &= \frac{1 - \sin^2\vartheta}{\sin^2\vartheta} \\
z_\mathrm{t} &= \frac{1}{\gamma\, \tan\vartheta}.
\end{aligned} \tag{2.6}$$

We see that the position of the point of reflection depends only on the gradient of the magnetic field and on the initial pitch angle.

2.2.2 Magnetic mirror of a different construction

Calculate the turning point for a charged particle in a magnetic mirror with induction given by

$$B(z) = B_0 \left(1 + (\gamma\, z)^4\right). \tag{2.7}$$

The initial pitch angle is ϑ.

$\left[z_\mathrm{t} = \left(\frac{1}{\gamma\, \tan\vartheta}\right)^{1/2}\right]$

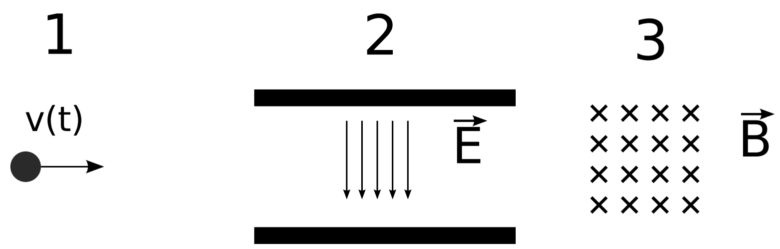

Figure 2.1: Sketch of the problem 2.2.3.

2.2.3 Electron in vacuum – three parts

(a) The time dependence of the position of the electron in the first part is expressed as $x(t) = \frac{1}{8}t^4 + \pi$. The electron remains in the first part for one second. Calculate the magnitude of the velocity v_x that the electron acquires at the end of the first part.
[$0.5\,\mathrm{m/s}$]

(b) After that, the electron, having the velocity v_x, enters the second part, where a transverse electric field $\overrightarrow{E}$ of the magnitude $10^{-10}\,\mathrm{V\,m^{-1}}$ is applied. This field is generated by the plates of a capacitor with the length $d = 1\,\mathrm{m}$. What is the vertical displacement of the electron with respect to the starting position at the end of the second part? First derive the general solution.
[$35.2\,\mathrm{m}$]

(c) Finally, the electron enters a homogeneous magnetic field $\overrightarrow{B}$ of the magnitude $20.6\,\mu\mathrm{T}$ (this is the magnitude of the horizontal component of the geomagnetic field induction in Brno). Calculate the Larmor radius, cyclotron frequency and the magnitude of the magnetic moment of the rotating electron.
[$r_c = 9.72 \times 10^{-6}\,\mathrm{m}$, $\Omega_c = 3.6 \times 10^{6}\,\mathrm{rad \cdot s^{-1}}$, $|\mathbf{m}| = 2.7 \times 10^{-23}\,\mathrm{A \cdot m^2}$]

(d) What would be the result for proton, neutron and positron? For illustration see 2.1.

2.2.4 $\mathbf{E} \times \mathbf{B}$ drift

Suppose we have a vacuum chamber with electric field $E = 1\,\mathrm{kV\,m^{-1}}$ perpendicular to magnetic field $B = 1\,\mathrm{mT}$. Calculate $\mathbf{E} \times \mathbf{B}$ drift speed for an electron inside the chamber.
[$\frac{E}{B}$]

2.2.5 Relativistic cyclotron frequency

What is the relativistic cyclotron frequency of an electron with velocity $0.8\,c$ (c denotes speed of light)?
[$\omega = \frac{6}{10}\,e\,B/m$]

2.2.6 Relativistic particle in an uniform magnetic field

Derive the gyroradius, angular gyrofrequency $\Omega_{\mathrm{c}}^{\mathrm{rel}}$, and energy of relativistic particle with speed v and charge q in an uniform magnetic field with magnitude of magnetic induction B.

Solution Gyroradius:

$$r = \frac{\gamma\beta m_0 c}{qB} \tag{2.8}$$

Angular gyrofrequency:

$$\Omega_{\mathrm{c}}^{\mathrm{rel}} = \frac{|q|B}{\gamma m_0} = \frac{\Omega_{\mathrm{c}}}{\gamma} = \Omega_{\mathrm{c}}\sqrt{1-\beta^2} = \Omega_{\mathrm{c}}\sqrt{1-\left(\frac{v}{c}\right)^2} \tag{2.9}$$

Energy:

$$E_{\mathrm{k}} = m\gamma c^2 - mc^2 = \frac{mc^2}{\sqrt{1-v^2/c^2}} - mc^2 \tag{2.10}$$

2.2.7 Law of conservation of electric charge

Derive continuity equation from Maxwell's Equations.
$\left[\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{J} = 0\right]$

2.2.8 Magnetostatic field

Proof, that in presence of magnetostatic field total kinetic energy of charged particle W_{k} remains constant.

2.2.9 Cyclotron frequency of electron

What is a cyclotron frequency (in Hz) of electron in homogenous magnetostatic field:
a) $|\vec{B}| = 0.01\,\mathrm{T}$
b) $|\vec{B}| = 0.1\,\mathrm{T}$
c) $|\vec{B}| = 1\,\mathrm{T}$
d) $|\vec{B}| = 5\,\mathrm{T}$
[a) 0.28 GHz ; b) 2.8 GHz; c) 28 GHz d) 140 GHz]

2.2.10 Cyclotron frequency of ionized hydrogen atom

What is a cyclotron frequency (in Hz) of ionized hydrogen atom in homogenous magnetostatic field:
a) $|\vec{B}| = 0.01\,\mathrm{T}$
b) $|\vec{B}| = 0.1\,\mathrm{T}$
c) $|\vec{B}| = 1\,\mathrm{T}$
d) $|\vec{B}| = 5\,\mathrm{T}$
[a) 0.15 MHz ; b) 1.5 MHz; c) 15 MHz d) 76 MHz]

2.2.11 Magnetic moment

Suppose a planar closed circular current loop has area $|S| = 10^{-3}\,\mathrm{m}^2$ and carries an electric current:
a) $I = 1\,\mathrm{A}$
b) $I = 2\,\mathrm{A}$
c) $I = 8\,\mathrm{A}$
Calculate the magnitude of its magnetic moment $|m|$.
[a) $|m| = 10^{-3}\,\mathrm{A\,m}^2$; b) $|m| = 2 \times 10^{-3}\,\mathrm{A\,m}^2$; c)$|m| = 8 \times 10^{-3}\,\mathrm{A\,m}^2$]

2.2.12 Magnetic moment 2

How can be written the magnitude of the magnetic moment $|\vec{m}|$, which is associated with the circulating current of charged particle (charge q, angular frequency $\vec{\Omega_c}$, mass m) in uniform magnetostatic field $\vec{B}$?
[$|\vec{m}| = \frac{|q|\,|\vec{\Omega_c}|}{2\,\pi}\,\pi\, r_c^2$; $|\vec{\Omega_c}| = \frac{|q|\,|\vec{B}|}{m}$]

2.2.13 Lorentz force

Suppose a magnetostatic field $\vec{B} = (1, 2, 0)\,\mathrm{T}$. The velocity of an electron is $\vec{v} = (0, 2, 1)\,\mathrm{m\,s}^{-1}$. Calculate Lorentz force.
[$\vec{F} = -e \cdot (-2, 1, -2)\,\mathrm{N}$]

Chapter 3

Elements of plasma kinetic theory

3.1 Theory

- Phase space is defined by six coordinations (x, y, z, v_x, v_y, v_z).
- The dynamical state of each particle is appropriately represented by a single point in this phase space.
- The distribution function in phase space, $f_\alpha(\vec{r}, \vec{v}, t)$, is defined as the density of representative points of the particles α in phase space:

$$f_\alpha(\vec{r}, \vec{v}, t) = N_\alpha^6(\vec{r}, \vec{v}, t)/(d^3r\, d^3v). \tag{3.1}$$

- The number density, $n_\alpha(\vec{r}, t)$, can be obtained by integrating $f_\alpha(\vec{r}, \vec{v}, t)$ over all of velocity space:

$$n_\alpha(\vec{r}, t) = \int_{\vec{v}} f_\alpha(\vec{r}, \vec{v}, t) d^3v \tag{3.2}$$

- The differential kinetic equation that is satisfied by the distribution function, is generally known as the Boltzmann kinetic equation:

$$\frac{\partial f_\alpha(\vec{r}, \vec{v}, t)}{\partial t} + \vec{v} \cdot \nabla_{\vec{r}} f_\alpha(\vec{r}, \vec{v}, t) + \vec{a} \cdot \nabla_{\vec{v}} f_\alpha(\vec{r}, \vec{v}, t) = \frac{\partial f_\alpha(\vec{r}, \vec{v}, t)}{\partial t} \Big|_{\text{collision}} \tag{3.3}$$

3.2 Problems

3.2.1 Uniform distribution function

Suppose we have system of particles uniformly distributed in space with constant particle number density n, which is characterised by one dimensional distribution function of speeds $F(v)$:

$$\begin{aligned} F(v) &= C \quad \text{for } v \leq v_0 \\ F(v) &= 0 \quad \text{otherwise,} \end{aligned}$$

where C is positive non-zero constant. Express C using n and v_0.
[**Solution**: By integration $n = C \int\limits_0^{v_0} \mathrm{d}v$ we will get the solution $C = \frac{n}{v_0}$.]

3.2.2 Linear distribution function

What is the normalizing constant C of the following distribution function of speeds?
$F(v) = C\,v$ for $v \in \langle 0, 1\rangle$ and $F(v) = 0$ otherwise.
[$C = 2\,n$ (n denotes the particle density)]

3.2.3 Quadratic distribution function

What is normalizing constant C of following distribution function of speeds?
$F(v) = C\,v^2$ for $v \in \langle 0, 3\rangle$ and $F(v) = 0$ otherwise.
[$C = n/9$ (n denotes the particle density)]

3.2.4 Sinusoidal distribution function

What is the normalizing constant C of the following distribution function of speeds?
$F(v) = C\,\sin(v)$ for $v \in \langle 0, \pi\rangle$ and $F(v) = 0$ otherwise.
[$C = n/2$ (n denotes the particle density)]

3.2.5 Boltzmann kinetic equation

Consider the motion of charged particles, in one dimension only, in the presence of an electric potential $\varphi(x)$. Show, by direct substitution, that a function of the form

$$f = f\,\left(\frac{1}{2}\,m\,v^2 + q\,\varphi(x)\right)$$

is a solution of the Boltzmann equation under steady state conditions.

Chapter 4

Average values and macroscopic variables

4.1 Theory

- The macroscopic variables, such as number density, flow velocity, kinetic pressure or thermal energy flux can be considered as average values of physical quantities, involving the collective behaviour of a large number of particles. These macroscopic variables are related to the various moments of the distribution function.

- With each particle in the plasma, we can associate some molecular property $\chi_\alpha(\vec{r}, \vec{v}, t)$. This property may be, for example, the mass, the velocity, the momentum, or the energy of the particle.

- The average value of the property $\chi_\alpha(\vec{r}, \vec{v}, t)$ for the particles of type α is defined by

$$\langle \chi_\alpha(\vec{r}, \vec{v}, t) \rangle = \frac{1}{n_\alpha(\vec{r}, t)} \int_{\vec{v}} \chi_\alpha(\vec{r}, \vec{v}, t) f_\alpha(\vec{r}, \vec{v}, t)\, d^3v. \tag{4.1}$$

- For example, the average velocity (or flow velocity) $\vec{u}_\alpha(\vec{r}, t)$ for the particles of type α is defined by

$$\vec{u}_\alpha(\vec{r}, t) = \langle v_\alpha(\vec{r}, t) \rangle = \frac{1}{n_\alpha(\vec{r}, t)} \int_{\vec{v}} \vec{v}\, f_\alpha(\vec{r}, \vec{v}, t)\, d^3v. \tag{4.2}$$

4.2 Problems

4.2.1 RMS speed

What is the rms speed of the following three electrons ($|v_1| = 1$, $|v_2| = 2$ and $|v_3| = 5$)?
[$\sqrt{10}$]

4.2.2 Mean speed of sinusoidal distribution

What is the mean speed of the following distribution function of speeds?
$f(v) = \frac{n}{2}\sin(v)$ for $v \in \langle 0, \pi\rangle$ and $f(v) = 0$ otherwise. n denotes the particle density.
[1]

4.2.3 Mean speed of quadratic distribution

What is the mean speed of the following distribution function of speeds?
$f(v) = 3\,n\,v^2$ for $v \in \langle 0, 1\rangle$ and $f(v) = 0$ otherwise n denotes the particle density.
[3/4]

4.2.4 The equilibrium temperature

Consider Maxwell-Boltzmann distributions in Fig. 4.1. Which one has the highest equilibrium temperature?
[c)]

4.2.5 Particle density

Consider Maxwell-Boltzmann distributions in Fig. 4.2. Which one has the highest particle density?
[c)]

4.2.6 Most probable speed of linear distribution

Consider the following distribution function of speeds $f(v) = n\,v$ for $v \in \langle 0, 1\rangle$ and $f(v) = 0$ otherwise.
What is the most probable speed of this distribution?
[1]

4.2.7 Most probable speed of sinusoidal distribution

Consider the following distribution function of speeds $f(v) = \frac{1}{2}\sin(v)$ for $v \in \langle 0, \pi\rangle$ and $f(v) = 0$ otherwise.

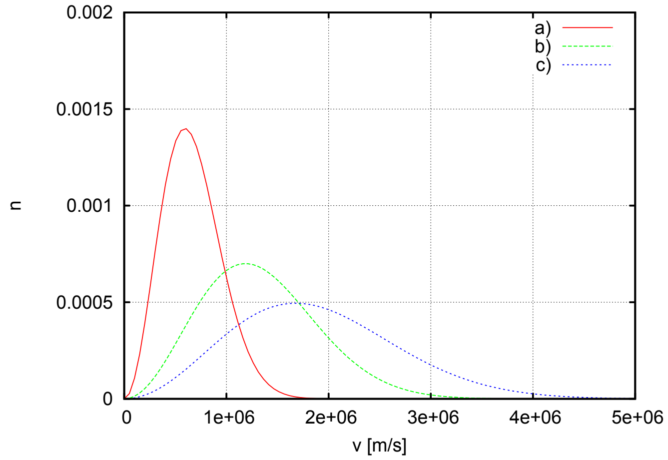

Figure 4.1: Diagram to the problem of the highest equilibrium temperature 4.2.4.

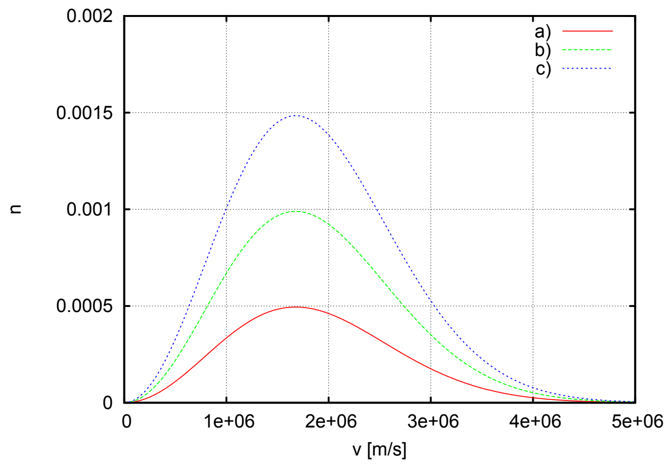

Figure 4.2: Diagram to the problem of the highest particle density 4.2.5.

What is the most probable speed of this distribution?
$[\frac{\pi}{2}]$

Chapter 5

The equilibrium state

5.1 Theory

- The equilibrium distribution function $f_\alpha^{Eq}(\vec{r}, \vec{v}, t)$ is the time-independent solution of the Boltzmann equation in the absence of external forces.
- In the equilibrium state the particle interactions do not cause any changes in $f_\alpha^{Eq}(\vec{r}, \vec{v}, t)$ with time and there are no spatial gradients in the particle number density.
- $f_\alpha^{Eq}(\vec{r}, \vec{v}, t)$ is known as the Maxwell–Boltzmann distribution or Maxwell distribution (see problems 5.2.2–5.2.4).

Math useful for calculations
The "Gaussian integral" is the integral of the Gaussian function e^{-x^2} over the entire real line. It is named after the German mathematician and physicist Carl Friedrich Gauss. The integral is (a, b denotes a constant):

$$\int_{-\infty}^{+\infty} e^{-x^2}\, dx = \sqrt{\pi}; \qquad \int_{-\infty}^{\infty} e^{-a(x+b)^2}\, dx = \sqrt{\frac{\pi}{a}}. \tag{5.1}$$

The gamma function $\Gamma(n)$ is an extension of the factorial function, with its argument shifted down by 1, to real and complex numbers. That is, if n is a positive integer:

$$\Gamma(n) = (n-1)! \tag{5.2}$$

Other important formulas:

$$\int_0^{\infty} x^n e^{-a\,x^2}\, dx = \frac{\Gamma(\frac{(n+1)}{2})}{2\,a^{\frac{(n+1)}{2}}}; \qquad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \tag{5.3}$$

5.2 Problems

5.2.1 Gamma function

Starting from the definition of a Gamma function show that, if n is a positive integer, then

$$\Gamma(n+1) = n!$$

Recipe: First, using integration by parts of $\Gamma(n+1) = \int_0^\infty x^n e^{-x}\, dx$ demonstrate that

$$\Gamma(a+1) = a\Gamma(a).$$

Next, it remains to show, that $\Gamma(1) = 1$.

5.2.2 1D Maxwell-Boltzmann distribution function

Gas composing of particles of one kind moving in only one dimension x is characterised by the following homogeneous isotropic one-dimensional Maxwell-Boltzmann distribution function:

$$f(v_x) = C \cdot \exp\left[-\frac{m\, v_x^2}{2kT}\right]. \tag{5.4}$$

(a) Calculate the constant C.

(b) Derive the 1D Maxwell-Boltzmann distribution function of speeds.

(c) Calculate the most probable speed.

(d) Calculate the mean speed.

(e) Derive the relation for the number of particles passing through a unit of length in a unit of time from one side (the flux of particles from one side).

Solution

(a) Integrate the distribution function over the whole velocity space. The condition that the integral equals the concentration of particles n yields

$$n = C \int\limits_{-\infty}^{\infty} \exp\left[-\frac{m\, v_x^2}{2kT}\right] \mathrm{d}v_x = C\sqrt{\frac{2kT\pi}{m}}. \tag{5.5}$$

$$C = n\sqrt{\frac{m}{2kT\pi}} \tag{5.6}$$

(b) Distribution of particle speeds $F(v)$ from the summation over the both possible directions is

$$F(v) = 2\,n\sqrt{\frac{m}{2kT\pi}}\exp\left[-\frac{m\,v^2}{2kT}\right] \tag{5.7}$$

(c) From the condition that the derivation of the distribution $F(v)$ must equal zero

$$0 = v\exp\left[-\frac{m\,v^2}{2kT}\right] \tag{5.8}$$

we will get that the most probable speed is zero.

(d)

$$\langle v\rangle = \int_0^\infty v\,F(v)\,\mathrm{d}v = \sqrt{\frac{2kT}{\pi m}} \tag{5.9}$$

(e)

$$\Gamma = \int_0^\infty v_x\,f(v_x)\mathrm{d}v_x = n\sqrt{\frac{kT}{2\pi m}} \tag{5.10}$$

5.2.3 Two-dimensional Maxwell-Boltzmann distribution function

Solve the tasks of the preceding problem with two-dimensional Maxwell-Boltzmann distribution function

$$f(v_x, v_y) = C\cdot\exp\left[-\frac{m\,(v_x^2+v_y^2)}{2kT}\right]. \tag{5.11}$$

Results:

(a) $C = \frac{m\,n}{2\,\pi\,k\,T}$

(b) $F(v) = 2\,\pi\,v\,f(v) = \frac{n\,m}{k\,T}v\,\exp\left[-\frac{m\,v^2}{2kT}\right]$

(c) Most probable speed $v = \sqrt{\frac{k\,T}{m}}$.

(d) Mean speed $\langle v\rangle = \sqrt{\frac{k\,T\,\pi}{2\,m}}$.

(e) $\Gamma = n\sqrt{\frac{k\,T}{2\,m\,\pi}}$

5.2.4 Three-dimensional Maxwell-Boltzmann distribution function

Solve the tasks of the preceding problem with three-dimensional Maxwell-Boltzmann distribution function

$$f(v_x, v_y, v_z) = C \cdot \exp\left[-\frac{m\,(v_x^2 + v_y^2 + v_z^2)}{2kT}\right]. \tag{5.12}$$

Results:

(a) $C = n\left(\frac{m}{2\,\pi\,k\,T}\right)^{3/2}$

(b) $F(v) = 4\pi\, n\,\left(\frac{m}{2\,\pi kT}\right)^{3/2} v^2\,\exp\left[-\frac{m\,v^2}{2kT}\right]$

(c) Most probable speed $v = \sqrt{\frac{2\,k\,T}{m}}$.

(d) Mean speed $\langle v\rangle = \sqrt{\frac{8\,k\,T}{\pi\,m}}$.

(e) $\Gamma = n\sqrt{\frac{k\,T}{2\,m\,\pi}}$

5.2.5 Exotic one-dimensional distribution function

Solve the tasks of the preceding problem with the following function (Cauchy/Lorentz distribution):

$$f(v) = \frac{C}{v^2 + \frac{kT}{m}}. \tag{5.13}$$

Results:

(a) $C = n\sqrt{\frac{kT}{m\pi^2}}$

(b) $F(v) = 2\,n\,\sqrt{\frac{kT}{m\pi^2}}\,\frac{1}{v^2+\frac{kT}{m}}$

(c) Most probable $v = 0$ speed.

(d) Mean speed v is not defined, [1] see Cauchy distribution.

(e) Not defined.

Chapter 6

Particle interactions in plasmas

6.1 Theory

Collisional phenomena can be divided into two categories:

- *elastic* - conservation of mass, momentum and energy is valid in such a way that there are no changes in the internal states of the particles involved and there is neither creation nor annihilation of particles.
- *inelastic* - the internal states of some or all of the particles involved are changed and particles may be created as well as destroyed. A charged particle may recombine with another to form a neutral particle or it can attach itself to a neutral particle to form a heavier charged particle. The energy state of an electron in an atom may be raised and electrons can be removed from their atoms resulting in ionization.

The total scattering cross section can be obtained by integrating $\sigma(\chi, \varepsilon)d\Omega$ over the entire solid angle:

$$\sigma_{\mathrm{t}} = \int_{\Omega} \sigma(\chi, \varepsilon)d\Omega. \tag{6.1}$$

In the special case, when the interaction potential is isotropic (e.g. Coulomb potential), we can get the total scattering cross section using the formula

$$\sigma_{\mathrm{t}} = 2\pi \int_{0}^{\pi} \sigma(\chi) \sin \chi d\chi. \tag{6.2}$$

For the same case, when the interaction potential is isotropic, we can get the momentum transfer cross section using the formula:

$$\sigma_{\mathrm{m}} = 2\pi \int_0^{\pi} (1 - \cos\chi)\sigma(\chi)\sin\chi d\chi. \tag{6.3}$$

6.2 Problems

6.2.1 Mean free path of Xe ions

Scattering cross section σ for elastic collisions of Xe^+ ions with Xe atoms is approximately independent on their energy with cross section value of $\sigma = 10^{-14}\,\mathrm{cm}^2$.

A) Calculate mean free path l of Xe^+ ions for elastic collisions in a weakly ionized plasma in xenon atmosphere at room temperature (20 °C) at the pressure:
a) 1000 Pa
b) 10 Pa
c) 0.1 Pa

B) How long is the time period between two subsequent collisions, if the mean temperature of Xe ions is T = 1000 K?

Solution:

A) The mean free path is defined as

$$\lambda = \frac{1}{n\,\sigma}.$$

Density of particles can be calculated from the equation of state $p = n\,k\,T$, so

$$\lambda = \frac{k\,T}{p\,\sigma}.$$

So the final results for given pressures are:

a) $4 \cdot 10^{-6}\,\mathrm{m}$ b) $4 \cdot 10^{-4}\,\mathrm{m}$ c) $4 \cdot 10^{-2}\,\mathrm{m}$.

B) The thermal velocity of ions is $v = \sqrt{\frac{3\,k\,T}{M}}$. Mass of Xe ion is approximately 131 amu (1 amu = $1.66 \cdot 10^{-27}\,\mathrm{kg}$). The time period between two subsequent collisions equals to the fraction of mean free path and thermal velocity:

$$\tau = \lambda \sqrt{\frac{m}{3\,k\,T}}.$$

So the results are: a) $17 \cdot 10^{-9}$ s b) $17 \cdot 10^{-7}$ s c) $17 \cdot 10^{-5}$ s.

6.2.2 Hard sphere model

What is the total scattering cross section for the hard sphere model (two elastic spheres, radius R_1 and R_2)?
$[\pi\,(R_1 + R_2)^2]$

6.2.3 Total scattering cross section

Differential cross section is given by

$$\sigma(\chi) = \frac{1}{2}\,\sigma_0(3\cos^2\chi + 1) \tag{6.4}$$

Calculate the total cross section and the momentum transfer cross section.
$[4\,\pi\,\sigma_0,\ 4\,\pi\,\sigma_0]$

Chapter 7

Macroscopic transport equations

7.1 Theory

From different moments of Boltzmann equation, the following macroscopic transport equations can be derived:

- From the condition of conservation of mass the continuity equation

$$\frac{\partial \rho_{m\alpha}}{\partial t} + \nabla \cdot (\rho_{m\alpha} \mathbf{u}_\alpha) = S_\alpha, \tag{7.1}$$

  where $\rho_{m\alpha}$ is the mass density of type-α particles and S_α describes the creation or destruction of particles due to collisions (ionization, recombination, etc.).

- From conservation of momentum the momentum transfer equation

$$\rho_{m\alpha} \frac{\mathrm{D}\mathbf{u}_\alpha}{\mathrm{D}t} = n_\alpha q_\alpha \left(\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}\right) + \rho_{m\alpha}\mathbf{g} - \nabla \cdot \mathcal{P}_\alpha + \mathbf{A}_\alpha - \mathbf{u}_\alpha S_\alpha \tag{7.2}$$

  $\mathbf{u}_\alpha$ is the mean velocity, $\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \mathbf{u}_\alpha \cdot \nabla$ is the total time derivative operator, n_α is the particle density, q_α is the charge of a single particle, $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields, $\mathbf{g}$ is the gravitational acceleration, $\mathcal{P}_\alpha$ is the kinetic pressure dyad,

$$\mathbf{A}_\alpha = -\rho_{m\alpha} \sum_\beta \nu_{\alpha\beta} (\mathbf{u}_\alpha - \mathbf{u}_\beta) \tag{7.3}$$

  is the collision term with $\nu_{\alpha\beta}$ being the collision frequency for the momentum transfer between the particles of type α and particles of type β. From conservation of momentum during a collision follows

$$\rho_{m\alpha}\, \nu_{\alpha\beta} = \rho_{m\beta}\, \nu_{\beta\alpha}. \tag{7.4}$$

- From the energy conservation the energy transport equation

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{3\,p_\alpha}{2}\right)+\frac{3\,p_\alpha}{2}\nabla\cdot\mathbf{u}_\alpha+(\mathcal{P}\cdot\nabla)\cdot\mathbf{u}_\alpha+\nabla\cdot\mathbf{q}_\alpha=$$
$$=M_\alpha-\mathbf{u}_\alpha\cdot\mathbf{A}_\alpha+\frac{1}{2}\,u_\alpha^2\,S_\alpha, \tag{7.5}$$

  where p_α is the scalar pressure, $\mathbf{q}_\alpha$ is the heat flux vector and M_α represents the rate of energy density change due to collisions.

7.2 Problems

7.2.1 Afterglow

Consider a homogeneous plasma afterglow consisting of electrons and one type of singly charged positive ions. In this case, the continuity equation is

$$\frac{\partial n_e}{\partial t}=-k_r\,n_e\,n_i, \tag{7.6}$$

where k_r is the rate coefficient for recombination. The spatial derivatives vanish because of the spatial uniformity. The concentration of electrons at $t=0$ is n_0. Calculate $n_e(t>0)$. Remember the quasineutrality condition. $\left[n_e(t)=\frac{n_0}{n_0\,k_r\,t+1}\right]$

7.2.2 Macroscopic collision term – momentum equation

Consider a uniform mixture of different fluids (all spatial derivatives vanish), with no external forces, so that the equation of motion for the α species reduces to

$$\frac{\mathrm{d}\mathbf{u}_\alpha}{\mathrm{d}t}=-\nu_{\alpha\beta}\,(\mathbf{u}_\alpha-\mathbf{u}_\beta). \tag{7.7}$$

Assume that the mass density of β species is much greater and thus neglect the temporal change of $\mathbf{u}_\beta$. Notice that at equilibrium ($\mathrm{d}\mathbf{u}_\alpha/\mathrm{d}t=0$) the velocities of all species must be the same.

solution The situation is identical in all spatial coordinates, thus, only the solution in the x direction will be presented.

$$\frac{\mathrm{d}u_{\alpha x}(t)}{\mathrm{d}t}+\nu_{\alpha\beta}u_{\alpha x}(t)=\nu_{\alpha\beta}\,u_{\beta x} \tag{7.8}$$

This simple differential equation can be solved by the method of variation of parameter. First, look for the particular solution of the homogeneous equation

$$\frac{\mathrm{d}u_{\alpha x,p}(t)}{\mathrm{d}t}+\nu_{\alpha\beta}u_{\alpha x,p}(t)=0 \tag{7.9}$$

This is obviously

$$u_{\alpha x,p}(t) = C\,\mathrm{e}^{-\nu_{\alpha\beta}t} \tag{7.10}$$

We now take the parameter C to be time-dependent $C = C(t)$ and calculate the deriative

$$\frac{\mathrm{d}u_{\alpha x}(t)}{\mathrm{d}t} = \frac{\mathrm{d}C(t)}{\mathrm{d}t}\,\mathrm{e}^{-\nu_{\alpha\beta}t} - C(t)\,\nu_{\alpha\beta}\mathrm{e}^{-\nu_{\alpha\beta}t} \tag{7.11}$$

inserting this into the original equation (7.8) yields

$$\frac{\mathrm{d}C(t)}{\mathrm{d}t}\,\mathrm{e}^{-\nu_{\alpha\beta}t} = \nu_{\alpha\beta}\,u_{\beta x}$$

from which we obtain by integrating

$$C(t) = u_{\beta x}\,\mathrm{e}^{\nu_{\alpha\beta}\,t} + K$$

where K is an arbitrary integration constant. The solution is then

$$u_{\alpha x}(t) = u_{\beta x} + K\,\mathrm{e}^{-\nu_{\alpha\beta}\,t} \tag{7.12}$$

And similarly for all three spatial components. The velocity $\mathbf{u}_\alpha$ will exponentially approach to the velocity $\mathbf{u}_\beta$ with the rate given by the collision frequency for momentum transfer $\nu_{\alpha\beta}$.

7.2.3 Macroscopic collision – momentum equation II

Recalculate the task of the previous problem without the assumption $\mathbf{u}_\beta = \text{const}$. In this case, the velocities $\mathbf{u}_\alpha$, $\mathbf{u}_\beta$ are described by a pair of coupled differential equations

$$\frac{\mathrm{d}\mathbf{u}_\alpha(t)}{\mathrm{d}t} = -\nu_{\alpha\beta}\,(\mathbf{u}_\alpha(t) - \mathbf{u}_\beta(t)). \tag{7.13}$$

$$\frac{\mathrm{d}\mathbf{u}_\beta(t)}{\mathrm{d}t} = -\frac{\rho_{m\alpha}}{\rho_{m\beta}}\,\nu_{\alpha\beta}\,(\mathbf{u}_\beta(t) - \mathbf{u}_\alpha(t)), \tag{7.14}$$

where $\rho_{m\alpha}$, $\rho_{m\beta}$ are the mass densities of particles α, β. Suppose that $\mathbf{u}_\alpha$ and $\mathbf{u}_\beta$ are parallel and $u_\alpha(t=0) = 2\,u_\beta(t=0)$.

(a) Calculate the time dependence of the difference $u = u_\alpha - u_\beta$.

(b) Calculate $u_\alpha(t)$ and $u_\beta(t)$.

Results:

(a) $u(t) = u_\alpha(0)\cdot\exp\left[\left(1+\frac{\rho_{m\alpha}}{\rho_{m\beta}}\right)\,t\right]$

(b) $u_\alpha(t) = \frac{u_\alpha(0)}{\rho_{m\alpha}+\rho_{m\beta}}\left(\rho_{m\beta}\cdot\exp\left[-\nu_{\alpha\beta}\left(1+\frac{\rho_{m\alpha}}{\rho_{m\beta}}\right)\,t\right]+\rho_{m\alpha}\right)$

$u_\beta(t) = u(t) + u_\alpha(t)$

7.2.4 Simplified heat flow equation

Suppose the simplified equation for heat flow in a stationary electron gas

$$\frac{5\,p_e}{2}\,\nabla\left(\frac{p_e}{\rho_{me}}\right)+\Omega_{ce}\,(\mathbf{q}_e\times\mathbf{B})=\left(\frac{\delta\mathbf{q_e}}{\delta t}\right)_{\rm coll}. \tag{7.15}$$

Assume the collision term given by the relaxation model

$$\left(\frac{\delta\mathbf{q_e}}{\delta t}\right)_{\rm coll}=-\nu\,(f_e-f_{e0}) \tag{7.16}$$

and the ideal gas law $p_e = n_e\,k\,T_e$. Show that the heat flow equation can be written as

$$\frac{\Omega_{ce}}{\nu}\,(\mathbf{q}_e\times\mathbf{B})=-K_0\nabla T_e+(f_e-f_{e0}), \tag{7.17}$$

where

$$K_0=\frac{5\,k\,p_e}{2\,m_e\,\nu} \tag{7.18}$$

is the thermal conductivity.

Chapter 8

Macroscopic equations for a conducting fluid

8.1 Theory

The equations governing the important physical properties of the plasma as a whole can be obtained by summing the terms for the particular species. If also several simplifying assumptions are made, the following set of so called *magnetohydrodynamic equations* can be derived:

- The continuity equation

$$\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \mathbf{u}) = 0 \tag{8.1}$$

- The momentum equation

$$\rho_m \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \mathbf{J} \times \mathbf{B} - \nabla p \tag{8.2}$$

- Generalised Ohm's law

$$\mathbf{J} = \sigma_0(\mathbf{E} + \mathbf{u} \times \mathbf{B}) - \frac{\sigma_0}{n\,e}\,\mathbf{J} \times \mathbf{B}. \tag{8.3}$$

The electric and magnetic fields are further bound by the Maxwell equations. In these equations, viscosity and thermal conductivity are neglected.

8.2 Problems

8.2.1 Electric current density

The mean velocity of plasma $\mathbf{u}$ is defined as a weighted average of the mean velocities of the particular species

$$\mathbf{u} = \sum_{\alpha} \frac{\rho_{m\alpha}}{\rho_m}\,\mathbf{u}_\alpha \tag{8.4}$$

where ρ_m is the total mass density of the plasma. Each species has concentration n_α, charge q_α and the so called diffusion velocity $\mathbf{w}_\alpha = \mathbf{u}_\alpha - \mathbf{u}$. Calculate the total electric current density $\mathbf{J}$ in terms of the total electric charge density ρ and the particular densities, charges and diffusion velocities. Note, that due to the definition of the mean velocity of plasma, the result is *not* simply $\mathbf{J} = \rho\,\mathbf{u}$.
$\left[\mathbf{J} = \rho\mathbf{u} + \sum_\alpha n_\alpha\, q_\alpha \mathbf{w}_\alpha\right]$

8.2.2 Fully ionised plasma

From the equation for electric current density in fully ionised plasma containing electrons and one type of ions with charge e

$$\mathbf{J} = \sum_\alpha n_\alpha\, q_\alpha\, \mathbf{u}_\alpha = e(n_i\, \mathbf{u}_i - n_e\, \mathbf{u}_e) \tag{8.5}$$

and form the equation for the mean velocity of the plasma as a whole

$$\mathbf{u} = \frac{1}{\rho_m}\left(\rho_{me}\, \mathbf{u}_e + \rho_{mi}\, \mathbf{u}_i\right) \tag{8.6}$$

derive the drift velocities $\mathbf{u}_i$ and $\mathbf{u}_e$.
$\left[\mathbf{u}_i = \frac{\mu}{\rho_{mi}}\left(\frac{\rho_m \mathbf{u}}{m_e} + \frac{\mathbf{J}}{e}\right), \quad \mathbf{u}_e = \frac{\mu}{\rho_{me}}\left(\frac{\rho_m \mathbf{u}}{m_i} - \frac{\mathbf{J}}{e}\right), \quad \mu = \frac{m_e\, m_i}{m_e + m_i}\right]$

8.2.3 Diffusion across the magnetic field

From the momentum conservation equation with the magnetohydrodynamic approximation

$$\rho_m \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \mathbf{J} \times \mathbf{B} - \nabla p \tag{8.7}$$

and the generalised Ohm's law in the simplified form and without considering the Hall effect term

$$\mathbf{J} = \sigma_0\left(\mathbf{E} + \mathbf{u} \times \mathbf{B}\right) \tag{8.8}$$

derive the equation for the fluid velocity $\mathbf{u}$.

Assume $E = 0$ and $p =$ const. and calculate the fluid velocity perpendicular to the magnetic field $\mathbf{B}$.

Solution The equation for $\mathbf{u}$ is

$$\rho_m \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \sigma_0\, \mathbf{E} \times \mathbf{B} + \sigma_0\left(\mathbf{u} \times \mathbf{B}\right) \times \mathbf{B} - \nabla p. \tag{8.9}$$

Assuming $E = 0$ and $p =$ const., it reduces to

$$\rho_m \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \sigma_0\left(\mathbf{u} \times \mathbf{B}\right) \times \mathbf{B} \tag{8.10}$$

To calculate the vector $(\mathbf{u} \times \mathbf{B}) \times \mathbf{B}$ we define the coordinates such that the z-axis is parallel with the magnetic field. In these coordinates, the cross product is

$$(\mathbf{u} \times \mathbf{B}) \times \mathbf{B} = (-u_x B^2,\ -u_y B^2,\ 0). \tag{8.11}$$

The equations for x and y component of the velocity are thus of the same form. Writing only the equation in x

$$\frac{\mathrm{D}u_x}{\mathrm{D}t} = \frac{-\sigma_0 B^2}{\rho_m} u_x \tag{8.12}$$

This has a form of a simple decay problem. The solution is

$$u_x(t) = u_x(0) \exp\left(-\frac{\sigma_0 B^2}{\rho_m} t\right). \tag{8.13}$$

The same holds for u_y, so the time dependence of the component of the velocity perpendicular to the magnetic field $u_\perp = \sqrt{u_x^2 + u_y^2}$ is

$$u_\perp(t) = u_\perp(0) \exp\left(-t/\tau\right), \tag{8.14}$$

where

$$\tau = \frac{\rho_m}{\sigma_0 B^2} \tag{8.15}$$

is the characteristic time for diffusion across the magnetic field lines.

Chapter 9

Plasma conductivity and diffusion

9.1 Theory

In weakly ionised cold plasma, the equation of motion for electrons takes the simple form of the so called Langevin equation

$$m_e \frac{\mathrm{D}\mathbf{u}_e}{\mathrm{D}t} = -e\left(\mathbf{E} + \mathbf{u}_e \times \mathbf{B}\right) - \nu_c\, m_e\, \mathbf{u}_e, \tag{9.1}$$

where ν_c is the collision frequency for momentum transfer between the electrons and the heavy particles.

In the absence of a magnetic field, the current produced by moving electrons is

$$\mathbf{J} = -e\, n_e\, \mathbf{u_e} \tag{9.2}$$

and the DC conductivity is

$$\sigma_0 = \frac{n_e\, e^2}{m_e\, \nu_c} \tag{9.3}$$

and the electron mobility

$$\mathcal{M}_e = -\frac{e}{m_e\, \nu_c} = -\frac{\sigma_0}{n_e\, e}. \tag{9.4}$$

When an external magnetic field is present, the plasma becomes anisotropic, and the DC conductivity and electron mobility are described by tensors (see problem 9.2.2).

In weakly ionised plasma with relatively high density of neutrals, the diffusion equation for charged species α is

$$\frac{\partial n_\alpha}{\partial t} = D\, \nabla^2 n_\alpha. \tag{9.5}$$

The diffusion coefficient D_e for electrons in an isotropic plasma with no internal electric fields is

$$D_e = \frac{k\,T_e}{m_e\,\nu_c}. \tag{9.6}$$

In the magnetised plasma, the D_e becomes a tensor similar to the DC conductivity or electron mobility.

In plasma, electrons usually diffuse faster than ions due to their lower mass and thus higher mobility. As a result, internal electric field is produced, slowing down the diffusion of electrons and speeding up the diffusion of ions. This effect is called ambipolar diffusion. If the relation between the ion concentration n_i and n_e is

$$n_i = C\,n_e \tag{9.7}$$

where C is a constant, the ambipolar diffusion coefficient D_a is

$$D_a = \frac{k\,(T_e + C\,T_i)}{m_e\,\nu_{ce} + C\;m_i\,\nu_{ci}}, \tag{9.8}$$

where ν_{ci}, ν_{ce} are the collision frequencies for momentum transfer between neutrals and ions or electrons, respectively.

9.2 Problems

9.2.1 DC plasma conductivity

From the Langevin equation for electrons in the absence of magnetic field and in the steady state

$$-\,e\,\mathbf{E} - m_e\,\nu_c\,\mathbf{u}_e = 0 \tag{9.9}$$

derive the expression for the DC conductivity of the plasma.

Solution The electric current density is defined as

$$\mathbf{J} = -e\,n_e\,\mathbf{u}_e \tag{9.10}$$

Inserting this into the Langevin equation (9.9), we obtain the expression for the current density $\mathbf{J}$

$$\mathbf{J} = \frac{n_e\,e^2}{m_e\,\nu_c}\,\mathbf{E} \tag{9.11}$$

The Ohm's law states

$$\mathbf{J} = \sigma_0\,\mathbf{E} \tag{9.12}$$

the DC conductivity is thus

$$\sigma_0 = \frac{n_e\,e^2}{m_e\,\nu_c}. \tag{9.13}$$

9.2.2 Mobility tensor for magnetised plasma

In magnetised plasma, the Ohm's law obtains a matrix form

$$\mathbf{J} = \mathcal{S} \cdot \mathbf{E} \tag{9.14}$$

$$\begin{pmatrix} J_x \\ J_y \\ J_z \end{pmatrix} = \begin{pmatrix} \sigma_\perp & -\sigma_H & 0 \\ \sigma_H & \sigma_\perp & 0 \\ 0 & 0 & \sigma_\parallel \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix},$$

where the components of the DC conductivity tensor $\mathcal{S}$ are

$$\sigma_\perp = \frac{\nu_c^2}{\nu_c^2 + \Omega_{ce}^2}\,\sigma_0 \tag{9.15}$$

$$\sigma_H = \frac{\nu_c\,\Omega_{ce}}{\nu_c^2 + \Omega_{ce}^2}\,\sigma_0 \tag{9.16}$$

$$\sigma_\parallel = \sigma_0 = \frac{n_e\,e^2}{m_e\,\nu_c}, \tag{9.17}$$

where ν_c is the collision frequency for momentum transfer between electrons and heavy particles and Ω_{ce} is the electron cyclotron frequency due to the external magnetic field. Find the components of the mobility tensor $\mathcal{M}_e$ defined as

$$\mathbf{u}_e = \mathcal{M}_e \cdot \mathbf{E}. \tag{9.18}$$

Results:

$$\mathcal{M}_e = \begin{pmatrix} \mathcal{M}_\perp & -\mathcal{M}_H & 0 \\ \mathcal{M}_H & \mathcal{M}_\perp & 0 \\ 0 & 0 & \mathcal{M}_\parallel \end{pmatrix}$$

$$\mathcal{M}_\perp = -\frac{\nu_c\,e}{m_e\,(\nu_c^2 + \Omega_{ce}^2)} \tag{9.19}$$

$$\mathcal{M}_H = -\frac{\Omega_{ce}\,e}{m_e\,(\nu_c^2 + \Omega_{ce}^2)} \tag{9.20}$$

$$\mathcal{M}_\parallel = -\frac{e}{m_e\,\nu_c} \tag{9.21}$$

9.2.3 Ohm's law with magnetic field

Consider the equation $\mathbf{J} = \mathcal{S} \cdot \mathbf{E}$ as in the preceding problem. Suppose that $\mathbf{E} = (E_\perp,\ 0,\ E_\parallel)$ and $\mathbf{B} = (0,\ 0,\ B_0)$. Calculate $\mathbf{J}$. Note what direction of the electric current is governed by what component of $\mathcal{S}$.
$\left[\mathbf{J} = (\sigma_\perp\,E_\perp,\ \sigma_H\,E_\perp,\ \sigma_\parallel\,E_\parallel)\right]$

9.2.4 Diffusion equation

Solve the diffusion equation for one spatial dimension

$$\frac{\partial n(x,t)}{\partial t} = D\,\frac{\partial^2 n(x,t)}{\partial x^2} \tag{9.22}$$

by separation of variables, assuming

$$n(x,t) = S(x)\,T(t). \tag{9.23}$$

Results:

- $T_k(t) = T_0\,\exp(-D\,k^2\,t)$
- $S(x) = c(k)\,\exp(i\,k\,x)$, k is a separation constant
- $n(x,t) = \int\limits_{-\infty}^{+\infty} c(k)\,\exp(-i\,k\,x - D\,k^2\,t)\,\mathrm{d}k$

Chapter 10

Some basic plasma phenomena

10.1 Theory

In a paper from 1923, an American chemist and physicist wrote, that electrons are repelaed from negative electrode, whereas positive ions are attracted towards it. Langmuir concluded, that around every negative electrode, a sheath of defined thickness containing only positive ions and neutral atoms exists. Morever, Langmuir observed, that also the glass wall of the discharge chamber is negatively charged and repels (or reflects) almost all electrons [16].

The fact, that insulated objects inside plasma are negatively charged (in respect to plasma) to floating potential, is caused by higher mobility of electrons than ions. The thermal velocity of electrons $(k_\mathrm{B}T_\mathrm{e}/m_\mathrm{e})^{1/2}$ is at least 100 times higher than the thermal velocity of ions $(k_\mathrm{B}T_\mathrm{i}/M_\mathrm{i})^{1/2}$ [17]. The first reason for different mobility is higher mass of ions. If we consider only proton (the lightest ion that can appear in a plasma), than the ratio between the mass of proton and electron $m_\mathrm{p}/m_\mathrm{e}$ is 1836. This ratio corresponds approximately to the ratio of masses of heavy bowling ball (5 kg) and ping pong ball (2,7 g). Another reason for higher thermal velocity of electrons in low-temperature plasma is their higher temperature in respect to the ions.

The slowest possible speed of ions at the sheath edge is called the Bohm speed u_B. Ionts are accelerated to this speed in a quasi-neutral pre-sheath, where small electric field exists. The Bohm criterion of plasma sheath is described by following equation

$$u_\mathrm{s}(0) \geq u_\mathrm{B} = \sqrt{\frac{k_\mathrm{B}\,T_\mathrm{e}}{M_i}}\ . \qquad (10.1)$$

10.2 Problems

10.2.1 Waves in non-magnetized plasma

Plasma of so called E layer of Earth's ionosphere has electron density approximately $10^5\,\mathrm{cm}^{-3}$ and is at altitude of approximatelyo 100 km.
a) Which electromagnetic waves can be reflected from this layer?
b) Calculate the dielectric constant of plasma for the waves with frequencies of 100 Mhz and 1000 Hz.
b) Calculate the skin depth of the wave with frequency of 1000 Hz.

Solution:

a) All electromagnetic waves with frequency lower than the plasma frequency (2 839 725 Hz) will be reflected.

b) The dielectric constant of plasma is defined as

$$\varepsilon = 1 - \frac{\omega_p^2}{\omega^2}$$

For 100 MHz $\varepsilon = 0.9991$ (positive value, elmag. waves propagate), for 1000 Hz $\varepsilon = -8064037$ (negative value, imaginary refraction index, reflection).
c) Skin depth δ appropriately equals to $c/\sqrt{\omega_p^2 - \omega^2}$, where c is speed of light. The skin depth for 1000 Hz is 16.8 m.

10.2.2 Floating potential

Explain why insulated object inserted to plasma will aquire a negative potential with respect to the plasma itself.

10.2.3 Bohm velocity

Calculate the Bohm velocity for hydrogen ion in plasma with electron temperature of $T_\mathrm{e} = 1\,\mathrm{eV}$.
[9 787.2 m/s]

10.2.4 Plasma frequency

When the macroscopic neutrality of plasma is instantaneously perturbed by external means, the electrons react in such a way as to give rise to oscillations at the electron plasma frequency. Consider these oscillations, but include also the motion of ions. Derive the natural frequency of oscillation of the net charge density in this case. Use the linearized equations of continuity and of momentum for each species, and Poisson equation, considering only

the electric force due to the internal charge separation.
[$\omega = (\omega_e^2 + \omega_i^2)^{1/2}$, where $\omega_i = \sqrt{\frac{n_e\, e^2}{\varepsilon_0\, M_i}}$]

Chapter 11

Boltzmann and Fokker-Planck collision terms

11.1 Theory

Under several simplifying assumptions (mainly homogeneous and isotropic distribution function of electronic velocities, molecular chaos, considering only binary collisions and ignoring external forces), so called Boltzmann collision integral can be derived

$$\left(\frac{\partial f}{\partial t}\right)_{\text{coll}} = \iint g\,\sigma(g,\Omega)\,[f_e(\mathbf{v}')\,f_1(\mathbf{v}'_1) - f_e(\mathbf{v})\,f_1(\mathbf{v}_1)]\,d\Omega\,d^3\mathbf{v}. \tag{11.1}$$

$g = |\mathbf{v} - \mathbf{v}_1|$ is the relative speed of the electron and its collision partner, σ is the differential cross section for this type of collisions, depending on the solid angle Ω. Two types of distribution functions are considered here – the electronic distribution function $f_e(\mathbf{v})$ and that of the particular kind of collision partners $f_1(\mathbf{v}_1)$. If more kinds of collision partners should be considered, the collision term is expressed as a sum of terms similar to eq. (11.1).

The first term expresses the amount of electrons with initial velocity $\mathbf{v}'$ that undergo collisions with the collision partner with velocity $\mathbf{v}'_1$. After this collision, the electrons have velocity $\mathbf{v}$ and their collision partners have velocity $\mathbf{v}_1$, i.e. they add to the electronic distribution function at the velocity $\mathbf{v}$. The second term expresses an inverse collision, which leads to loss of particles of the velocity $\mathbf{v}$ and is thus negative.

If only collisions leading to small-angle deflections are considered, as expected for long-range Coulomb interactions, the Fokker-Planck collision term can be derived

$$\left(\frac{\delta\, f_\alpha}{\delta t}\right)_{\text{coll}} = -\sum_i \frac{\partial}{\partial v_i}(f_\alpha \langle \Delta v_i \rangle_{av}) + \frac{1}{2}\sum_{ij}\frac{\partial^2}{\partial v_i \partial v_j}(f_\alpha \langle \Delta v_i \Delta v_j \rangle_{av}), \tag{11.2}$$

where

$$\langle \Delta v_i \rangle_{av} = \int_\Omega \int_{v1} \Delta v_i \, g \, \sigma(\Omega) \mathrm{d}\Omega f_{\beta 1} \, \mathrm{d}^3 v_1 \tag{11.3}$$

$$\langle \Delta v_i \Delta v_j \rangle_{av} = \int_\Omega \int_{v_1} \Delta v_i \Delta v_j \, g \, \sigma(\Omega) \mathrm{d}\Omega f_{\beta 1} \, \mathrm{d}^3 v_1 \tag{11.4}$$

are the coefficients of dynamical friction and diffusion in velocity space, respectively.

11.2 Problems

11.2.1 Collisions for Maxwell-Boltzmann distribution function

Consider a plasma in which the electrons and the ions are characterised, respectively, by the following distribution functions

$$f_e = n_0 \left(\frac{m_e}{2 \pi k T_e} \right)^{3/2} \exp \left[- \frac{m_e (\mathbf{v} - \mathbf{u}_e)^2}{2 k T_e} \right] \tag{11.5}$$

$$f_i = n_0 \left(\frac{m_i}{2 \pi k T_i} \right)^{3/2} \exp \left[- \frac{m_i (\mathbf{v} - \mathbf{u}_i)^2}{2 k T_i} \right] \tag{11.6}$$

(a) Calculate the difference $(f_e(\mathbf{v}') \, f_i(\mathbf{v}_1') - f_e(\mathbf{v}) \, f_i(\mathbf{v}_1))$.

(b) Show that this plasma of electrons and ions will be in the equilibrium state, that is, the difference $(f_e(\mathbf{v}') \, f_i(\mathbf{v}_1') - f_e(\mathbf{v}) \, f_i(\mathbf{v}_1))$ will vanish if and only if $\mathbf{u}_e = \mathbf{u}_i$ and $T_e = T_i$.

Solution

(a)

$$\begin{aligned}(f_e(\mathbf{v}') \, f_i(\mathbf{v}_1') - f_e(\mathbf{v}) \, f_i(\mathbf{v}_1)) = n_0^2 \left(\frac{1}{2 \pi k} \right)^3 \left(\frac{m_e \, m_i}{T_e \, T_i} \right)^{3/2} \times \\ \times \left(\exp \left[- \frac{m_e \, (\mathbf{v}' - \mathbf{u}_e)^2}{2 k T_e} - \frac{m_i \, (\mathbf{v}_1' - \mathbf{u}_i)^2}{2 k T_i} \right] - \right. \\ \left. - \exp \left[- \frac{m_e \, (\mathbf{v} - \mathbf{u}_e)^2}{2 k T_e} - \frac{m_i \, (\mathbf{v}_1 - \mathbf{u}_i)^2}{2 k T_i} \right] \right)\end{aligned} \tag{11.7}$$

(b) For the difference to vanish, the term in the parentheses must equal zero. This will happen if the arguments of the exponentials will be equal. Let us rewrite the arguments, omitting the factor $-(2 k)^{-1}$:

$$\frac{m_e}{T_e} \, (v'^2 - 2 \, \mathbf{v}' \cdot \mathbf{u}_e + u_e^2) + \frac{m_i}{T_i} (v_1'^2 - 2 \, \mathbf{v}_1' \cdot \mathbf{u}_i + u_i^2) \tag{11.8}$$

$$\frac{m_e}{T_e}(v^2 - 2\,\mathbf{v}\cdot\mathbf{u}_e + u_e^2) + \frac{m_i}{T_i}(v_1^2 - 2\,\mathbf{v}_1\cdot\mathbf{u}_i + u_i^2) \tag{11.9}$$

From the derivation of the Boltzmann collision term follows, that the pairs of velocities $\mathbf{v}$, $\mathbf{v}_1$ and $\mathbf{v}'$, $\mathbf{v}_1'$ can be considered as pairs of velocities before and after an elastic two-body collision. Thus, they are bound by the conservation laws:

$$\frac{m_e\,v^2 + m_i\,v_1^2}{2} = \frac{m_e\,v'^2 + m_i\,v_1'^2}{2} \tag{11.10}$$

$$m_e\,\mathbf{v} + m_i\,\mathbf{v}_1 = m_e\,\mathbf{v}' + m_i\,\mathbf{v}_1' \tag{11.11}$$

It is now obvious from the last four equations, that the collision term will vanish if and only if $T_e = T_i$ and $\mathbf{u}_e = \mathbf{u}_i$. In other words, the distribution function f_e will be changed by the collisions only if the plasma is out of equilibrium – the collisions tend to bring the plasma to the state of equilibrium.

11.2.2 Collisions for different distributions

Recalculate the task (a) of the preceding problem with Druyvesteyn-like distribution function for electrons and Maxwell-Boltzmann-like distribution for ions (C_e, a_e and C_i are constants)

$$f_e = C_e\,\exp[-a_e\,m_e^2\,(\mathbf{v}-\mathbf{u}_e)^4] \tag{11.12}$$

$$f_i = C_i\,\exp\left[-\frac{m_i\,(\mathbf{v}_1'-\mathbf{u}_i)^2}{2\,k\,T_i}\right] \tag{11.13}$$

Will the difference $(f_e(\mathbf{v}')\,f_i(\mathbf{v}_1') - f_e(\mathbf{v})\,f_i(\mathbf{v}_1))$ be zero for $\mathbf{u}_e = \mathbf{u}_i$?

11.2.3 Collisions for Druyvesteyn distribution

Recalculate task (a) of the first problem for Druyvesteyn-like distribution for both electron and ion velocities (see eq. (11.12)). Can the collision term be equal zero for $\mathbf{u}_e = \mathbf{u}_i$? Is it possible to find equilibrium state of plasma described by the Boltzmann kinetic equation with Boltzmann collision term in form of Druyvesteyn-like distribution?

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