evropský sociální fond v ČR EVROPSKÁ UNIE MINISTERSTVO ŠKOLSTVÍ, MLÁDEŽE A TĚLOVÝCHOVY OP Vzdělávání pro konkurenceschopnost

INVESTICE DO ROZVOJE VZDĚLÁVÁNÍ

Two country-models in international economics: modeling, applications, and solution

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Lecture given at the Masaryk University, October, 2011

Outline of the Lecture

1. Motivation
2. International Economics
    - International Trade
    - Balassa-Samuelson effect
3. Application to Central European Countries: Brůha-Podpiera model
4. Numerical Techniques

Goal of the lecture

Goal of the lecture

During this lecture, I will introduce some models from international economics, which may be useful for understanding real convergence, trade flows, or external balance of open economies.

One can investigate these phenomena from different perspectives, such as:

- business-cycle dynamics,
- **trends**,
- .....

Goal of the lecture /2

I will concentrate on modeling **trends**. Hence, most models will be casted in a **perfect-foresight** framework with no aggregate uncertainty. This is distinct from DSGE models in:

- **Goal:** understanding of trends rather than business cycle fluctuations
- **Approach:** perfect foresight rather than rational expectations;
- **Solution:**
  - most DSGE – dynamics around BGP, where trends are exogenous (sometimes even around steady state)
  - this kind of models – dynamics of trends

International trade

The main issues:

1. **Why** there is trade?
2. **What** is traded?
3. **Who** trade with **whom**?
4. At which **price**?

Selected frameworks:

- Comparative advantages (David Ricardo)
- Intra-industry trade (Paul Krugman)
- Intra-industry trade + heterogenous firms (Jacques Melitz)

Ricardian theory of trade

- Countries differ in their technology.
- Key assumption: it is easier to move goods than technologies.
- Motive for trade – it is statically efficient to trade if technologies are different (so-called comparative advantages.)

This theory predicts that:

- Most trade will occur between countries with different technologies (North-South trade should dominate)
- As countries converge, motives for trade fall

Modern version of the model: Eaton and Kortum (2002)

Heckscher-Ohlin model of trade (1933)

- Countries differ in their factor endowments.
- Key assumption: it is easier to trade goods than factors of production.
- Key finding: trade alone may equalize factor prices.
- Motive for trade: endogenous differences in technology.

Countries **must differ** in order to trade:

- **Ricardo model** – technologies differ;
- **HO model** – factor endowments differ.

Empirical challenges to Ricardo and Heckscher-Ohlin

- Countries with similar technologies trade.
- Countries with similar factor endowments trade.
- $\Rightarrow$ North-North trade dominates trade flows (technologically advanced countries, capital abundance)
- A large fraction of trade is two-way **intra-industry trade**.

Krugman model of trade (1980)

Very elegant model, which can explain why countries with **identical technology** and **preferences** trade.

Key ingredients

- monopolistic competition;
- increasing-returns-to scale (product specialization);
- love-for-variety (consumers want to consume all possible goods).

The model relied by the then advances in modeling of imperfect competition (Dixit-Stiglitz approach).

Krugman model – stylized exposition /1

Consumers: utility maximization:

$$\left(\sum_i x_i^{\frac{\theta-1}{\theta}}\right)^{\frac{\theta}{\theta-1}},$$

s.t.

$$\sum_i p_i x_i = \text{Income}.$$

Parameter $\theta > 1$ measures the elasticity of substitution (if $\theta \to \infty$), goods are perfect substitutes (perfect competition).

Krugman model – stylized exposition /2

Demand function:

$$x_i = \left(\frac{p_i}{P}\right)^{-\theta},$$

$$P = \left(\sum_i p_i^{1-\theta}\right)^{\frac{1}{1-\theta}}.$$

Note:

1. P does not depend on x_i;
2. If $p_i = \bar{p}$, then $P = \bar{p} n^{\frac{1}{1-\theta}}$ – this is called **love-for-variety**.

Krugman model – stylized exposition /3

Firms: Total costs = marginal cost (constant for simplicity) + fixed costs of production:

$$\mathrm{TC}_i = q_i \frac{w}{a} + f,$$

(a is technology, f is fixed costs).

Resulting optimal **supply**:

$$p_i = \frac{\theta}{\theta - 1} \frac{w}{a}.$$

Without trade:

$$\mathrm{Profit}_i = \mathrm{TR}_i - \mathrm{TC}_i = p_i q_i - q_i \frac{w}{a} - f,$$

$$\frac{\mathrm{Profit}_i}{P} = \left(\frac{a}{w/P}\right)^{\theta-1} \frac{1}{\theta} \left(\frac{\theta-1}{\theta}\right)^{\theta-1} - f,$$

and the zero-profit condition yields the equilibrium real factor price $w/P \propto \frac{a}{f^{\frac{1}{\theta-1}}}$.

Krugman model – stylized exposition /5

Trade: iceberg costs – a fraction of goods sent is lost during transportation $\mathbf{t}$.

Domestic price: $p_i = \frac{\theta - 1}{\theta} \frac{w}{a}$;

Foreign price: $p_i^x = (1 + \mathbf{t}) \frac{\theta - 1}{\theta} \frac{w}{a}$

Results:

1. all goods are traded even if countries are perfectly symmetric (love-for-variety effect);
2. specialization (each country produces a subset of goods);
3. trade gains: increase the number of products (increase of profits);
4. decrease in $\mathbf{t}$: effect of P, but not on average of p_i.

Krugman model – stylized exposition /6

Asymmetric countries (n a large market (or in a country with better technology, i.e., lower marginal costs):

- lower price index P, but higher average price $\bar{P}$;
- consumers are less willing to import additional unit of foreign varieties (due to constant elasticity of the demand);
- relative factor price increases (aka currency appreciation)
- higher nominal income, lower price index P – higher real income.

Interesting implications in the economic geography.

Krugman model – empirical problems

- There is a lot of heterogeneity across firms, within any sector.
- Very few firms export (or engage in FDI).
- Exporters are very different from non exporters (usually bigger and more productive).

Melitz model

Heterogeneity:

- Firms differ in productivity

Trade barriers:

- Iceberg costs
- Fixed entry cost to export market

Extensions

- In the original Melitz model, countries are symmetric
- In the original Melitz model, firms differ only by productivity

All these assumptions can be relaxed

Melitz model – implications

Implications:

Three sets of firms:

- non-producers;
- those who produce only for the domestic market,
- exporters.

Sorting is based on productivity.

Original model has labor only, but if capital is added, then exporters would be larger than non-exporters.

Trade liberalization:

- Aggregate productivity is increasing;
- Reallocation to more productive firms;
- The effect of the liberalization can be seen even **before** the liberalization actually happens.

CES preferences

CES preferences are used in most international-trade models:

- Simplicity
- Constant-elasticity of the demand
- No choke prices (even with very large price, there is some demand)

Alternative: linear-quadratic utility:
$U = \alpha \sum_i q_i - \beta \sum_i q_i^2 - \gamma \left(\sum_i q_i\right)^2$

- Demand: $q_i = a - b * p_i + c * P$, with $P = \sum_i p_i$.
- There is a choke price: $p_i = \frac{a + c * P}{b}$
- Elasticity of demand increases with price
- Complicated

Comparison of IT models – based on Baldwin and Harrigan (2007)

Model	Pr (export=0)		
	distance	importer size	remoteness
Eaton-Kortum	+	+	+
Mon. comp. (CES)	0	0	0
Mon. comp. (linear demand)	+	0	+
Hetero. firms (CES)	+	-	+
Hetero. firms (linear demand)	+	+	+
Hetero. firms (CES + quality)	+	-	+

Comparison of IT models – based on Baldwin and Harrigan (2007) / 2

Model	Export price		
	distance	importer size	remoteness
Eaton-Kortum	-	0	+
Mon. comp. (CES)	0	0	0
Mon. comp. (linear demand)	-	0	+
Hetero. firms (CES)	-	-	+
Hetero. firms (linear demand)	-	-	+
Hetero. firms (CES + quality)	+	-	-

Open issues in international trade

Open issues:

- Why trade has increased faster than the GDP?
- The Interplay between FDI and trade?
- Why did trade collapse during the recent recession.

Real exchange rates – some definitions:

Real exchange rate = nominal FX + foreign price level - domestic price level in logs: $q = e + p^* - p$,

Two sectors: tradable and non-tradable. Domestic price level: $p = a * p^T + (1 - a) * p^{NT}$. Hence:

$$q = e + (p^{*T} - p^T) + [(1 - a)(p^{NT} - p^T) - (1 - a)(p^{*NT} - p^{*T})],$$

If PPP holds in the tradable sector, then $e + (p^{*T} - p^T) = 1$, i.e., real terms-of-trade: $q^T = e + (p^{*T} - p^T)$

Supply side with two sectors:

$Y_T = A_T F(K_T, L_T)$ and $Y_{NT} = A_{NT} G(K_{NT}, L_{NT})$.

If F and G are constant-return-to-scale, then in per capita terms ($y_T = Y_T/L_T = f(k_T) = 1/L_T * F(K_T/L_T, 1)$ and so on):
$y_T = A_T f(k_T)$ and $y_{NT} = A_{NT} f(k_{NT})$.

The F.O.C. are given as: $P_T A_T f'(k_T) = r$, $P_{NT} A_{NT} f'(k_{NT}) = r$,
and hence: $k_T = k_T(\underbrace{A_T}_{+}, \underbrace{r}_{-})$, $k_{NT} = k_{NT}(\underbrace{A_{NT}}_{+}, \underbrace{r}_{-})$

$P_T A_T[f(k_T) - f'(k_T)k_T] = w$,
$P_{NT} A_{NT}[f(k_{NT}) - f'(k_{NT})k_{NT}] = w$.

Factor price equalization

If the interest rate r is exogenous (world price) and both factors can freely move across sectors, then: $w = p_T w_T(\underbrace{A_T}_{+}, \underbrace{r}_{-})$ and

$w = p_{NT} w_{NT}(\underbrace{A_{NT}}_{+}, \underbrace{r}_{-})$ and hence:

$$\frac{P_{NT}}{P_T} = \frac{w_T(A_T, r)}{w_{NT}(A_{NT}, r)},$$

i.e., just the relative productivity in both sectors determines the relative price $\frac{P_{NT}}{P_T}$.

This result **does not depend on the demand side** of the model.

Log-linearization implies:

$$p^{NT} - p^T = \frac{\text{Labor share in NT}}{\text{Labor share in T}} a^T - a^{NT}.$$

BS effect:

Recall:

$$q = e + (p^{*T} - p^T) + [(1-a)(p^{NT} - p^T) - (1-a)(p^{*NT} - p^{*T})],$$

and plug in

$$p^{NT} - p^T = \frac{\text{Labor share in NT}}{\text{Labor share in T}} a^T - a^{NT}.$$

If the technological progress is **relatively** biased towards tradable sector, then the real FX rate will appreciate.

Pitfalls:

- Why should be technological progress biased towards the tradable sector?
- The RER is explained by the movements in the non-tradable prices: implications for Terms-of-Trade.

BS effect – evidence for CEE countries

The upper estimates suggest that about 1/3 of the observed RER appreciation is explained by the BS effect.

Explanations:

- Administrative and regulated prices
- Initial undervaluation
- Appreciation in the tradable sector

Motivation

Brůha-Podpiera two-country models

Motivation:

- to mimic a strong pace of the real exchange rate appreciation observed in transition countries,
- to inquire about the necessary model ingredients,

The model aims at long-run trends, not medium frequency deviations, so it is formulated as a perfect-foresight DGE model.

Stylized Facts

Stylized facts related to V4 countries:

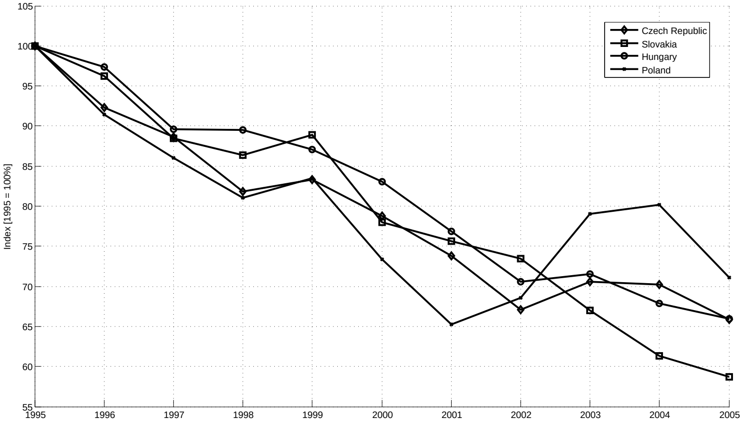
Economic convergence towards the EU average the convergence in GDP per capita towards the EU average about 1 p.p. a year

Trade integration an increase in the export/GDP ratio about 2 p.p. a year

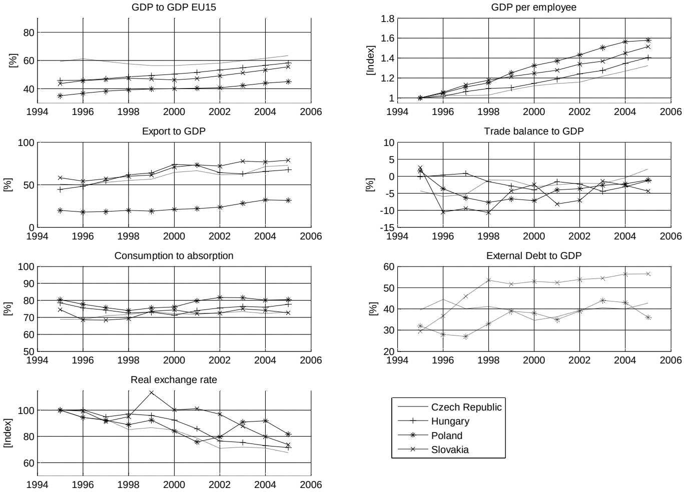
Real exchange rate appreciation about 2% a year (also in the *subindex of manufacturing*).

High-tech production share has gained from 1.5 - 2 p.p. a year

RER appreciation



Stylized facts



How to generate the RER appreciation?

It is not trivial to generate the RER appreciation after an uniform increase in productivity.

Why?

Because of the downward sloping demand curve!

Possible approaches:

1. Horizontal investment (expansion in new varieties)
2. Harrod-Balassa-Samualson story
3. Vertical investment (quality)

Horizontal investments

Love-for-variety

The horizontal investment explanation is based on a dichotomy between welfare-theoretical price indexes and 'average' observable price indexes.

A more productive country has ceteris paribus higher *average* prices, but welfare-theoretical price index is lower because of expansion in varieties.

Krugman (1980), Melitz (2003)

Export Eligibility

The productivity increase may be biased towards tradable goods, then the usual HBS effect causes the RER appreciation.

Why should be productivity biased towards tradables?

The self-selection mechanism, Bergin, Glick, Taylor (2006).

Data – very limited scope for the HBS in the V4 countries: Podpiera, Cincibuch (2006), Égert (2007).

Vertical Investment

The productivity increase vertical margin (quality investment), which implies that more goods can be sell for higher prices.

The RER appreciation after a productivity increase is based on dichotomy between quality- adjusted and quality- unadjusted prices. Price indexes are rarely adjusted for quality: Ahnert, Kenny (2004).

Task

is to integrate the vertical margin in a two-country DGE model and to inquire whether implications are consistent with the facts outlined above.

Framework

- Two countries in discrete time
- Each country endowed with a representative consumer and heterogeneous firms
- Foreign country – big and advanced
- Domestic country – small and converging
  - A metaphor for a transition country (domestic country) versus the Euro area (foreign country)

Vertical Investment Margin

We consider the following production function:

$$q_{jt} = A_t z_j k^\alpha l^{1-\alpha},$$

where A_t is the TFP, z_j is the idiosyncratic productivity, k is the quality input, l is labor and $\alpha \in [0\ 1)$.

If $\alpha = 0$, the production function is linear and all types goods have the same quality (as is standard e.g. in Ghironi, Melitz 2005).

If $\alpha > 0$, then it is optimal to choose $k > 0$. The optimal amount of invested capital $k = k(\underbrace{A_t}_{+}, \underbrace{z_j}_{+})$.

Firms

Firms are NPV optimizers and choose:

- labor input (variable);
- export eligibility (fixed at entry, sunk costs);
- quality level (fixed at entry).

Think of firms as of projects!

Backward induction used for solution of firms' problem:

1. labor is chosen as to equalize MPL with real wage;
2. the quality level is increasing in z_j and is higher for exporters;
3. there is a cut-off of z_j, which determines the exporter status.

Market structure – Dixit-Stiglitz

The aggregate good is defined as:

$$Q_t = \left(\sum_{\tau \le t} (1-\delta)^{t-\tau} \left[n_\tau \int {q^d_{j\tau t}}^{\frac{\theta-1}{\theta}} \, \mathrm{d}G(j) + n^*_\tau \int \mathbf{1}^{x*}_{j\tau} {q^m_{j\tau t}}^{\frac{\theta-1}{\theta}} \, \mathrm{d}G(j) \right] \right)^{\frac{\theta}{\theta-1}},$$

where n_τ is the number of entrants.
The market structure implies the aggregate price index:

$$P_t = \left(\sum_{\tau \le t} (1-\delta)^{t-\tau} \left[n_\tau \int {p^d_{j\tau t}}^{1-\theta} \, \mathrm{d}G(j) + n^*_\tau \int \mathbf{1}^{x*}_{j\tau} {p^m_{j\tau t}}^{1-\theta} \, \mathrm{d}G(j) \right] \right)^{\frac{1}{1-\theta}}.$$

Today, I would experiment with the linear-quadratic utility.

Households

The household maximizes

$$\max U = \sum_{t=0}^{\infty} \beta^t u(C_t),$$

subject to

$$B_t = (1+r^*_{t-1})B_{t-1} + \frac{-1}{\eta_t}(C_t - \mathbb{W}_t \mathcal{L}) + \frac{1}{\eta_t}(\Xi_t - \widetilde{c}_t n_t) - \frac{\Psi_B}{2}B_t^2 + \mathcal{T}_t,$$

$$\Xi_t = \sum_{s \le t} (1-\delta)^{t-s} n_s \widetilde{\mathbb{P}}_{s,t}.$$

FOC: $(1 + \Psi_B B_t) = \frac{\eta_{t+1}}{\eta_t}(1 + r^*_t)\mu_t^{t+1}$,
$\widetilde{c}_t = \sum_{v \ge 0} (1-\delta)^v \mu_t^{t+v} \widetilde{\mathbb{P}}_{t,t+v}$.

General Equilibrium

General Equilibrium

is a sequence of prices and quantities such that all agents maximize and all market clears.

- Labor Markets clear
- Goods Markets clear (GDP identity in the two countries)
- Consistency of Portfolios

Computational experiments

We use a computer-intensive sampling scheme to understand the implications of the various modeling assumptions.

Parameter	Lower bound	Upper bound
exit shock δ	0.050	0.750
CES parameter θ	3.500	7.500
icebergs $\mathbf{t}$	0.025	0.150
investment cost c^n	2.000	10.00
export-eligibility costs c^e	1.050	5.000

Implications

Is there a combination of parameters which could generate the reasonable REER appreciation?

No

under the standard assumptions (i.e. $\alpha = 0$).

Yes

if the model framework is extended by the quality investments.

Implications of Different Investment Margins

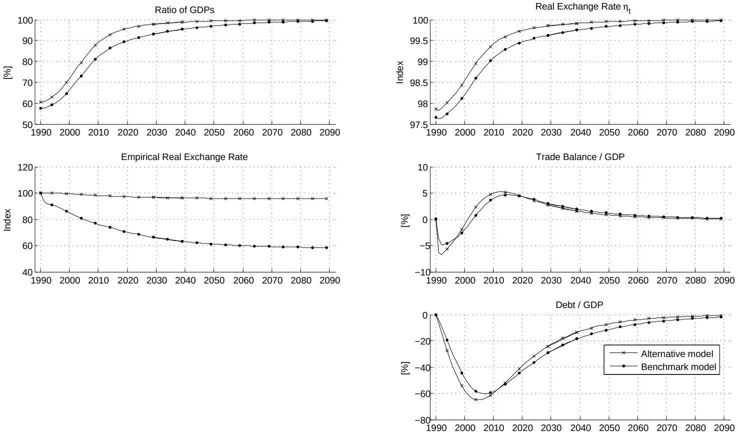
Export self-selection and horizontal margin helps ...

Export self-selectiveness can explain why more productive economies have higher price levels and help to explain why the 'observed' real FX rate of a converging economy is expected to appreciate.

... but they are alone insufficient

Quality investment needed to explain the observed pace.

The convergent trajectory



Applications

The modeling framework has been applied in a different context:

The assessment of the EMU inflation criterion by Brůha and Podpiera (2007), ECB WP 740

The calibration of the Czech economy by Brůha, Podpiera and Polák (2010), The Convergence Dynamics of a Transition Economy: The Case of the Czech Republic, Economic Modelling 27, January 2010, pp. 116-124.

The assessment of the EMU inflation criterion

RER decomposition:

$$\widehat{\eta}_t^e = \widehat{s}_t + \pi_t^* - \pi_t,$$

Conditional on stable nominal exchange rate $\widehat{s}_t = 0$, and the price stability of the EA, $\pi_t^* = 0.02$, we evaluate the dynamic path for the trend inflation of the converging country as follows:$\pi_t = \pi_t^* - \widehat{\eta}_t^e$.

The path can be in turn compared against the benchmark inflation (average inflation in the three best performing EU Member states plus 1.5 percentage points), i.e., $\pi_t^{**} = \pi_t^* + 0.015$

Probability of fulfillment of the criterion: $\text{Prob}(\pi_t^{**} > \pi_t | \sigma, \widehat{s}_t = 0, \widehat{\eta}_t^e)$. Historical evaluation using detrended (Hodrick-Prescott filter $\lambda = 100$) inflation (CPI index) over period 1995-2010.

Table: Parameters of the model

Parameter		CZ	HU	PO	SK
Elasticity of intra. subst.	θ	6.32			
Utility function	ϵ	0.50			
Production function	α	0.20			
Exit shock	δ	0.05			
Iceberg costs	$\mathbf{t}$	0.27			
Sunk cost of exporting	c^x	0.50			
Portfolio adj. costs	ψ_B	10.0			
Productivity	m	1.72	1.79	2.31	1.18
Productivity	n	6.28	7.37	8.97	6.58
Productivity	A^*	1.35	1.35	1.23	1.43
Productivity	τ	9.33	9.33	11.70	9.33
Relative country size	$\mathcal{L}^*/\mathcal{L}$	30	30	10	60

Domestic productivity: $A_t = A^* \frac{1+m\exp(-(t-1995)/\tau)}{1+n\exp(-(t-1995)/\tau)}$.

Figure: Czech Republic

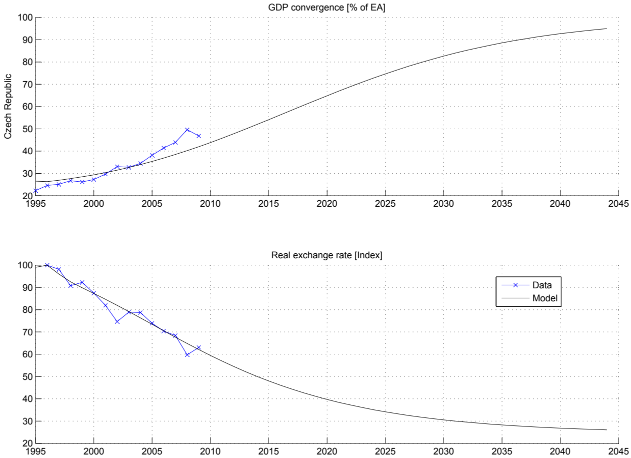


Figure: Hungary

Figure: Poland

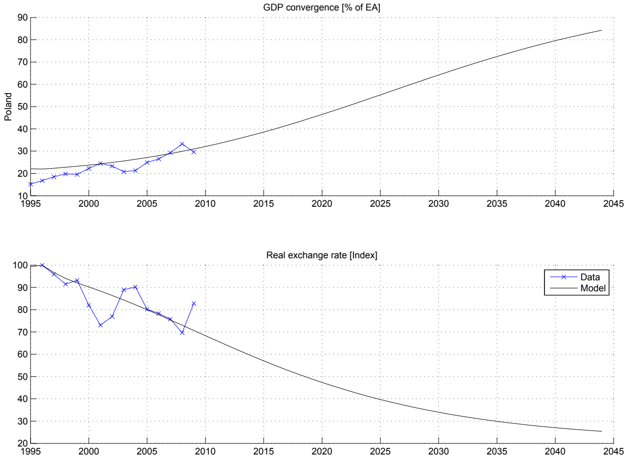


Figure: Slovakia

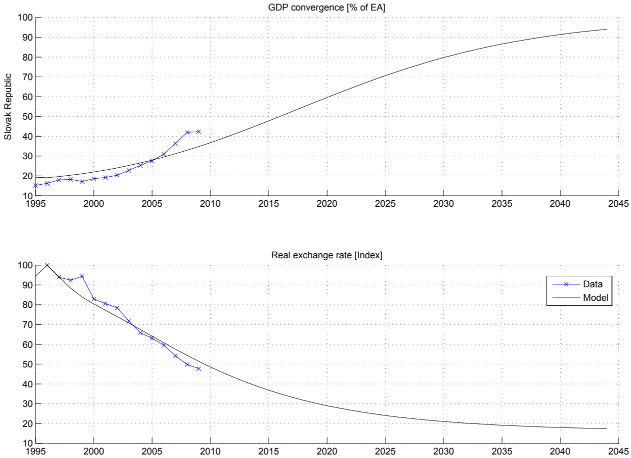
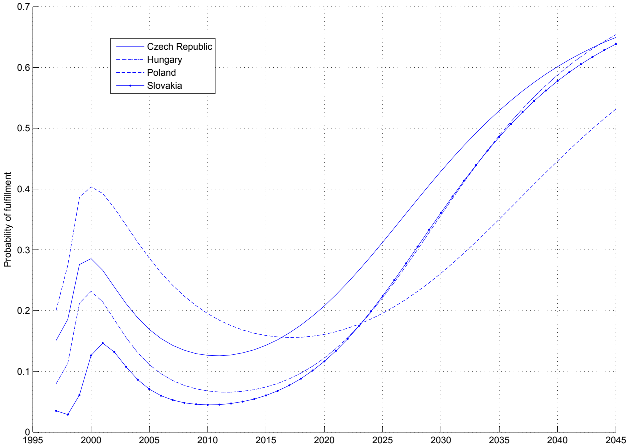


Figure: Probability of fulfillment of the inflation criterium



How to solve perfect-foresight models

This part of the lecture will overview selected solution techniques for perfect-foresight discrete-time economic models.

Problem statement

Two-point boundary value problem (with infinite horizon)

Two difficult points:

- **perfect-foresight:** what agents do today depends on the current state (what they did yesterday) and their expectations on what would happen tomorrow (what they will do in future);
- **infinite-horizon:** equilibrium is an infinite-dimensional system (policy function is of no help, if the model is not autonomous).

Problem statement

General problem statement:

1. Initial condition for state variables (e.g., capital and technology): k_1, A_1 given;
2. Law of motion for exogenous states (e.g. productivity): $\{A_t\}_{t=1}^{\infty}$ – agents know this;
3. Law of motion for endogenous states (such as capital accumulation: $k_{t+1} = (1-\delta)k_t + I_t$);
4. Equilibrium conditions (agents' decisions, market clearing) $F(k_t, c_t, A_t) = 0$ for all $t \in \mathbb{Z}_+$;
5. Transversality conditions (usually in the form of $\lim_{t\to\infty} \beta^t u(c_t, k_t) = 0$).

The goal is to find $\{k_t\}_{t=1}^{\infty}$ and $\{c_t\}_{t=1}^{\infty}$ consistent with conditions above.

Simple example – a growth in an open economy: model

- Two countries in discrete time;
- One country big and advanced, the other country small and converging;
- In each country, there is a representative consumer with recursive utilities: $U_t = \sum_{\tau=t}^{\infty} \beta^{\tau-t} u(c_t)$,
- Budget constraint:
  $C_t = (1 + r_t)W_t - W_{t+1} - \mathcal{T}(\Delta W_{t+1}) + Y_t - i_t$
- Production technology $Y_t = f(k_t, A_t)$, the market clearing $Y_t = c_t + i_t + x_t$;
- Capital accumulation $k_{t+1} = (1 - \delta)k_t + i_t$;
- Balance-of-payments $W_{t+1} = (1 + r_t)W_t + x_t$;
- Initial conditions k_1, W_1.
- Terminal conditions $\lim_{t\to\infty} \beta^t u'(c_t)k_t = 0$, $\lim_{t\to\infty} \beta^t u'(c_t)w_t = 0$.

Simple example – a growth in an open economy: equilibrium equations

- Optimal investments($u'(c_t) = \beta u'(c_{t+1})[f_k(k_{t+1}, A_{t+1}) + (1-\delta)]$, $(1 + \mathcal{T}'(\Delta W_{t+1}) = \beta(1 + r_{t+1})\frac{u'(c_{t+1})}{u'(c_t)}$
- Production technology $Y_t = f(k_t, A_t)$, the market clearing $Y_t = c_t + i_t + x_t$;
- Market clearings $x_t = -x_t^*$ and $W_t = -W_t^*$
- Capital accumulation $k_{t+1} = (1-\delta)k_t + i_t$;
- Balance-of-payments $W_{t+1} = (1 + r_t)W_t + x_t$;
- Initial conditions k_1, W_1, k_1^*, W_1^*.
- Terminal conditions $\lim_{t\to\infty} \beta^t u'(c_t)k_t = 0$, $\lim_{t\to\infty} \beta^t u'(c_t)w_t = 0$.

Three possible approaches

1. Domain-truncation techniques
   1. First-order iterations (Fair-Taylor)
   2. Quasi-Newton techniques (L-B-J)
2. Projection techniques

Domain truncation techniques solve the model for T periods with the hope that for $t > T$, endogenous variables will be at the constant levels (hence the infinite dimensionality is approximated by the dynamics with finite horizon).

Projection techniques approximate the equilibrium dynamics by a (linear) combination of few elements (basic functions).

Fair-Taylor approach

Fair-Taylor:

1. choose T and guess $\{k_t^0, c_t^0\}_{t=1}^T$
2. set $i = 1$ and for $t = 1, \ldots T$, compute k_t^i and c_t^i using k_{t-1}^i and c_{t-1}^i and k_{t+1}^{i-1} and c_{t+1}^{i-1};
3. check the convergence, if the convergence is not achieved, increase $i \leftarrow i + 1$ and go to 2.

Advantages:

- economic intuition – learning;

Disadvantages:

- it may not converge – Gauss-Seidel method;
- sometimes a dampening factor is helpful ($k_t^i = \mu k_t^{i*} + (1 - \mu) k_t^{i-1}$);
- even if it converges, it is slow (linear convergence).

L-B-J approach

L-B-J (due to Lafargue, 1990, Boucekkine, 1995, and Juillard et al., 1998):

1. choose T and form a huge (really huge) system $H(k_1, c_1, \ldots, k_t, c_t, \ldots, k_T, c_T) = 0$ (and set k_{T+1} equal to k_T when appropriate.
2. apply a (quasi-) Newton techniques.
3. if you are clever, you can make this approach efficient (the Jacobian is usually tri-diagonal, clever ways of updating of the Jacobian, ...)

Advantages:

- if it converges, it is fast (quadratic convergence);

Disadvantages:

- it is really a huge system: a system of equations with TM unknowns (M being the number of endogenous variables);
- How to choose T? T should be much larger than the horizon of projection.

Projection techniques /1

Projection techniques (due to Judd, 2002):

- Approximate the path of endogenous variables by a (linear) combination of basis functions: $k_t \cong \sum_i a_i^k f_i(t)$.
- Choose a_i^k so that equilibrium conditions are satisfied.
- The infinite dimensional problem is reduced to find coefficients a_i^k.
- Basis functions can be: (orthogonal) polynomials, splines, radial basis functions, finite elements, ...

Judd (2002) recommends:

$$k_t \cong e^{-\lambda t}\left(k_0 + \sum_i a_i^k f_i(t)\right) + (1 - e^{-\lambda t})k_{SS},$$

where $f_i(t) = L_i(2\lambda t)e^{-\lambda t}$ and L_i are Laguerre polynomials, λ governs the speed of convergence to the new steady state k_{SS} and could (actually should) be computed based on the linearization of the model.

Projection techniques /2

How to choose coefficients a?

- Set residual function $R(t,a)$.
- **Brut force:** solve the optimization problem $\min_a \sum_{t=1}^{T} \|R(t,a)\|_p$ for suitable p.
- If $p=2$, then you solve a non-linear least-square problem.
- you still have to truncate the time to compute the sum, but instead of T coefficients, you need only I.

It is possible to combine L-B-J with projection techniques:

- If the trajectory of endogenous variables is not smooth (abrupt, unexpected changes), then it is hard to approximate it with smooth basis functions (such as polynomials) – you would need a large I.
- The idea is to approximate for first t by L-B-J and then use projection.

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Projection techniques /3

There are better ways to chose the coefficients a: **Galerkin method**

- consider the integral $\int_0^\infty R(t,a)\psi_j(t)dt$, where $\psi_j(t)$ are test functions.
- if you choose $\psi(t) = R(t,a)$ you are back to non-linear least-square problem.
- Hope is that if you chose test functions $\psi_j(t)$ cleverly, then $\int_0^\infty R(t,a)\psi_j(t)dt$ will be zero if $R(t,a)$ is.
- use a quadrature to approximate $\int_0^\infty R(t,a)\psi_j(t)dt \cong \sum_k R(t_k,a)\psi_j(t_k)w_k$.
- Therefore, you need not to compute the residual function $R(t,a)$ for all $t = 1,\ldots,T$, but only for (rounded) values t_k.

Not always applicable.

Application to Brůha-Podpiera model

The model is rewritten into the first-order form and the idea is to rewrite all variables in term of 7 endogenous variables – a great reduction in the dimensionality of the problem. It has its costs as the Jacobian for L-B-J is no longer tridiagonal and all $1 \leq t \leq T$ should be computed even for the Galerkin method.

- **Fair-Taylor**: the method failed;
- **L-B-J**: in general it works, but it is relatively slow during first iterations ;
- **Projections**: safe and method, but sometimes difficult to obtain precise results (slow last iterations);
- The best way seems to use projections to get relatively accurate results (error about 10^{-6}) and then use L-B-J if further accuracy is required.