FUNDAMENTALS OF
PLASMA PHYSICS

THIRD EDITION

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PREFACE

This text is intended as a general introduction to plasma physics and was designed with the main purpose of presenting a comprehensive, logical and unified treatment of the fundamentals of plasma physics based on statistical kinetic theory. It should be useful primarily for advanced undergraduate and first year graduate students meeting the subject of plasma physics for the first time and presupposes only a basic elementary knowledge of vector analysis, differential equations and complex variables, as well as courses on classical mechanics and electromagnetic theory beyond sophomore level. Some effort has been made to make the book self-contained by including in the text developments of the fluid mechanics and kinetic theory that is needed.

Throughout the text the emphasis is on clarity, rather than formality. The various derivations are explained in detail and, wherever possible, the physical interpretations are emphasized. The equations are presented in such a way that they connect together, without requiring the reader to do extensive algebra to bridge the gap. The features of clarity and completeness make the book suitable for self-learning and for self-paced courses.

The structure of this book is as follows. The first chapter consists of a basic introduction to plasma physics, at a descriptive level, intended to give the reader an overall view of the subject. The motion of charged particles under the influence of specified electric and magnetic fields is treated in detail in Chapters 2, 3 and 4. In the next five chapters the fundamental equations necessary for an elementary description of plasma

phenomena are developed. Chapter 5 introduces the concepts of phase space and distribution function, and derives the basic differential kinetic equation which governs the evolution of the distribution function in phase space. The definitions of the macroscopic variables in terms of the phase space distribution function are presented in Chapter 6 and their physical interpretations are discussed. The Maxwell-Boltzmann equilibrium distribution function is introduced in Chapter 7 as the equilibrium solution of the Boltzmann equation and its kinetic properties are analyzed in some detail. In Chapter 8 the macroscopic transport equations for a plasma considered as a mixture of various interpenetrating fluids are derived, whereas the macroscopic transport equations for the whole plasma as a single conducting fluid are developed in Chapter 9.

The remaining of the book is devoted to applications of these basic equations in the description of a variety of important phenomena in plasmas. The problems of electrical conductivity and diffusion in plasmas are analyzed in Chapter 10 and other basic plasma phenomena, such as electron plasma oscillations and Debye shielding, are treated in Chapter 11. Simple applications of the magnetohydrodynamic equations, such as in plasma confinement by magnetic fields and the pinch effect, are presented in Chapters 12 and 13. The subject of wave phenomena in plasmas is organized in the next six chapters. A review of the basic concepts related to electromagnetic wave propagation in free space is given in Chapter 14. The propagation of very low frequency waves in a highly conducting fluid is analyzed in Chapter 15, under the title of magnetohydrodynamic waves. The various modes of wave propagation in cold and warm plasmas are considered in Chapters 16 and 17, respectively. In Chapters 18 and 19 the various properties of wave propagation in hot nonmagnetized plasmas and in hot magnetized plasmas, respectively, are analyzed. Collisional phenomena in plasmas are treated in Chapter 20 and the derivations of the Boltzmann collision integral and of the Fokker-Planck collision term are presented in Chapter 21. Finally, in Chapter 22 some applications of the Boltzmann equation to the analysis of transport phenomena in plasmas are presented.

A number of problems is provided at the end of each chapter, which illustrate additional applications of the theory and supplement the textual material. Most of the problems are designed in such a way as to provide a guideline for the student, including intermediate steps and answers in their statements.

The numbering of the equations, within each chapter, starts over again at each section. When reference is made to an equation using three

numbers, the first number indicates the chapter and the last two numbers indicate the section and the equation, respectively. Within the same chapter the first number is omitted. Vector quantities are represented by bold face type letters (such as $\mathbf{r}$) and unit vectors by a circunflex above the corresponding letter (such as $\hat{\mathbf{r}}$). Dyadic and triadic quantities are represented by calligraphic type letters (such as $\mathcal{Q}$).

The system of units used in this text is the rationalized MKSA.

The book contains more material than what can normally be covered in one semester. This permits some freedom in the selection of topics depending on the level and desired emphasis of the course, and on the interests of the students. The whole text can also be adequately covered within two semesters.

In this, as in any introductory book, the topics included clearly do not cover all areas of plasma physics. No attempt was made to present the experimental aspects of the subject. Moreover, there are some important theoretical topics which are covered only very briefly and some which have been left for more advanced courses on plasma physics, such as plasma instabilities, plasma radiation, nonlinear plasma theory and plasma turbulence.

I am grateful to the many people who contributed to this book, both directly and indirectly, and especially to the many students to whom I had the opportunity to test my ideas in the various courses I taught over the last twenty five years. The amount of digitalized information in a book such as this is truly enormous and some errors may be bound to occur. Further feedback from readers will be appreciated. I wish to thank the many professors, students and researchers who have used the first two editions of this book, all over the world, and contributed to its improvement.

J. A. Bittencourt

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1

INTRODUCTION

1. GENERAL PROPERTIES OF PLASMAS

1.1 Definition of a Plasma

The term *plasma* is used to describe a wide variety of macroscopically neutral substances containing many interacting free electrons and ionized atoms or molecules, which exhibit collective behavior due to the long-range Coulomb forces. Not all media containing charged particles, however, can be classified as plasmas. For a collection of interacting charged and neutral particles to exhibit plasma behavior it must satisfy certain conditions, or *criteria*, for plasma existence. These criteria will be discussed in some detail in the next section.

The word plasma comes from the Greek and means *something molded.* It was applied for the first time by Tonks and Langmuir, in 1929, to describe the inner region, remote from the boundaries, of a glowing ionized gas produced by electric discharge in a tube, the ionized gas as a whole remaining electrically neutral.

1.2 Plasma as the Fourth State of Matter

From a scientific point of view, matter in the known Universe is often classified in terms of four states: *solid, liquid, gaseous* and *plasma.* The basic distinction between solids, liquids and gases lies in the difference between the strength of the bonds that hold their constituent particles

together. These binding forces are relatively strong in a solid, weak in a liquid and essentially almost absent in the gaseous state. Whether a given substance is found in one of these states depends on the *random kinetic energy* (thermal energy) of its atoms or molecules i.e. on its temperature. The equilibrium between this particle thermal energy and the interparticle binding forces determines the state.

By heating a solid or liquid substance the atoms or molecules acquire more thermal kinetic energy until they are able to overcome the binding potential energy. This leads to *phase transitions*, which occur at a constant temperature for a given pressure. The amount of energy required for the phase transition is called the *latent heat.*

If sufficient energy is provided, a molecular gas will gradually dissociate into an atomic gas as a result of collisions between those particles whose thermal kinetic energy exceeds the molecular binding energy. At sufficiently elevated temperatures an increasing fraction of the atoms will possess enough kinetic energy to overcome, by collisions, the binding energy of the outermost orbital electrons, and an *ionized gas* or *plasma* results. However, this transition from a gas to a plasma is not a phase transition in the thermodynamic sense, since it occurs gradually with increasing temperature.

1.3 Plasma Production

A plasma can be produced by raising the temperature of a substance until a reasonably high fractional ionization is obtained. Under thermodynamic equilibrium conditions, the degree of ionization and the electron temperature are closely related. This relation is given by the *Saha equation* (see Chapter 7). Although plasmas in local thermodynamic equilibrium are found in many places in nature, as is the case for many astrophysical plasmas, they are not very common in the laboratory.

Plasmas can also be generated by ionization processes that raise the degree of ionization much above its thermal equilibrium value. There are many different methods of creating plasmas in the laboratory and, depending on the method, the plasma may have a high or low density, high or low temperature, it may be steady or transient, stable or unstable, and so on. In what follows, a brief description is presented of the most commonly known processes of *photoionization* and *electric discharge* in gases.

In the *photoionization* process, ionization occurs by absorption of incident photons whose energy is equal to, or greater than, the ionization

potential of the absorbing atom. The excess energy of the photon is transformed into kinetic energy of the electron-ion pair formed. For example, the ionization potential energy for the outermost electron of atomic oxygen is 13.6 eV, which can be supplied by radiation of wavelength smaller than about 91 nm i.e. in the far ultraviolet. Ionization can also be produced by X-rays or gamma-rays, which have much smaller wavelengths. The Earth's ionosphere, for example, is a natural photoionized plasma (see Section 3).

In a *gas discharge*, an electric field is applied across the ionized gas, which accelerates the free electrons to energies sufficiently high to ionize other atoms by collisions. One characteristic of this process is that the applied electric field transfers energy much more efficiently to the light electrons than to the relatively heavy ions. The electron temperature in gas discharges is therefore usually higher than the ion temperature, since the transfer of thermal energy from the electrons to the heavier particles is very slow.

When the ionizing source is turned off, the ionization decreases gradually because of recombination until it reaches an equilibrium value consistent with the temperature of the medium. In the laboratory the recombination usually occurs so fast that the plasma completely disappears in a small fraction of a second.

1.4 Particle Interactions and Collective Effects

The properties of a plasma are markedly dependent upon the *particle interactions*. One of the basic features which distinguish the behavior of plasmas from that of ordinary fluids and solids is the existence of *collective effects*. Due to the long-range of electromagnetic forces, each charged particle in the plasma interacts *simultaneously* with a considerable number of other charged particles, resulting in important collective effects which are responsible for the wealth of physical phenomena that take place in a plasma.

The particle dynamics in a plasma is governed by the internal fields due to the nature and motion of the particles themselves, and by externally applied fields. The basic particle interactions are electromagnetic in character. Quantum effects are negligible, except for some cases of close collisions.

In a plasma we must distinguish between *charge-charge* and *charge-neutral* interactions. A charged particle is surrounded by an electric field and interacts with the other charged particles according to the *Coulomb*

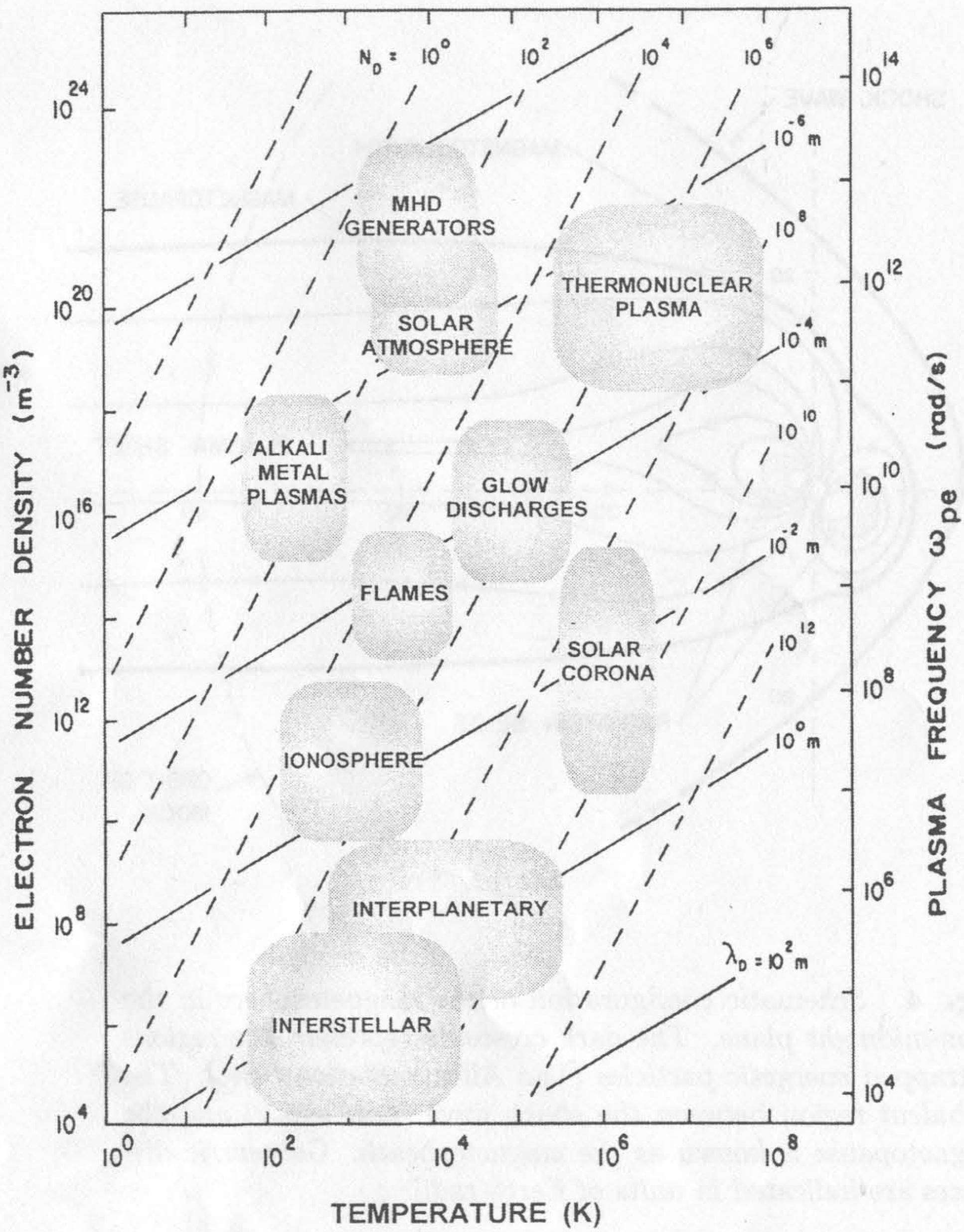


Fig. 2 *Ranges of temperature and electron density for several laboratory and cosmic plasmas and their characteristic physical parameters: Debye length λ_D, plasma frequency ω_{pe} and number of electrons N_D in a Debye sphere.*

3.1 The Sun and Its Atmosphere

The *Sun*, which is our nearest star and upon which the existence of life on Earth fundamentally depends, is a plasma phenomenon. Its energy output is derived from thermonuclear fusion reactions of protons forming helium ions deep in its interior, where temperatures exceed 1.2×10^7 K. The high temperature of its interior and the consequent thermonuclear reactions keep the entire Sun gaseous. Due to its large mass (2×10^{30} kg), the Sun's gravitational force is sufficient to prevent the escape of all but the most energetic particles and, of course, radiation from the hot solar plasma.

There is no sharp boundary surface to the Sun. Its visible part is known as the solar atmosphere, which is divided into three general regions or layers. The *photosphere*, with a temperature of about 6000 K, comprises the visible disk, the layer in which the gases become opaque, and is a few hundred kilometers thick. Surrounding the photosphere there is a reddish ring called the *chromosphere*, approximately ten thousand kilometers thick, above which flame-like prominences rise with temperatures of the order of 100,000 K. Surrounding the chromosphere there is a tenuous hot plasma, extending millions of kilometers into space, known as the *corona*. A steep temperature gradient extends from the chromosphere to the hotter corona, where the temperature exceeds 10^6 K.

The Sun possesses a variable magnetic field which at its surface is typically of the order of 10^{-4} tesla, but in the regions of *sunspots* (regions of relatively cooler gases) the solar magnetic field rises to about 0.1 tesla.

3.2 The Solar Wind

A highly conducting tenuous plasma called the *solar wind*, composed mainly of protons and electrons, is continuously emitted by the Sun at very high speeds into interplanetary space, as a result of the supersonic expansion of the hot solar corona. The solar magnetic field tends to remain frozen in the streaming plasma due to its very high conductivity. Because of solar rotation, the field lines are carried into Archimedean spirals by the radial motion of the solar wind (see Fig. 3). Typical values of the parameters in the solar wind are: electron density $n_e \simeq 5 \times 10^6$ m^{-3}, electron and ion temperatures $T_e \simeq 5 \times 10^4$ K, $T_i \simeq 10^4$ K, magnetic field $B \simeq 5 \times 10^{-9}$ tesla and drift velocity $u_e \simeq 3 \times 10^5$ m/s.

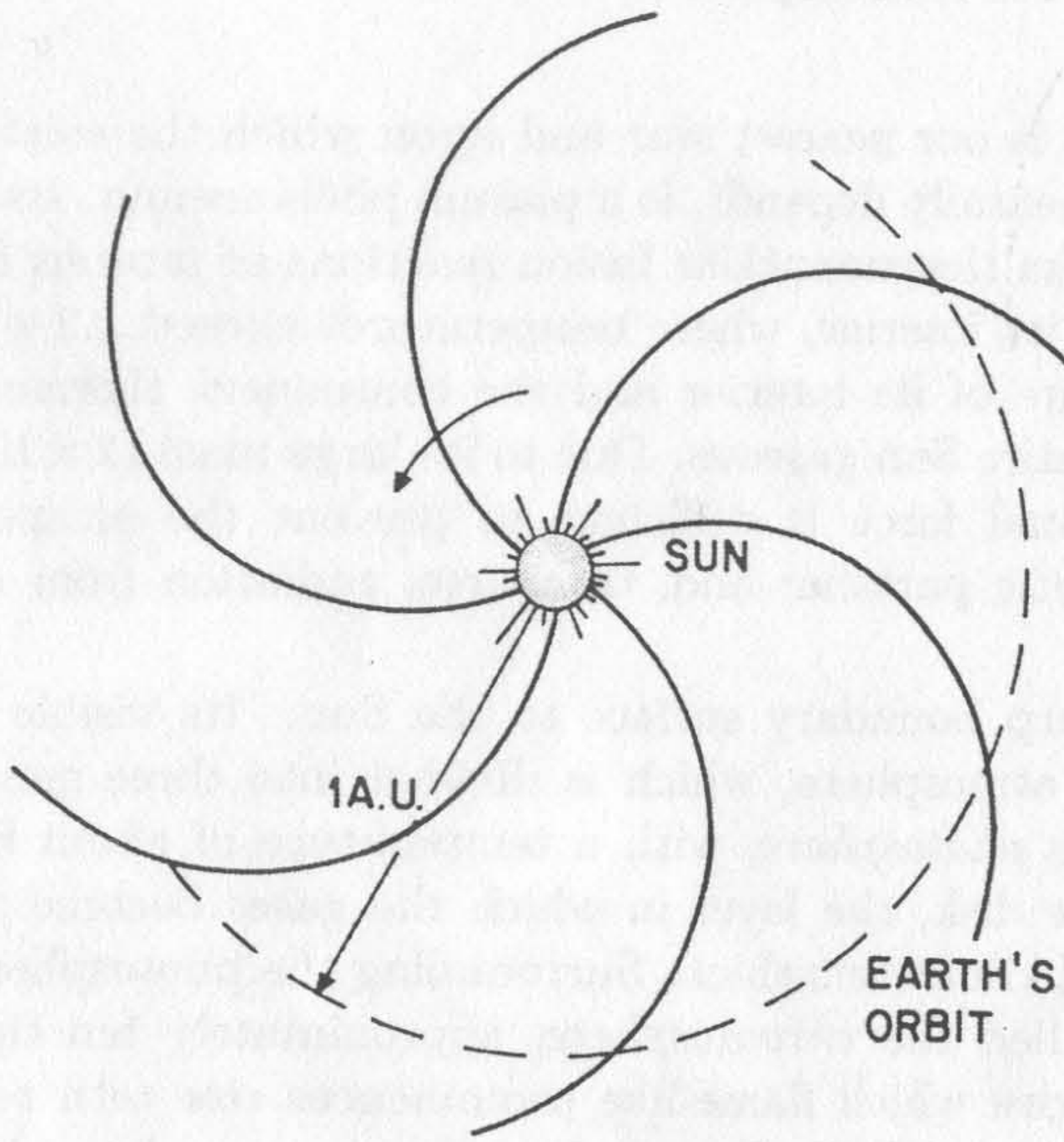


Fig. 3 *Schematic representation of the Archimedes spiral structure of the interplanetary magnetic field in the ecliptic plane.*

3.3 The Magnetosphere and the Van Allen Radiation Belts

As the highly conducting solar wind impinges on the Earth's magnetic field, it compresses the field on the sunward side and flows around it at supersonic speeds. This creates a boundary, called the *magnetopause*, which is roughly spherical on the sunward side and roughly cylindrical in the anti-sun direction (see Fig. 4). The inner region, from which the solar wind is excluded and which contains the compressed Earth's magnetic field, is called the *magnetosphere*.

Inside the magnetosphere we find the *Van Allen radiation belts*, in which energetic charged particles (mainly electrons and protons) are trapped into regions where they execute complicated trajectories which spiral along the geomagnetic field lines and, at the same time, drift slowly around the Earth. The origin of the inner belt is ascribed to cosmic rays, which penetrate into the atmosphere and form proton-electron pairs which are then trapped by the Earth's magnetic field. The outer belt is considered to be due to and maintained by streams of plasma consisting mainly

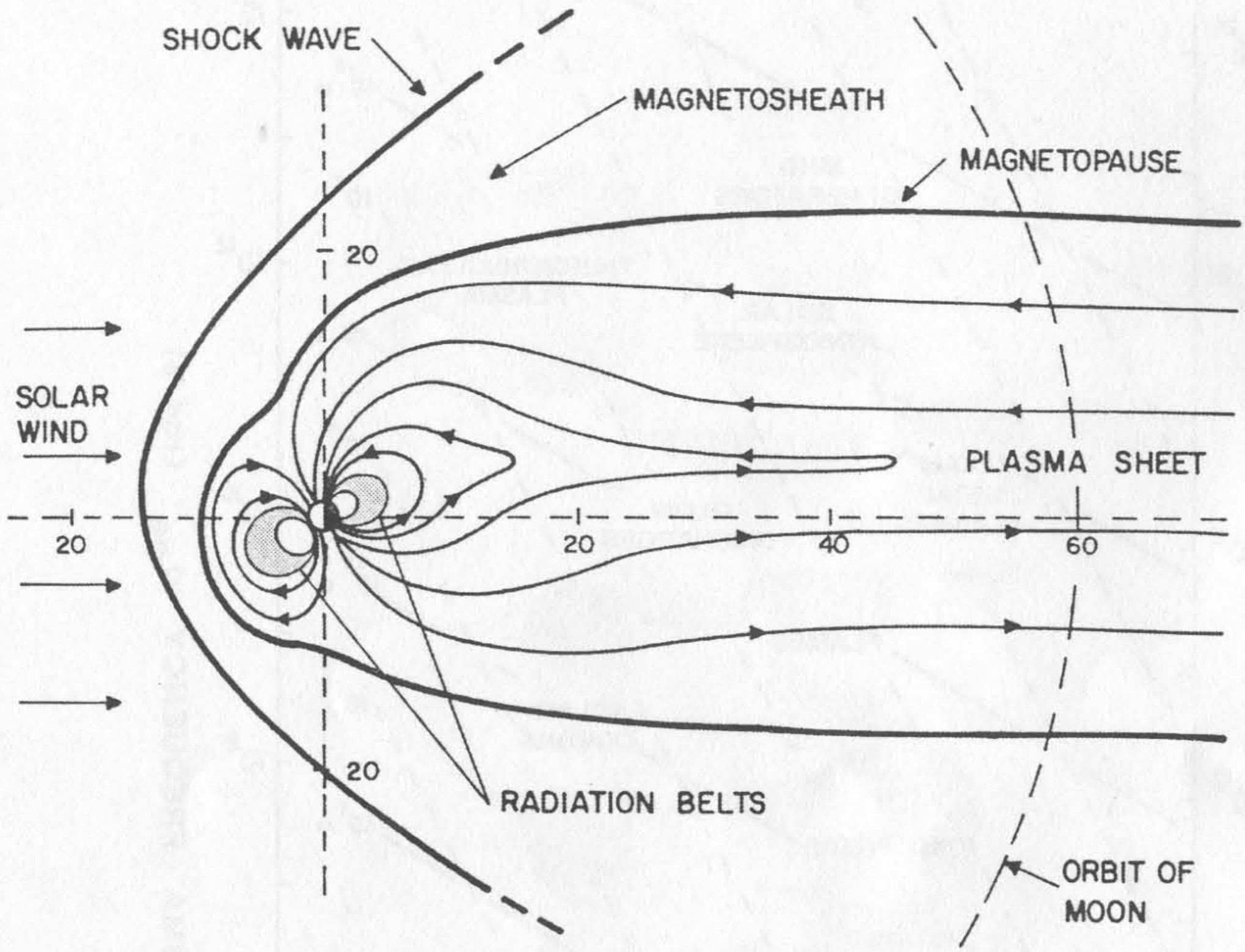


Fig. 4 *Schematic configuration of the magnetosphere in the noon-midnight plane. The dark crescents represent the regions of trapped energetic particles (Van Allen radiation belts). The turbulent region between the shock wave (bow shock) and the magnetopause is known as the magnetosheath. Geocentric distances are indicated in units of Earth radii.*

of protons and electrons which are ejected from time to time by the Sun. Depending on solar activity, particularly violent solar eruptions may occur with the projection of hot streams of plasma material into space. The separation into inner and outer belts reflects only an altitude-dependent energy spectrum, rather than two separate trapping regions.

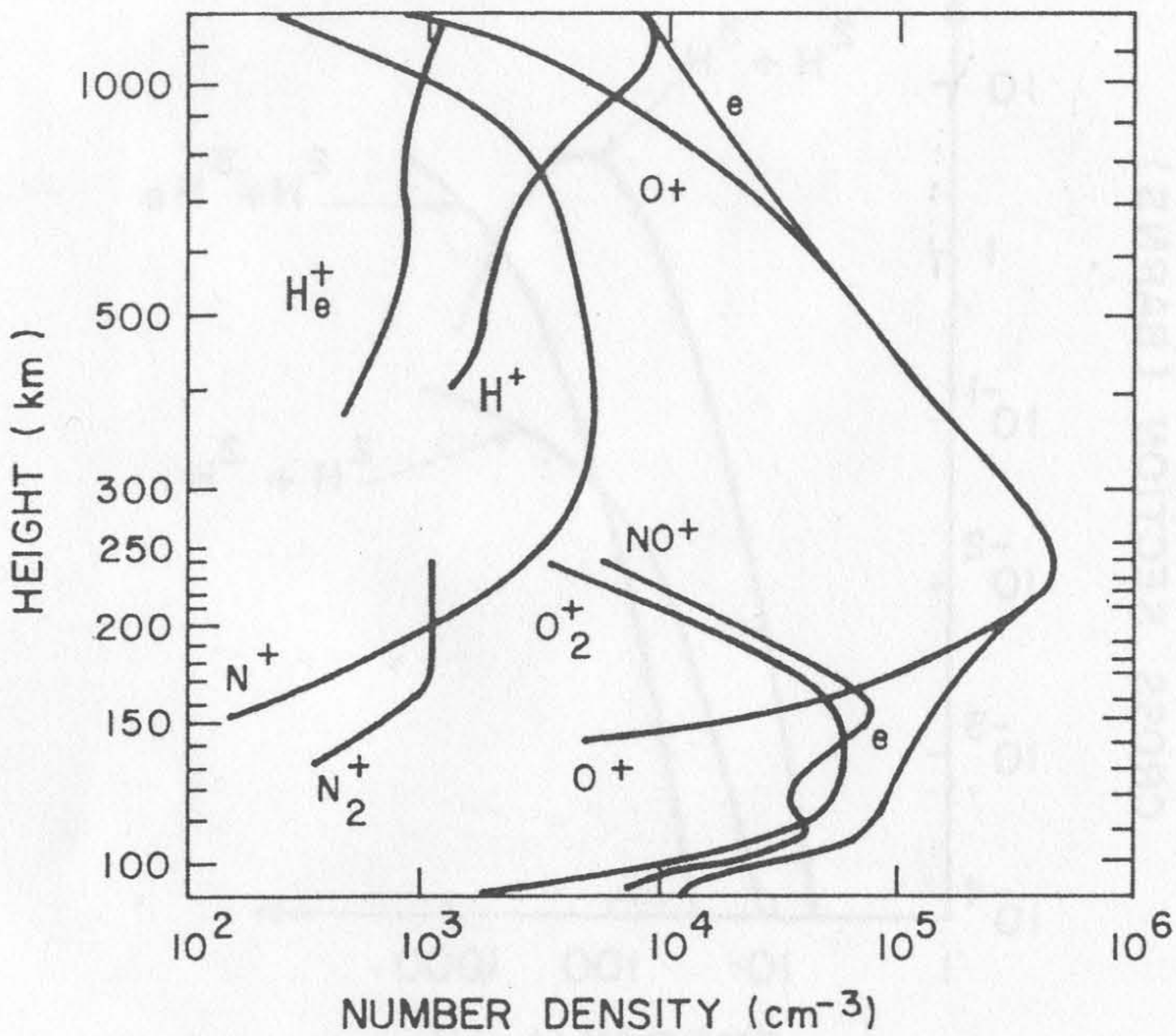


Fig. 5 *Height distribution of the electrons and of the principal positive ions, typical of the daytime ionosphere during solar minimum conditions.*

3.4 The Ionosphere

The large natural blanket of plasma in the atmosphere, which envelopes the Earth from an altitude of approximately 60 km to several thousands of kilometers, is called the *ionosphere.* The ionized particles in the ionosphere are produced during the daytime through absorption of solar extreme ultraviolet and X-ray radiation by the atmospheric species. As the ionizing radiation from the Sun penetrates deeper and deeper into the Earth's atmosphere it encounters a larger and larger density of gas particles, producing more and more electrons per unit volume. However, since radiation is absorbed in this process, there is a height where the rate of electron production reaches a maximum. Below this height the rate of electron production decreases, in spite of the increase in atmospheric density, since most of the ionizing radiation was already absorbed at the higher altitudes.

Fig. 5 provides some information on the relative concentration and altitude distribution of the electrons and of the principal positive ions, typical of the daytime ionosphere during solar minimum conditions. The Earth's magnetic field exerts a great influence on the dynamic behavior of the ionospheric plasma. An interesting phenomenon which occurs in the ionospheric polar regions is the *aurora*. It consists of electromagnetic radiation emitted by the atmospheric species and induced by energetic particles of solar and cosmic origin which penetrate into the atmosphere along the geomagnetic field lines near the poles.

3.5 Plasmas Beyond the Solar System

Beyond the solar system we find a great variety of natural plasmas in *stars*, *interstellar space*, *galaxies*, *intergalactic space*, and far beyond to systems quite unknown before the start of astronomy from space vehicles. There we find a variety of phenomena of great cosmological and astrophysical significance, including interstellar shock waves from remote *supernova* explosions, rapid variations of X-ray fluxes from *neutron stars* with densities like that of atomic nuclei, pulsating radio stars or *pulsars* (which are theoretically pictured as rapidly rotating neutron stars with plasmas emitting synchrotron radiation from the surface) and the plasma phenomena around the remarkable *black holes* (which are considered to be singular regions of space into which matter has collapsed, possessing such a powerful gravitational field that nothing, whether material objects or even light itself, can escape from them).

The behavior of plasmas in the Universe involves the interaction between plasmas and magnetic fields. The Crab nebula, for example, is a rich source of plasma phenomena because it contains a magnetic field. The widespread existence of magnetic fields in the Universe has been demonstrated by independent measurements, and a wide range of field magnitudes has been found, varying from 10^{-9} tesla in interstellar space to 1 tesla on the surface of magnetic variable stars.

4. APPLICATIONS OF PLASMA PHYSICS

A wide variety of plasma experiments have been perfomed in the laboratory to aid in the understanding of plasmas, as well as to test and help expand plasma theory. The progress in plasma research has lead to a wide range of plasma applications. A brief description of some important practical applications of plasma physics is presented in this section.

4.1 Controlled Thermonuclear Fusion

The most important application of man-made plasmas is in the control of *thermonuclear fusion reactions*, which holds a vast potential for the generation of power. Nuclear fusion is the process whereby two light nuclei combine to form a heavier one, the total final mass being slightly less than the total initial mass. The mass difference (Δm) appears as energy (E) according to Einstein's famous law $E = (\Delta m)c^2$, where c denotes the speed of light. The nuclear fusion reaction is the source of energy in the stars, including the Sun. The confinement of the hot plasma in this case is provided by the self-gravity of the stars.

In the nuclear fusion of hydrogen the principal reactions involve the deuterium (^2H) and tritium (^3H) isotopes of hydrogen, as follows:

$$^2H + {}^2H \longrightarrow {}^3He + {}^1n + 3.27 \text{ MeV} \tag{4.1a}$$

$$^2H + {}^2H \longrightarrow {}^3H + {}^1H + 4.03 \text{ MeV} \tag{4.1b}$$

$$^2H + {}^3H \longrightarrow {}^4He + {}^1n + 17.58 \text{ MeV} \tag{4.1c}$$

$$^2H + {}^3He \longrightarrow {}^4He + {}^1H + 18.34 \text{ MeV} \tag{4.1d}$$

where ^1n represents a neutron. The basic problem in achieving controlled fusion is to generate a plasma at very high temperatures (with thermal energies at least in the 10 keV range) and hold its particles together long enough for a substantial number of fusion reactions to take place. The need for high temperatures comes from the fact that, in order to undergo fusion, the positively charged nuclei must come very close together (within a distance of the order of 10^{-14} m), which requires sufficient kinetic energy to overcome the electrostatic Coulomb repulsion.

Fig. 6 presents the cross sections, as a function of the incident particle energy, for the nuclear fusion reactions of hydrogen given in (4.1). They are appreciable only for incident particles with energies above at least 10 keV. This means that the plasma must have temperatures of the order of 10^8 K. Other fusion reactions involving nuclei with larger values of the atomic number Z require even higher energies to overcome the Coulomb repulsion.

Many confinement schemes have been suggested and built which use some type of magnetic field configuration. The main experimental efforts for achieving plasma conditions for fusion can be grouped into four approaches: (1) open systems (magnetic mirrors); (2) closed systems (toruses); (3) theta pinch devices; and (4) laser-pellet fusion.

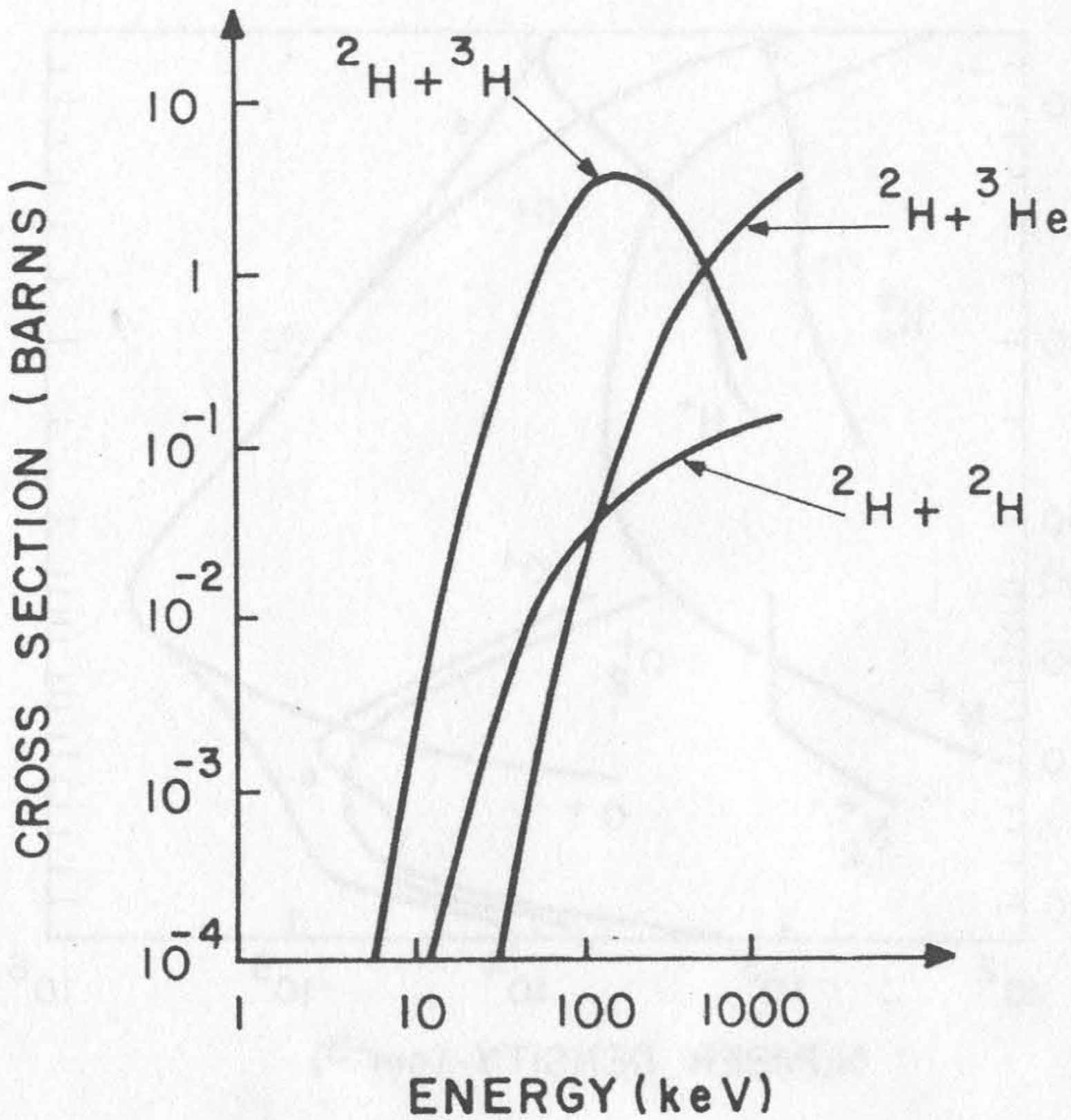


Fig. 6 *Fusion cross sections, in barns (1 barn = 10^{-28} m^2), as a function of energy, in keV, for the hydrogen reactions given in (4.1).*

The *mirror machines* are linear devices with an axial magnetic field to keep the particles away from the wall, and with magnetic mirrors (regions of converging magnetic field lines) at the ends to reduce the number of particles escaping at each end (see Fig.7).

The four principal *toroidal systems* differ in the way they twist the magnetic field lines. They are the stellarators (in which the twisting of the field lines is produced by external helical conductors), the tokamaks (in which a poloidal field produced by an internal plasma current is superposed on the toroidal field), the multipoles (which have their magnetic field lines primarilly in the poloidal direction and produced by internal conductors), and the Astron (in which internal relativistic particle beams modify a mirror field into a form having stable confinement regions with closed lines of force).

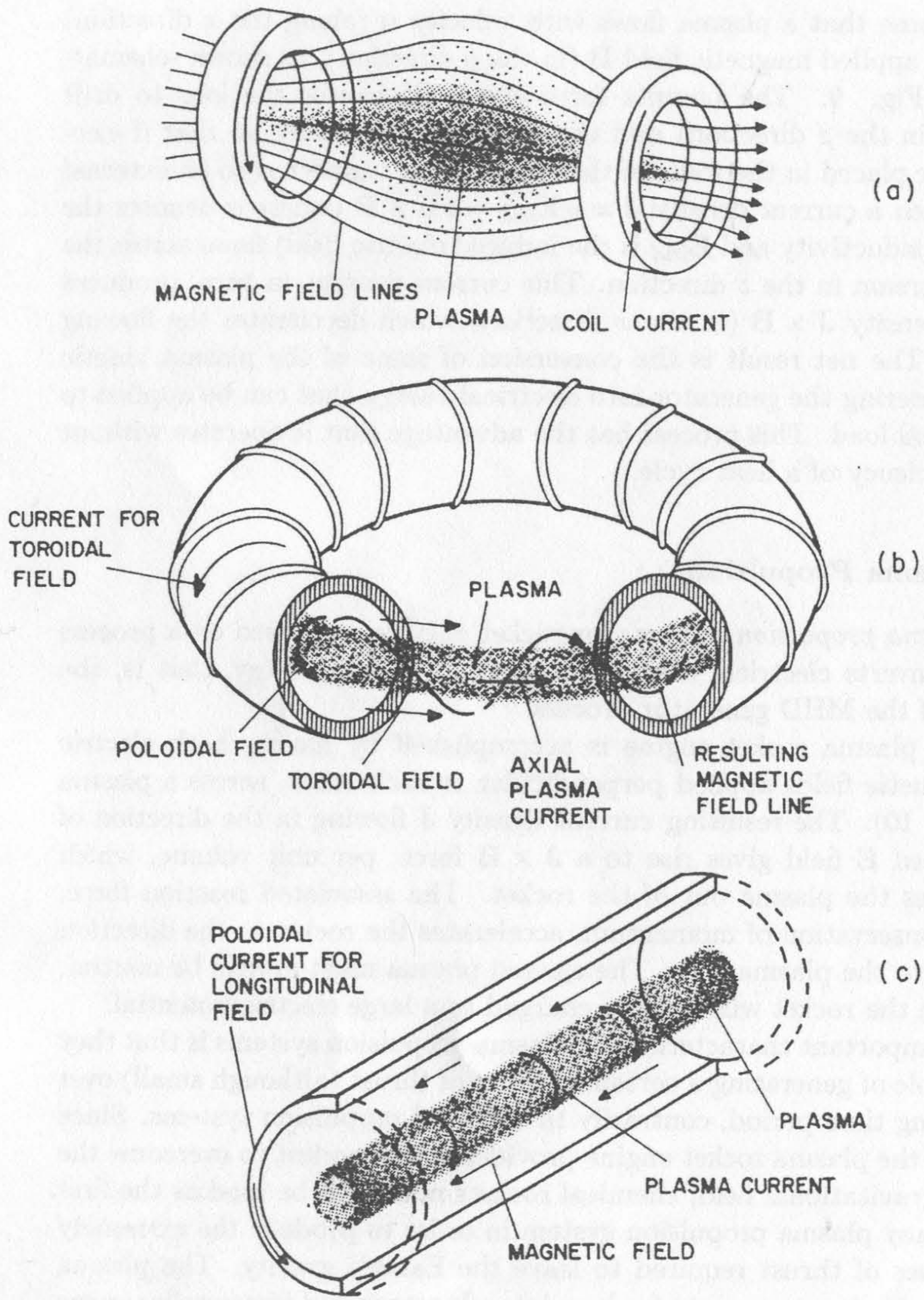


Fig. 7 *Schematic illustration showing the magnetic field configurations of some basic schemes for plasma confinement. (a) Magnetic mirror system. (b) Tokamak. (c) Linear θ pinch.*

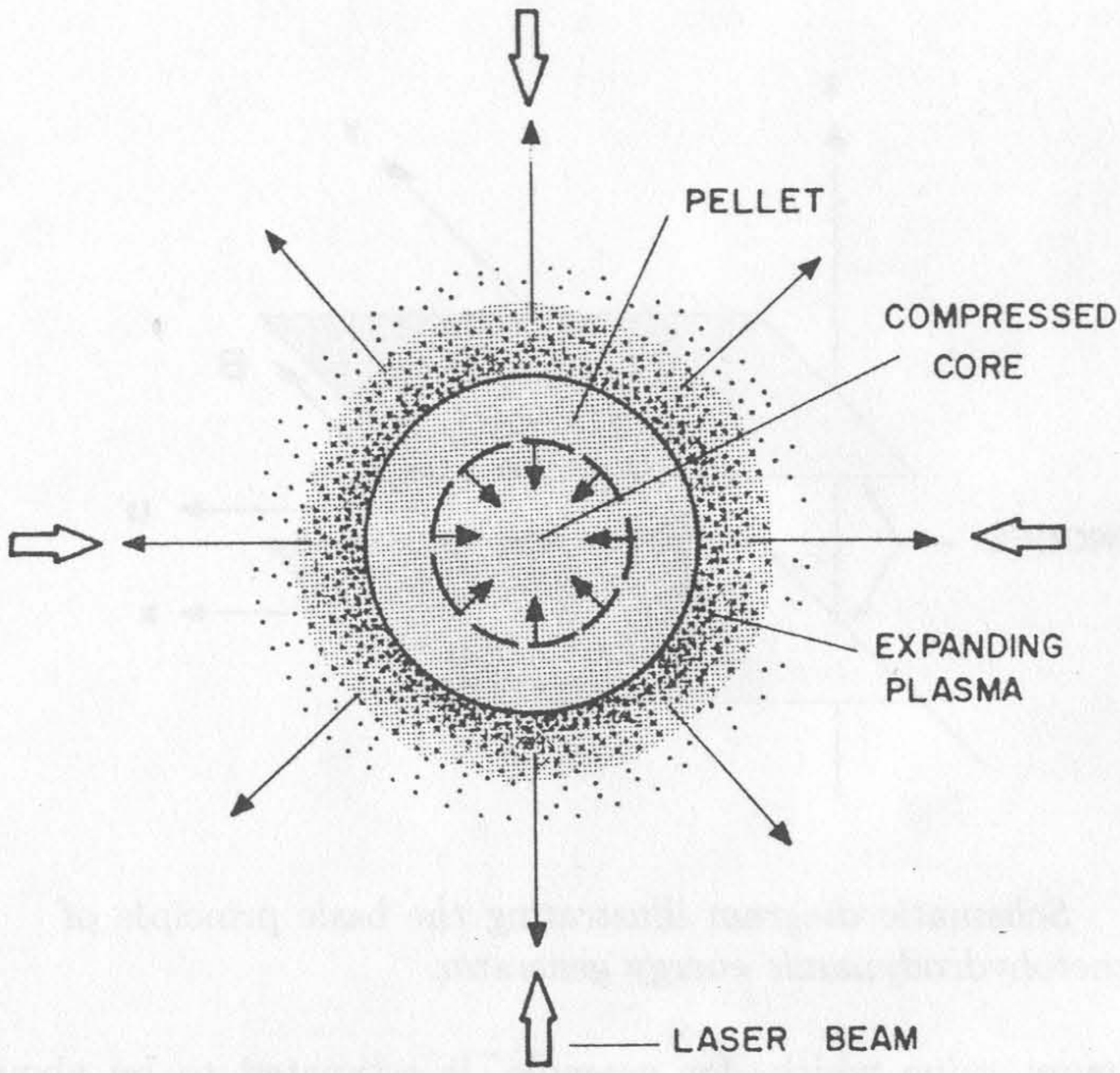


Fig. 8 *Illustrating laser-pellet fusion.*

In the *theta pinch devices*, a plasma current in the azimuthal direction and a longitudinal magnetic field produce a force which compresses the cross-sectional area of the plasma.

Finally, the scheme to ignite a fusion reaction using *pulsed lasers* consists in focusing converging laser beams on a small pellet of solid deuterium-tritium material producing a rapid symmetrical heating of the plasma, followed by an expansion of the heated surrouding shell and compression of the pellet core by the recoil (see Fig. 8).

In addition to the plasma heating and confinement problems, attention must be given to the energy loss by radiation (predominantly electron-ion bremsstrahlung and electron cyclotron radiation). These radiation losses constitute a serious problem in maintaining a self-sustaining fusion device. To generate more energy by fusion than is required to heat and confine the plasma, and to supply the radiation losses, a condition is imposed on the plasma density (n) and the confinement time (τ), as well as on the temperature. It turns out that the product $n\tau$ must be higher

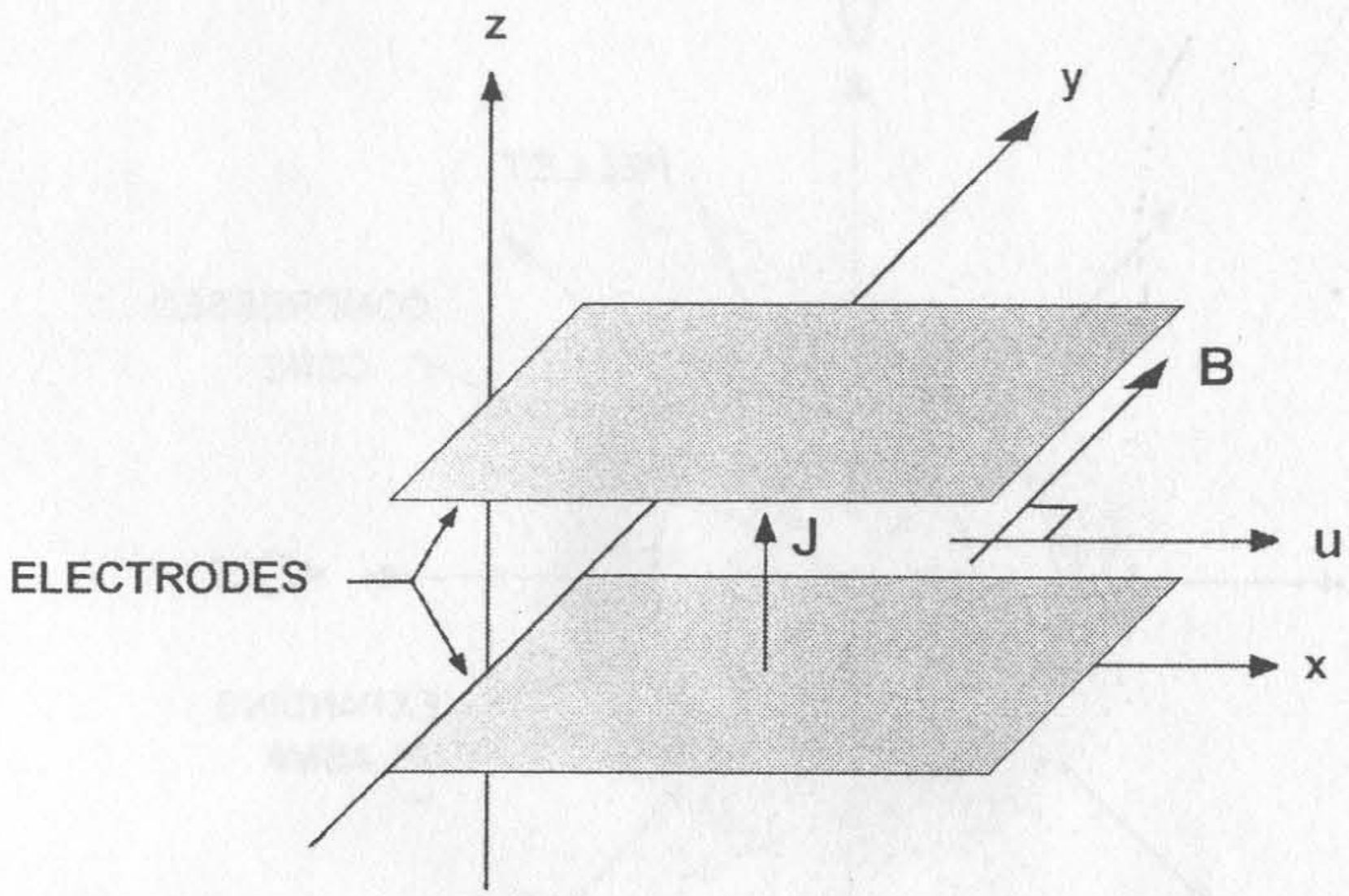


Fig. 9 *Schematic diagram illustrating the basic principle of the magnetohydrodynamic energy generator.*

than a minimum value which, for example, is estimated to be about 10^{20} m^{-3}s for deuterium-tritium (with $T > 10^7$ K) and about 10^{22} m^{-3}s for deuterium-deuterium (with $T > 10^8$ K). This condition is known as the *Lawson criterion.* Consequently, controlled fusion can be achieved either by having a large number density of hot plasma particles confined for a short period of time, or by having a smaller number density of particles confined for a longer period of time. For this reason some fusion experiments operate in the regime of high density and short confinement time utilizing a pulsed mode of operation.

Since controlled nuclear fusion can provide an almost limitless source of energy, it is certainly one of the most important scientific challenges man faces today, and its achievement will cause an enourmous impact on our civilization.

4.2 The Magnetohydrodynamic Generator

The *magnetohydrodynamic (MHD) energy generator* converts the kinetic energy of a dense plasma flowing across a magnetic field into electrical energy. While a rigorous discussion of this device becomes quite involved, its basic principle is quite simple.

Suppose that a plasma flows with velocity $\mathbf{u}$ (along the x direction) across an applied magnetic field $\mathbf{B}$ (in the y direction), as shown schematically in Fig. 9. The Lorentz force $q(\mathbf{u} \times \mathbf{B})$ causes the ions to drift upward (in the z direction) and the electrons downward, so that if electrodes are placed in the walls of the channel and connected to an external circuit then a current density $\mathbf{J} = \sigma \mathbf{E}_{ind} = \sigma \mathbf{u} \times \mathbf{B}$ (where σ denotes the plasma conductivity and $\mathbf{E}_{ind}$ is the induced electric field) flows across the plasma stream in the z direction. This current density, in turn, produces a force density $\mathbf{J} \times \mathbf{B}$ (in the x direction) which decelerates the flowing plasma. The net result is the conversion of some of the plasma kinetic energy entering the generator into electrical energy that can be applied to an external load. This process has the advantage that it operates without the inefficiency of a heat cycle.

4.3 Plasma Propulsion

Plasma propulsion systems for rocket engines are based on a process which converts electrical energy into plasma kinetic energy, that is, the reverse of the MHD generator process.

The plasma rocket engine is accomplished by having both electric and magnetic fields applied perpendicular to each other, across a plasma (see Fig. 10). The resulting current density $\mathbf{J}$ flowing in the direction of the applied $\mathbf{E}$ field gives rise to a $\mathbf{J} \times \mathbf{B}$ force, per unit volume, which accelerates the plasma out of the rocket. The associated reaction force, due to conservation of momentum, accelerates the rocket in the direction opposite to the plasma flow. The ejected plasma must always be neutral, otherwise the rocket will become charged to a large electric potential.

An important characteristic of plasma propulsion systems is that they are capable of generating a certain amount of thrust (although small) over a very long time period, contrarily to chemical propulsion systems. Since the force the plasma rocket engine provides is too modest to overcome the Earth's gravitational field, chemical rockets must still be used as the first stage of any plasma propulsion system in order to produce the extremely high values of thrust required to leave the Earth's gravity. The plasma rocket engine is appropriate for long interplanetary and insterstellar space travel.

4.4 Other Plasma Devices

A number of other practical applications of plasma physics should be

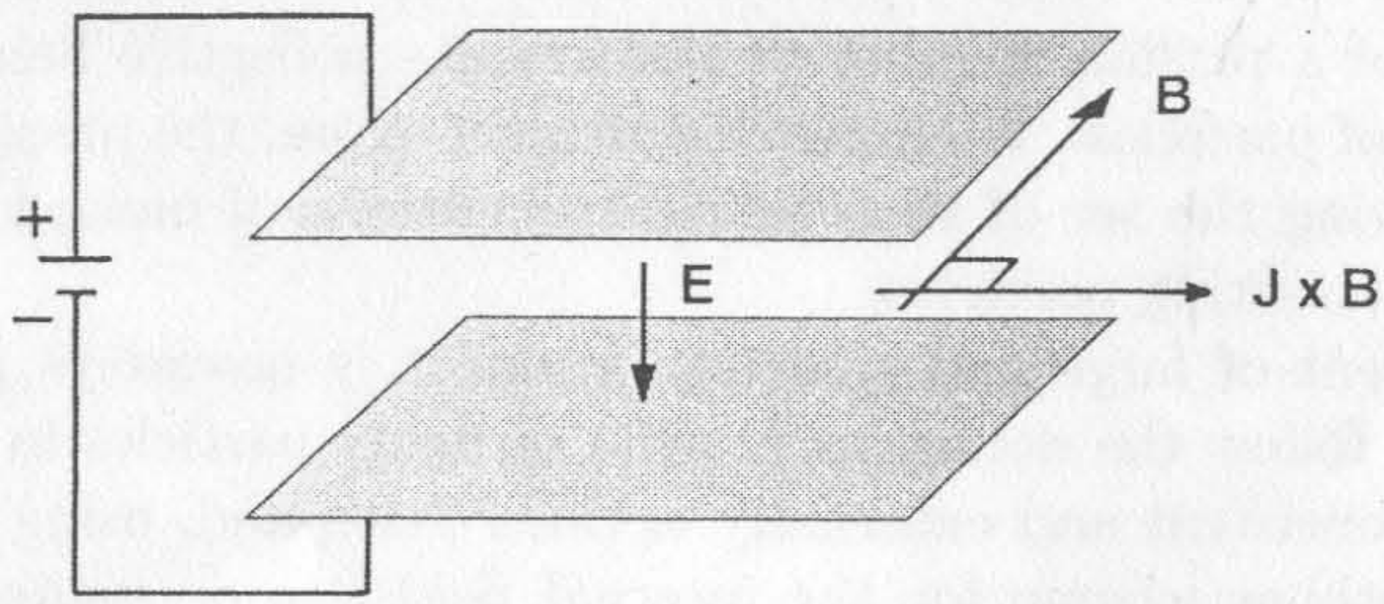


Fig. 10 *Schematic diagram illustrating the basic principle of the plasma rocket engine.*

mentioned in addition to controlled fusion, MHD energy conversion and plasma propulsion.

The *thermionic energy converter* is a device which utilizes a cesium plasma between two electrodes to convert thermal energy into electrical energy. The cathode is heated, so that electrons are emitted from the surface, and the anode is cooled. Due to the presence of the cesium plasma, very large electrical currents can be produced at the expense of a significant fraction of the thermal energy applied to the cathode.

Examples of applications involving *gas discharges* include the ordinary fluorescent tubes and neon lights used for illumination and for signs, mercury rectifiers, spark gaps, a number of specialized tubes like the hydrogen thyratrons and the ignitrons, which are used for switching, and the arc discharges or plasma jets, which are the source of temperatures two or more times as high as the hottest gas flames and which are used in metallurgy for cutting, melting and welding metals.

Two major applications in the area of *communications* are the long-distance radio wave propagation by reflection in the ionospheric plasma and the communication with a space vehicle through the plasma layer which forms around it during the reentry period into the Earth's atmosphere.

Finally, there is the realm of *solid state plasmas.* If the usual lattice temperature is considered, it can be easily verified that solids do not satisfy the plasma shielding criterion $N_D \gg 1$. Nevertheless, quantum mechanical effects, associated with the uncertainty principle, give some solids an effective electron temperature high enough to make N_D sufficiently large,

so that plasma behavior can be observed. It has been demonstrated that the free electrons and holes in appropriate solid materials, particularly semiconductors, exhibit the same sort of oscillations and instabilities as gaseous plasmas. The most likely application of solid state plasmas is in electronic circuitry.

5. THEORETICAL DESCRIPTION OF PLASMA PHENOMENA

The dynamic behavior of a plasma is governed by the interaction between the plasma particles and the internal fields produced by the particle themselves, and the externally applied fields. As the charged particles in a plasma move around, they can generate local concentrations of positive or negative charges, which give rise to electric fields. Their motion can also generate electric currents and therefore magnetic fields. The particle dynamics in a plasma is adequately described by the laws of classical (non-quantum) mechanics. Generally, the momentum of the plasma particles is high and the density low enough to keep their De Broglie wavelengths much smaller than the interparticle distance. Quantum effects turn out to be important only at very high densities and very low temperatures.

5.1 General Considerations on a Self-Consistent Formulation

The interaction of charged particles with electromagnetic fields is governed by the *Lorentz force.* For a typical particle of charge q and mass m, moving with velocity $\mathbf{v}$, in the presence of electric ($\mathbf{E}$) and magnetic induction ($\mathbf{B}$) fields, the equation of motion is

$$\frac{d\mathbf{p}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{5.1}$$

where $\mathbf{p} = m\mathbf{v}$ denotes the particle momentum. It is conceivable, at least in principle, to describe the dynamics of a plasma by solving the equations of motion for each particle in the plasma under the combined influence of the externally applied fields and the internal fields generated by all the other plasma particles.

If the total number of particles is N, we will have N nonlinear coupled differential equations of motion to solve simultaneously. A *self-consistent formulation* must be used since the fields and the particle trajectories are intrinsically coupled, that is, the internal fields associated with the presence and motion of the plasma particles influence their motions

which, in turn, modify the internal fields. The electromagnetic fields obey *Maxwell equations*

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{5.2}$$

$$\nabla \times \mathbf{B} = \mu_o \left(\mathbf{J} + \epsilon_o \frac{\partial \mathbf{E}}{\partial t} \right) \tag{5.3}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_o} \tag{5.4}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{5.5}$$

where ρ, $\mathbf{J}$, ϵ_o and μ_o denote, respectively, the total charge density, the total electric current density, the electric permittivity and the magnetic permeability of free space. The plasma charge and current densities can be expressed, respectively, as

$$\rho_p = \frac{1}{\delta V} \sum_i q_i \tag{5.6}$$

$$\mathbf{J}_p = \frac{1}{\delta V} \sum_i q_i \mathbf{v}_i \tag{5.7}$$

where the summation is over all charged particles contained inside a suitably chosen small volume element δV. Note that since we are dealing with a discrete distribution of charges and therefore also of current densities, ρ_p and $\mathbf{J}_p$ should actually be expressed in terms of *Dirac delta functions.* If point charges are considered, the problem gets even more complicated because the fields become singular at the particle positions. However, if δV is chosen big enough to contain a fairly large number of particles, then (5.6) and (5.7) should give smooth functions for ρ_p and $\mathbf{J}_p$ which are suitable for analytical calculations.

Although this self-consistent approach is conceivable in principle, it cannot be carried out in practice without introducing some averaging scheme, in view of the extremely large number of variables involved. According to the laws of classical mechanics, in order to determine the position and velocity of each particle in the plasma as a function of time under the action of known forces, it is necessary to know the initial position and velocity of each particle. For a system consisting of a very large number of interacting particles these initial conditions are obviously unknown. Furthermore, in order to explain and predict the macroscopic phenomena

observed in nature and in the laboratory, it is not of interest to know the detailed individual motion of each particle, since the observable macroscopic properties of a plasma are due to the average collective behavior of a large number of particles. We must discard, therefore, the possibility of analytically solving the set of simultaneous equations of motion for a large number of interacting particles.

With the advent of large and fast computers it is nowadays possible to numerically follow the nonlinear motion of many particles in their own internal self-consistent and externally applied fields and, using some averaging or smoothing scheme for the internal fields, to determine the macroscopic variables from the individual particle motion properties. This method, known as *plasma computer simulation via particles*, provides a profound view of plasma phenomena in the microscopic and macroscopic levels, and complements the analytical theoretical models and experimental observations.

5.2 Theoretical Approaches

For the theoretical description of plasma phenomena, there are basically four principal approaches with several different choices of approximations, each of which applies to different circumstances.

One useful approximation, known as *particle orbit theory*, consists in studying the motion of each charged particle in the presence of specified electric and magnetic fields. This approach is not really plasma theory, but rather the dynamics of a charged particle in given fields. Nevertheless it is important, since it provides some physical insight for a better understanding of the dynamical processes in plasmas. It has proven to be useful for predicting the behavior of very low density plasmas, which is determined primarily by the interaction of the particles with external fields. This is the case, for example, of the highly rarefied plasmas of the Van Allen radiation belts and the solar corona, as well as of cosmic rays, high energy accelerators and cathode ray tubes.

Since a plasma consists of a very large number of interacting particles, in order to provide a macroscopic description of plasma phenomena it is appropriate to adopt a *statistical approach*. This implies a great reduction in the amount of information to be handled. In the *kinetic theory* statistical description it is necessary to know only the *distribution function* for the system of particles under consideration. The problem consists in solving the appropriate *kinetic equations* which govern the evolution of the distribution function in phase space. One example of differential

kinetic equation is the *Vlasov equation*, in which the interaction between the charged particles is described by smeared out internal electromagnetic fields consistent with the distributions of electric charge density and current density inside the plasma, and the effects of short-range correlations (close collisions) are neglected.

When collisions between the plasma particles are very frequent, so that each species is able to maintain a *local equilibrium* distribution function, then each species can be treated as a fluid described by a local density, local macroscopic velocity and local temperature. In this case the plasma is treated as a mixture of two or more interpenetrating fluids. This theory is called *two-fluid* or *many-fluid theory*, depending on the number of different species considered. In addition to the usual electrodynamic equations, there is a set of hydrodynamic equations expressing conservation of mass, of momentum and of energy for each particle species in the plasma.

Another approach consists in treating the whole plasma as a *single conducting fluid* using lumped macroscopic variables and their corresponding hydrodynamic conservation equations. This theory is usually referred to as the *one-fluid theory*. An appropriately simplified form of this theory, applicable to the study of very low frequency phenomena in highly conducting fluids immersed in magnetic fields, is usually referred to as the *magnetohydrodynamic (MHD) approximation.*

PROBLEMS

1.1 The interatomic or intermolecular forces are usually represented in terms of a potential energy function $V(r)$ such that $F(r) = -dV(r)/dr$. For *neutral* particles, at large internuclear distances, there is a slight attractive potential between the particles called the *van der Waals potential* (which is the long-range part of the Lennard-Jones potential). For like atoms or molecules in like states the van der Waals interaction potential can be represented by

$$V(r) = -C(a_o/r)^6 \; Ry$$

where C is a constant (which depends on the type of particle), a_o is the Bohr radius (0.0529 nm) and Ry denotes the Rydberg energy unit (13.605 eV). Calculate the van der Waals force of attraction betwen two hydrogen molecules (for which $C = 24.0$), and compare with the Coulomb

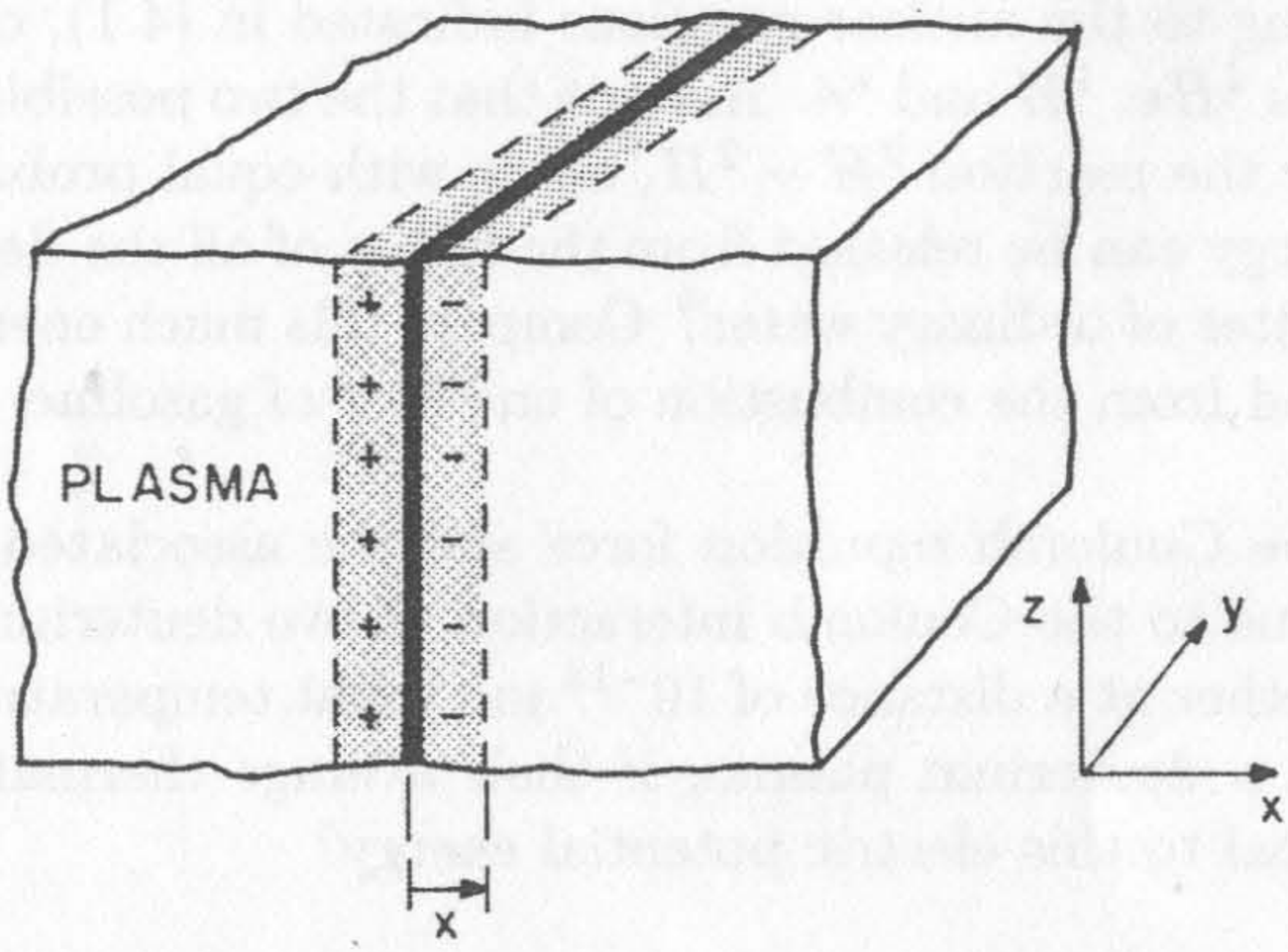


Fig. 11 *One-dimensional perturbation in which the electrons, in the y-z plane, are displaced by a small amount x.*

force between a proton and an electron at a distance $r = Na_o$, where $N \gg 1$.

1.2 Consider an initially uniform plasma in which the electron and ion number densities are each equal to n. By some external means, let a one-dimensional perturbation occur such that the electrons in an infinite plane (y, z plane) are displaced by a small amount x, as indicated in Fig. 11. (a) Using Gauss' law show that the electric field which appears across the perturbed plane is given by

$$E = \left(\frac{ne}{\epsilon_o}\right)x$$

(b) Show that the equation of motion (Newton's second law) for each electron, under the action of this electric field, is

$$\frac{d^2x}{dt^2} + \left(\frac{ne^2}{m\epsilon_o}\right)x = 0$$

Verify that this is the equation of a harmonic oscillator of frequency

$$\omega_{pe} = \left(\frac{ne^2}{m\epsilon_o}\right)^{1/2}$$

1.3 (a) Calculate the amount of energy released by the fusion of 1 g of deuterium according to the nuclear reactions indicated in (4.1), considering as end products ^4He, ^1H and ^1n. Assume that the two possible results shown in (4.1), for the reaction $^2H + {}^2H$, occur with equal probabilities. (b) How much energy can be released from the fusion of all the deuterium that exists in one liter of ordinary water? Compare this much energy with the energy obtained from the combustion of one liter of gasoline.

1.4 Calculate the Coulomb repulsion force and the associated electric potential energy due to the Coulomb interaction of two deuterium nuclei when brought together at a distance of 10^{-14} m. What temperature must have the nuclei in a deuterium plasma, if their average thermal kinetic energy is to be equal to this electric potential energy?

1.5 In a MHD generator a plasma of conductivity σ is driven with velocity $\mathbf{u}$ (in the x direction) across a magnetic field $\mathbf{B}$ (in the y direction). Two electrode plates, each of area A and separated by a distance d, are placed oriented parallel to the (x, y) plane, as shown in Fig. 9 of Chapter 1.
(a) Show that the open-circuit electric potential difference between the two electrode plates is given by

$$\phi = Bud$$

(b) If an external load of resistance R_L is connected between the electrodes, show that the current that flows is given by

$$I = \frac{Bud}{(R_L + R_p)}$$

where R_p denotes the internal plasma resistance.
(c) Show that the power delivered to the load is

$$P_L = \frac{B^2 u^2 d^2 R_L}{(R_L + R_p)^2}$$

Verify that this power has a maximum ($dP_L/dR_L = 0$) when $R_L = R_p$ and show that the maximum power that can be delivered to the load is given by

$$P_{L,max} = \frac{1}{4} B^2 u^2 dA\sigma$$

(d) Determine numerical results for items (a), (b) and (c) when $B = 1$ tesla, $u = 100$ m/s, $\sigma = 100$ mho/m, $d = 0.1$ m and $A = 1$ m^2.

1.6 Consider a rocket once it is beyond the Earth's gravitational field. Let:

v = constant velocity of the exhaust gas relative to the rocket.
$u(t)$ = instantaneous velocity of the rocket.
$M(t)$ = instantaneous total mass of the rocket.
$-dM/dt$ = constant time rate of decrease of $M(t)$, that is, the mass expelled per unit time.

(a) Verify that the equation of motion (Newton's second law) for the rocket is

$$\frac{d}{dt}[M(t)u(t)] = \frac{dM}{dt}[v - u(t)]$$

and show that the instantaneous acceleration of the rocket is

$$\frac{du}{dt} = -\frac{v}{M(t)}\frac{dM}{dt}$$

(b) Integrate the equation of motion to show that

$$u(t) = u(t_o) + v \ln [M(t_o)/M(t)]$$

(c) If the rocket burns for a time interval $\delta t = t - t_o$ and if $M(t) \ll M(t_o)$, show that the initial acceleration of the rocket is

$$\left(\frac{du}{dt}\right)_{t_o} = \frac{v}{M(t_o)}\frac{M(t_o) - M(t)}{\delta t} \simeq \frac{v}{\delta t}$$

(d) Calculate numerically $(du/dt)_{t_o}$ and $u(t)$ for a chemical rocket with $v = 10^3$ m/s and $\delta t = 10$ s; and also for a plasma propulsion system with $v = 10^4$ m/s and $\delta t = 100$ days. For the calculation of $u(t)$ consider $u(t_o) = 0$ and $M(t_o) = 10\ M(t)$.

1.7 Using Maxwell equations (5.3) and (5.4) derive the charge conservation equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

This result shows that conservation of electric charge is already implied by Maxwell equations.

1.8 From Maxwell curl equation (5.2) derive the equation

$$\nabla \cdot \mathbf{B} = constant$$

Therefore, (5.5) can be considered as an *initial condition* for (5.2) since, if $\nabla \cdot \mathbf{B} = 0$ at a given initial time, then (5.2) implies that this condition will remain satisfied for all subsequent times.

1.9 Using Maxwell equations derive the following energy conservation law for electromagnetic fields, known as *Poynting's theorem*,

$$\frac{\partial}{\partial t}\int_V \left(\tfrac{1}{2}\epsilon E^2 + \tfrac{1}{2}\mu H^2\right) d^3r + \oint_S (\mathbf{E}\times\mathbf{H})\cdot d\mathbf{S} = -\int_V (\mathbf{J}\cdot\mathbf{E}) d^3r$$

for a linear isotropic medium, for which $\mathbf{D} = \epsilon\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$. Give the physical interpretation for each term in this equation. What are the physical dimensions of these terms?

1.10 Consider the following Maxwell equations:

$$\nabla \times \mathbf{B} = \mu_o\left(\mathbf{J}_t + \epsilon_o \frac{\partial \mathbf{E}}{\partial t}\right)$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho_t}{\epsilon_o}$$

$$\nabla \cdot \mathbf{D} = \rho$$

For a general medium for which

$$\mathbf{D} = \epsilon_o \mathbf{E} + \mathbf{P}$$

$$\mathbf{B} = \mu_o(\mathbf{H} + \mathbf{M})$$

where $\mathbf{P}$ is the *polarization* vector and $\mathbf{M}$ is the *magnetization* vector, show that the *total* electric charge density (ρ_t) and current density ($\mathbf{J}_t$) are given by

$$\rho_t = \rho - \nabla \cdot \mathbf{P}$$

$$\mathbf{J}_t = \mathbf{J} + \frac{\partial \mathbf{P}}{\partial t} + \nabla \times \mathbf{M}$$

Explain why $\mathbf{E}$ and $\mathbf{B}$ are considered as *fundamental* fields, whereas $\mathbf{D}$ and $\mathbf{H}$ are *partial* fields.

2

CHARGED PARTICLE MOTION IN CONSTANT AND UNIFORM ELECTROMAGNETIC FIELDS

1. INTRODUCTION

In this and in the following two chapters we investigate the motion of charged particles in the presence of electric and magnetic fields known as functions of position and time. Thus, the electric and magnetic fields are assumed to be prescribed and are not affected by the charged particles. This chapter, in particular, considers the fields to be constant in time and spatially uniform. This subject is considered in some detail, since many of the more complex situations, considered in Chapters 3 and 4, can be treated as perturbations to this problem.

The study of the motion of charged particles in specified fields is important, since it provides a good physical insight for understanding some of the dynamical processes in plasmas. It also permits to obtain information on some macroscopic phenomena which are due to the collective behavior of a large number of particles. Not all of the components of the detailed microscopic particle motion contribute to macroscopic effects, but it is possible to isolate the components of the individual motion that contribute to the collective plasma behavior. Nevertheless, the macroscopic parameters can be obtained much more easily and conveniently from the macroscopic transport equations presented in Chapters 8 and 9.

The equation of motion for a particle of charge q, under the action of the Lorentz force $\mathbf{F}$ due to electric ($\mathbf{E}$) and magnetic induction ($\mathbf{B}$) fields, can be written as

$$\frac{d\mathbf{p}}{dt} = \mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{1.1}$$

where $\mathbf{p}$ represents the momentum of the particle and $\mathbf{v}$ its velocity. This equation is relativistically correct if we take

$$\mathbf{p} = \gamma m \mathbf{v} \tag{1.2}$$

where m is the rest mass of the particle and γ is the so-called *Lorentz factor* defined by

$$\gamma = (1 - v^2/c^2)^{-1/2} \tag{1.3}$$

where c is the speed of light in vacuum. In the relativistic case, (1.1) can also be written in the form

$$\gamma m \frac{d\mathbf{v}}{dt} + q\left(\frac{\mathbf{v}}{c^2}\right)(\mathbf{v} \cdot \mathbf{E}) = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{1.4}$$

noting that the time rate of change of the total relativistic energy ($U = \gamma m c^2$) is given by $dU/dt = q(\mathbf{v} \cdot \mathbf{E})$ and that $d\mathbf{p}/dt = d(U\mathbf{v}/c^2)/dt$.

In many situations of pratical interest, however, the term v^2/c^2 is neglibible compared to unity. For $v^2/c^2 \ll 1$ we have $\gamma \simeq 1$ and m can be considered constant (independent of v), so that (1.4) reduces to the following nonrelativistic expression

$$m \frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{1.5}$$

If the velocity obtained from (1.5) does not satisfy the condition $v^2 \ll c^2$, then the corresponding result is not valid and the relativistic expression (1.4) must be used instead of (1.5). Relativistic effects become important only for highly energetic particles (a 1 MeV proton, for instance, has a velocity of 1.4×10^7m/s, with $v^2/c^2 \simeq 0.002$). For the situations to be considered here it is assumed that the restriction $v^2 \ll c^2$, implicit in (1.5), is not violated. Also, all radiation effects are neglected.

2. ENERGY CONSERVATION

In the absence of an electric field ($\mathbf{E} = 0$), the equation of motion (1.5) reduces to

$$m \frac{d\mathbf{v}}{dt} = q(\mathbf{v} \times \mathbf{B}) \tag{2.1}$$

Since the magnetic force is perpendicular to $\mathbf{v}$, it does no work on the particle. Taking the dot product of (2.1) with $\mathbf{v}$ and noting that for any

vector $\mathbf{v}$, we have $(\mathbf{v} \times \mathbf{B}) \cdot \mathbf{v} = 0$, we obtain

$$m\frac{d\mathbf{v}}{dt} \cdot \mathbf{v} = \frac{d}{dt}\left(\tfrac{1}{2}mv^2\right) = 0 \tag{2.2}$$

which shows that the particle kinetic energy $(mv^2/2)$ and the magnitude of its velocity (speed v) are both constants . Therefore, a *static* magnetic field does not change the particle kinetic energy. This result is valid whatever the spatial dependence of the magnetic flux density $\mathbf{B}$. However, if $\mathbf{B}$ varies with time then, according to Maxwell equations, an electric field such that $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$ is also present which does work on the particle changing its kinetic energy.

When both *magnetostatic* and *electrostatic* fields are present, we obtain from (1.5)

$$\frac{d}{dt}\left(\tfrac{1}{2}mv^2\right) = q(\mathbf{E} \cdot \mathbf{v}) \tag{2.3}$$

Since $\nabla \times \mathbf{E} = 0$, we can express the electrostatic field in terms of the electrostatic potential according to $\mathbf{E} = -\nabla\phi$, so that

$$\frac{d}{dt}\left(\tfrac{1}{2}mv^2\right) = -q(\nabla\phi) \cdot \mathbf{v} = -q(\nabla\phi) \cdot \frac{d\mathbf{r}}{dt} = -q\frac{d\phi}{dt} \tag{2.4}$$

This result can be rearranged in the following conservation form

$$\frac{d}{dt}\left(\tfrac{1}{2}mv^2 + q\phi\right) = 0 \tag{2.5}$$

which shows that the sum of the particle kinetic and electric potential energies remains constant in the presence of *static* electromagnetic fields. Note that the electric potential ϕ can be considered as the potential energy per unit charge.

When the fields are time-dependent we have $\nabla \times \mathbf{E} \neq 0$ and $\mathbf{E}$ is not the gradient of a scalar function. But, since $\nabla \cdot \mathbf{B} = 0$, we can define a magnetic vector potential $\mathbf{A}$ by $\mathbf{B} = \nabla \times \mathbf{A}$ and write (1.5.2) as

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = \nabla \times \mathbf{E} + \frac{\partial}{\partial t}(\nabla \times \mathbf{A}) = \nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = 0 \tag{2.6}$$

Hence, we can express the electric field in the form

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \tag{2.7}$$

In this case the system is not conservative in the usual sense and there is no energy integral, but the analysis may be performed using a *Lagrangian function* L for a charged particle in electromagnetic fields, defined by

$$L = \tfrac{1}{2}mv^2 - U \tag{2.8}$$

where U is a velocity-dependent potential energy given by

$$U = q(\phi - \mathbf{v} \cdot \mathbf{A}) \tag{2.9}$$

The energy considerations presented in this section assume that the particle energy changes only as a result of the work done by the fields. This assumption is not strictly correct since every charged particle when accelerated irradiates energy in the form of electromagnetic waves. For the situations to be considered here this effect is usually small and can be neglected.

3. UNIFORM ELECTROSTATIC FIELD

According to (1.1) the motion of a charged particle in an electric field obeys the following differential equation

$$\frac{d\mathbf{p}}{dt} = q\mathbf{E} \tag{3.1}$$

For a constant $\mathbf{E}$ field, (3.1) can be integrated directly giving

$$\mathbf{p}(t) = q\mathbf{E}t + \mathbf{p}_o \tag{3.2}$$

where $\mathbf{p}_o = \mathbf{p}(0)$ denotes the initial particle momentum. Using the non-relativistic expression

$$\mathbf{p} = m\mathbf{v} = m\frac{d\mathbf{r}}{dt} \tag{3.3}$$

and performing a second integration in (3.2), we obtain the following expression for the particle position as a function of time

$$\mathbf{r}(t) = \tfrac{1}{2}\left(\frac{q\mathbf{E}}{m}\right)t^2 + \mathbf{v}_o t + \mathbf{r}_o \tag{3.4}$$

where $\mathbf{r}_o$ denotes the particle initial position and $\mathbf{v}_o$ its initial velocity. Therefore, the particle moves with a constant acceleration, $q\mathbf{E}/m$, in the direction of $\mathbf{E}$ if $q > 0$, and in the opposite direction if $q < 0$. In a direction perpendicular to the electric field there is no acceleration and the particle state of motion remains unchanged.

4. UNIFORM MAGNETOSTATIC FIELD

4.1 Formal Solution of the Equation of Motion

For a particle of charge q and mass m, moving with velocity $\mathbf{v}$ in a region of space where there is only a magnetic induction $\mathbf{B}$ (no electric field $\mathbf{E}$), the equation of motion is

$$m\frac{d\mathbf{v}}{dt} = q(\mathbf{v} \times \mathbf{B}) \tag{4.1}$$

It is convenient to separate $\mathbf{v}$ in components parallel ($\mathbf{v}_\parallel$) and perpendicular ($\mathbf{v}_\perp$) to the magnetic field,

$$\mathbf{v} = \mathbf{v}_\parallel + \mathbf{v}_\perp \tag{4.2}$$

as indicated in Fig. 1. Substituting (4.2) into (4.1) and noting that ($\mathbf{v}_\parallel \times \mathbf{B} = 0$) we obtain

$$\frac{d\mathbf{v}_\parallel}{dt} + \frac{d\mathbf{v}_\perp}{dt} = \frac{q}{m}(\mathbf{v}_\perp \times \mathbf{B}) \tag{4.3}$$

Since the term ($\mathbf{v}_\perp \times \mathbf{B}$) is perpendicular to $\mathbf{B}$, the *parallel* component equation can be written as

$$\frac{d\mathbf{v}_\parallel}{dt} = 0 \tag{4.4}$$

and the *perpendicular* component equation as

$$\frac{d\mathbf{v}_\perp}{dt} = \frac{q}{m}(\mathbf{v}_\perp \times \mathbf{B}) \tag{4.5}$$

Eq. (4.4) shows that the particle velocity *along* $\mathbf{B}$ does not change and is equal to the particle initial velocity. For motion in the plane *perpendicular* to $\mathbf{B}$, we can write (4.5) in the form

$$\frac{d\mathbf{v}_\perp}{dt} = \mathbf{\Omega}_c \times \mathbf{v}_\perp \tag{4.6}$$

where $\mathbf{\Omega}_c$ is a vector defined by

$$\mathbf{\Omega}_c = -\frac{q\mathbf{B}}{m} = \frac{|q|\ B}{m}\widehat{\mathbf{\Omega}}_c = \Omega_c\widehat{\mathbf{\Omega}}_c \tag{4.7}$$

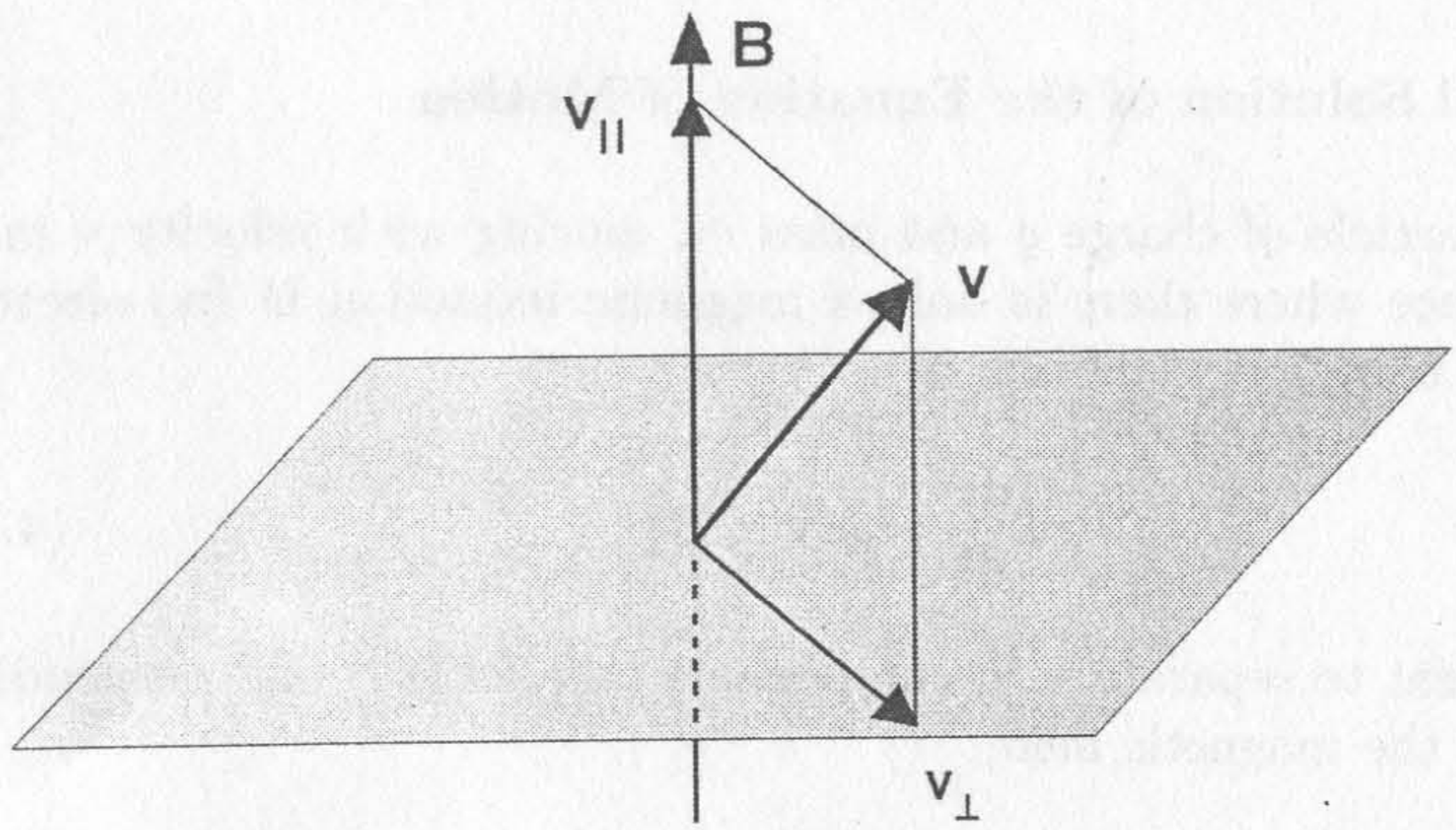


Fig. 1 *Decomposition of the velocity vector into components parallel* ($\mathbf{v}_{\parallel}$) *and perpendicular* ($\mathbf{v}_{\perp}$) *to the magnetic field.*

Thus, $\widehat{\mathbf{\Omega}}_c$ points in the direction of $\mathbf{B}$ for a *negatively* charged particle ($q < 0$) and in the opposite direction for a *positively* charged particle ($q > 0$). Its magntitude Ω_c is always positive ($\Omega_c = |q|\ B/m$). The unit vector $\widehat{\mathbf{\Omega}}_c$ points along $\mathbf{\Omega}_c$.

Since $\mathbf{\Omega}_c$ is constant and, from conservation of kinetic energy, $v_{\perp}$ (the magnitude of $\mathbf{v}_{\perp}$) is also constant, (4.6) shows that the particle acceleration is constant in magnitude and its direction is perpendicular to both $\mathbf{v}_{\perp}$ and $\mathbf{B}$. Thus, this acceleration corresponds to a rotation of the velocity vector $\mathbf{v}_{\perp}$ in the plane perpendicular to $\mathbf{B}$ with the constant angular velocity $\mathbf{\Omega}_c$. We can integrate (4.6) directly, noting that $\mathbf{\Omega}_c$ is constant and taking $\mathbf{v}_{\perp} = d\mathbf{r}_c/dt$, to obtain

$$\mathbf{v}_{\perp} = \mathbf{\Omega}_c \times \mathbf{r}_c \tag{4.8}$$

where the vector $\mathbf{r}_c$ is interpreted as the particle position vector with respect to a point G (the center of gyration) in the plane perpendicular to $\mathbf{B}$ which contains the particle. Since the particle speed $v_{\perp}$ is constant, the magnitude r_c of the position vector is also constant. Therefore, (4.8) shows that the velocity $\mathbf{v}_{\perp}$ corresponds to a rotation of the position vector $\mathbf{r}_c$ about the point G in the plane perpendicular to $\mathbf{B}$ with constant angular velocity $\mathbf{\Omega}_c$. The component of the motion in the plane perpendicular to $\mathbf{B}$ is therefore a circle of radius r_c. The *instantaneous* center of gyration of the particle (the point G at the distance r_c from the particle) is called the

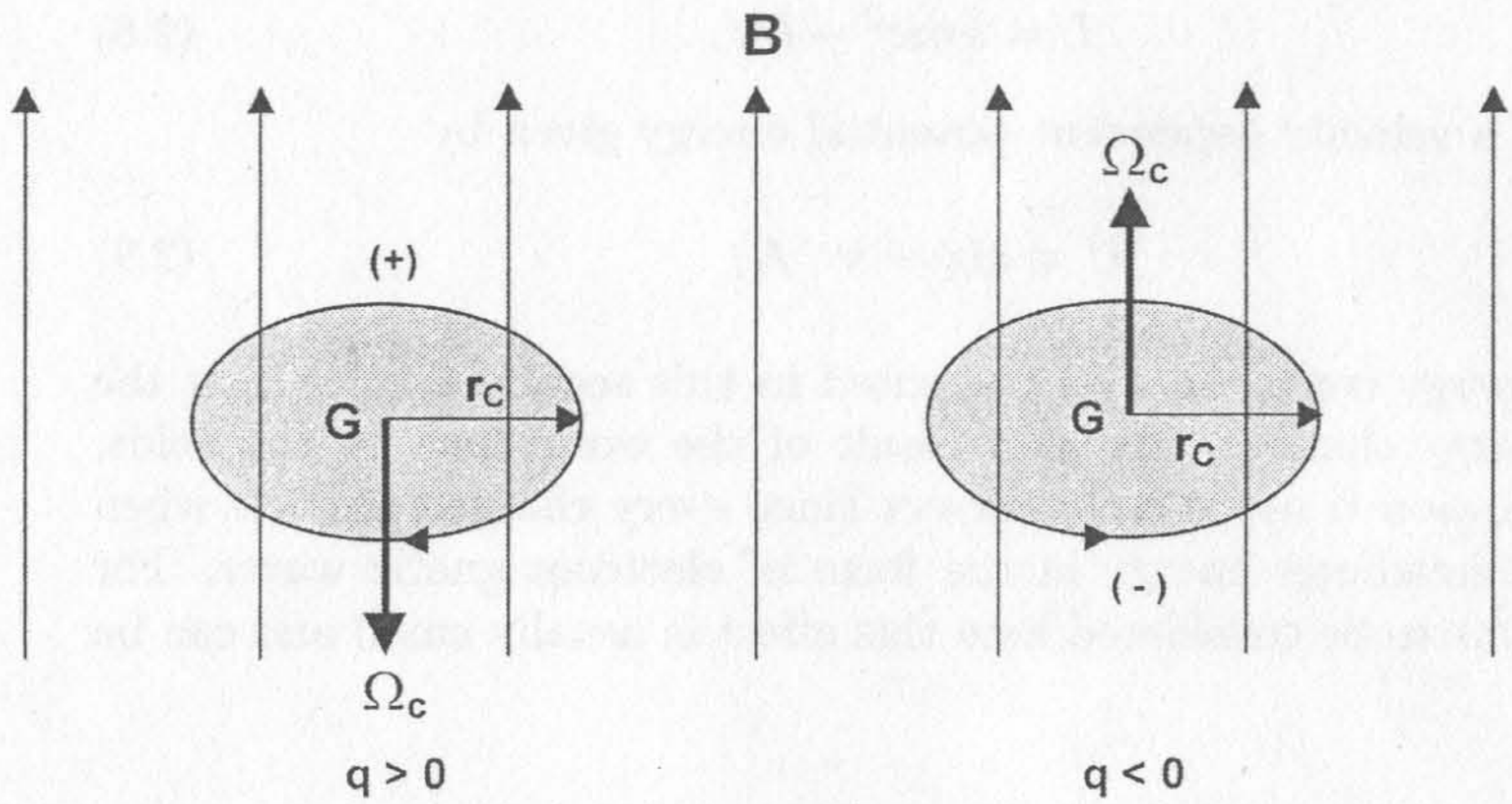


Fig. 2 *Circular motion of a charged particle about the guiding center in a uniform magnetostatic field.*

guiding center. This circular motion about the guiding center is illustrated in Fig. 2.

Note that according to the definition of $\mathbf{\Omega}_c$, given in (4.7), $\widehat{\mathbf{\Omega}}_c$ always points in the same direction as the particle *angular momentum vector* ($\mathbf{r}_c \times \mathbf{p}$), irrespective of its charge.

The resulting trajectory of the particle is given by the superposition of a *uniform* motion *along* $\mathbf{B}$ (with the constant velocity $\mathbf{v}_\parallel$) and a *circular* motion in the plane *normal* to $\mathbf{B}$ (with the constant speed $v_\perp$). Hence, the particle describes a *helix* (see Fig. 3). The angle between $\mathbf{B}$ and the direction of motion of the particle is called the *pitch angle* and is given by

$$\alpha = \sin^{-1}\left(\frac{v_\perp}{v}\right) = \tan^{-1}\left(\frac{v_\perp}{v_\parallel}\right) \tag{4.9}$$

where v is the total speed of the particle ($v^2 = v_\parallel^2 + v_\perp^2$). When $v_\parallel = 0$ but $v_\perp \neq 0$, we have $\alpha = \pi/2$ and the particle trajectory is a circle in the plane normal to $\mathbf{B}$. On the other hand, when $v_\perp = 0$ but $v_\parallel \neq 0$, we have $\alpha = 0$ and the particle moves along $\mathbf{B}$ with the velocity $\mathbf{v}_\parallel$.

The magnitude of the angular velocity,

$$\Omega_c = \frac{|q|\ B}{m} \tag{4.10}$$

is known as the *angular frequency of gyration*, and is also called the *gyrofrequency*, or *cyclotron frequency* or *Larmor frequency*. For an electron $|q| = 1.602 \times 10^{-19}$ coulomb and $m = 9.109 \times 10^{-31}$ kg, so that

$$\Omega_c(electron) = 1.76 \times 10^{11} B \quad \text{(rad/s)} \tag{4.11}$$

with B in tesla (or, equivalently, weber/m^2). Similarly, for a proton $m = 1.673 \times 10^{-27}$ kg, so that

$$\Omega_c(proton) = 9.58 \times 10^{7} B \quad \text{(rad/s)} \tag{4.12}$$

The radius of the circular orbit, given by

$$r_c = \frac{v_\perp}{\Omega_c} = \frac{m v_\perp}{|q|\, B} \tag{4.13}$$

is called the *radius of gyration*, or *gyroradius*, or *cyclotron radius*, or *Larmor radius*. It is important to note that Ω_c is directly proportional to B. Consequently, as B increases, the gyrofrequency increases and the radius decreases. Also, the smaller the particle mass the larger will be its gyrofrequency and the smaller its gyroradius. Multiplying (4.13) by B gives

$$B r_c = \frac{m v_\perp}{|q|} = \frac{p_\perp}{|q|} \tag{4.14}$$

which shows that the magnitude of B times the particle gyroradius is equal to the particle momentum per unit charge. This quantity is often called the *magnetic rigidity*.

4.2 Solution in Cartesian Coordinates

The treatment presented so far in this section was not related to any particular frame of reference. Consider now a Cartesian coordinate system (x, y, z) such that $\mathbf{B} = B\widehat{\mathbf{z}}$. In this case, the cross product between $\mathbf{v}$ and $\mathbf{B}$ can be written as

$$\mathbf{v} \times \mathbf{B} = \det \begin{vmatrix} \widehat{\mathbf{x}} & \widehat{\mathbf{y}} & \widehat{\mathbf{z}} \\ v_x & v_y & v_z \\ 0 & 0 & B \end{vmatrix} = B(v_y \widehat{\mathbf{x}} - v_x \widehat{\mathbf{y}}) \tag{4.15}$$

and the equation of motion (4.1) becomes

$$\frac{d\mathbf{v}}{dt} = \frac{qB}{m}(v_y \widehat{\mathbf{x}} - v_x \widehat{\mathbf{y}}) = \pm \Omega_c (v_y \widehat{\mathbf{x}} - v_x \widehat{\mathbf{y}}) \tag{4.16}$$

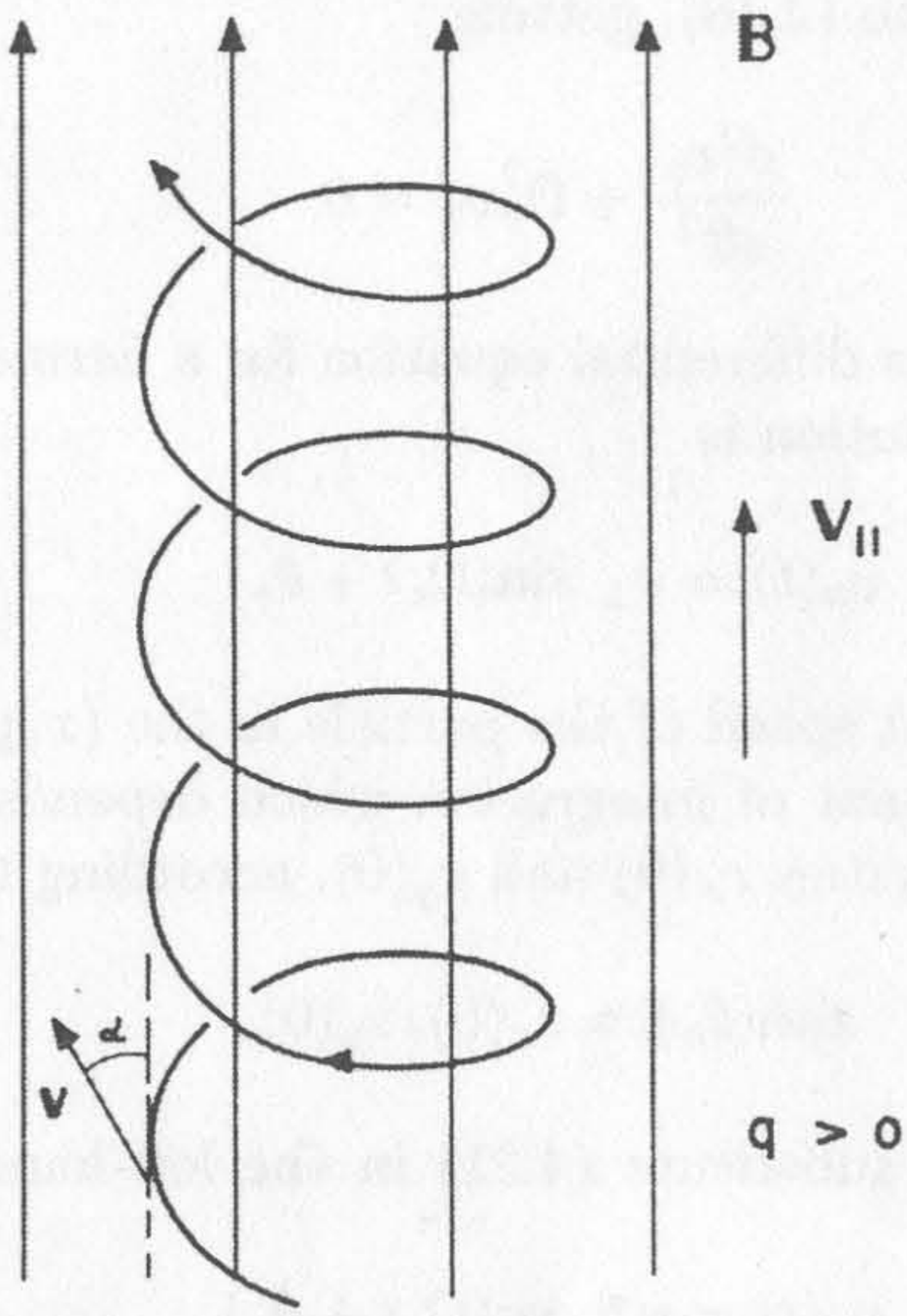


Fig. 3 *Helicoidal trajectory of a positively charged particle in a uniform magnetostatic field.*

The (+) sign in front of Ω_c applies to a positively charged particle ($q > 0$) and the (−) sign to a negatively charged particle ($q < 0$), since Ω_c is always positive, according to its definition given in (4.10). In what follows we shall consider a positively charged particle. The results for a negative charge can be obtained by changing the sign of Ω_c in the results for the positive charge.

The Cartesian components of (4.16) are (for $q > 0$)

$$\frac{dv_x}{dt} = \Omega_c v_y \tag{4.17}$$

$$\frac{dv_y}{dt} = -\Omega_c v_x \tag{4.18}$$

$$\frac{dv_z}{dt} = 0 \tag{4.19}$$

The last of these equations gives $v_z(t) = v_z(0) = v_{\parallel}$, which is the initial value of the velocity component parallel to **B**. To obtain the solution of (4.17) and (4.18), we take the derivative of (4.17) with respect to time and

substitute this result into (4.18), getting

$$\frac{d^2 v_x}{dt^2} + \Omega_c^2 v_x = 0 \tag{4.20}$$

This is the homogeneous differential equation for a *harmonic oscillator* of frequency Ω_c, whose solution is

$$v_x(t) = v_\perp \sin(\Omega_c t + \theta_o) \tag{4.21}$$

where $v_\perp$ is the constant speed of the particle in the (x, y) plane (normal to $\mathbf{B}$) and θ_o is a constant of integration which depends on the relation between the initial velocities $v_x(0)$ and $v_y(0)$, according to

$$\tan(\theta_o) = v_x(0)/v_y(0) \tag{4.22}$$

To determine $v_y(t)$ we substitute (4.21) in the left-hand side of (4.17), obtaining

$$v_y(t) = v_\perp \cos(\Omega_c t + \theta_o) \tag{4.23}$$

Note that $v_x^2 + v_y^2 = v_\perp^2$. The equations for the components of $\mathbf{v}$ can be further integrated with respect to time, yielding

$$x(t) = -\frac{v_\perp}{\Omega_c} \cos(\Omega_c t + \theta_o) + X_o \tag{4.24}$$

$$y(t) = \frac{v_\perp}{\Omega_c} \sin(\Omega_c t + \theta_o) + Y_o \tag{4.25}$$

$$z(t) = v_\parallel t + z_o \tag{4.26}$$

where we have defined

$$X_o = x_o + \frac{v_\perp}{\Omega_c} \cos(\theta_o) \tag{4.27}$$

$$Y_o = y_o - \frac{v_\perp}{\Omega_c} \sin(\theta_o) \tag{4.28}$$

The vector $\mathbf{r} = x_o\widehat{\mathbf{x}} + y_o\widehat{\mathbf{y}} + z_o\widehat{\mathbf{z}}$ gives the initial particle position. From (4.24) and (4.25) we see that

$$(x - X_o)^2 + (y - Y_o)^2 = (v_\perp/\Omega_c)^2 = r_c^2 \tag{4.29}$$

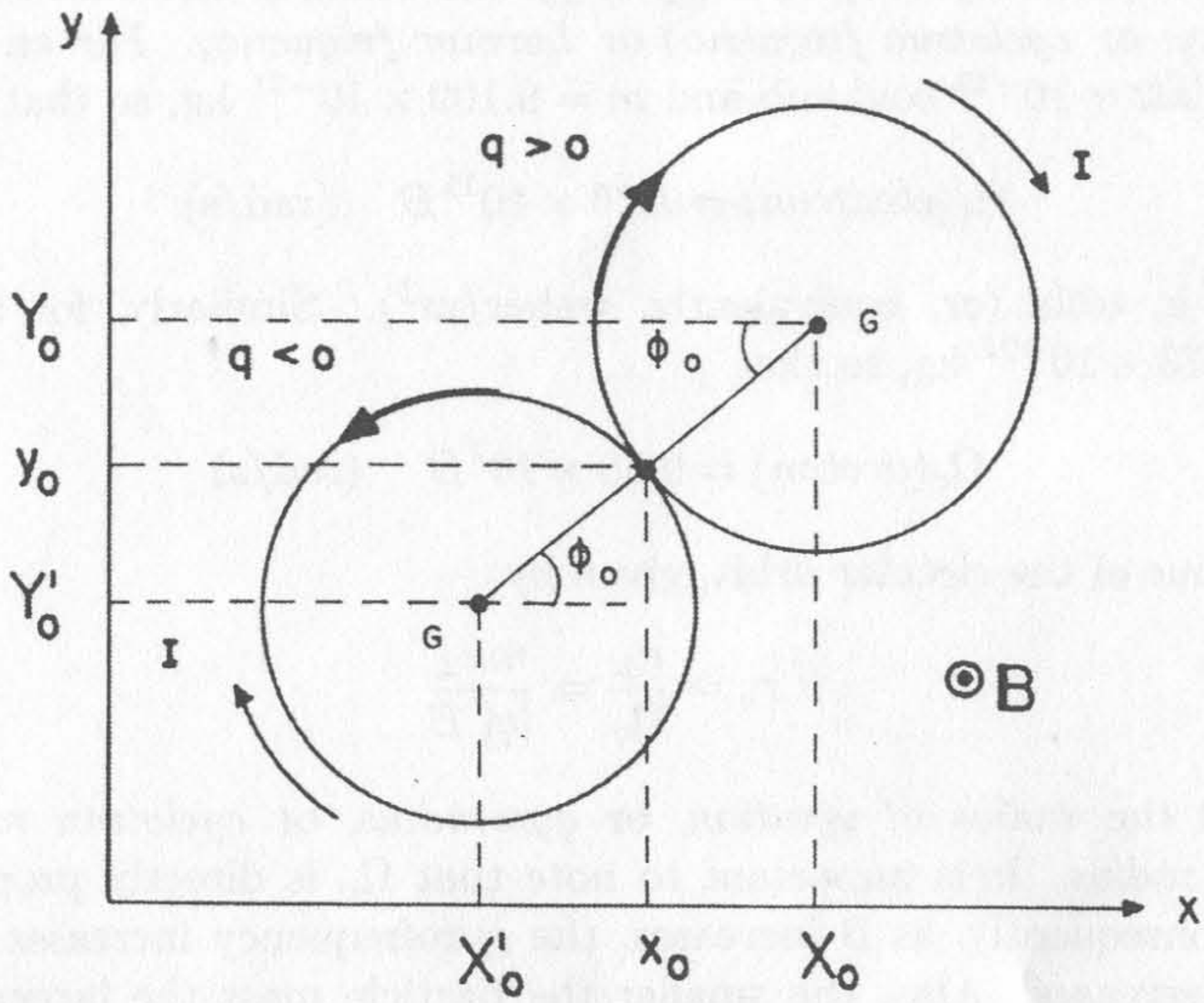


Fig. 4 *Circular trajectory of a charged particle in a uniform and constant* **B** *field (directed out of the paper), and the direction of the associated electric current.*

The particle trajectory in the plane normal to **B** is therefore a circle with center at (X_o, Y_o) and radius equal to $(v_\perp/\Omega_c)$. The motion of the point $[X_o, Y_o, z(t)]$, at the instantaneous center of gyration, corresponds to the trajectory of the guiding center. Thus, the guiding center moves with constant velocity $v_\parallel$ along **B**.

In the (x, y) plane, the argument $\phi(t)$, defined by

$$\phi(t) = \tan^{-1}\frac{(y - Y_o)}{(x - X_o)} = -(\Omega_c t + \theta_o) \quad ; \quad \phi_o = -\theta_o \tag{4.30}$$

decreases with time for a positively charged particle. For a magnetic field pointing towards the observer, a *positive* charge describes a circle in the *clockwise* direction. For a negatively charged particle Ω_c must be replaced by $-\Omega_c$ in the results of this sub-section. Hence, (4.30) shows that for a *negative* charge $\phi(t)$ increases with time and the particle moves in a circle in the *counterclockwise* direction, as shown in Fig. 4. The resulting particle motion is a *cylindrical helix* of constant pitch angle. Fig. 5 shows the parameters of the helix with reference to a Cartesian coordinate system.

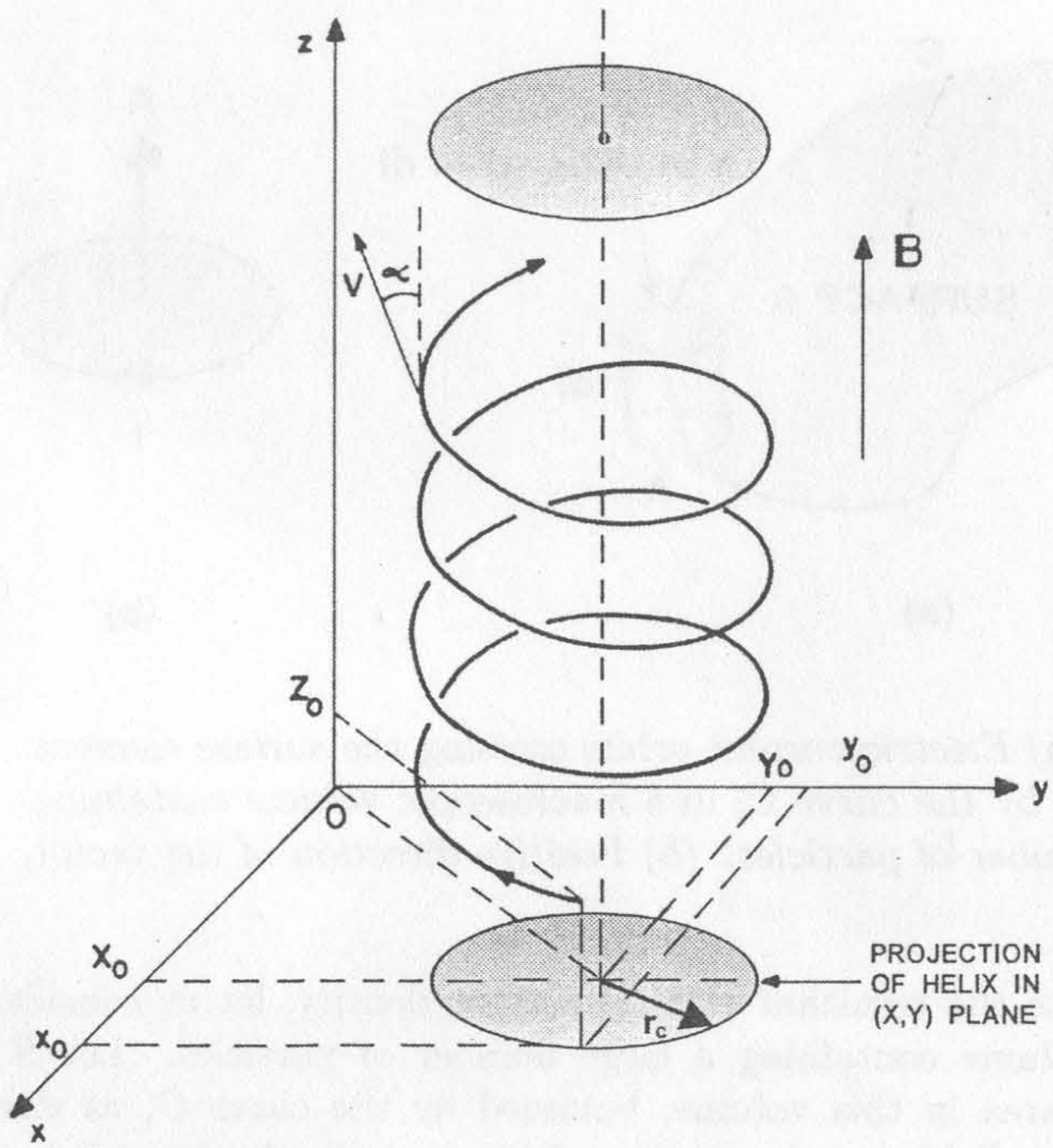


Fig. 5 *Parameters of the helicoidal trajectory of a positively charged particle with reference to a Cartesian coordinate system.*

4.3 Magnetic Moment

To the circular motion of a charged particle in a magnetic field there is associated a circulating electric current I. This current flows in the clockwise direction for a **B** field pointing towards the observer (Fig. 4). From Ampere's law, the direction of the magnetic field associated with this circulating current is given by the *right-hand rule* i.e. with the right thumb pointing in the direction of the current I, the right fingers curl in the direction of the associated magnetic field. Therefore, the **B** field produced by the circular motion of a charged particle is *opposite* to the externally applied **B** field *inside* the particle orbit, but in the same direction outside the orbit. The magnetic field generated by the ring current I, at distances much larger than r_c, is similar to that of a *dipole* (Fig. 6).

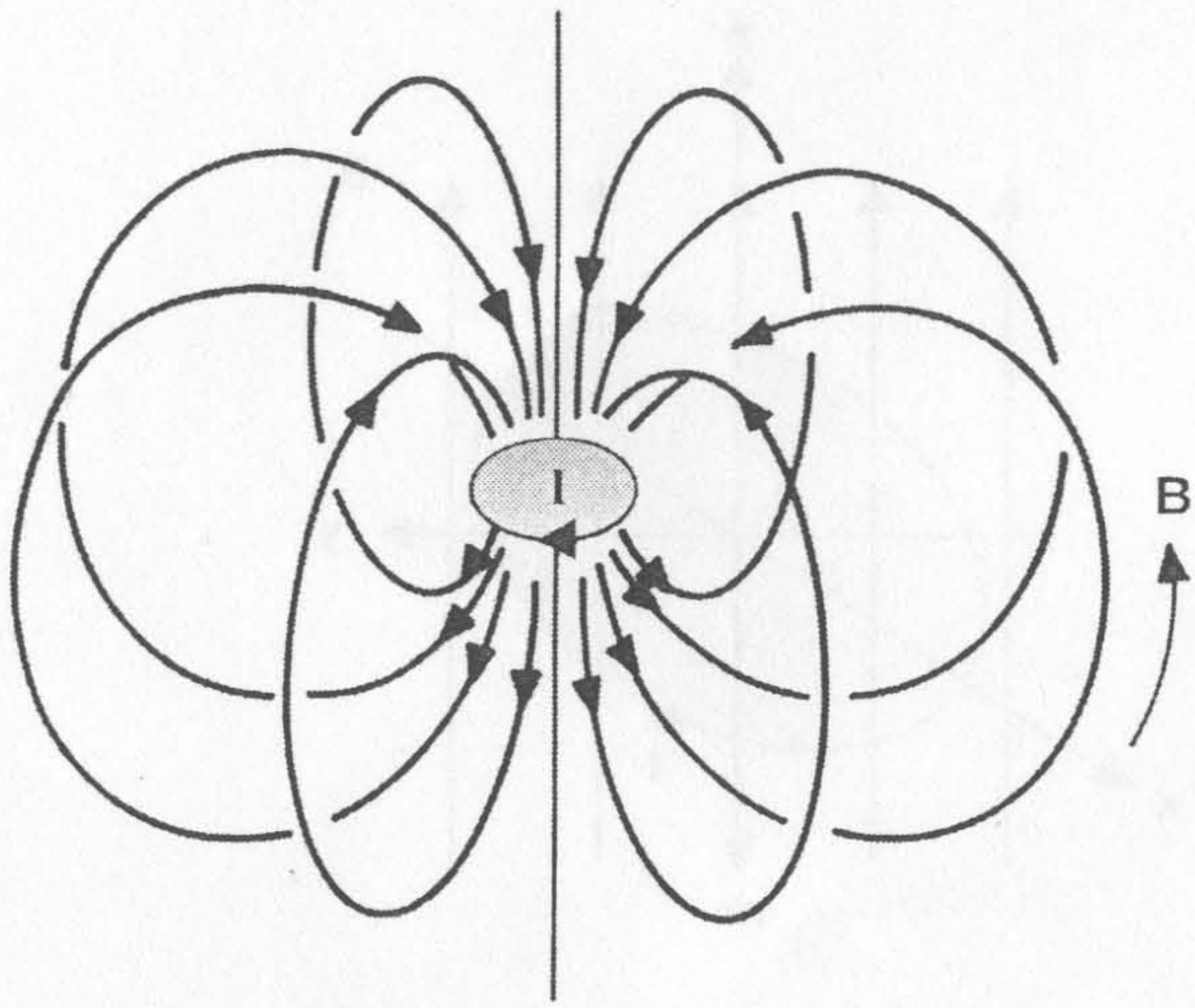


Fig. 6 *The magnetic field generated by a small ring current is that of a magnetic dipole.*

Since a plasma is a collection of charged particles, it possesses therefore *diamagnetic* properties.

The *magnetic moment* $\mathbf{m}$ associated with the circulating current is normal to the area A bounded by the particle orbit and points in the direction opposite to the externally applied $\mathbf{B}$ field, as shown in Fig. 7. Its magnitude is given by

$$|\mathbf{m}| = (current) \cdot (orbital \ \ area) = IA \tag{4.31}$$

This circulating current corresponds to a flow of charge and is given by

$$I = \frac{|q|}{T_c} = \frac{|q| \ \Omega_c}{2\pi} \tag{4.32}$$

where $T_c = 2\pi/\Omega_c$ is the period of the particle orbit, known as the *cyclotron period* or *Larmor period.* The magnitude of $\mathbf{m}$ is therefore

$$|\mathbf{m}| = \frac{|q| \ \Omega_c}{2\pi} \pi r_c^2 = \tfrac{1}{2}|q| \ \Omega_c r_c^2 \tag{4.33}$$

Using the relations $\Omega_c = |q| \ B/m$ and $r_c = v_\perp/\Omega_c$, (4.33) becomes

$$|\mathbf{m}| = \frac{\frac{1}{2}mv_\perp^2}{B} = \frac{W_\perp}{B} \tag{4.34}$$

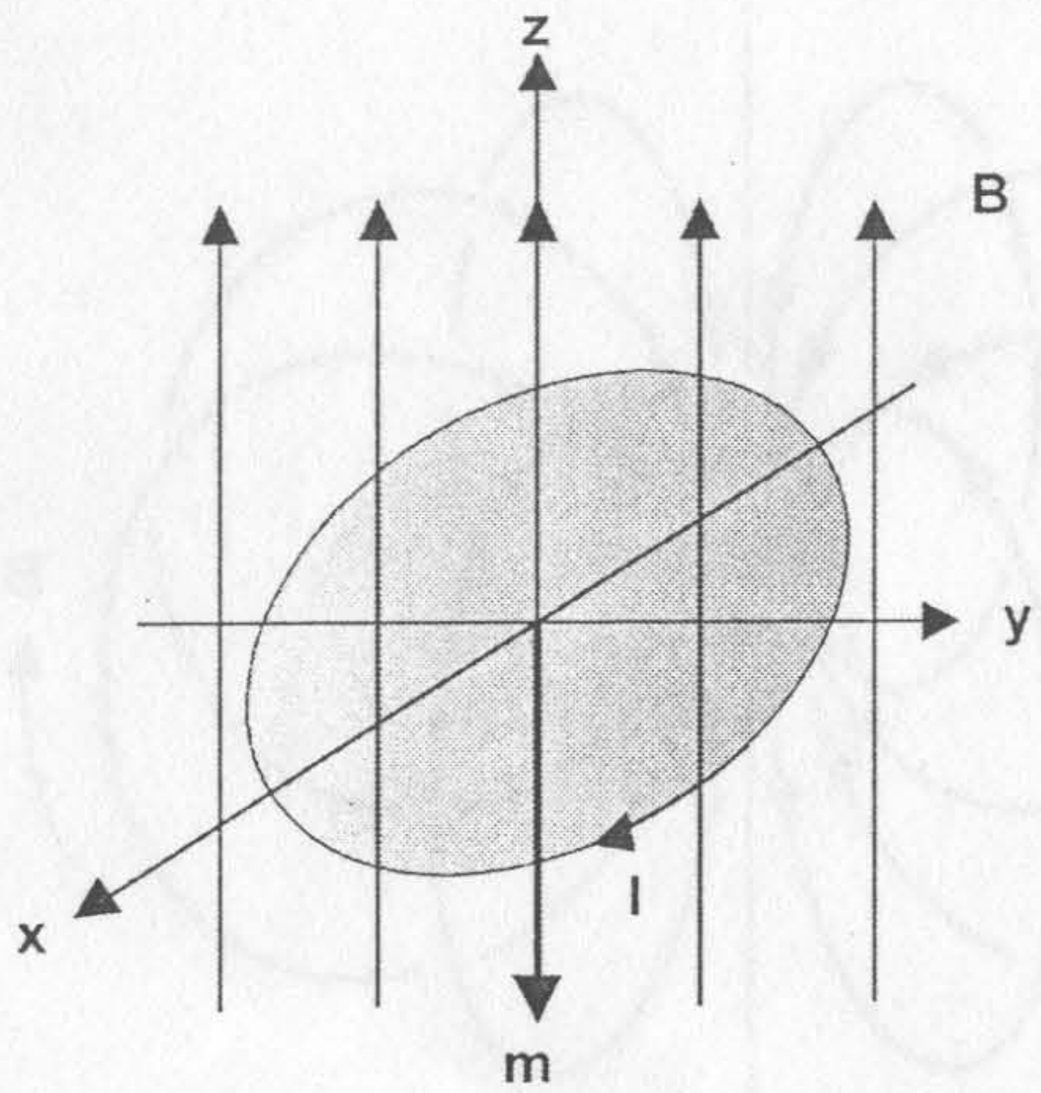


Fig. 7 *Magnetic moment* **m** *associated with a circulating current due to the circular motion of a charged particle in an external* **B** *field.*

where $W_\perp$ denotes the part of the particle kinetic energy associated with the transverse velocity $v_\perp$. Thus, in vector form,

$$\mathbf{m} = -\frac{W_\perp}{B^2}\mathbf{B} \tag{4.35}$$

4.4 Magnetization Current

Consider now a collection of charged particles, positive and negative in equal numbers (in order to have no internal macroscopic electrostatic fields), instead of just one single particle. For instance, consider the case of a low-density plasma in which the particle collisions can be neglected (collisionless plasma). The condition for this is that the average time between collisions be much greater than the cyclotron period. This condition is fulfilled for many space plasmas, for example. For a collisionless plasma in an external magnetic field, the magnetic moments due to the orbital motion of the charged particles act together, giving rise to a resultant magnetic field which may be strong enough to appreciably change the externally applied **B** field. The magnetic field produced by the orbital motion of the charged particles can be determined from the *net* electric current density associated with their motion.

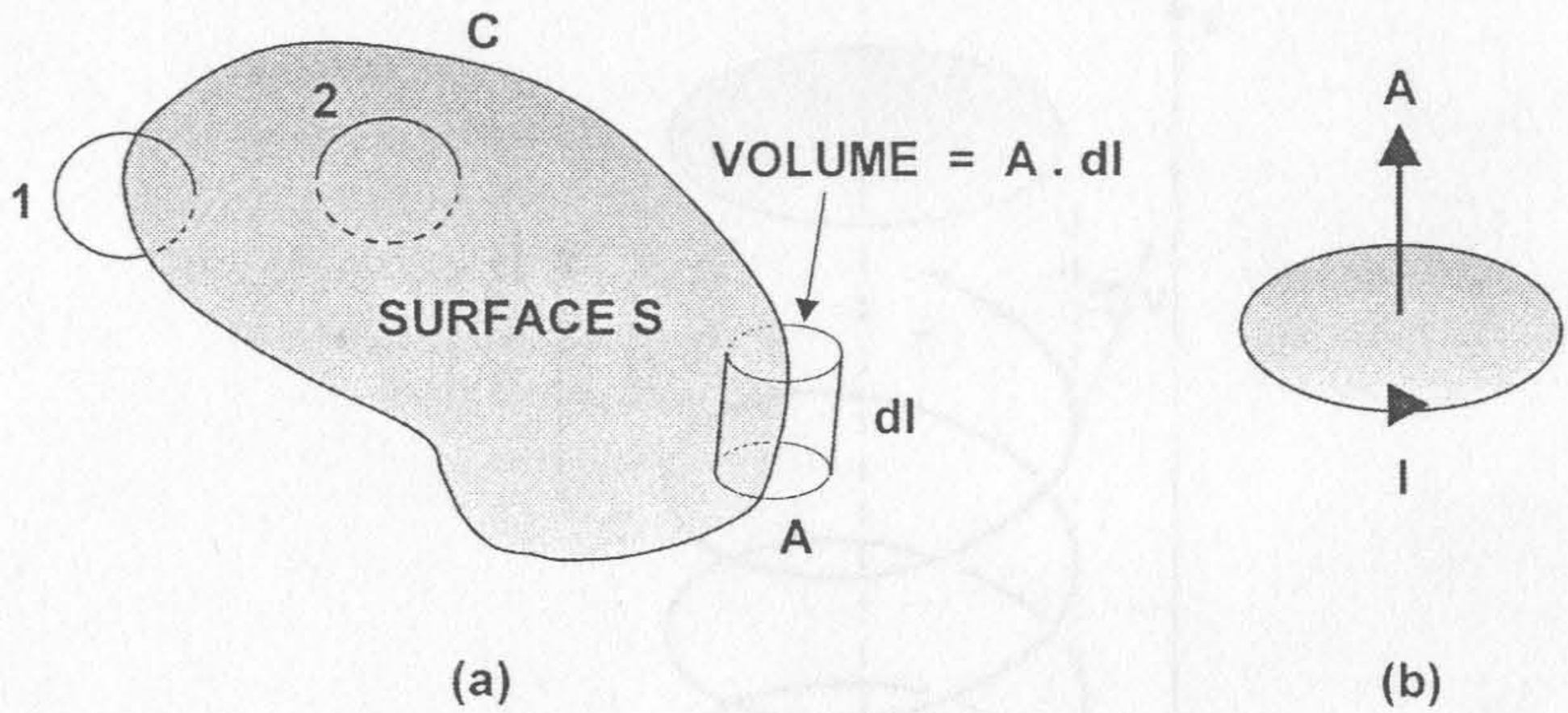


Fig. 8 *(a) Electric current orbits crossing the surface element S bounded by the curve C, in a macroscopic volume containing a large number of particles. (b) Positive direction of the vector area* **A**.

To calculate the resultant electric current density, let us consider a macroscopic volume containing a large number of particles. Let S be an element of area in this volume, bounded by the curve C, as shown in Fig. 8 (a). Orbits such as (1), which encircle the bounded surface only once, contribute to the resultant current, whereas orbits such as (2), which cross the surface twice, do not contribute to the net current. If $d\mathbf{l}$ is an element of arc along the curve C, the number of orbits encircling $d\mathbf{l}$ is given by $n\mathbf{A} \cdot d\mathbf{l}$, where n is the number of orbits of current I, per unit volume, and $\mathbf{A}$ is the vector area bounded by each orbit. The direction of $\mathbf{A}$ is that of the normal to the orbital area A, the positive sense being related to the sense of circulation in the way the linear motion of a right-hand screw is related to its rotary motion. Thus, $\mathbf{A}$ points in the direction of the observer when I flows counterclockwise, as shown in Fig. 8 (b). The net resultant current crossing S is therefore given by the current encircling $d\mathbf{l}$ integrated along the curve C,

$$I_n = \oint In\mathbf{A} \cdot d\mathbf{l} \tag{4.36}$$

Since $\mathbf{m} = I\mathbf{A}$, the magnetic moment, per unit volume, $\mathbf{M}$, (also called the *magnetization vector*) is given by

$$\mathbf{M} = n\mathbf{m} = nI\mathbf{A} \tag{4.37}$$

Hence, (4.36) can be written as

$$I_n = \oint \mathbf{M} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{M}) \cdot d\mathbf{S} \tag{4.38}$$

where we have applied *Stokes' theorem.* We may define an average *magnetization current density*, $\mathbf{J}_M$, crossing the surface S, by

$$I_n = \int_S \mathbf{J}_M \cdot d\mathbf{S} \tag{4.39}$$

Consequently, from (4.38) and (4.39) we obtain the magnetization current density as

$$\mathbf{J}_M = \nabla \times \mathbf{M} \tag{4.40}$$

where, from (4.37) and (4.35),

$$\mathbf{M} = n\mathbf{m} = -\Big(\frac{nW_\perp}{B^2}\Big)\mathbf{B} \tag{4.41}$$

and $nW_\perp$ denotes the kinetic energy, per unit volume, associated with the transverse particle velocity.

The charge density ρ_M associated with the magnetization current density $\mathbf{J}_M$ can be deduced from the equation of continuity,

$$\frac{\partial \rho_M}{\partial t} + \nabla \cdot \mathbf{J}_M = 0 \tag{4.42}$$

Since $\mathbf{J}_M = \nabla \times \mathbf{M}$ and since for any vector $\mathbf{a}$, we have $\nabla \cdot (\nabla \times \mathbf{a}) = 0$, it follows that the charge density ρ_M is constant.

In the following Maxwell equation

$$\nabla \times \mathbf{B} = \mu_o\Big(\mathbf{J} + \epsilon_o \frac{\partial \mathbf{E}}{\partial t}\Big) \tag{4.43}$$

we can separate the *total* current density $\mathbf{J}$ into two parts: a magnetization current density $\mathbf{J}_M$ and a current density $\mathbf{J}'$ due to other sources,

$$\mathbf{J} = \mathbf{J}_M + \mathbf{J}' \tag{4.44}$$

Expressing $\mathbf{J}_M$ in terms of $\mathbf{M}$, through (4.40), and substituting in (4.43), we obtain

$$\nabla \times \mathbf{B} = \mu_o\Big(\nabla \times \mathbf{M} + \mathbf{J}' + \epsilon_o \frac{\partial \mathbf{E}}{\partial t}\Big) \tag{4.45}$$

which can be rearranged as

$$\nabla \times \left(\frac{1}{\mu_o}\mathbf{B} - \mathbf{M}\right) = \mathbf{J}' + \epsilon_o \frac{\partial \mathbf{E}}{\partial t} \tag{4.46}$$

Defining an *effective* magnetic field $\mathbf{H}$ by the relation

$$\mathbf{B} = \mu_o(\mathbf{H} + \mathbf{M}) \tag{4.47}$$

we can write (4.46) as

$$\nabla \times \mathbf{H} = \mathbf{J}' + \epsilon_o \frac{\partial \mathbf{E}}{\partial t} \tag{4..48}$$

Thus, the effective magnetic field $\mathbf{H}$ is related to the current due to other sources $\mathbf{J}'$, in the way $\mathbf{B}$ is related to the total current $\mathbf{J}$. Eqs. (4.40) and (4.47) constitute the basic relations for the classical treatment of magnetic materials.

A simple linear relation between $\mathbf{B}$ and $\mathbf{H}$ exists when $\mathbf{M}$ is proportional to $\mathbf{B}$ or $\mathbf{H}$,

$$\mathbf{M} = \chi_m \mathbf{H} \tag{4.49}$$

where the constant χ_m is called the *magnetic susceptibility* of the medium. However, for a plasma we have seen that $M \propto 1/B$ [see (4.41)], so that the relation between $\mathbf{H}$ and $\mathbf{B}$ (or $\mathbf{M}$) is not linear. Within this context it is generally not convenient to treat a plasma as a magnetic medium.

5. UNIFORM ELECTROSTATIC AND MAGNETOSTATIC FIELDS

5.1 Formal Solution of the Equation of Motion

We consider now the motion of a charged particle in the presence of both electric and magnetic fields which are constant in time and spatially uniform. The nonrelativistic equation of motion is

$$m\frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{5.1}$$

Taking components parallel and perpendicular to $\mathbf{B}$,

$$\mathbf{v} = \mathbf{v}_{\parallel} + \mathbf{v}_{\perp} \tag{5.2}$$

$$\mathbf{E} = \mathbf{E}_{\parallel} + \mathbf{E}_{\perp} \tag{5.3}$$

we can resolve (5.1) into two component equations

$$m\frac{d\mathbf{v}_\parallel}{dt} = q\mathbf{E}_\parallel \tag{5.4}$$

$$m\frac{d\mathbf{v}_\perp}{dt} = q(\mathbf{E}_\perp + \mathbf{v}_\perp \times \mathbf{B}) \tag{5.5}$$

Eq. (5.4) is similar to (3.1) and represents a motion with constant acceleration $q\mathbf{E}_\parallel/m$ along the $\mathbf{B}$ field. Hence, according to (3.2) and (3.4),

$$\mathbf{v}_\parallel(t) = \left(\frac{q\mathbf{E}_\parallel}{m}\right)t + \mathbf{v}_\parallel(0) \tag{5.6}$$

$$\mathbf{r}_\parallel(t) = \tfrac{1}{2}\left(\frac{q\mathbf{E}_\parallel}{m}\right)t^2 + \mathbf{v}_\parallel(0)t + \mathbf{r}_\parallel(0) \tag{5.7}$$

To solve (5.5) it is convenient to separate $\mathbf{v}_\perp$ into two components

$$\mathbf{v}_\perp(t) = \mathbf{v}'_\perp(t) + \mathbf{v}_E \tag{5.8}$$

where $\mathbf{v}_E$ is a constant velocity in the plane normal to $\mathbf{B}$. Hence, $\mathbf{v}'_\perp$ represents the particle velocity as seen by an observer in a frame of reference moving with the constant velocity $\mathbf{v}_E$. Substituting (5.8) into (5.5), and writing the component of the electric field perpendicular to $\mathbf{B}$ in the form (see Fig. 9)

$$\mathbf{E}_\perp = -\left(\frac{\mathbf{E}_\perp \times \mathbf{B}}{B^2}\right) \times \mathbf{B} \tag{5.9}$$

we obtain

$$m\frac{d\mathbf{v}'_\perp}{dt} = q\left(\mathbf{v}'_\perp + \mathbf{v}_E - \frac{\mathbf{E}_\perp \times \mathbf{B}}{B^2}\right) \times \mathbf{B} \tag{5.10}$$

This equation shows that in a coordinate system moving with the constant velocity

$$\mathbf{v}_E = \frac{\mathbf{E}_\perp \times \mathbf{B}}{B^2} \tag{5.11}$$

the particle motion in the plane normal to $\mathbf{B}$ is governed entirely by the magnetic field, according to

$$m\frac{d\mathbf{v}'_\perp}{dt} = q(\mathbf{v}'_\perp \times \mathbf{B}) \tag{5.12}$$

Thus, in this frame of reference, the electric field component $\mathbf{E}_\perp$ is transformed away, whereas the magnetic field is left unchanged. Eq. (5.12) is

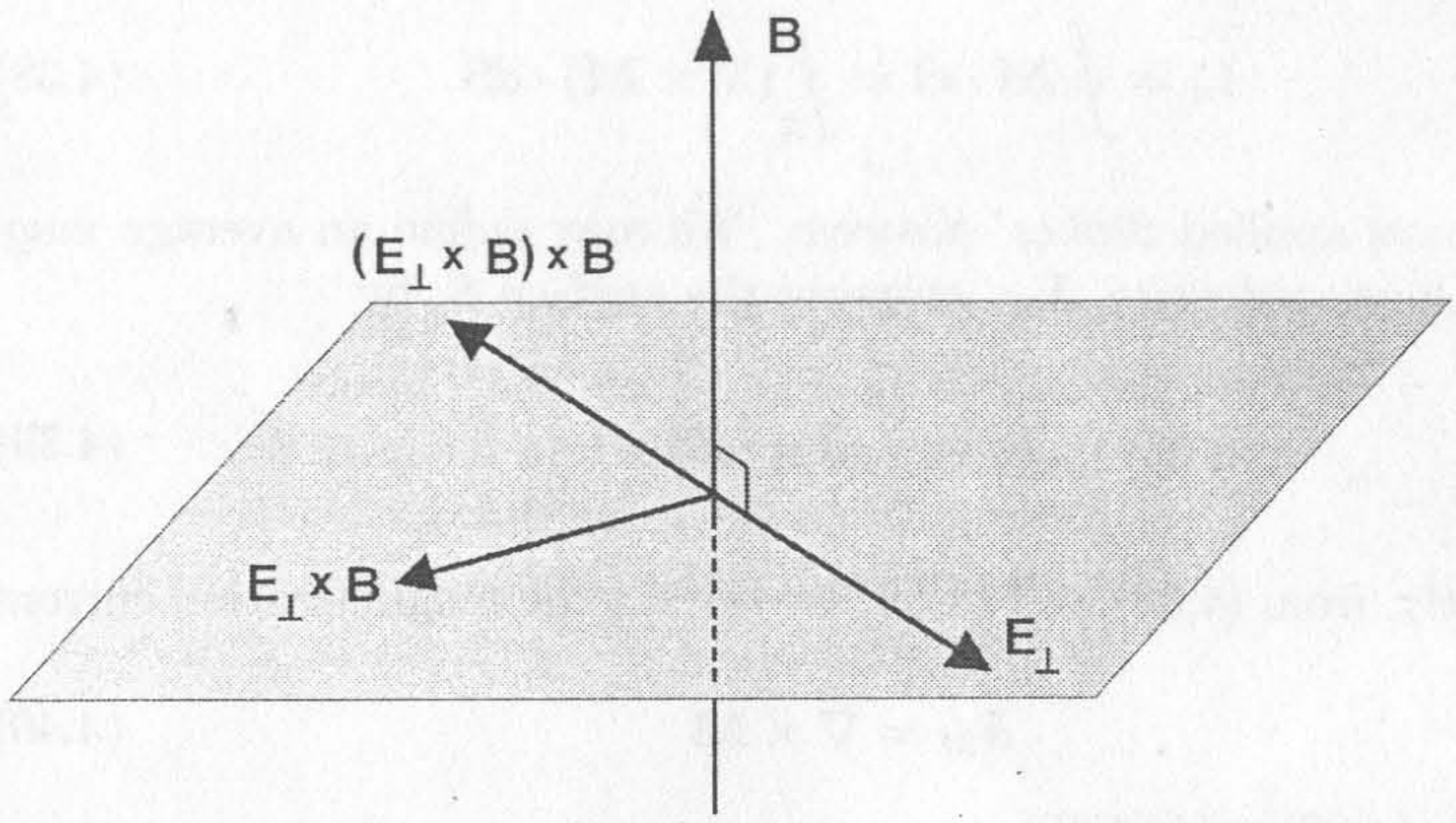


Fig. 9 *Vector products appearing in Eq. (5.9)* ($\widehat{\mathbf{B}} = \mathbf{B}/B$).

identical to (4.5) and implies that in the reference system moving with the constant velocity $\mathbf{v}_E$, given by (5.11), the particle describes a circular motion at the cyclotron frequency Ω_c with radius r_c,

$$\mathbf{v}'_\perp = \mathbf{\Omega}_c \times \mathbf{r}_c \tag{5.13}$$

The results obtained so far indicate that the resulting particle motion is described by a superposition of a circular motion in the plane normal to $\mathbf{B}$, with a uniform motion with the constant velocity $\mathbf{v}_E$ perpendicular to both $\mathbf{B}$ and $\mathbf{E}_\perp$, plus a uniform acceleration $q\mathbf{E}_\parallel/m$ along $\mathbf{B}$. The particle velocity can be expressed in vector form, independently of a coordinate system, as

$$\mathbf{v}(t) = \mathbf{\Omega}_c \times \mathbf{r}_c + \frac{\mathbf{E}_\perp \times \mathbf{B}}{B^2} + \frac{q\mathbf{E}_\parallel}{m}t + \mathbf{v}_\parallel(0) \tag{5.14}$$

The first term in the right-hand side of (5.14) represents the cyclotron circular motion, and the following ones represent, respectively, the drift velocity of the guiding center (perpendicular to both $\mathbf{E}_\perp$ and $\mathbf{B}$), the constant acceleration of the guiding center along $\mathbf{B}$, and the initial velocity parallel to $\mathbf{B}$.

Note that the velocity $\mathbf{v}_E$ is independent of the mass and of the sign of the charge and therefore is the same for both positive and negative

particles. It is usually called the *plasma drift velocity* or the *electromagnetic plasma drift*. Since $\mathbf{E}_{\parallel} \times \mathbf{B} = 0$, (5.11) can also be written as

$$\mathbf{v}_E = \frac{\mathbf{E} \times \mathbf{B}}{B^2} \tag{5.15}$$

The resulting motion of the particle in the plane normal to $\mathbf{B}$ is, in general, a *cycloid*, as shown in Fig. 10. The physical explanation for this cycloidal motion is a follows. The electric force $q\mathbf{E}_{\perp}$, acting simultaneously with the magnetic force, accelerates the particle so as to increase or decrease its velocity, depending on the relative direction of the particle motion with respect to the direction of $\mathbf{E}_{\perp}$ and on the charge sign. According to (4.13) the radius of gyration increases with velocity and hence the radius of curvature of the particle path varies under the action of $\mathbf{E}_{\perp}$. This results in a cycloidal trajectory with a net drift in the direction perpendicular to both $\mathbf{E}$ and $\mathbf{B}$. Different trajectories are obtained, depending on the initial conditions and on the magnitude of the applied electric and magnetic fields.

The ions are much more massive than the electrons and therefore the Larmor radius for ions is correspondingly greater and the Larmor frequency correspondingly smaller than for electrons. Consequently, the arcs of cycloid for ions are greater than for electrons, but there is a larger number of arcs of cycloid per second for electrons, such that the drift velocity is the same for both species.

In a collisionless plasma the drift velocity does not imply an electric current, since both positive and negative particles move together. When collisions between charged and neutral particles are important, this drift gives rise to an electric current, since the ion-neutral collision frequency is greater than the electron-neutral collision frequency, causing the ions to move slower than the electrons. This current is normal to both $\mathbf{E}$ and $\mathbf{B}$, and is in the direction opposite to $\mathbf{v}_E$. It is known as the *Hall current*.

5.2 Solution in Cartesian Coordinates

Let us choose a Cartesian coordinate system with the z axis pointing in the direction of $\mathbf{B}$, so that

$$\mathbf{B} = B\widehat{\mathbf{z}} \tag{5.16}$$

$$\mathbf{E} = E_x\widehat{\mathbf{x}} + E_y\widehat{\mathbf{y}} + E_z\widehat{\mathbf{z}} \tag{5.17}$$

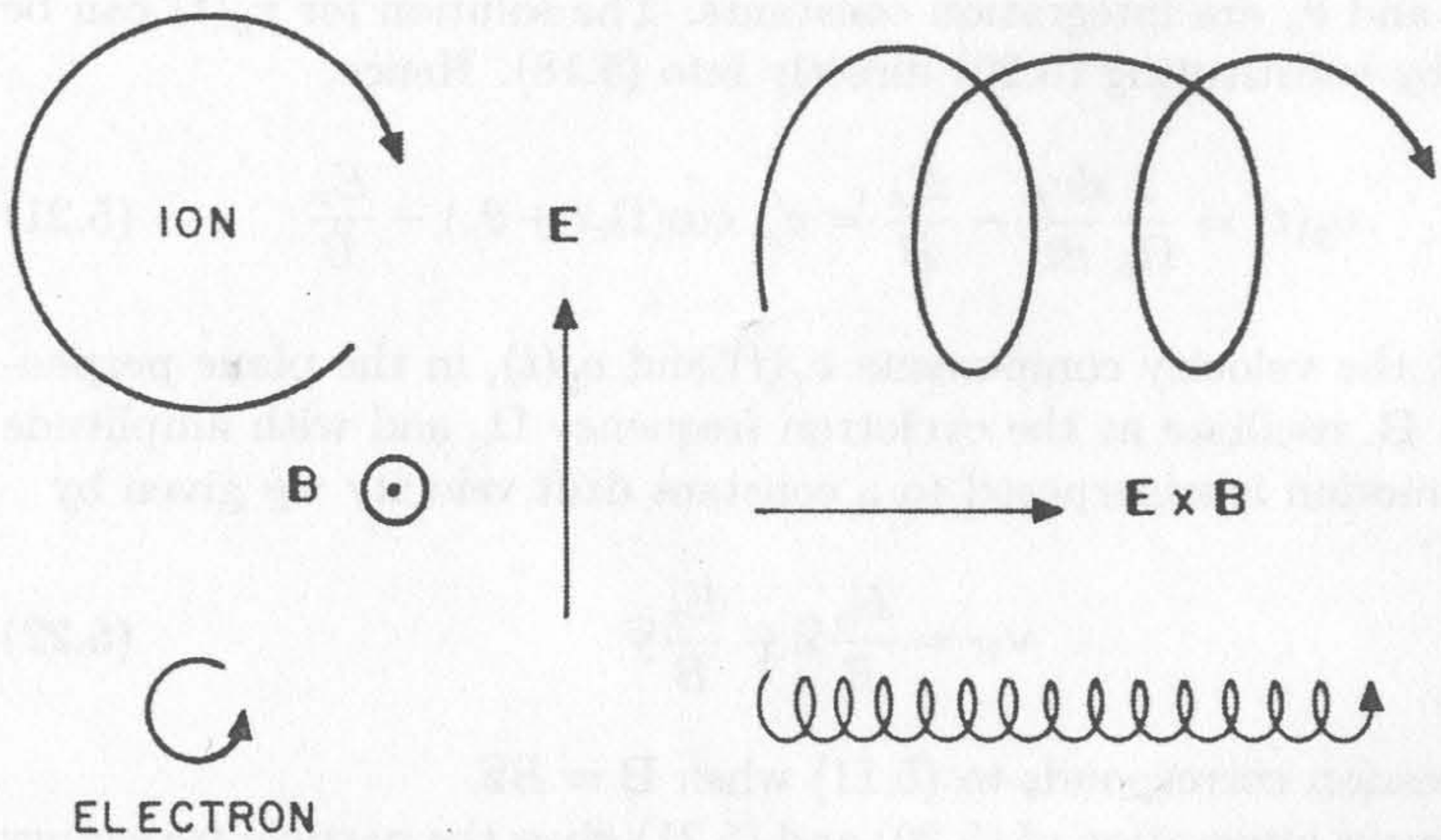


Fig. 10 *Cycloidal trajectories described by ions and electrons in crossed electric and magnetic fields. The electric field* **E** *acting together with the magnetic flux density* **B** *gives rise to a drift velocity in the direction given by* $\mathbf{E} \times \mathbf{B}$.

Using (4.15), the equation of motion (5.1) can be written as

$$\frac{d\mathbf{v}}{dt} = \frac{q}{m}[(E_x + v_y B)\widehat{\mathbf{x}} + (E_y - v_x B)\widehat{\mathbf{y}} + E_z\widehat{\mathbf{z}}] \tag{5.18}$$

As before, we consider, in what follows, a *positive* charge. The results for a negative charge can be obtained by changing the sign of Ω_c in the results for the positive charge.

The z component of (5.18) can be integrated directly and gives the same results expressed in (5.6) and (5.7). For the x and y components, we first take the derivative of dv_x/dt with respect to time and substitute the expression for dv_y/dt, which gives

$$\frac{d^2 v_x}{dt^2} + \Omega_c^2 v_x = \Omega_c^2 \frac{E_y}{B} \tag{5.19}$$

This is the *inhomogeneous* differential equation for a *harmonic oscillator* of frequency Ω_c. Its solution is given by the sum of the homogeneous equation solution, given in (4.21), with a particular solution (which is clearly given by E_y/B). Thus,

$$v_x(t) = v'_\perp \sin(\Omega_c t + \theta_o) + \frac{E_y}{B} \tag{5.20}$$

where $v'_\perp$ and θ_o are integration constants. The solution for $v_y(t)$ can be obtained by substituting (5.20) directly into (5.18). Hence,

$$v_y(t) = \frac{1}{\Omega_c}\frac{dv_x}{dt} - \frac{E_x}{B} = v'_\perp \cos(\Omega_c t + \theta_o) - \frac{E_x}{B} \tag{5.21}$$

Therefore, the velocity components $v_x(t)$ and $v_y(t)$, in the plane perpendicular to $\mathbf{B}$, oscillate at the cyclotron frequency Ω_c and with amplitude $v'_\perp$. This motion is superposed to a constant drift velocity $\mathbf{v}_E$ given by

$$\mathbf{v}_E = \frac{E_y}{B}\widehat{\mathrm{x}} - \frac{E_x}{B}\widehat{\mathrm{y}} \tag{5.22}$$

This expression corresponds to (5.11) when $\mathbf{B} = B\widehat{\mathbf{z}}$.

One more integration of (5.20) and (5.21) gives the particle trajectory in the (x, y) plane

$$x(t) = -\frac{v'_\perp}{\Omega_c}\cos(\Omega_c t + \theta_o) + \frac{E_y}{B}t + X_o \tag{5.23}$$

$$y(t) = \frac{v'_\perp}{\Omega_c}\sin(\Omega_c t + \theta_o) - \frac{E_x}{B}t + Y_o \tag{5.24}$$

where X_o and Y_o are defined according to (4.27) and (4.28), but with $v_\perp$ replaced by $v'_\perp$.

In summary, the motion of a charged particle in uniform electrostatic and magnetostatic fields consists of three components:

(a) A constant acceleration $q\mathbf{E}_\parallel/m$ along the $\mathbf{B}$ field. If $\mathbf{E}_\parallel = 0$, the particle moves along $\mathbf{B}$ with its initial velocity.

(b) A rotation in the plane normal to $\mathbf{B}$ at the cyclotron frequency $\Omega_c = |q|\ B/m$ and radius $r_c = v'_\perp/\Omega_c$.

(c) An electromagnetic drift velocity $\mathbf{v}_E = (\mathbf{E}\times\mathbf{B})/B^2$, perpendicular to both $\mathbf{B}$ and $\mathbf{E}$.

6. DRIFT DUE TO AN EXTERNAL FORCE

If some additional force $\mathbf{F}$ (gravitational force or inertial force, if the motion is considered in a noninertial system, for example) is present, the equation of motion (1.5) must be modified to include this force,

$$m\frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B}) + \mathbf{F} \tag{6.1}$$

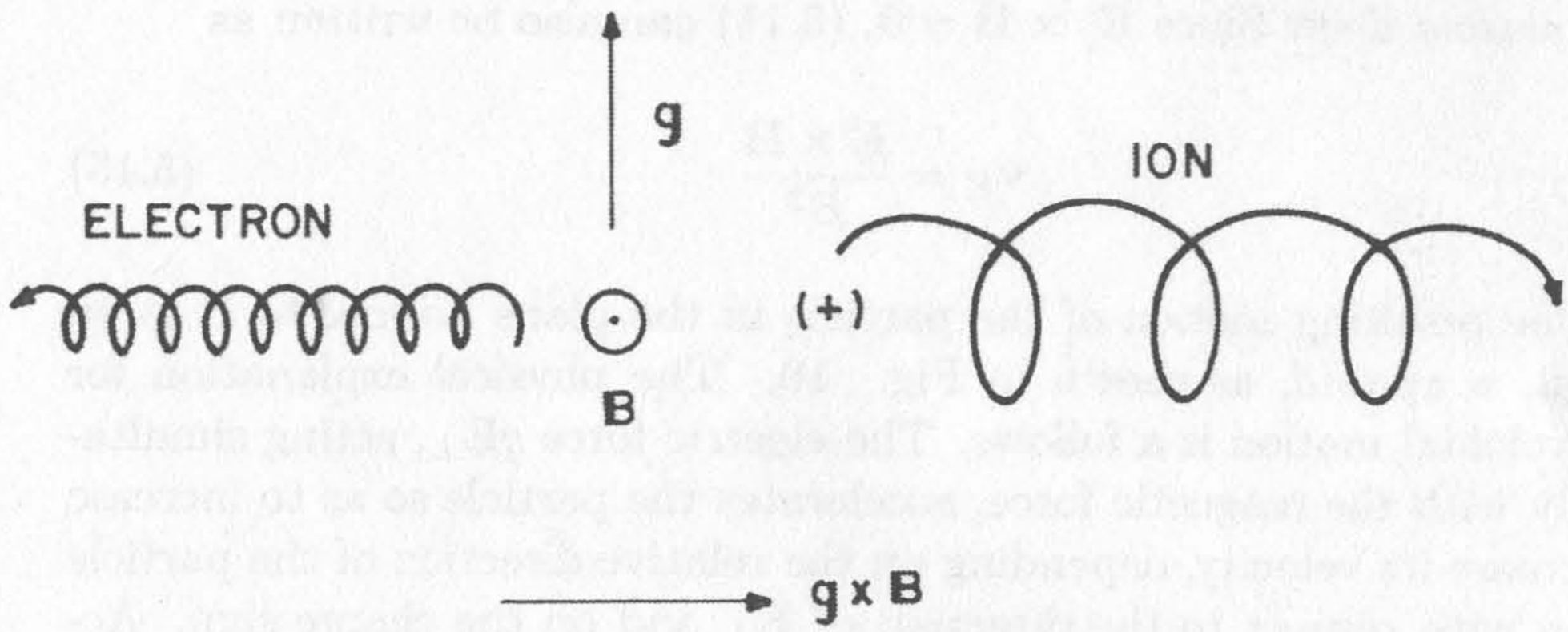


Fig.11 *Drift of a gyrating particle in crossed gravitational and magnetic fields.*

The effect of this force is, in a formal sense, analogous to the effect of the electric field. We assume here that $\mathbf{F}$ is uniform and constant. In analogy with the electromagnetic drift velocity $\mathbf{v}_E$, given in (5.15), the drift produced by the force $\mathbf{F}$ having a component normal to $\mathbf{B}$ is given by

$$\mathbf{v}_F = \frac{\mathbf{F} \times \mathbf{B}}{qB^2} \tag{6.2}$$

In the case of a uniform gravitational field, for example, we have $\mathbf{F} = m\mathbf{g}$, where $\mathbf{g}$ is the acceleration due to gravity, and the drift velocity is given by

$$\mathbf{v}_g = \frac{m}{q}\frac{\mathbf{g} \times \mathbf{B}}{B^2} \tag{6.3}$$

This drift velocity depends on the ratio m/q and therefore it is in opposite directions for particles of opposite charge (Fig. 11). We have seen that in a coordinate system moving with the velocity $\mathbf{v}_E$, the electric field component $\mathbf{E}_\perp$ is transformed away leaving the magnetic field unchanged. The gravitational field however cannot, in this context, be transformed away.

In a collisionless plasma, associated with the gravitational drift velocity there is an electric current density, $\mathbf{J}_g$, in the direction of $\mathbf{g} \times \mathbf{B}$, which can be expressed as

$$\mathbf{J}_g = \frac{1}{\delta V}\sum_i q_i \mathbf{v}_{gi} \tag{6.4}$$

where the summation is over all charged particles contained in a suitably chosen small volume element δV. Using (6.3) we obtain

$$\mathbf{J}_g = \frac{1}{\delta V}\Big(\sum_i m_i\Big)\frac{\mathbf{g}\times\mathbf{B}}{B^2} = \rho_m \frac{\mathbf{g}\times\mathbf{B}}{B^2} \tag{6.5}$$

where ρ_m denotes the total mass density of the charged particles.

A comment on the validity of (6.2) is appropriate here. Since we have used the nonrelativistic equation of motion, there is a limitation on the magnitude of the force $\mathbf{F}$ in order that (6.2) be applicable. The magnitude of the transverse drift velocity is given by

$$v_D = \frac{F_\perp}{qB} \tag{6.6}$$

Hence, for the nonrelativistic equation of motion to be applicable we must have

$$\frac{F_\perp}{qB} \ll c \tag{6.7}$$

or, if $\mathbf{F}$ is due to an electrostatic field $\mathbf{E}$,

$$\frac{E_\perp}{B} \ll c \tag{6.8}$$

For a magnetic field of 1 tesla, for example, (6.2) may be used as long as $E_\perp$ is much less than 10^8 volts/m. If these conditions are not satisfied, the problem must be treated using the relativistic equation of motion. Although the relativistic equation of motion can be integrated exactly for constant $\mathbf{B}$, $\mathbf{E}$ and $\mathbf{F}$, we shall not analyze this problem here. It is left as an exercise for the reader.

PROBLEMS

2.1 Calculate the cyclotron frequency and the cyclotron radius for:
(a) An electron in the Earth's ionosphere at 300 km altitude, where the magnetic flux density $B \simeq 0.5 \times 10^{-4}$ tesla, considering that the electron moves at the thermal velocity (kT/m), with $T = 1000$ K, where k is Boltzmann's constant.
(b) A 50 MeV proton in the Earth's inner Van Allen radiation belt at

about 1.5 R_E (where $R_E = 6370$ km is the Earth's radius) from the center of the Earth in the equatorial plane, considering $B \simeq 10^{-5}$ tesla.
(c) A 1 MeV electron in the Earth's outer Van Allen radiation belt at about 4 R_E from the center of the Earth in the equatorial plane, where $B \simeq 10^{-7}$ tesla.
(d) A proton in the solar wind with a streaming velocity of 100 km/s, in a magnetic flux density $B \simeq 10^{-9}$ tesla.
(e) A 1 MeV proton in a sunspot region of the solar photosphere, considering $B \simeq 0.1$ tesla.

2.2 For an electron and an oxygen ion O^+ in the Earth's ionosphere, at 300 km altitude in the equatorial plane, where $B \simeq 0.5 \times 10^{-4}$ tesla, calculate:
(a) The gravitational drift velocity $\mathbf{v}_g$.
(b) the gravitational current density $\mathbf{J}_g$, considering $n_e = n_i = 10^{12}$ m^{-3}.
Assume that $\mathbf{g}$ is perpendicular to $\mathbf{B}$.

2.3 Consider a particle of mass m and charge q moving in the presence of constant and uniform electromagnetic fields given by $\mathbf{E} = E_o\widehat{\mathbf{y}}$ and $\mathbf{B} = B_o\widehat{\mathbf{z}}$. Assuming that initially ($t = 0$) the particle is at rest at the origin of a Cartesian coordinate system, show that it moves on the *cycloid*

$$x(t) = \frac{E_o}{B_o}\left[t - \frac{1}{\Omega_c}\sin(\Omega_c t)\right]$$

$$y(t) = \frac{E_o}{B_o}\frac{1}{\Omega_c}[1 - \cos(\Omega_c t)]$$

Plot the trajectory of the particle in the $z = 0$ plane for $q > 0$ and for $q < 0$, and consider the cases when $v_\perp > v_E$, $v_\perp = v_E$ and $v_\perp < v_E$, where $v_\perp$ denotes the particle cyclotron motion velocity and v_E is the electromagnetic drift velocity.

2.4 In general the trajectory of a charged particle in crossed electric and magnetic fields is a cycloid. Show that, if $\mathbf{v} = v_o\widehat{\mathbf{x}}$, $\mathbf{B} = B_o\widehat{\mathbf{z}}$ and $\mathbf{E} = E_o\widehat{\mathbf{y}}$, then for $v_o = E_o/B_o$ the path is a straight line. Explain how this situation can be exploited to design a mass spectrometer.

2.5 Derive the relativistic equation of motion in the form (1.4), starting from (1.1) and the relation (1.2).

2.6 Write down, in vector form, the relativistic equation of motion for a charged particle in the presence of a uniform magnetostatic field $\mathbf{B} = B_o\widehat{\mathbf{z}}$, and show that its Cartesian components are given by

$$\frac{d}{dt}(\gamma v_x) = \left(\frac{qB_o}{m}\right)v_y$$

$$\frac{d}{dt}(\gamma v_y) = -\left(\frac{qB_o}{m}\right)v_x$$

$$\frac{d}{dt}(\gamma v_z) = 0$$

where

$$\gamma = \frac{1}{(1-\beta^2)^{1/2}}$$

and where $\beta = v/c$. Show that the velocity and trajectory of the charged particle are given by the same formulas as in the nonrelativistic case, but with Ω_c replaced by $|q|\ B_o/(m\gamma)$.

2.7 Analyze the motion of a relativistic charged particle in the presence of crossed electric ($\mathbf{E}$) and magnetic ($\mathbf{B}$) fields which are constant in time and uniform in space. What coordinate transformation must be made in order to transform away the transversal electric field? Derive equations for the velocity and trajectory of the charged particle.

3

CHARGED PARTICLE MOTION IN NONUNIFORM MAGNETOSTATIC FIELDS

1. INTRODUCTION

When the fields are spatially nonuniform, or when they vary with time, the integration of the equation of motion (2.1.1) can be a mathematical problem of great difficulty. In this case, since the equation of motion is nonlinear, the theory may become extremely involved and rigorous analytic expressions for the charged particle trajectory cannot, in general, be obtained in closed form. Numerical methods of integration must be used in order to obtain all the details of the motion.

There is one particularly important case, however, in which it becomes possible to obtain an approximate, but otherwise general, solution whithout recourse to numerical integration, if the details of the particle motion are not of interest. This is the case when the magnetic field is strong and *slowly varying* in both *space* and *time*, and when the electric field is weak. In a wide variety of situations of interest the fields are *approximately* constant and uniform, at least on the distance and time scales seen by the particle during one gyration about the magnetic field. This is the case for many laboratory plasmas, including those of relevance to the problem of controlled thermonuclear reactions, and also for a great number of astrophysical plasmas.

In this chapter we investigate the motion of a charged particle in a static magnetic field *slightly* inhomogeneous in space. The word slightly here means that the spatial variation of the magnetic field inside the particle orbit is small compared to the magnitude of **B**. In other words, we shall consider only magnetostatic fields whose spatial change in a distance

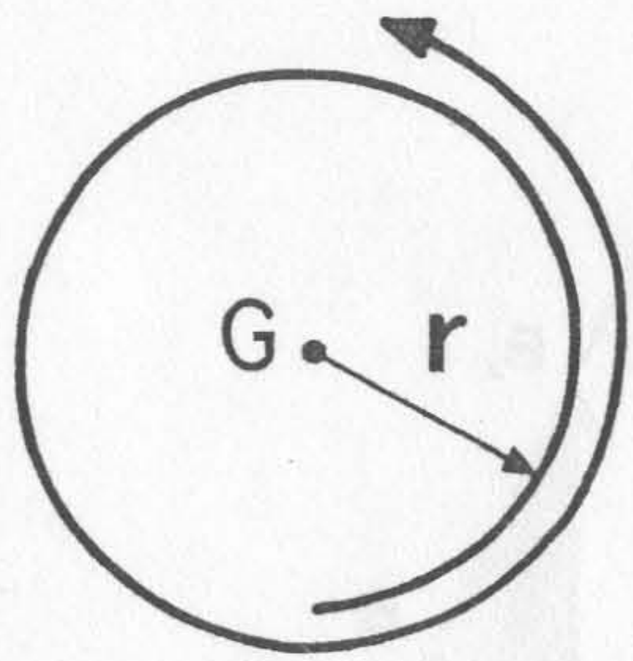


Fig. 1 *The motion of a charged particle in a slightly inhomogeneous magnetic field is nearly circular.*

of the order of the Larmor radius, r_c, is much smaller than the magnitude of the field itself.

To specify more quantitatively this assumption, let δB represent the spatial change in the magnitude of $\mathbf{B}$ in a distance of the order of r_c, that is, $\delta B = r_c \, |\nabla B|$, where ∇B is the gradient of the magnitude of $\mathbf{B}$. It is assumed therefore that $\delta B \ll B$. Consequently, in what follows we limit our discussion to problems where the deviations from uniformity are small and solve for the particle trajectory only in the first order approximation. The analysis of charged particle motion in stationary fields based on this approximation in often referred to as the *first order orbit theory.* This theory was first used systematically by the Swedish scientist Alfvén, and it is also known as the *Alfvén approximation* or the *guiding center approximation.*

The concept of guiding center is of great utility in the development of this theory. We have seen that in a uniform magnetic field the particle motion can be regarded as a superposition of a circular motion about the direction of $\mathbf{B}$, with a motion of the guiding center along $\mathbf{B}$. In the case of a nonuniform $\mathbf{B}$ field, satisfying the condition (1.2), the value of $\mathbf{B}$ at the particle position differs only slightly from its value at the guiding center. The component of the particle motion, in a plane normal to the field line that passes through the guiding center instantaneous position, will still be nearly circular (Fig. 1). However, due to the spatial variation of $\mathbf{B}$, we expect in this case a gradual drift of the guiding center across $\mathbf{B}$, as well as a gradual change of its velocity along $\mathbf{B}$.

The rapid gyrations of the charged particle about the direction of $\mathbf{B}$ are not usually of great interest and it is convenient to eliminate them

from the equations of motion, and focus attention on the guiding center motion. In the motion of the guiding center, the small oscillations (of amplitude small compared to the cyclotron radius) occurring during one gyration period may be averaged out, since they represent the effect of perturbations due to the spatial variation of the magnetic field. The problem is thus reduced to the calculation of the average values over one gyration period (and not the instantaneous values) of the guiding center *transverse drift velocity* and *parallel acceleration.*

2. SPATIAL VARIATION OF THE MAGNETIC FIELD

Any of the three components of the magnetic flux density, $\mathbf{B} = B_x\widehat{\mathbf{x}} + B_y\widehat{\mathbf{y}} + B_z\widehat{\mathbf{z}}$, may vary with respect to the three coordinates x, y, and z. Consequently, nine parameters are needed to completely specify the spatial variation of $\mathbf{B}$. These parameters can be conveniently represented by the dyad (or tensor) $\nabla\mathbf{B}$, which can be written in matrix form as

$$\nabla\mathbf{B} = (\widehat{\mathbf{x}} \quad \widehat{\mathbf{y}} \quad \widehat{\mathbf{z}}) \begin{pmatrix} \partial B_x/\partial x & \partial B_y/\partial x & \partial B_z/\partial x \\ \partial B_x/\partial y & \partial B_y/\partial y & \partial B_z/\partial y \\ \partial B_x/\partial z & \partial B_y/\partial z & \partial B_z/\partial z \end{pmatrix} \begin{pmatrix} \widehat{\mathbf{x}} \\ \widehat{\mathbf{y}} \\ \widehat{\mathbf{z}} \end{pmatrix} \tag{2.1}$$

Of these nine components only eight are independent, since the following Maxwell equation

$$\nabla \cdot \mathbf{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0 \tag{2.2}$$

shows that only two of the divergence terms are independent.

If the condition $\mathbf{J} = 0$ is also satisfied, in the region where the particle is moving, then other restrictions are imposed on the number of independent components of $\mathbf{B}$ since, under these circumstances, the relation $\nabla \times \mathbf{B} = 0$ holds. This means that, in regions where there are no electric currents, $\mathbf{B}$ can be written as the gradient of a *scalar magnetic potential,*

$$\mathbf{B} = \nabla\phi_m \tag{2.3}$$

where the magnetic potential ϕ_m satisfies the Laplace equation

$$\nabla^2\phi_m = 0 \tag{2.4}$$

In regions where an electric current density exists, we have $\nabla \times \mathbf{B} = \mu_0\mathbf{J}$ and we cannot define a scalar magnetic potential ϕ_m as indicated. The

number of independent components of $\nabla\mathbf{B}$ cannot, in this case, be reduced without knowing the electric current density $\mathbf{J}$.

Let us consider a Cartesian coordinate system such that at the origin the magnetic field is in the z direction,

$$\mathbf{B}(0,0,0) = \mathbf{B}_0 = B_0\widehat{\mathbf{z}} \tag{2.5}$$

The nine components of $\nabla\mathbf{B}$ can be conveniently grouped into four categories:

(a) Divergence terms:

$$\partial B_x/\partial x, \quad \partial B_y/\partial y, \quad \partial B_z/\partial z \tag{2.6a}$$

(b) Gradient terms:

$$\partial B_z/\partial x, \quad \partial B_z/\partial y \tag{2.6b}$$

(c) Curvature terms:

$$\partial B_x/\partial z, \quad \partial B_y/\partial z \tag{2.6c}$$

(d) Shear terms:

$$\partial B_x/\partial y, \quad \partial B_y/\partial x \tag{2.6d}$$

2.1 Divergence Terms

We shall initially discuss the magnetic field line geometry corresponding to the *divergence* terms of $\nabla\mathbf{B}$. The presence of a small variation in the component B_z in the z direction (i.e. $\partial B_z/\partial z \neq 0$), implies that at least one of the terms $\partial B_x/\partial x$ or $\partial B_y/\partial y$ is also present, as can be seen from (2.2). It is of great utility to make use here of the concept of *magnetic flux lines* which, at any point, are parallel to the $\mathbf{B}$ field at that point and whose density at each point is proportional to the local magnitude of $\mathbf{B}$. To determine the differential equation of a line of force, let

$$d\mathbf{s} = dx\,\widehat{\mathbf{x}} + dy\,\widehat{\mathbf{y}} + dz\,\widehat{\mathbf{z}} \tag{2.7}$$

be an element of arc along the magnetic field line. Then, we must have

$$d\mathbf{s} \times \mathbf{B} = 0 \tag{2.8}$$

since $d\mathbf{s}$ is parallel to $\mathbf{B}$, which gives by expansion of the cross product,

$$\frac{dx}{B_x} = \frac{dy}{B_y} = \frac{dz}{B_z} \tag{2.9}$$

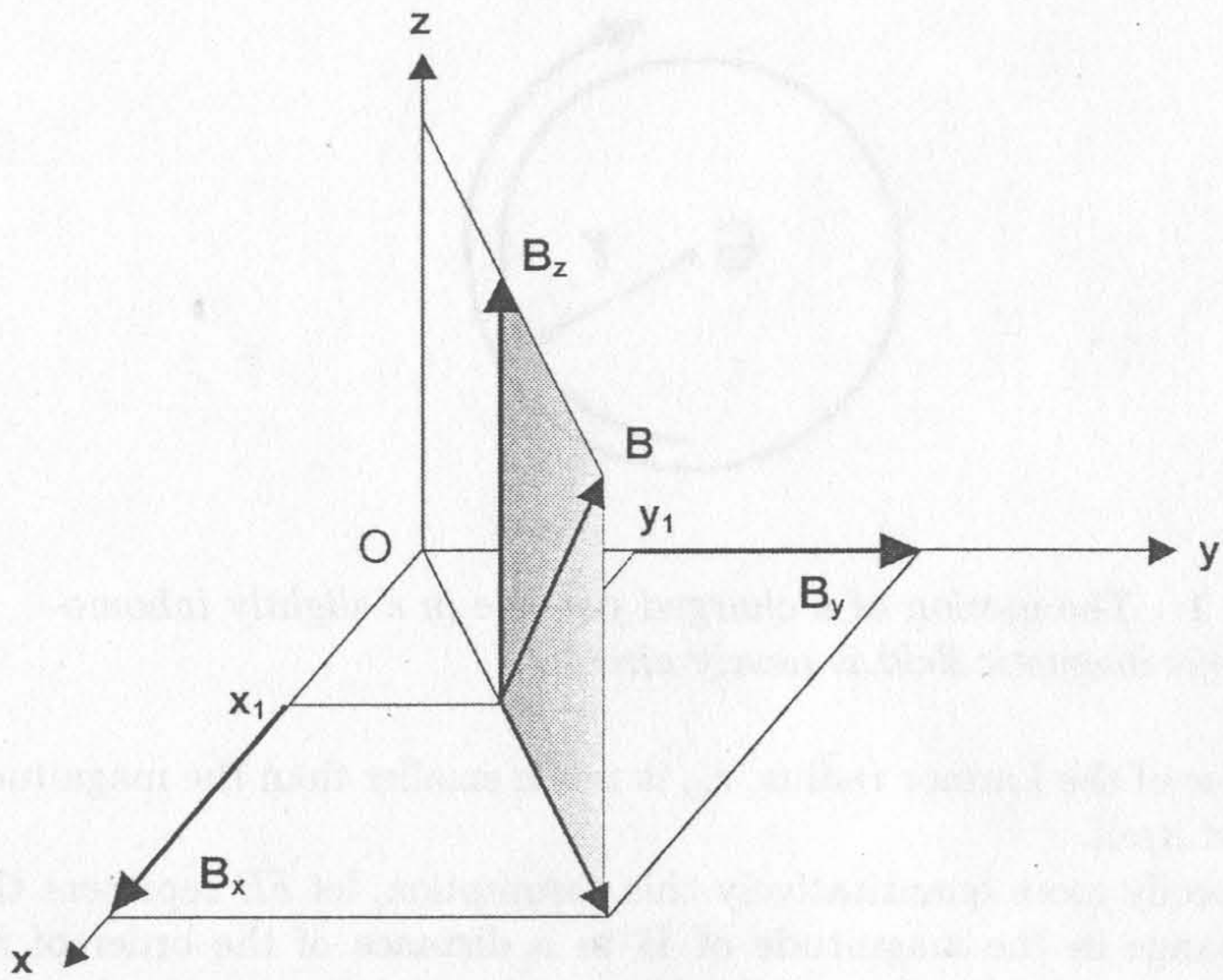


Fig. 2 *The magnetic field components B_x and B_y at the points $(x_1, 0, 0)$ and $(0, y_1, 0)$, near the origin.*

Since we are focusing attention only on the divergence terms of **B**, and since in the region of interest the field is considered to be mainly in the z direction, we may expand B_x and B_y in a Taylor series about the origin (see Fig. 2), as follows,

$$B_x(x_1, 0, 0) = B_x(0, 0, 0) + \left(\frac{\partial B_x}{\partial x}\right)x_1 = \left(\frac{\partial B_x}{\partial x}\right)x_1 \tag{2.10}$$

$$B_y(0, y_1, 0) = B_y(0, 0, 0) + \left(\frac{\partial B_y}{\partial y}\right)y_1 = \left(\frac{\partial B_y}{\partial y}\right)y_1 \tag{2.11}$$

where the second and higher order terms were neglected. Note that at the origin $B_x = B_y = 0$. Therefore, the magnetic field line crossing the $z = 0$ plane at the point $(x_1, y_1, 0)$, when projected on the (x, z) plane $(y = 0)$ and on the (y, z) plane $(x = 0)$, satisfies the following differential equations, respectively,

$$\frac{dx}{dz} = \frac{B_x}{B_z} = \frac{1}{B_z}\left(\frac{\partial B_x}{\partial x}\right)x_1 \qquad (y = 0) \tag{2.12}$$

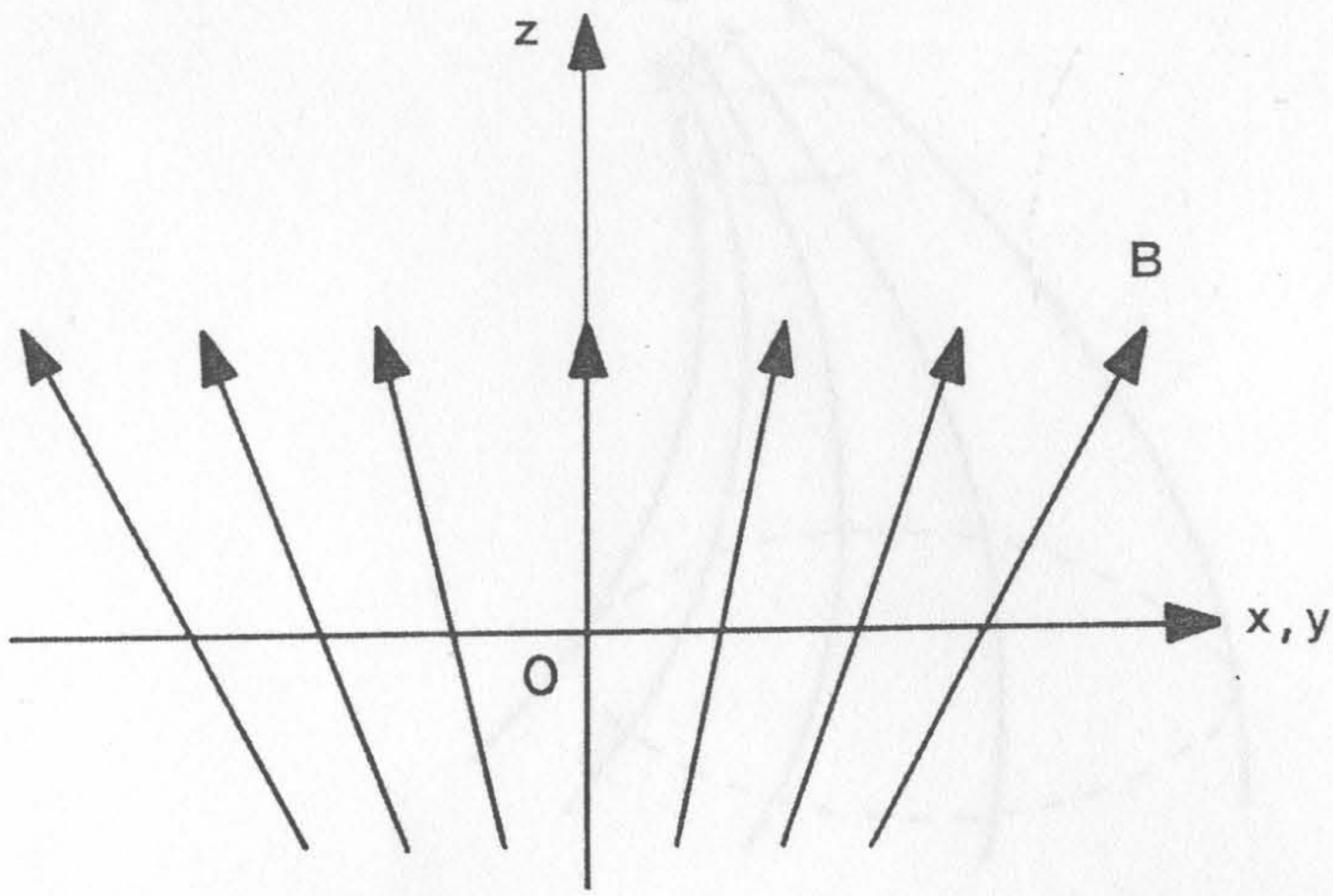


Fig. 3 *Geometry of the magnetic field lines corresponding to the divergence terms $\partial B_x/\partial x$ or $\partial B_y/\partial y$, when they are positive.*

$$\frac{dy}{dz} = \frac{B_y}{B_z} = \frac{1}{B_z}\left(\frac{\partial B_y}{\partial y}\right)y_1 \qquad (x = 0) \tag{2.13}$$

These equations show that the field lines converge or diverge in the (x, z) plane or in the (y, z) plane, depending on the sine of the divergence terms of $\mathbf{B}$. Fig. 3 illustrates the field line geometry when $\partial B_x/\partial x$ and $\partial B_y/\partial y$ are positive.

2.2 Gradient and Curvature Terms

The following vector field has a *gradient* in the x direction, as shown schematically in Fig. 4,

$$\mathbf{B} = B_z\widehat{\mathbf{z}} = B_0(1 + \alpha x)\widehat{\mathbf{z}} \tag{2.14}$$

We must note however that in a region where $\mathbf{J} = 0$ this vector field does not satisfy the Maxwell equation $\nabla \times \mathbf{B} = 0$, so that we must add to (2.14) a term of *curvature*, given by $B_x\widehat{\mathbf{x}} = B_0\alpha z\widehat{\mathbf{x}}$. Therefore, a magnetic field having gradient and curvature terms, and which satisfies $\nabla \times \mathbf{B} = 0$, is

$$\mathbf{B} = B_0[\alpha z\widehat{\mathbf{x}} + (1 + \alpha x)\widehat{\mathbf{z}}] \tag{2.15}$$

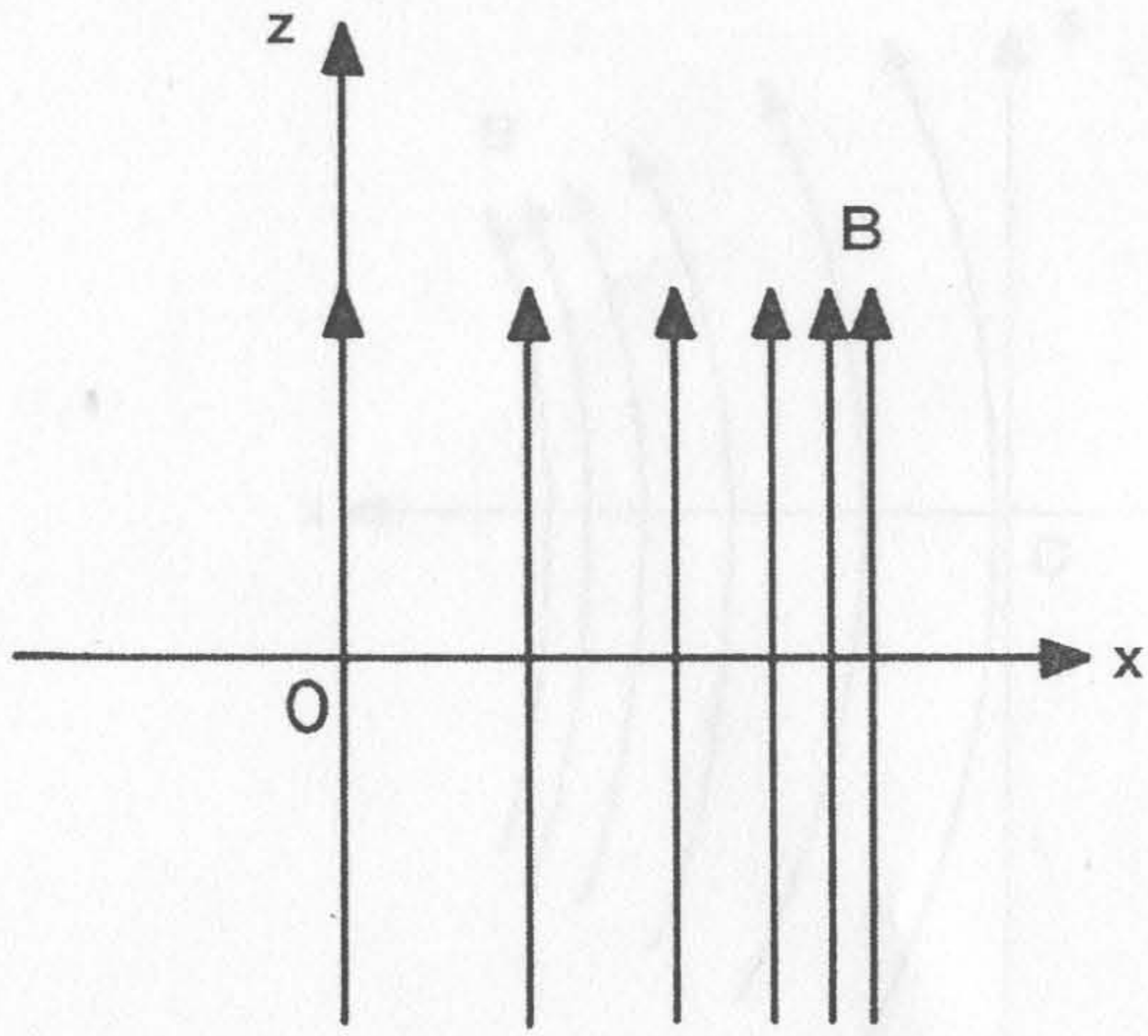


Fig. 4 *Geometry of the magnetic field lines when* **B** *has a gradient in the x direction, according tp Eq. (2.14). This field geometry does not satisfy* $\nabla \times \mathbf{B} = 0$.

The geometry of the magnetic field lines corresponding to this expression is schematically indicated in Fig. 5.

Generally, all terms corresponding to divergence, gradient and curvature are simultaneously present. Fig. 6 illustrates a **B** field having divergence, gradient and curvature. An example is the Earth's magnetic field (refer to Fig. 1.4). Later in this section we will investigate separately the effects of each one of these terms on the charged particle motion. Since in the first order approximation the equations are linear, the net effect will be the sum of the effects due to each one of them.

2.3 Shear Terms

The *shear terms* of (2.6) enter into the z component of $\nabla \times \mathbf{B}$, that is, into $\mathbf{B} \cdot (\nabla \times \mathbf{B})$, and cause twisting of the magnetic field lines about each other. They do not produce any first order drifts, although the shape of the orbit can be slightly changed. They do not give rise to any particularly interesting effects on the motion of charged particles and will not be considered any further.

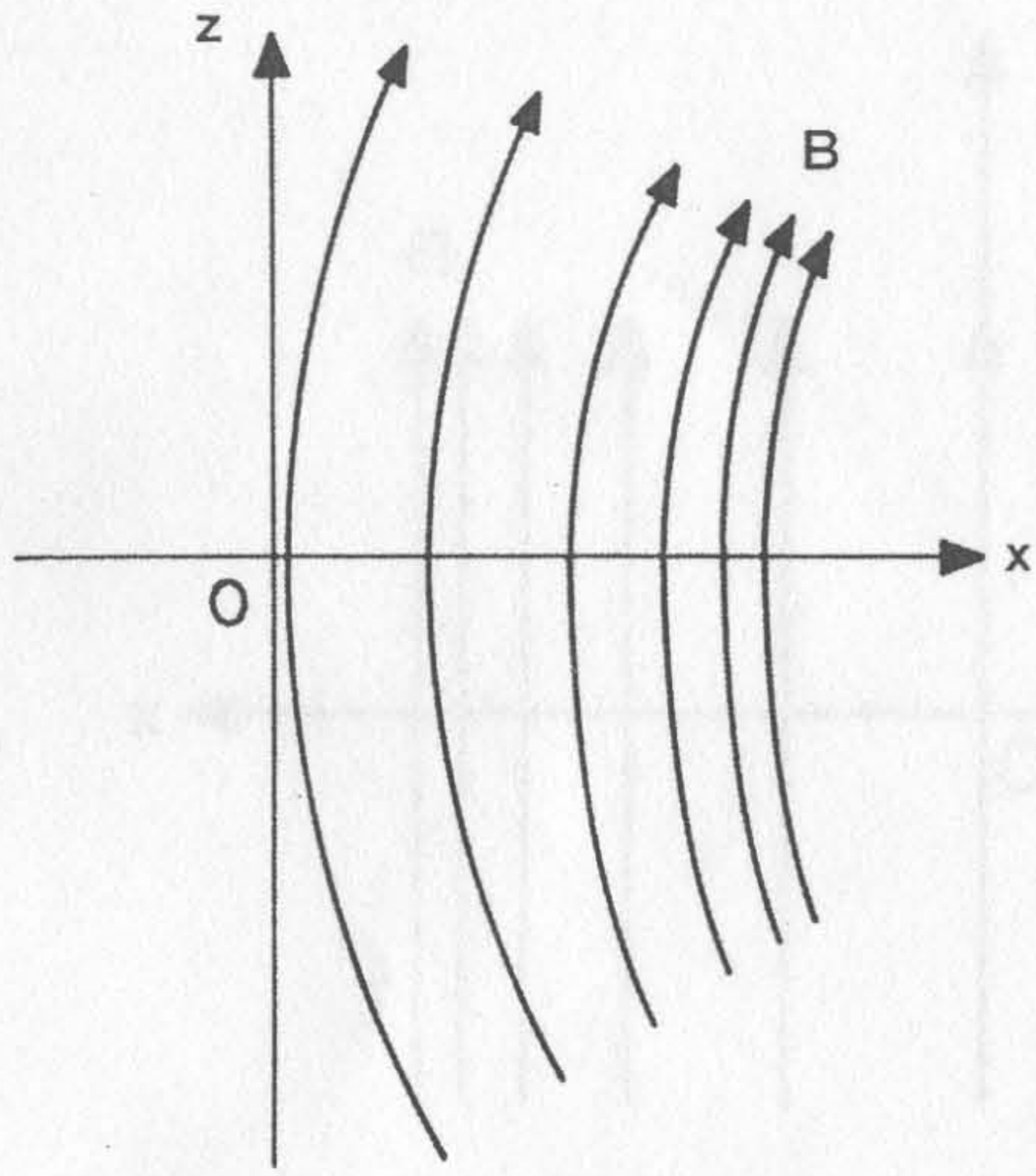


Fig. 5 *Geometry of the magnetic field lines corresponding to (2.15), with gradient and curvature terms.*

3. EQUATION OF MOTION IN THE FIRST ORDER APPROXIMATION

We consider that the magnetic field $\mathbf{B}_0$ which exists at the origin in the guiding center coordinate system is in the z direction,

$$\mathbf{B}(0,0,0) \equiv \mathbf{B}_0 = B_0\widehat{\mathbf{z}} \tag{3.1}$$

The particle motion in the neighborhood of the origin can be described by considering only a linear approximation to the magnetic field near the origin. Let $\mathbf{r}$ be the momentary position vector of the particle in the guiding center coordinate system (see Fig. 1). In the region of interest (near the origin) the magnetic field can be expressed by a Taylor expansion about the origin,

$$\mathbf{B}(\mathbf{r}) = \mathbf{B}_0 + \mathbf{r}\cdot(\nabla\mathbf{B}) + \quad \cdots \tag{3.2}$$

where the derivatives of $\mathbf{B}$ are to be calculated at the origin. Note that the instantaneous position of the particle guiding center actually moves slightly during one period of rotation, while the origin is kept fixed during this time.

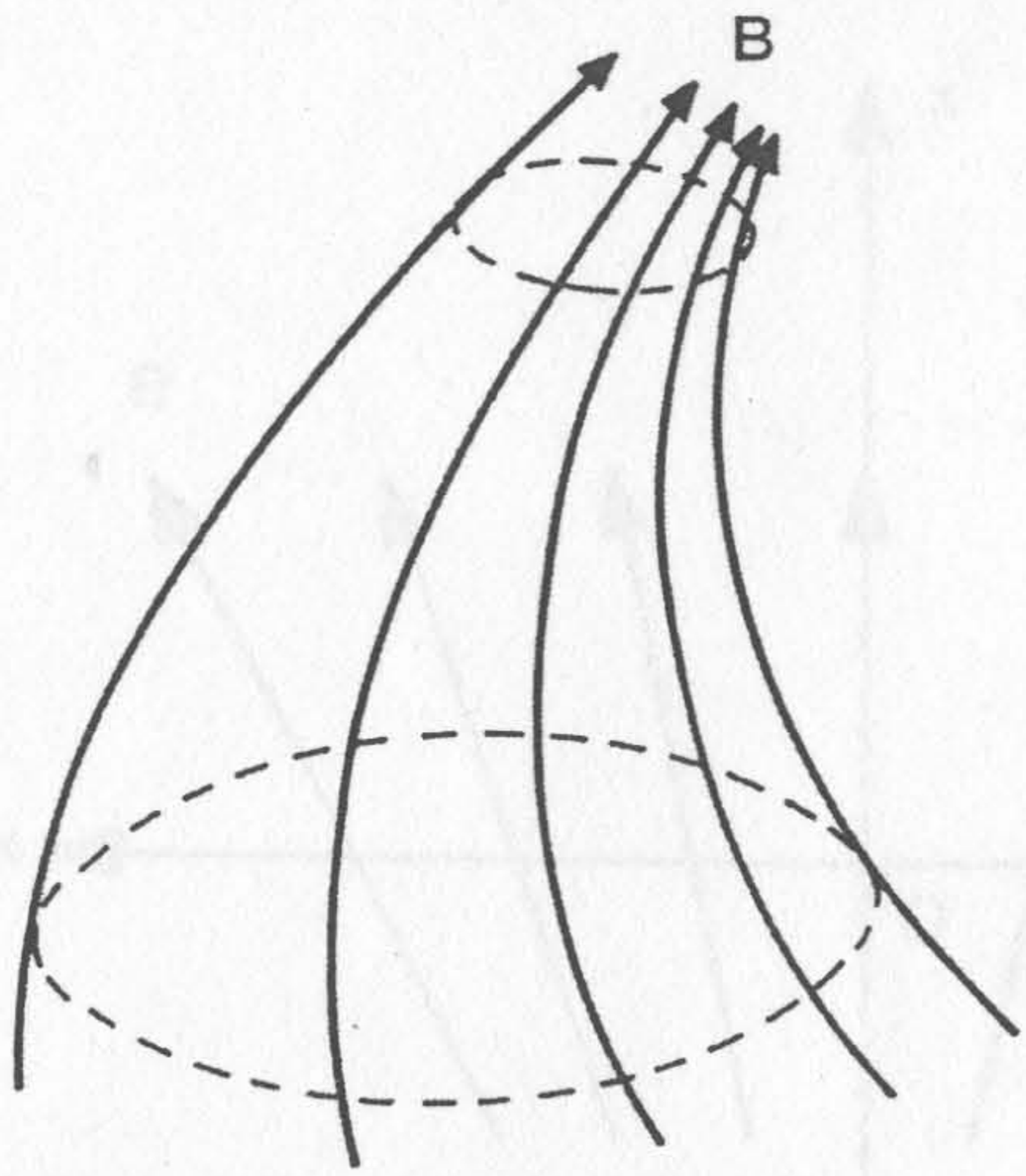


Fig. 6 *Schematic representation of a magnetic field having divergence, gradient and curvature terms.*

Since we are assuming that the spatial variation of $\mathbf{B}$ in a distance of the order of the Larmor radius is much smaller than the magnitude of $\mathbf{B}$ itself, the higher order terms of (3.2) can be neglected. The condition

$$\delta B = |\mathbf{r} \cdot (\nabla \mathbf{B})| \ll |\mathbf{B}_0| \tag{3.3}$$

is clearly met (see section 1). Thus, the magnetic field at the particle position differs only slightly from that existing at the guiding center. The first order term $\mathbf{r} \cdot (\nabla \mathbf{B})$ can be written explicitly as

$$\mathbf{r} \cdot (\nabla \mathbf{B}) = (\mathbf{r} \cdot \nabla)\mathbf{B} = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)\mathbf{B} =$$

$$\left(x\frac{\partial B_x}{\partial x} + y\frac{\partial B_x}{\partial y} + z\frac{\partial B_x}{\partial z}\right)\widehat{\mathbf{x}} + \left(x\frac{\partial B_y}{\partial x} + y\frac{\partial B_y}{\partial y} + z\frac{\partial B_y}{\partial z}\right)\widehat{\mathbf{y}} +$$

$$\left(x\frac{\partial B_z}{\partial x} + y\frac{\partial B_z}{\partial y} + z\frac{\partial B_z}{\partial z}\right)\widehat{\mathbf{z}} \tag{3.4}$$

where the partial derivatives are to be calculated at the origin. Substituting (3.2) into the equation of motion (2.1.5), with $\mathbf{E} = 0$, gives

$$m\frac{d\mathbf{v}}{dt} = q(\mathbf{v} \times \mathbf{B}_0) + q\mathbf{v} \times [\mathbf{r} \cdot (\nabla \mathbf{B})] \tag{3.5}$$

The last term in the right-hand side is of first-order compared to the first one. The particle velocity can be written as a superposition,

$$\mathbf{v} = \mathbf{v}^{(0)} + \mathbf{v}^{(1)} = \frac{d\mathbf{r}^{(0)}}{dt} + \frac{d\mathbf{r}^{(1)}}{dt} \tag{3.6}$$

where $\mathbf{v}^{(1)}$ is a first-order perturbation

$$|\mathbf{v}^{(1)}| \ll |\mathbf{v}^{(0)}| \tag{3.7}$$

and $\mathbf{v}^{(0)}$ is the solution of the zero-order equation

$$m\frac{d\mathbf{v}^{(0)}}{dt} = q(\mathbf{v}^{(0)} \times \mathbf{B}_0) \tag{3.8}$$

which has already been discussed in section 4 of Chapter 2. Neglecting second-order terms we can write, therefore,

$$\mathbf{v} \times [\mathbf{r} \cdot (\nabla \mathbf{B})] = \mathbf{v}^{(0)} \times [\mathbf{r}^{(0)} \cdot (\nabla \mathbf{B})] \tag{3.9}$$

The equation of motion (3.5) becomes, under these approximations,

$$m\frac{d\mathbf{v}}{dt} = q(\mathbf{v} \times \mathbf{B}_0) + q\mathbf{v}^{(0)} \times [\mathbf{r}^{(0)} \cdot (\nabla \mathbf{B})] \tag{3.10}$$

The second term in the right-hand side constitutes the force term of (2.6.1). This additional force, however, is not constant since it depends on the *instantaneous* particle position. Thus, small oscillations occur during one period of gyration. Since we are interested in the smoothed motion of the guiding center we shall eliminate these small oscillations by averaging this force term over one gyration period. Therefore, in what follows we will be involved in calculating the average value over one gyration period of the force term $q\mathbf{v}^{(0)} \times [\mathbf{r}^{(0)} \cdot (\nabla \mathbf{B})]$, which will allow us to determine the *parallel acceleration* of the guiding center and its *transverse drift velocity* using (2.6.2).

4. AVERAGE FORCE OVER ONE GYRATION PERIOD

Consider initially the case when the particle initial velocity along $\mathbf{B}$ is zero, so that the particle path differs but little from a circle. In a uniform magnetic field this would be equivalent to observing the particle motion in a coordinate system moving with the guiding center velocity $\mathbf{v}_{\parallel}$. However,

when the field lines are bent, a coordinate system gliding along $\mathbf{B}$ is not an inertial system. The curvature of the field lines give rise to inertial forces and therefore to a *curvature drift* of the particle. This effect will be investigated later in section 7. For the moment we will assume that the field lines are not curved and that the coordinate system moves with velocity $\mathbf{v}_{\parallel}$.

Under the conditions indicated above, the zero-order variables, $\mathbf{v}^{(0)}$ and $\mathbf{r}^{(0)}$, are seen to be situated in the (x, y) plane. The force term

$$\mathbf{F} = q\mathbf{v}^{(0)} \times [\mathbf{r}^{(0)} \cdot (\nabla \mathbf{B})] \tag{4.1}$$

can be separated into a component $\mathbf{F}_{\parallel}$ along $\mathbf{B}_0$ (z axis) and a component $\mathbf{F}_{\perp}$ normal to $\mathbf{B}_0$, in the (x, y) plane. Using a local cylindrical coordinate system (r, θ, z) with the z axis pointing along $\mathbf{B}_0$ at the origin (refer to Fig. 7), we have

$$\mathbf{r}^{(0)} \cdot (\nabla \mathbf{B}) = r^{(0)} \frac{\partial \mathbf{B}}{\partial r} \tag{4.2}$$

Of the three components of $\mathbf{B} = B_r \widehat{\mathbf{r}} + B_\theta \widehat{\Theta} + B_z \widehat{\mathbf{z}}$, the θ-component is parallel to $\mathbf{v}^{(0)}$ and therefore gives no contribution to $\mathbf{F}$, while $B_r \widehat{\mathbf{r}}$ contributes to $\mathbf{F}_{\parallel}$ and $B_z \widehat{\mathbf{z}}$ contributes to $\mathbf{F}_{\perp}$. Hence, from (4.1) and (4.2),

$$\mathbf{F}_{\parallel} = q(\mathbf{v}^{(0)} \times \widehat{\mathbf{r}})\, r^{(0)} \frac{\partial B_r}{\partial r} = |q|\, v^{(0)} r^{(0)} \frac{\partial B_r}{\partial r} \widehat{\mathbf{z}} \tag{4.3}$$

$$\mathbf{F}_{\perp} = q(\mathbf{v}^{(0)} \times \widehat{\mathbf{z}})\, r^{(0)} \frac{\partial B_z}{\partial r} = -|q|\, v^{(0)} r^{(0)} \frac{\partial B_z}{\partial r} \widehat{\mathbf{r}} \tag{4.4}$$

Note that if $q > 0$ we have $\mathbf{v}^{(0)} \times \widehat{\mathbf{r}} = v^{(0)} \widehat{\mathbf{z}}$, whereas if $q < 0$ we have $\mathbf{v}^{(0)} \times \widehat{\mathbf{r}} = -v^{(0)} \widehat{\mathbf{z}}$. Now, $r^{(0)}$ is the cyclotron radius corresponding to $\mathbf{B}_0$,

$$r^{(0)} = \frac{v^{(0)}}{\Omega_c} = \frac{m v^{(0)}}{|q|\, B_0} \tag{4.5}$$

and using the expression for the magnitude of the magnetic moment (2.4.34), we can write (4.3) and (4.4) as

$$\mathbf{F}_{\parallel} = 2\, |\mathbf{m}|\, \frac{\partial B_r}{\partial_r} \widehat{\mathbf{z}} \tag{4.6}$$

$$\mathbf{F}_{\perp} = -2\, |\mathbf{m}|\, \frac{\partial B_z}{\partial r} \widehat{\mathbf{r}} \tag{4.7}$$

There results apply to both positively and negatively charged particles.

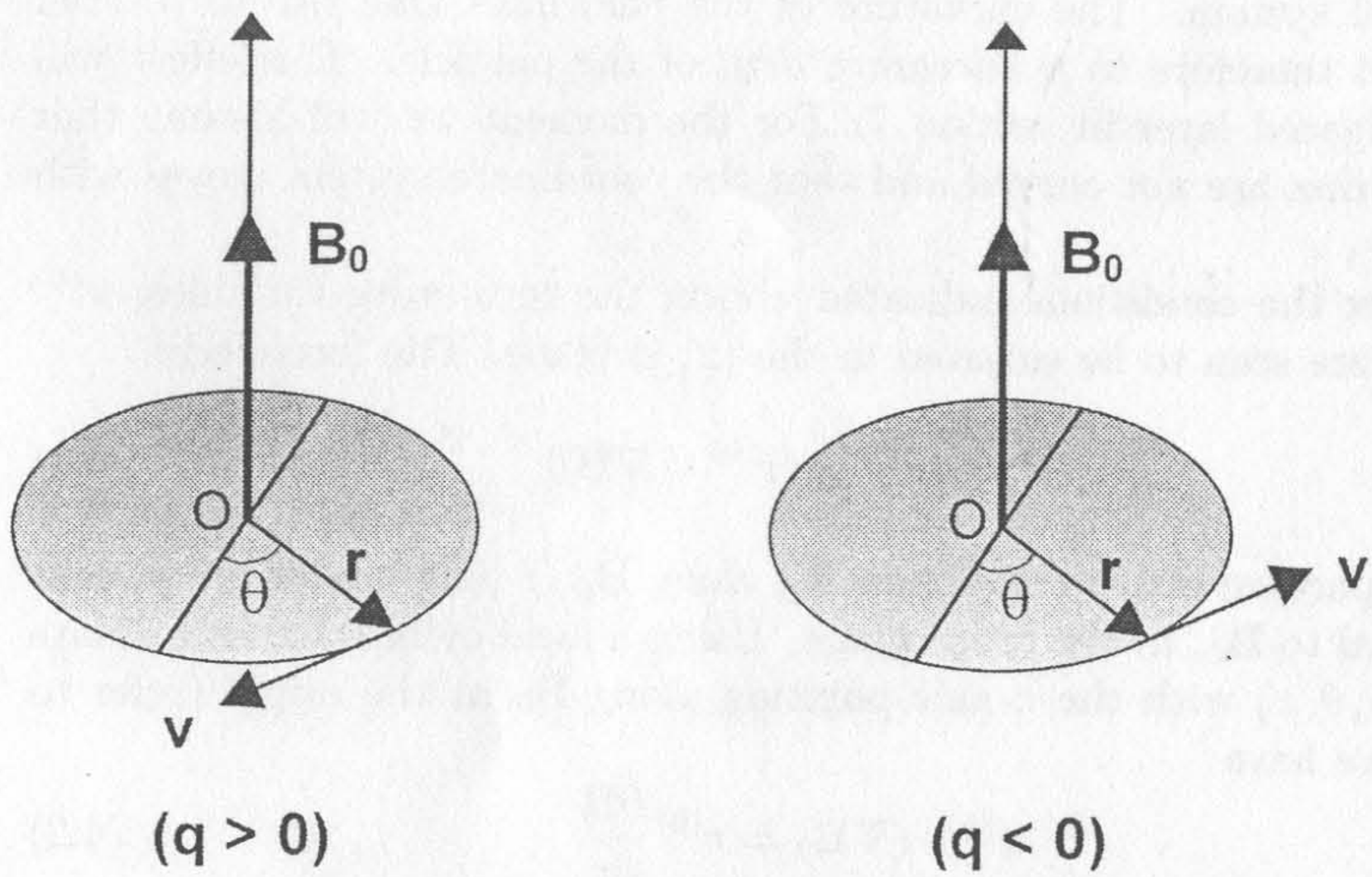


Fig. 7 *Local cylindrical coordinate system with the z axis pointing in the direction of the field* $\mathbf{B}_0$ *at the origin.*

The average values of $\mathbf{F}_\parallel$ and $\mathbf{F}_\perp$ over one gyration period are given by

$$< \mathbf{F}_\parallel > = 2\,|\mathbf{m}|\,\widehat{\mathbf{z}}\Big(\frac{1}{2\pi}\oint \frac{\partial B_r}{\partial r} d\theta\Big) = 2\,|\mathbf{m}|\,\widehat{\mathbf{z}}\;<\Big(\frac{\partial B_r}{\partial r}\Big)> \tag{4.8}$$

$$< \mathbf{F}_\perp > = -2\,|\mathbf{m}|\,\Big(\frac{1}{2\pi}\oint \frac{\partial B_z}{\partial r}\widehat{\mathbf{r}} d\theta\Big) = -2\,|\mathbf{m}|\;<\widehat{\mathbf{r}}\Big(\frac{\partial B_z}{\partial r}\Big)> \tag{4.9}$$

The average force $< \mathbf{F}_\parallel >$, given in (4.8), produces the guiding center *parallel acceleration*, while $< \mathbf{F}_\perp >$, given in (4.9), is responsible for the guiding center *transverse drift velocity*. The first one is the result of the *divergence* terms of $\mathbf{B}$, and the second one of the *gradient* terms. We proceed now to evaluate each force term separately.

4.1 Parallel Force

Note that from $\nabla \cdot \mathbf{B} = 0$ we have, in cylindrical coordinates,

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(B_\theta) + \frac{\partial}{\partial z}(B_z) = 0 \tag{4.10}$$

The first term can be expanded as

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_r) = \frac{\partial B_r}{\partial r} + \frac{B_r}{r} \tag{4.11}$$

Since at $r = 0$ we have $B_r = 0$, and since near the origin B_r changes only very slightly with r, we can take

$$\frac{B_r}{r} = \frac{\partial B_r}{\partial r} \tag{4.12}$$

Consequently, from (4.12) and (4.11)

$$\frac{\partial B_r}{\partial r} = -\frac{1}{2}\Big(\frac{1}{r}\frac{\partial B_\theta}{\partial \theta} + \frac{\partial B_z}{\partial z}\Big) \tag{4.13}$$

Hence, taking the average over one gyration period,

$$< \Big(\frac{\partial B_r}{\partial r}\Big) > = -\frac{1}{2} < \frac{1}{r}\Big(\frac{\partial B_\theta}{\partial \theta}\Big) > -\frac{1}{2} < \Big(\frac{\partial B_z}{\partial z}\Big) > \tag{4.14}$$

Now, since B is single-valued,

$$< \frac{1}{r}\Big(\frac{\partial B_\theta}{\partial \theta}\Big) > = \frac{1}{2\pi}\oint \frac{1}{r}\Big(\frac{\partial B_\theta}{\partial \theta}\Big)d\theta = 0 \tag{4.15}$$

Furthermore, since $\partial B_z/\partial z$ is a very slowly varying function inside the particle orbit, it can be taken outside the integral sign, so that we have approximately,

$$< \Big(\frac{\partial B_z}{\partial z}\Big) > = \frac{1}{2\pi}\oint \Big(\frac{\partial B_z}{\partial z}\Big)d\theta = \frac{\partial B_z}{\partial z} = \frac{\partial B}{\partial z} \tag{4.16}$$

It is justifiable to replace B_z by B in (4.16), since all the spatial variations of the magnetic field in the region of interest are very small. Therefore, we have finally from (4.14), (4.15) and (4.16),

$$< \Big(\frac{\partial B_r}{\partial r}\Big) > = -\frac{1}{2}\Big(\frac{\partial B}{\partial z}\Big) \tag{4.17}$$

Using this result, the parallel force (4.8) becomes

$$< \mathbf{F}_\parallel > = -\,|\mathbf{m}|\,\frac{\partial B}{\partial z}\widehat{\mathbf{z}} = -\,|\mathbf{m}|\,(\nabla B)_\parallel \tag{4.18}$$

or, equivalently,

$$< \mathbf{F}_\parallel > = (\mathbf{m}\cdot\nabla)B\widehat{\mathbf{z}} = -\frac{|\mathbf{m}|}{B}[(\mathbf{B}\cdot\nabla)\mathbf{B}]_\parallel \tag{4.19}$$

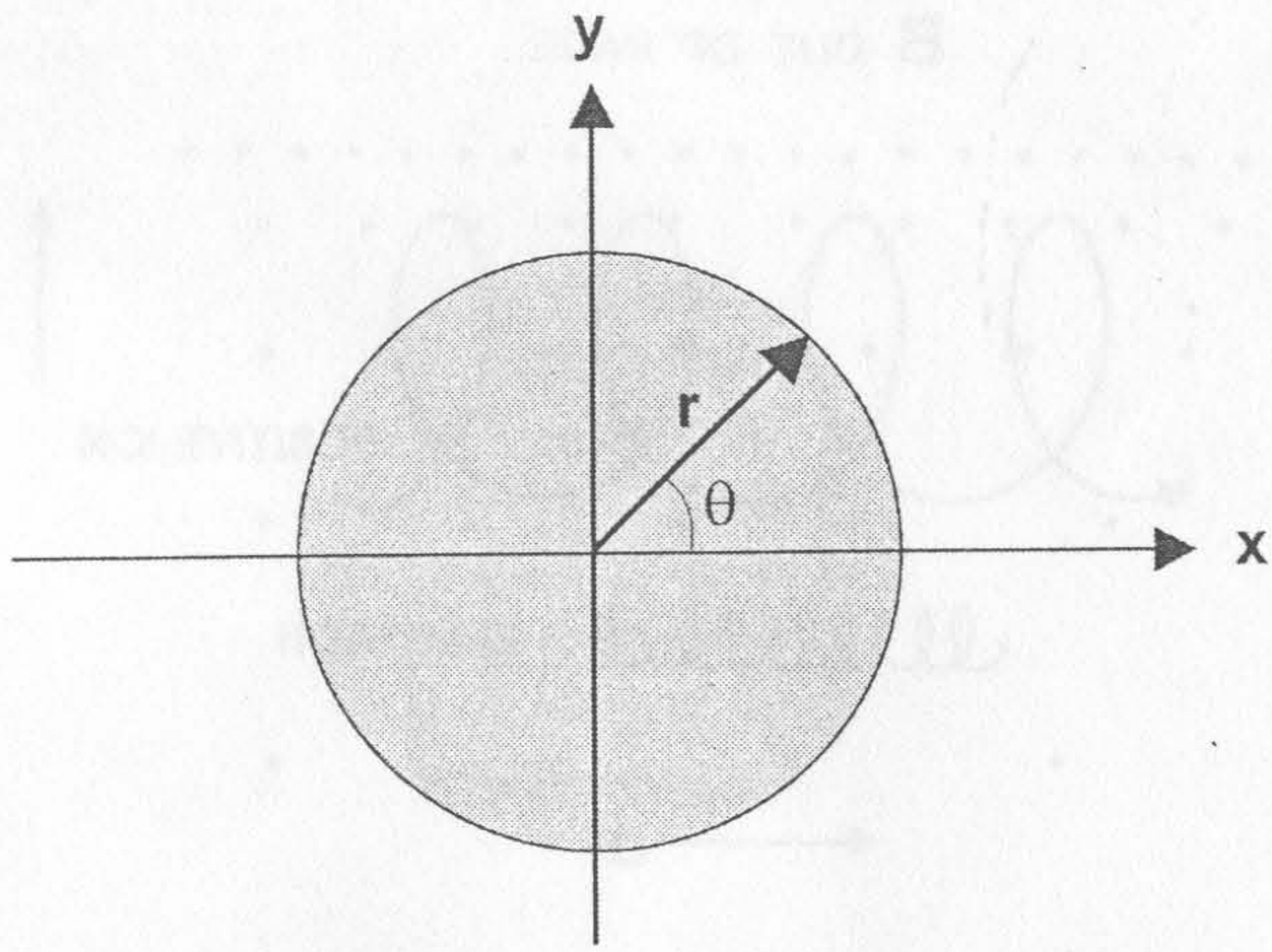


Fig. 8 *Two-dimensional coordinate system in the perpendicular plane, used in the evaluation of* $< \mathbf{F}_\perp >$.

since $\mathbf{m} = -|\mathbf{m}|\ \widehat{\mathbf{z}} = -|\mathbf{m}|\ \mathbf{B}/B$, and where the derivatives are evaluated at the origin.

4.2 Perpendicular Force

It is convenient to consider a two-dimensional Cartesian coordinate system (x, y) in the perpendicular plane, such that $x = r\ \cos(\theta)$ and $y = r\ \sin(\theta)$, as shown in Fig. 8. Hence,

$$\widehat{\mathbf{r}} = \cos(\theta)\widehat{\mathbf{x}} + \sin(\theta)\widehat{\mathbf{y}} \tag{4.20}$$

$$\frac{\partial}{\partial r} = \frac{dx}{dr}\frac{\partial}{\partial x} + \frac{dy}{dr}\frac{\partial}{\partial y} = \cos(\theta)\frac{\partial}{\partial x} + \sin(\theta)\frac{\partial}{\partial y} \tag{4.21}$$

Therefore, we obtain

$$< \widehat{\mathbf{r}}\Big(\frac{\partial B_z}{\partial r}\Big) > = < [\cos(\theta)\widehat{\mathbf{x}} + \sin(\theta)\widehat{\mathbf{y}}]\ \Big[\cos(\theta)\frac{\partial B_z}{\partial x} + \sin(\theta)\frac{\partial B_z}{\partial y}\Big] >$$

$$= < \cos^2(\theta)\frac{\partial B_z}{\partial x}\widehat{\mathbf{x}} > \ + \ < \sin(\theta)\ \cos(\theta)\frac{\partial B_z}{\partial x}\widehat{\mathbf{y}} > \ +$$

$$+ < \cos(\theta)\,\sin(\theta)\frac{\partial B_z}{\partial y}\widehat{\mathrm{x}} > \; + \; < \sin^2(\theta)\frac{\partial B_z}{\partial y}\widehat{\mathrm{y}} > \tag{4.22}$$

Next we approximate $(\partial B_z/\partial x)$ by $(\partial B/\partial x)$, and $(\partial B_z/\partial y)$ by $(\partial B/\partial y)$, since these terms are slowly varying functions inside the particle orbit, so that they can be taken outside the integral sign contained in the average values of (4.22). Noting that $< \sin(\theta)\,\cos(\theta) > = 0$ and $< \cos^2(\theta) > = < \sin^2(\theta) > = 1/2$, we obtain

$$< \widehat{\mathbf{r}}\frac{\partial B_z}{\partial r} > = \frac{1}{2}\frac{\partial B}{\partial x}\widehat{\mathrm{x}} + \frac{1}{2}\frac{\partial B}{\partial y}\widehat{\mathrm{y}} \tag{4.23}$$

Substituting this result into (4.9), yields

$$< \mathbf{F}_\perp > = -|\mathbf{m}|\left(\frac{\partial B}{\partial x}\widehat{\mathrm{x}} + \frac{\partial B}{\partial y}\widehat{\mathrm{y}}\right) = -|\mathbf{m}|\,(\nabla B)_\perp \tag{4.24}$$

4.3 Total Average Force

We proceed now to write down a general expression for the *total average force* $< \mathbf{F} > = < \mathbf{F}_\parallel > \; + \; < \mathbf{F}_\perp >$. From (4.18) and (4.24) we have

$$< \mathbf{F} > = -|\mathbf{m}|\,(\nabla B)_\parallel \; - |\mathbf{m}|\,(\nabla B)_\perp = -|\mathbf{m}|\,\nabla B \tag{4.25}$$

Alternatively, we can use the vector identity

$$(\nabla \times \mathbf{B}) \times \mathbf{B} = (\mathbf{B}\cdot\nabla)\mathbf{B} - \nabla(\tfrac{1}{2}B^2) \tag{4.26}$$

and write (4.25) in the form

$$< \mathbf{F} > = -\frac{|\mathbf{m}|}{B}[(\mathbf{B}\cdot\nabla)\mathbf{B} - (\nabla \times \mathbf{B}) \times \mathbf{B}] \tag{4.27}$$

Since $\mathbf{m} = -|\mathbf{m}|\;\mathbf{B}/B$, we have

$$< \mathbf{F} > = (\mathbf{m}\cdot\nabla)\mathbf{B} + \mathbf{m} \times (\nabla \times \mathbf{B}) \tag{4.28}$$

This is the usual expression for the force acting on a small ring current immersed in a magnetic field with spatial variation. The first term on the right-hand side of (4.28) alone gives the force acting on a magnetic dipole.

5. GRADIENT DRIFT

From (2.6.2) and (4.24) we see that $< \mathbf{F}_\perp >$ causes the guiding center to drift with the velocity

$$\mathbf{v}_G = \frac{< \mathbf{F}_\perp > \times \mathbf{B}}{qB^2} = -\frac{|\mathbf{m}|}{q}\frac{(\nabla B) \times \mathbf{B}}{B^2} \tag{5.1}$$

This gradient drift is perpendicular to $\mathbf{B}$ and to the field gradient, and its direction depends on the charge sign. Thus, positive and negative charges drift in opposite directions, giving rise to an electric current (see Fig. 9).

The physical reason for this gradient drift can be seen as follows. Since the Larmor radius of the particle orbit decreases as the magnetic field increases, the radius of curvature of the orbit is smaller in the regions of stronger $\mathbf{B}$ field. The positive ions gyrate in the clockwise direction for $\mathbf{B}$ pointing towards the observer, while the electrons gyrate in the counterclockwise direction, as shown in Fig. 9, so that the positive ions drift to the left and the electrons to the right.

In the case of a collisionless plasma, associated with this gradient drift across $\mathbf{B}$ there is a *magnetization current density* $\mathbf{J}_G$, given by

$$\mathbf{J}_G = \frac{1}{\delta V}\sum_i q_i \mathbf{v}_{Gi} \tag{5.2}$$

where the summation is over all charged particles contained in a suitably chosen element of volume δV. From (5.1) and (5.2), we have

$$\mathbf{J}_G = -\Big(\frac{1}{\delta V}\sum_i |\mathbf{m}_i|\Big)\frac{(\nabla B) \times \mathbf{B}}{B^2} \tag{5.3}$$

6. PARALLEL ACCELERATION OF THE GUIDING CENTER

The expression (4.18) for $< \mathbf{F}_\parallel >$ shows that, when the magnetic field has a longitudinal variation (i.e. convergence or divergence of the field lines along the z direction, as shown in Fig. 3), an axial force along z accelerates the particle in the direction of decreasing magnetic field, irrespective of whether the particle is positively or negatively charged. This is illustrated in Fig. 10. There are several important consequences of this repulsion of gyrating charges from a region of converging magnetic field lines, which we proceed to dicuss.

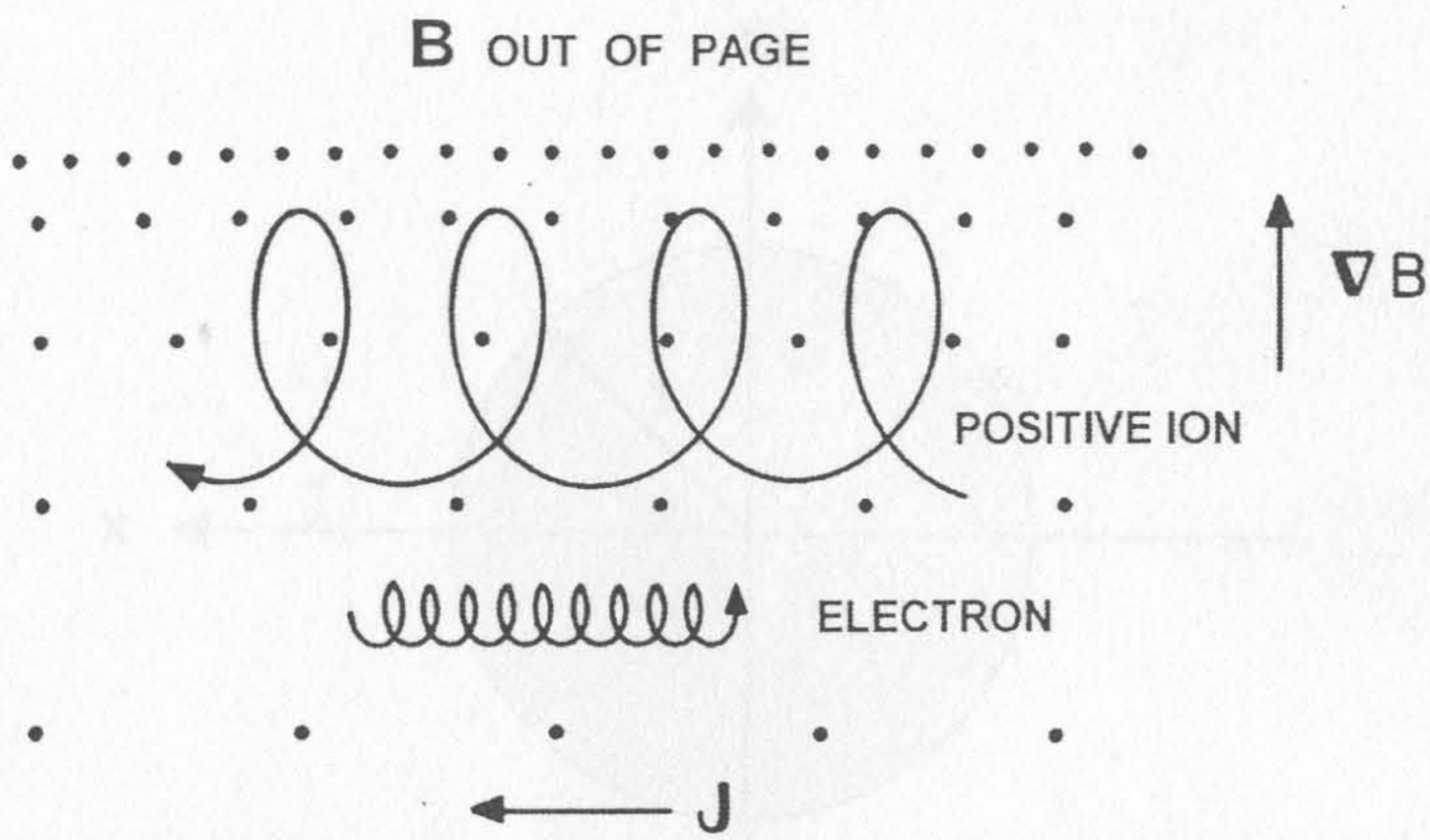


Fig. 9 *Charged particle drifts due to a* **B** *field gradient perpendicular to* **B**.

6.1 Invariance of the Orbital Magnetic Moment and of the Magnetic Flux

Using (4.18), the component of the equation of motion along **B** can be written as

$$m\frac{dv_\parallel}{dt}\widehat{\mathbf{z}} = <\mathbf{F}_\parallel> = -|\mathbf{m}|\,\frac{\partial B}{\partial z}\widehat{\mathbf{z}} \tag{6.1}$$

If we multiply both sides of this equation by $v_\parallel = dz/dt$, we obtain (replacing $|\mathbf{m}|$ by $W_\perp/B$),

$$mv_\parallel\frac{dv_\parallel}{dt} = \frac{d}{dt}(\tfrac{1}{2}mv_\parallel^2) = -\frac{W_\perp}{B}\frac{\partial B}{\partial z}\frac{dz}{dt} \tag{6.2}$$

where $W_\perp = mv_\perp^2/2$ denotes the part of the particle kinetic energy associated with its transverse velocity. Since the total kinetic energy of a charged particle in a magnetostatic field is constant, that is,

$$W_\parallel + W_\perp = \text{constant}, \tag{6.3}$$

it follows that

$$\frac{d}{dt}(W_\perp) = -\frac{d}{dt}(W_\parallel) = -\frac{d}{dt}(\tfrac{1}{2}mv_\parallel^2) \tag{6.4}$$

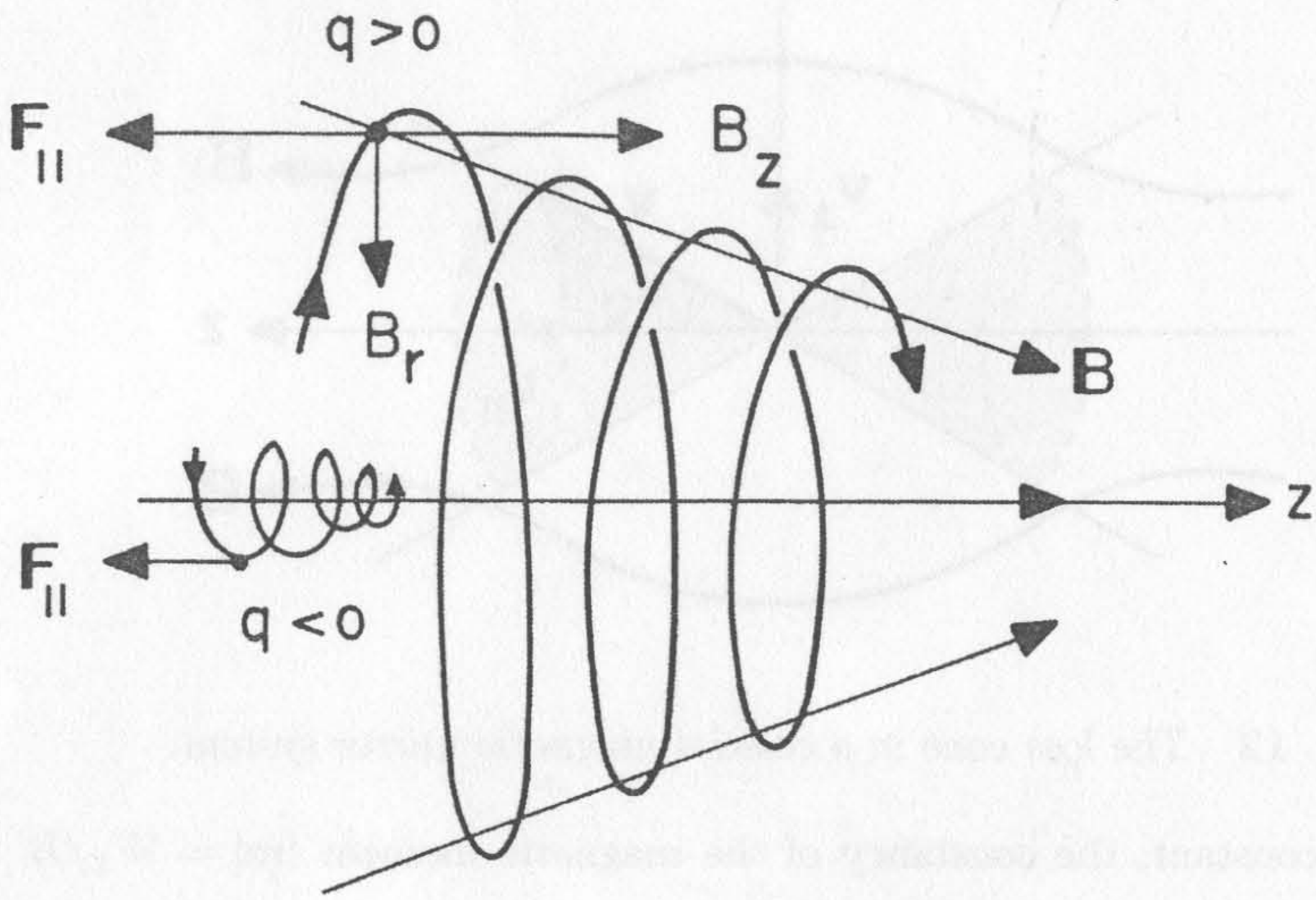


Fig. 10 *Repulsion of gyrating charges from a region of converging magnetic field lines.*

Therefore, from (6.2) and (6.4),

$$\frac{d}{dt}(W_\perp) = \frac{W_\perp}{B}\frac{\partial B}{\partial z}\frac{dz}{dt} = \frac{W_\perp}{B}\frac{dB}{dt} \tag{6.5}$$

where dB/dt represents the rate of change of B as seen by the particle as it moves in the spatially varying magnetic field (i.e. in the particle frame of reference). Comparing this result with the following identity

$$\frac{d}{dt}(W_\perp) = \frac{d}{dt}\left(\frac{W_\perp B}{B}\right) = \frac{W_\perp}{B}\frac{dB}{dt} + B\frac{d}{dt}\left(\frac{W_\perp}{B}\right) \tag{6.6}$$

we conclude that

$$\frac{d}{dt}\left(\frac{W_\perp}{B}\right) = 0 \tag{6.7}$$

or, equivalently,

$$|\mathbf{m}| = \frac{W_\perp}{B} = \text{constant}. \tag{6.8}$$

Therefore, as the particle moves into regions of converging or diverging **B** its cyclotron radius changes, but the magnetic moment remains constant. This constancy of the particle magnetic moment holds only within the approximation used, that is, when the spatial variation of **B** inside the particle orbit is small compared to the magnitude of **B**. Consequently, the orbital magnetic moment is said to be an *adiabatic invariant.* It is usually referred to as the *first adiabatic invariant.*

The *magnetic flux*, Φ_m, enclosed by one orbit of the particle is given by

$$\Phi_m = \int_S \mathbf{B} \cdot d\mathbf{S} = \pi r_c^2 B = \pi \frac{m^2 v_\perp^2}{q^2 B^2} B = \frac{2\pi m}{q^2}\left(\frac{W_\perp}{B}\right) \tag{6.9}$$

Therefore,

$$\frac{d}{dt}(\Phi_m) = \frac{2\pi m}{q^2}\frac{d}{dt}|\mathbf{m}| = 0 \tag{6.10}$$

in view of the invariance of $|\mathbf{m}|$. Hence, as the charged particle moves in a region of converging **B** field, it will orbit with increasingly smaller radius, so that the magnetic flux enclosed by the orbit remains constant.

6.2 Magnetic Mirror Effect

As a consequence of the adiabatic invariance of $|\mathbf{m}|$ and Φ_m, as the particle moves into a region of converging magnetic field lines its transverse kinetic energy $W_\perp$ increases, while its parallel kinetic energy $W_\parallel$ decreases, in order to keep $|\mathbf{m}|$ and the total energy constant. Ultimately, if the **B** field becomes strong enough, the particle velocity in the direction of increasing field may eventually come to zero and then be reversed. After reversion, the particle is speeded up in the direction of decreasing field, while its transverse velocity diminishes. Thus, the particle is *reflected* from the region of converging magnetic field lines. This phenomenon is called the *magnetic mirror effect* and is the basis for one of the primary schemes of plasma confinement.

When two coaxial magnetic mirrors are considered, as illustrated in Fig. 11, the charged particles may be reflected by the magnetic mirrors and may travel back and forth in the space between them, becoming trapped. This trapping region has been called a *magnetic bottle* and it has been used in laboratory for plasma confinement.

The trapping in a magnetic mirror system is not perfect, however. The effectiveness of a coaxial magnetic mirror system in the trapping of charged particles can be measured by the *mirror ratio* B_m/B_0, where B_m

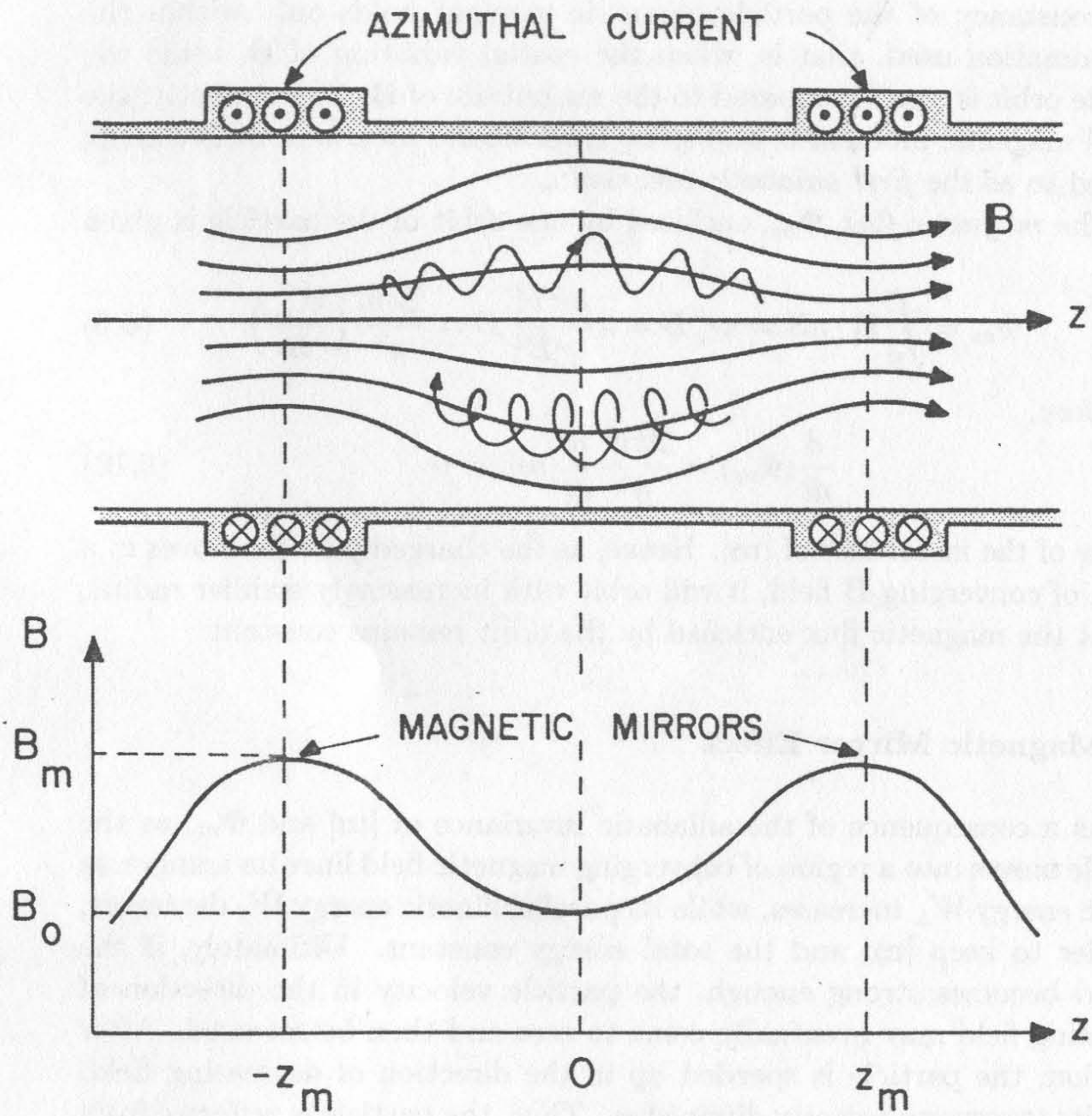


Fig. 11 *Schematic diagram showing the arrangement of coils to produce two coaxial magnetic mirrors facing each other, for plasma confinement, and the relative intensity variation of the magnetic field.*

is the intensity of the magnetic field at the point of reflection (where the *pitch angle* of the particle is $\pi/2$) and B_0 is the intensity of the magnetic field at the center of the magnetic bottle.

Consider a charged particle having a pitch angle α_0 at the center of the magnetic bottle. If v is the particle speed, which in a static magnetic field

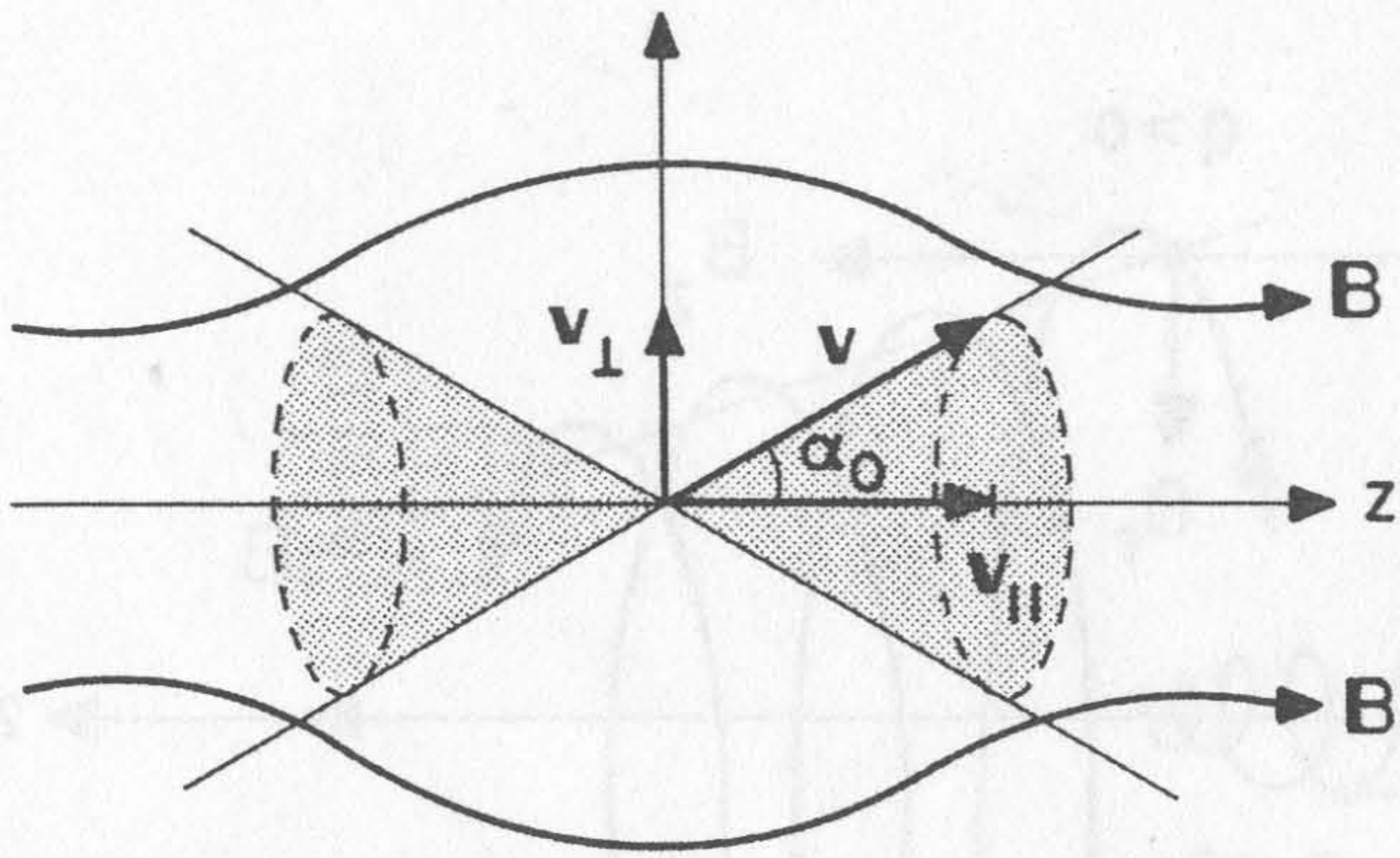


Fig. 12 *The loss cone in a coaxial magnetic mirror system.*

remains constant, the constancy of the magnetic moment $|\mathbf{m}| = W_\perp/B$ leads to

$$\tfrac{1}{2}mv^2(\sin^2\alpha)/B = \tfrac{1}{2}mv^2(\sin^2\alpha_0)/B_0 \tag{6.11}$$

where α is the particle pitch angle at a position where the magnetic field intensity is B. Thus, at any point inside the magnetic bottle, for this particle,

$$\frac{\sin^2\alpha(z)}{B(z)} = \frac{\sin^2\alpha_0}{B_0} \tag{6.12}$$

Suppose now that this particle is reflected at the *throat* of the mirror, that is, $\alpha = \pi/2$ for $B(z) = B_m$. Therefore, from (6.12),

$$(\sin^2\alpha_0)/B_0 = 1/B_m \tag{6.13}$$

This means that a particle having a pitch angle α_0 given by

$$\alpha_0 = \sin^{-1}[(B_0/B_m)^{1/2}] = \sin^{-1}(v_\perp/v)_0 \tag{6.14}$$

at the center of the bottle, is reflected at a point where the intensity of the field is B_m. Therefore, for a magnetic bottle with a fixed mirror ratio B_m/B_0, the plasma particles having a pitch angle at the center *greater* than α_0, as given by (6.14), will be reflected *before* the ends of the magnetic bottle. On the other hand, if the pitch angle of the particle at the center is

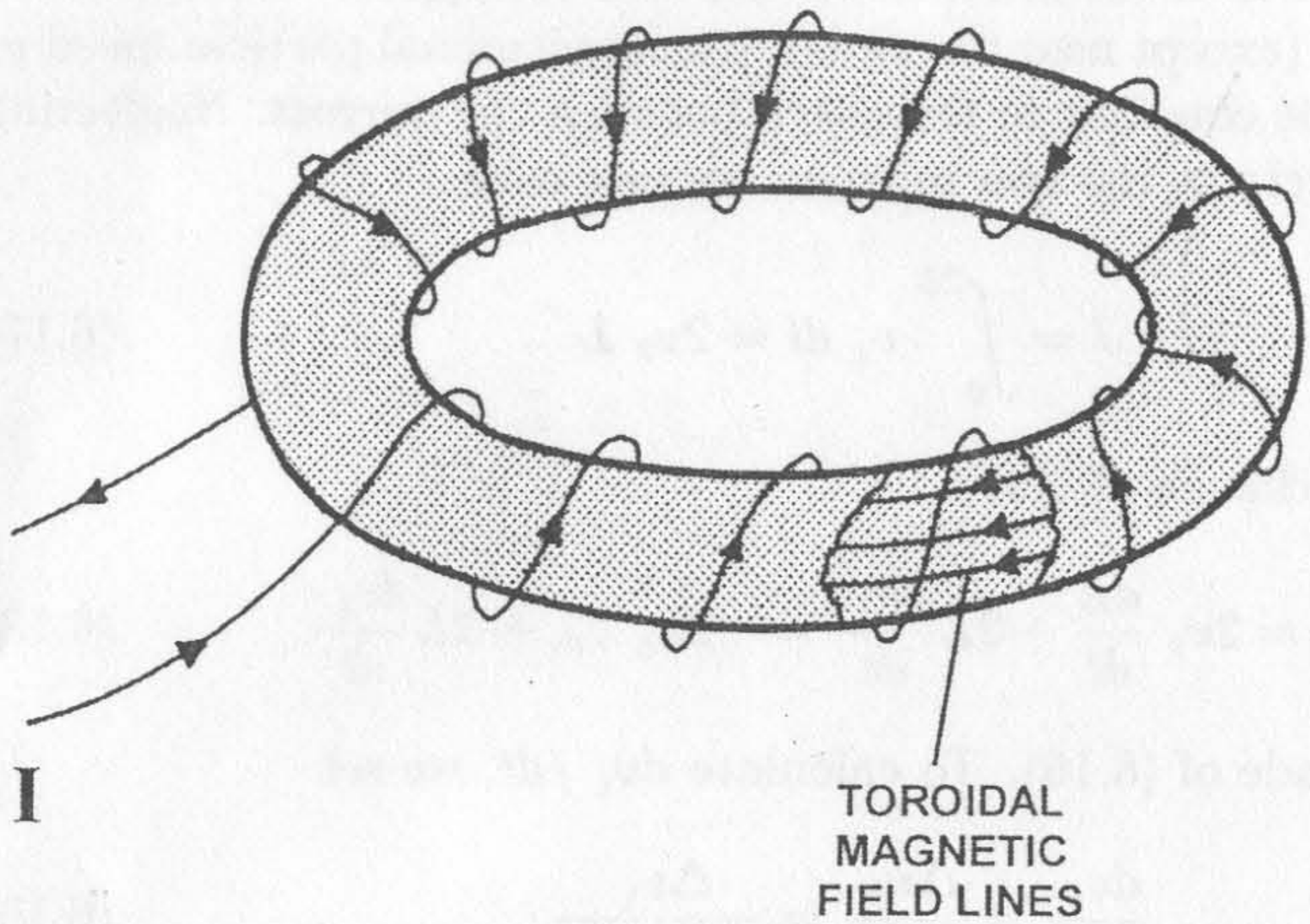


Fig. 13 *Magnetic field with toroidal geometry.*

less than α_0, its pitch angle will never reach the value $\pi/2$, which implies that at the ends of the bottle the particle has a non-vanishing parallel velocity and hence escapes through the ends of the mirror system. There is therefore a *loss cone*, a cone of half-angle α_0 with its vertex at the center, as shown in Fig. 12, where particles which have velocity vectors with a pitch angle falling inside it are not trapped. The loss cone is determined by the mirror ratio B_m/B_0, according to (6.14).

Devices that have no ends, with geometries such that the magnetic field lines close on themselves, offer many advantages for plasma confinement. Toroidal geometries (Fig. 13) for example have no ends, but it turns out that confinement of a plasma inside a toroidal magnetic field does not provide a plasma equilibrium situation, because of the radial inhomogeneity of the field. In this case a poloidal magnetic field is normally superposed on the toroidal field, resulting in *helical* field lines (as in the *Tokamak*). The major problem in most plasma confinement schemes, however, is that instabilities and small fluctuations from the desired equilibrium configuration are always present, which lead to a rapid escape of the particles from the magnetic bottle. This instability problem is a fundamental one, and it is likely to occur in any conceivable magnetic confinement scheme.

A good example of a natural magnetic bottle is the Earth's magnetic field, which traps charged particles of solar and cosmic origin. These

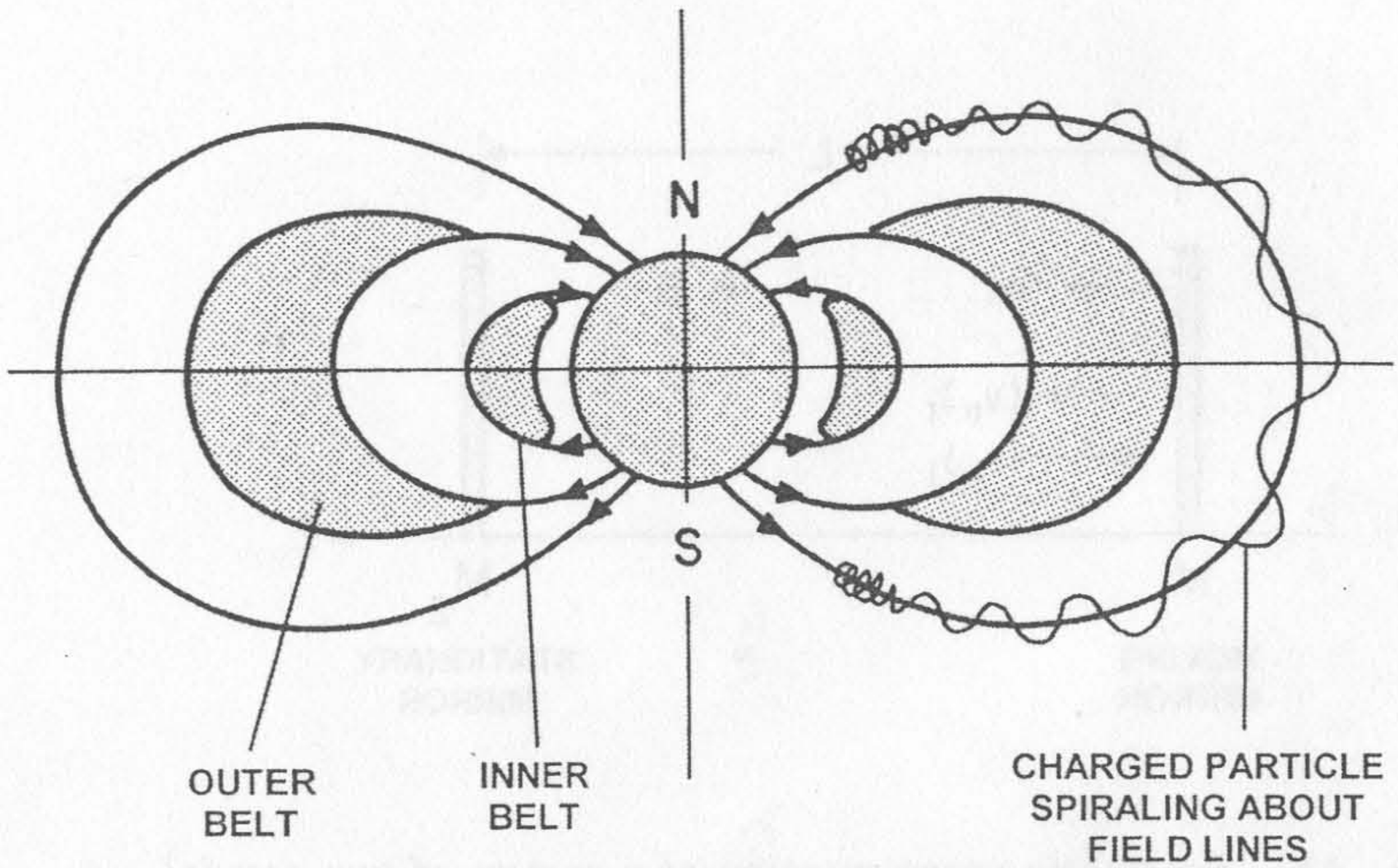


Fig. 14 *Dipole approximation of the Earth's magnetic field. The distance of the Van Allen radiation belts from the center of the Earth, at the equator, is about 1.5 Earth radii for the high energy protons and about 3 to 4 Earth radii for the high energy electrons.*

charged particles trapped in the Earth's magnetic field constitute the so called *Van Allen radiation belts.* As shown in Fig. 14, the geomagnetic field near the Earth is approximately that of a dipole, with the field lines converging towards the north and south magnetic poles.

The electrons and protons which are trapped in the Van Allen radiation belts spiral in almost helical paths along the field lines, and towards the magnetic poles, where they are eventually reflected. These particles bounce back and forth between the poles. In addition to this bouncing motion, these trapped charged particles are also subject to a gradient drift and a curvature drift in the east-west direction, to be discussed later in this chapter.

6.3 The Longitudinal Adiabatic Invariant

Consider a particle trapped between two magnetic mirrors and bouncing between them. Suppose that the separation distance between the two

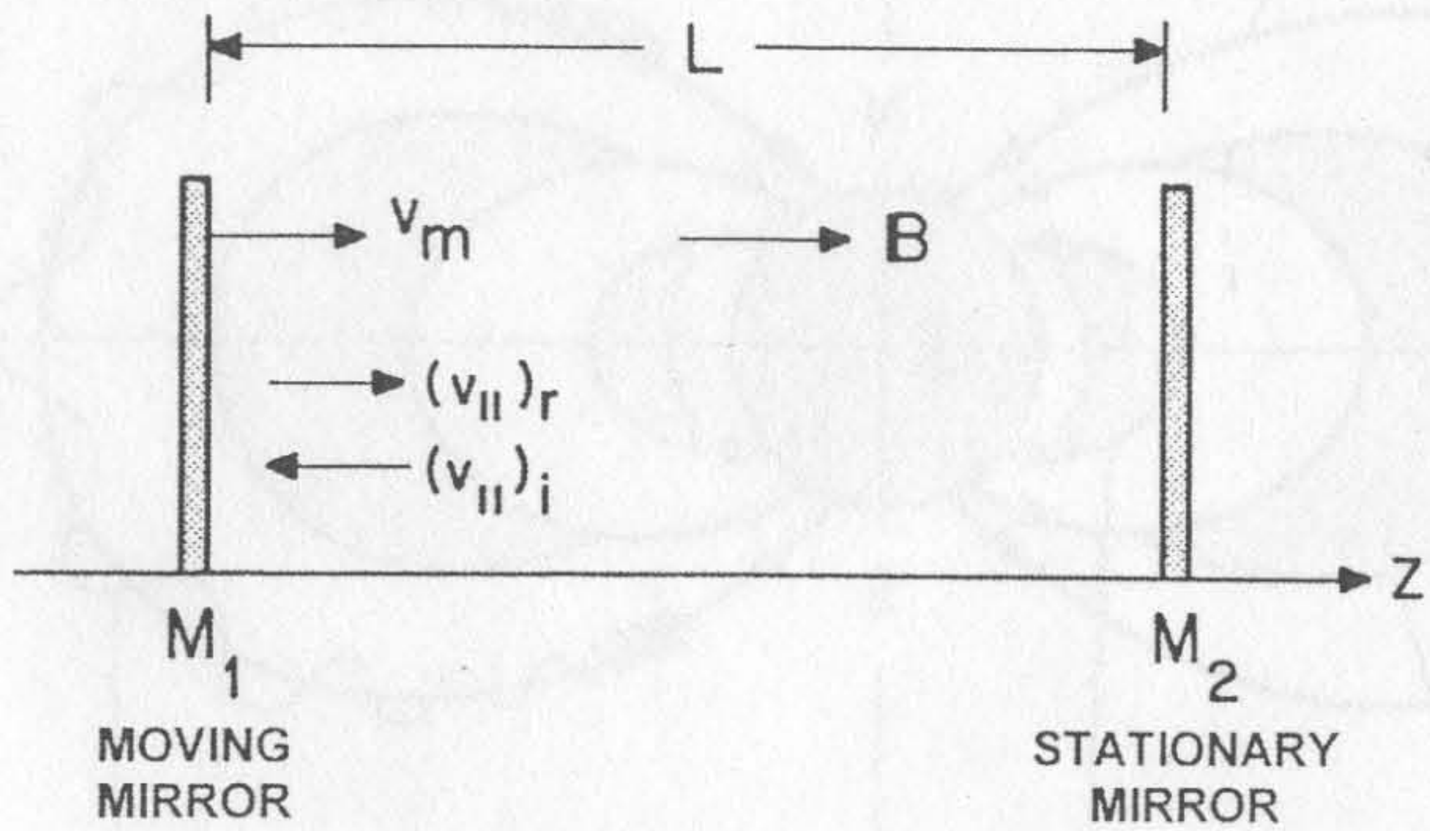


Fig. 15 *Schematic representation of a system of two coaxial magnetic mirrors approaching each other.*

mirrors changes very slowly in time as compared to the bounce period. With the periodic motion of the particle between the two magnetic mirrors (whose separation varies slowly in time) there is associated an adiabatic invariant called the *longitudinal adiabatic invariant*, defined by the integral

$$J = \oint \mathbf{v} \cdot d\mathbf{l} = \oint v_{\parallel}\, dl \tag{6.15}$$

taken over one period of oscillation of the particle back and forth between the mirror points.

For a simple proof of the adiabatic invariance of J, consider the idealized situation illustrated in Fig. 15, where the existing $\mathbf{B}$ field in the z direction is uniform in space, except near the points M_1 and M_2, where the field increases to form the two mirrors separated by a distance L. Suppose that the mirror M_1 approaches the other one with velocity

$$v_m = -\frac{dL}{dt} \tag{6.16}$$

the negative sign being due to the fact that L decreases with time. It is assumed that this velocity is much smaller than the longitudinal component of the particle velocity, that is, $v_m \ll v_{\parallel}$. Thus, the distance moved by the mirror M_1 during one period of oscillation of the particle is small compared to the distance L between the mirrors.

Further, since **B** is assumed to be uniform throughout the space between the mirrors (except near the ends), the longitudinal particle speed $v_\parallel$ may be taken to be constant in the space between the mirrors. Neglecting the small end effects at the two mirrors, we can take

$$J = \int_0^{2L} v_\parallel \, dl = 2v_\parallel \, L \tag{6.17}$$

The time rate of change of J is

$$\frac{dJ}{dt} = 2v_\parallel \, \frac{dL}{dt} + 2L\frac{dv_\parallel}{dt} = -2v_\parallel \, v_m + 2L\frac{dv_\parallel}{dt} \tag{6.18}$$

where use was made of (6.16). To calculate $dv_\parallel \, /dt$, we set

$$\frac{dv_\parallel}{dt} = \frac{\Delta v_\parallel}{\Delta t} = \frac{\Delta v_\parallel}{(2L/v_\parallel \,)} \tag{6.19}$$

where $\Delta v_\parallel$ denotes the change in the particle speed $v_\parallel$ on reflection from the moving mirror, and $\Delta t = (2L/v_\parallel)$ is the period of oscillation between the mirrors. In order to find $\Delta v_\parallel$ it is convenient to transform to a coordinate system moving with the magnetic mirror M_1, at the speed v_m. Let us denote this moving coordinate system by a prime and the incident and reflected particle speeds by subscripts i and r, respectively. Thus,

$$(v_\parallel)_i^{'} = (v_\parallel)_i + v_m \tag{6.20}$$

$$(v_\parallel)_r^{'} = (v_\parallel)_r - v_m \tag{6.21}$$

which gives for the change in the particle speed, in one reflection,

$$\Delta v_\parallel \; = (v_\parallel)_r - (v_\parallel)_i = 2v_m \tag{6.22}$$

since in the moving coordinate system $(v_\parallel)_i^{'} = (v_\parallel)_r^{'}$ with only their directions reversed. Therefore, (6.19) becomes

$$\frac{dv_\parallel}{dt} = \frac{2v_m}{(2L/v_\parallel)} = \frac{v_m v_\parallel}{L} \tag{6.23}$$

On substituting this result into (6.18) we find

$$\frac{dJ}{dt} = \frac{d}{dt}(2v_\parallel \, L) = 0 \tag{6.24}$$

which shows that J is an adiabatic invariant. This quantity is also referred to as the *second adiabatic invariant.* The parallel kinetic energy of a charged particle trapped between the two mirrors is (taking $J = 2v_{\parallel}\ L$)

$$W_{\parallel} = \tfrac{1}{2}mv_{\parallel}^2 = \frac{mJ^2}{8L^2} \tag{6.25}$$

which increases rapidly as L decreases. The Italian physicist Fermi suggested this process as as mechanism for the acceleration of charged particles in order to explain the origin of high energy cosmic rays. Fermi proposed that two stellar clouds moving towards each other, and having a magnetic field greater than in the space between them, may trap and accelerate the cosmic charged particles. There is a limit, however, in the particle longitudinal speed increase, since the direction of the particle velocity at the center of the mirror system may eventually enter the loss cone and escape through the ends of the system. It should be noted that a magnetic mirror moving towards a stationary one involves in fact time-varying **B** fields and, consequently, electric fields, which can lead to a change in the particle kinetic energy.

7. CURVATURE DRIFT

So far the effects associated with the curvature of the magnetic field lines have not been considered. As stated previously, a **B** field with only curvature terms does not satisfy the equation $\nabla \times \mathbf{B} = 0$, so that in practice the gradient and the curvature drifts will always be present simultaneously. In first order orbit theory the effects corresponding to each of the components of **B** are additive.

We investigate now the effect of the curvature terms $\partial B_x/\partial z$ and $\partial B_y/\partial z$, referred to in (2.6c), on the motion of a charged particle. We will assume that these terms are so small that the radius of curvature of the magnetic field lines is very large compared to the particle cyclotron radius. Let us introduce a local coordinate system gliding along the magnetic field line with the particle longitudinal velocity $\mathbf{v}_{\parallel}$. Since this is not an inertial system because of the curvature of the field lines, a centrifugal force will be present. This local coordinate system can be specified by the orthogonal set of unit vectors $\widehat{\mathbf{B}}$, $\widehat{\mathbf{n}}_1$ and $\widehat{\mathbf{n}}_2$, where $\widehat{\mathbf{B}}$ is along the field line, $\widehat{\mathbf{n}}_1$ is along the principal normal to the field line, and $\widehat{\mathbf{n}}_2$ is along the binormal to the curved magnetic field line, as indicated in Fig. 16.

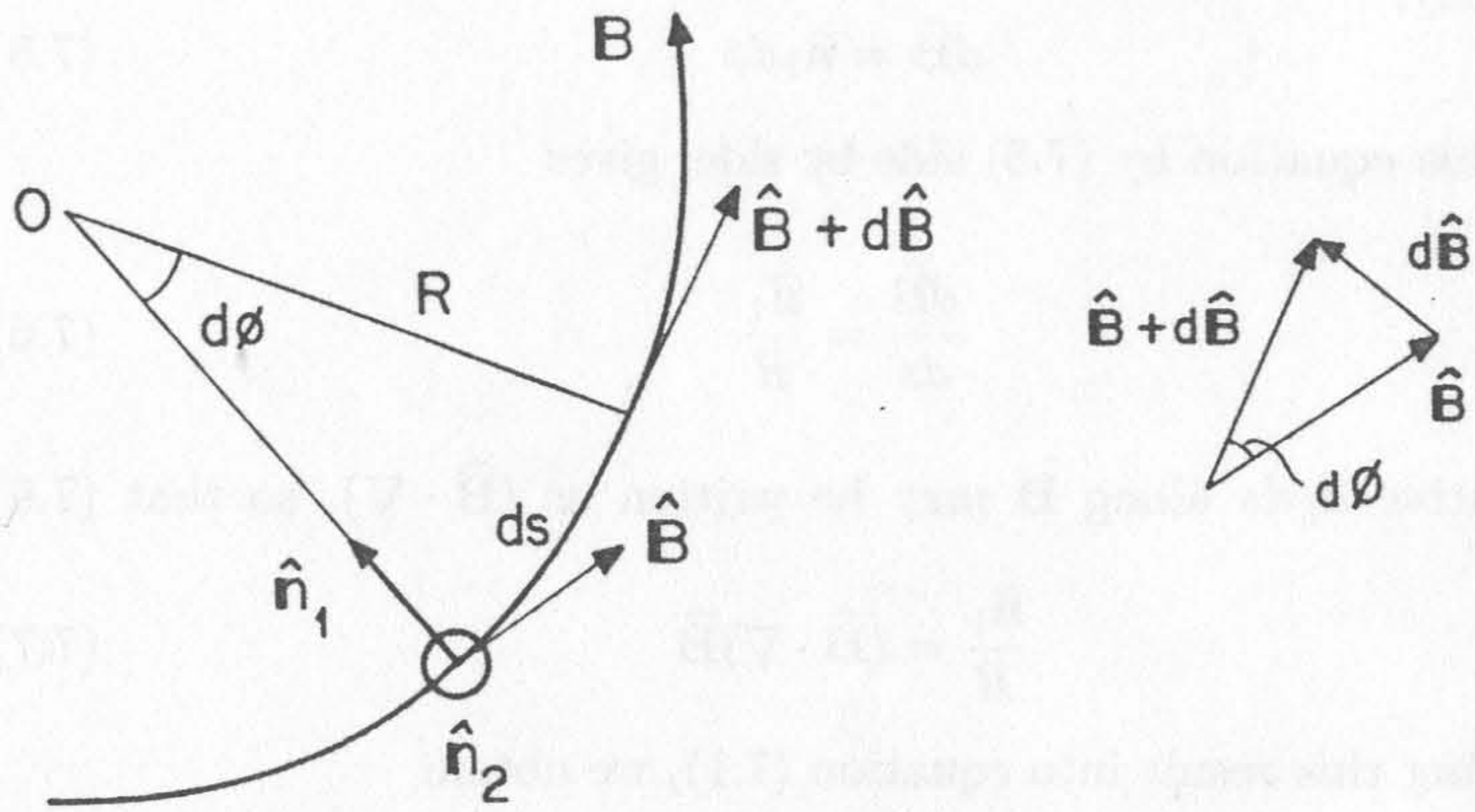


Fig. 16 *Curved magnetic field line showing the unit vector* $\widehat{\mathbf{B}}$ *along the field line, the principal normal* $\widehat{\mathbf{n}}_1$ *and the binormal* $\widehat{\mathbf{n}}_2$*, at an arbitrary point. Note that* $\widehat{\mathbf{n}}_1 \times \widehat{\mathbf{n}}_2 = \widehat{\mathbf{B}}$*. The local radius of curvature is* R*.*

The centrifugal force $\mathbf{F}_c$ acting on the particle, as seen from this noninertial system, is given by

$$\mathbf{F}_c = -\frac{mv_\parallel^2}{R}\widehat{\mathbf{n}}_1 \tag{7.1}$$

where R denotes the local radius of curvature of the magnetic field line and $v_\parallel$ is the particle instantaneous longitudinal speed. From (2.6.2) the curvature drift associated with this force is

$$\mathbf{v}_C = \frac{\mathbf{F}_c \times \mathbf{B}}{qB^2} = -\frac{mv_\parallel^2}{RqB^2}(\widehat{\mathbf{n}}_1 \times \mathbf{B}) \tag{7.2}$$

To express the unit vector $\widehat{\mathbf{n}}_1$ in terms of the unit vector $\widehat{\mathbf{B}}$ along the magnetic field line, we let ds represent and element of arc along the field line subtending an angle $d\phi$,

$$ds = Rd\phi \tag{7.3}$$

If $d\widehat{\mathbf{B}}$ denotes the change in $\mathbf{B}$ due to the displacement ds (see Fig. 16), then $d\widehat{\mathbf{B}}$ is in the direction of $\widehat{\mathbf{n}}_1$ and its magnitude is

$$|d\widehat{\mathbf{B}}| = |\widehat{\mathbf{B}}|\, d\phi = d\phi \tag{7.4}$$

Consequently,

$$d\widehat{\mathbf{B}} = \widehat{\mathbf{n}}_1 d\phi \tag{7.5}$$

Dividing this equation by (7.3) side by side, gives

$$\frac{d\widehat{\mathbf{B}}}{ds} = \frac{\widehat{\mathbf{n}}_1}{R} \tag{7.6}$$

The derivative d/ds along $\mathbf{B}$ may be written as $(\widehat{\mathbf{B}} \cdot \nabla)$, so that (7.6) becomes

$$\frac{\widehat{\mathbf{n}}_1}{R} = (\widehat{\mathbf{B}} \cdot \nabla)\widehat{\mathbf{B}} \tag{7.7}$$

Incorporating this result into equation (7.1), we obtain

$$\mathbf{F}_c = -\, mv_{\parallel}^2 (\widehat{\mathbf{B}} \cdot \nabla)\widehat{\mathbf{B}} \tag{7.8}$$

This force is obviously perpendicular to the magnetic field $\mathbf{B}$, since it is in the $-\widehat{\mathbf{n}}_1$ direction as indicated in (7.1), and gives rise to a curvature drift whose velocity is

$$\mathbf{v}_C = -\frac{mv_{\parallel}^2}{qB^2}[(\widehat{\mathbf{B}} \cdot \nabla)\widehat{\mathbf{B}}] \times \mathbf{B} \tag{7.9}$$

Since $\mathbf{B} = B\widehat{\mathbf{B}}$ and writing $W_{\parallel} = mv_{\parallel}^2/2$ for the particle longitudinal kinetic energy, (7.8) and (7.9) can be written, respectively, as

$$\mathbf{F}_c = -\frac{2W_{\parallel}}{B^2}[(\mathbf{B} \cdot \nabla)\mathbf{B}]_{\perp} \tag{7.10}$$

$$\mathbf{v}_C = -\frac{2W_{\parallel}}{qB^4}[(\mathbf{B} \cdot \nabla)\mathbf{B}] \times \mathbf{B} \tag{7.11}$$

Thus, at each point, the curvature drift is perpendicular to the osculating plane of the magnetic field line, as shown in Fig. 17.

An electric current is associated with the curvature drift, since it is in opposite directions for particles of opposite sign. From (7.11) and from the definition of the electric current density, we obtain for the curvature drift current density

$$\mathbf{J}_C = \frac{1}{\delta V}\sum_i (q_i \mathbf{v}_{Ci}) = -\,2\Big(\frac{1}{\delta V}\sum_i W_{\parallel i}\Big)\frac{[(\mathbf{B} \cdot \nabla)\mathbf{B}] \times \mathbf{B}}{B^4} \tag{7.12}$$

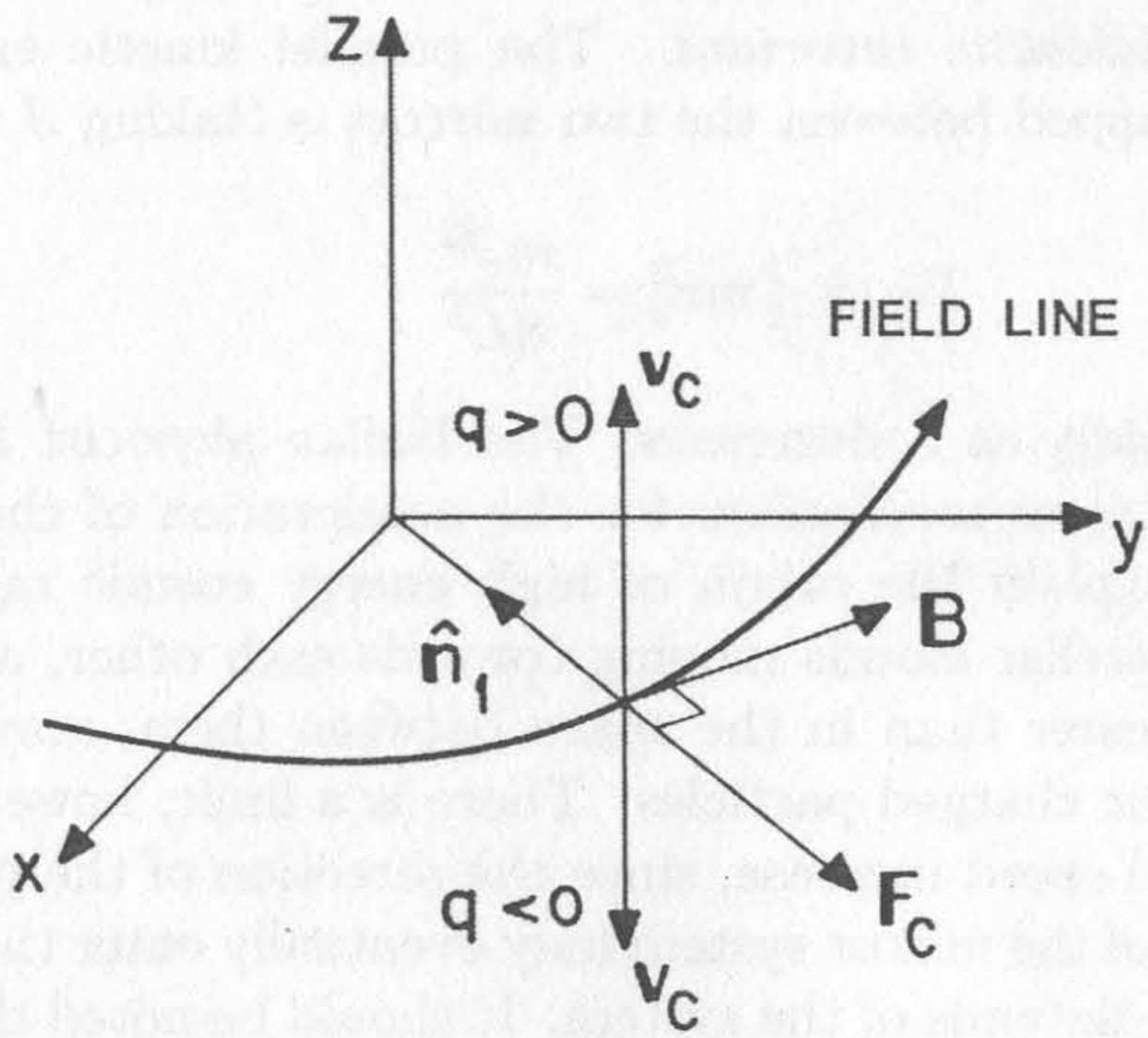


Fig. 17 *Relative direction of the particle guiding center drift velocity* $\mathbf{v}_C$*, due to the curvature of the magnetic field line.*

where the summation extends over all charged particles contained in the small volume element δV.

8. COMBINED GRADIENT-CURVATURE DRIFT

The curvature drift and the gradient drift always appear together and both point in the same direction, since the term ∇B points in the direction opposite to $\mathbf{F}_c$ (see Fig. 5). These two drifts can therefore be added up to form the combined gradient-curvature drift. Thus, from (5.1) and (7.11),

$$\mathbf{v}_{GC} = \mathbf{v}_G + \mathbf{v}_C = -\frac{\frac{1}{2}mv_\perp^2}{qB^3}(\nabla B) \times \mathbf{B} - \frac{mv_\parallel^2}{qB^4}[(\mathbf{B}\cdot\nabla)\mathbf{B}] \times \mathbf{B} \qquad (8.1)$$

When volume currents are not present (in a vacuum field, for example) so that $\nabla \times \mathbf{B} = 0$, the vector identity (4.26) allows the expression (8.1) to be written in the compact form

$$\mathbf{v}_{GC} = -\frac{m}{qB^4}\left(v_\parallel^2 + \tfrac{1}{2}v_\perp^2\right)\left(\nabla \tfrac{1}{2}B^2\right) \times \mathbf{B} \qquad (8.2)$$

In the Earth's magnetosphere, near the equatorial plane, both the curvature and the gradient drifts (B decreases with altitude) cause the

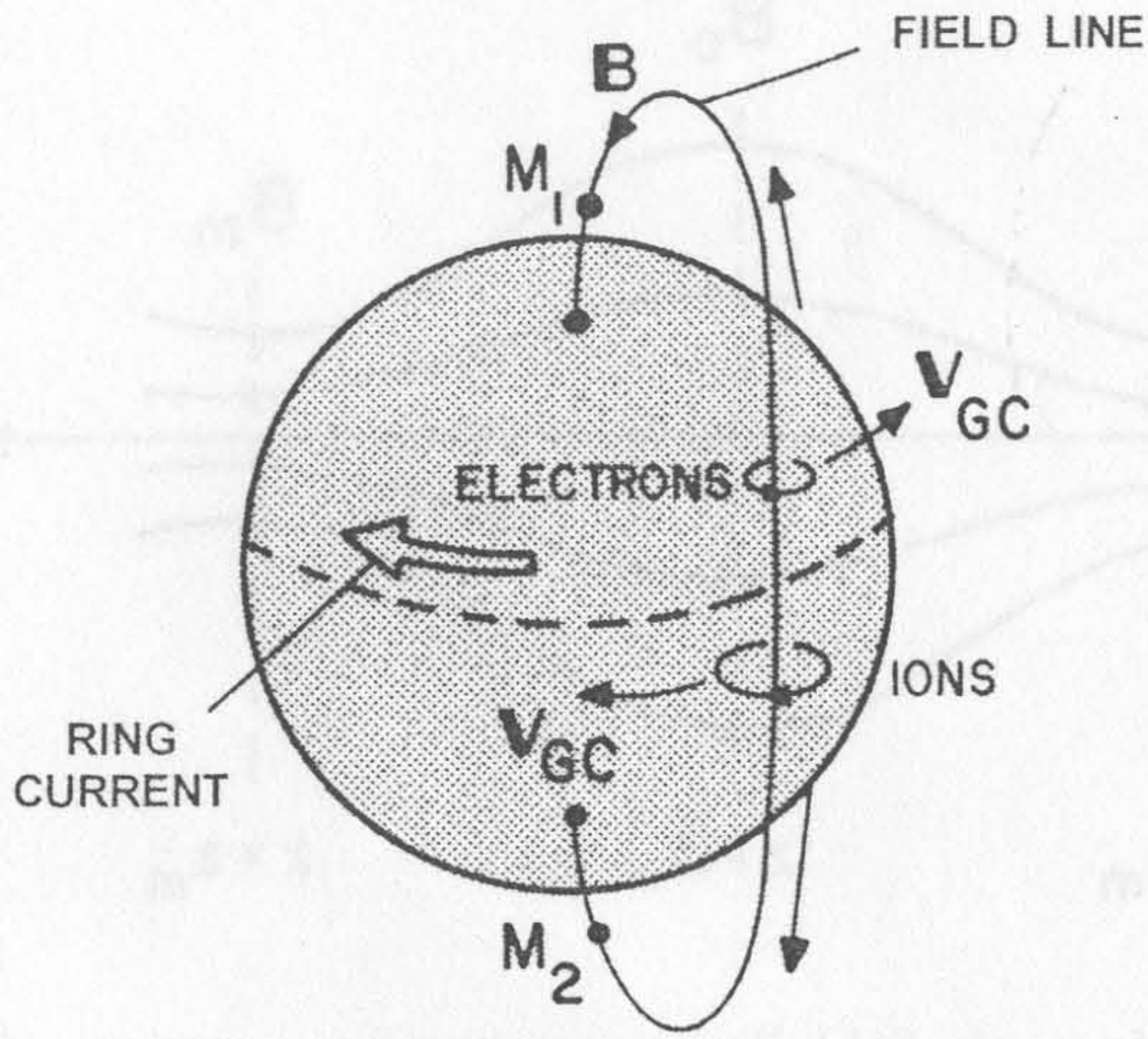


Fig. 18 *Sketch indicating the motion of a charged particle in the Earth's magnetic field. The longitudinal drift velocity* $\mathbf{v}_{GC}$, *due to the gradient and curvature of* $\mathbf{B}$, *results in an east to west current called the ring current.*

positively charged particles to slowly drift westward and the negative ones eastward, resulting in an east to west current, known as the *ring current.* Fig. 18 illustrates schematically the motion of a charged particle trapped in the Earth's magnetic field. The particle bounces back and forth along the field line between the mirror points M_1 and M_2, and drifts in longitude as a result of the gradient and curvature of the field lines. The trajectory described by the particle is therefore contained in a tire-shaped shell encircling the Earth (Fig. 19). This tire-shaped shell encircling the Earth defines a surface on which the particle guiding center drifts slowly around the Earth.

Connected with the periodic motion of the particle on this drift surface there is an adiabatic invariant, called the *third adiabatic invariant,* which is the total magnetic flux enclosed by the drift surface. Clearly, in a static situation this flux is obviously constant. The significant fact here is that the total magnetic flux, Φ_m, enclosed by the drift surface, remains invariant when the field varies slowly in time, that is, when the period of motion of the particle on the drift surface is small compared to the time

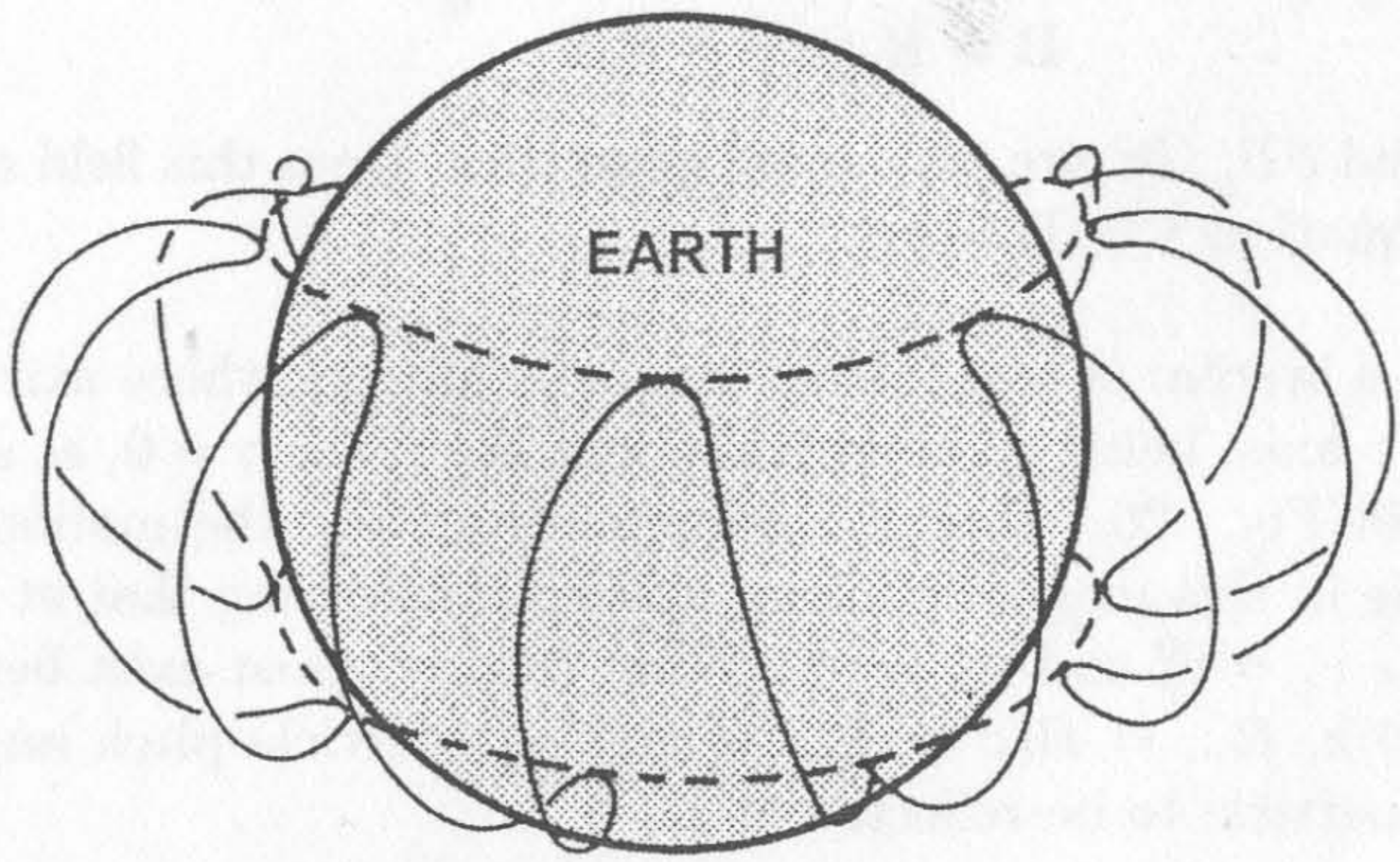


Fig. 19 *Schematic representation of the longitudinal drift of charged particles around the Earth.*

scale for the magnetic field to change significantly. This invariant has few applications because most fluctuations of **B** occur on a time scale short compared to the drift period.

PROBLEMS

3.1 Describe semiquantitatively the motion of an electron under the presence of a constant electric field in the x direction,

$$\mathbf{E} = E_0\widehat{\mathrm{x}}$$

and a space varying magnetic field given by

$$\mathbf{B} = B_0\alpha(x+z)\widehat{\mathrm{x}} + B_0[1+\alpha(x-z)]\widehat{\mathrm{z}}$$

where E_0, B_0 and α are positive constants, $|\alpha x| \ll 1$ and $|\alpha z| \ll 1$. Assume that initially the electron moves with constant velocity in the z direction, $\mathbf{v}(t=0) = v_0\widehat{\mathrm{z}}$.. Verify if this magnetic field satisfies the Maxwell equation $\nabla \times \mathbf{B} = 0$.

3.2 Verify if there is any drift velocity for a charged particle in a magnetic field given by

$$\mathbf{B} = B_y(x)\widehat{\mathbf{y}} + B_0\widehat{\mathbf{z}}$$

where $B_y(x)$ and $\partial B_y/\partial x$ are very small quantities. Does this field satisfy the Maxwell equation $\nabla \times \mathbf{B} = 0$?

3.3 Consider a system of two coaxial magnetic mirrors whose axis coincides with the z axis, being symmetrical about the plane $z = 0$, as shown schematically in Fig. 20. Describe semiquantitatively the motion of a charged particle in this magnetic mirror system considering that at $z = 0$ the particle has $v_\parallel = v_\parallel^0$ and $v_\perp = v_\perp^0$. What relation must exist between $\mathbf{B}_0 = B(z = 0)\widehat{\mathbf{z}}$, $\mathbf{B}_m = B(z = \pm z_m)\widehat{\mathbf{z}}$ and α_0 (particle pitch angle at $z = 0$) for the particle to be reflected at z_m ?

3.4 For the magnetic mirror system of Problem 3.3 suppose that the axial magnetic field changes in time, that is $\mathbf{B}_{axial} = B(z,t)\widehat{\mathbf{z}}$. Considering that the magnetic moment

$$|\mathbf{m}| = \frac{\frac{1}{2}mv_\perp^2(z,t)}{B(z,t)}$$

is an *adiabatic invariant* (note that its value is the same at $z = 0$ and at $z = \pm z_m$, and that $v^2 = v_\parallel^2 + v_\perp^2$), show that the *longitudinal adiabatic invariant* can be written in the form

$$\int_{-z_m}^{z_m} [B(z_m,t) - B(z,t)]^{1/2} dz = \text{constant}$$

3.5 Consider the magnetic mirror system shown in Fig. 20. Suppose that the axial magnetic field is given by

$$B(z) = B_0[1 + (z/a_0)^2]$$

where B_0 and a_0 are positive constants, and that the mirroring planes are given by $z = -z_m$ and $z = z_m$.
(a) For a charged particle trapped in this mirror system, show that the z component of the particle velocity is given by

$$v_\parallel(z) = \left(\frac{2\,|\mathbf{m}|\,B_0}{m}\right)^{1/2}\left[\left(\frac{z_m}{a_0}\right)^2 - \left(\frac{z}{a_0}\right)^2\right]^{1/2}$$

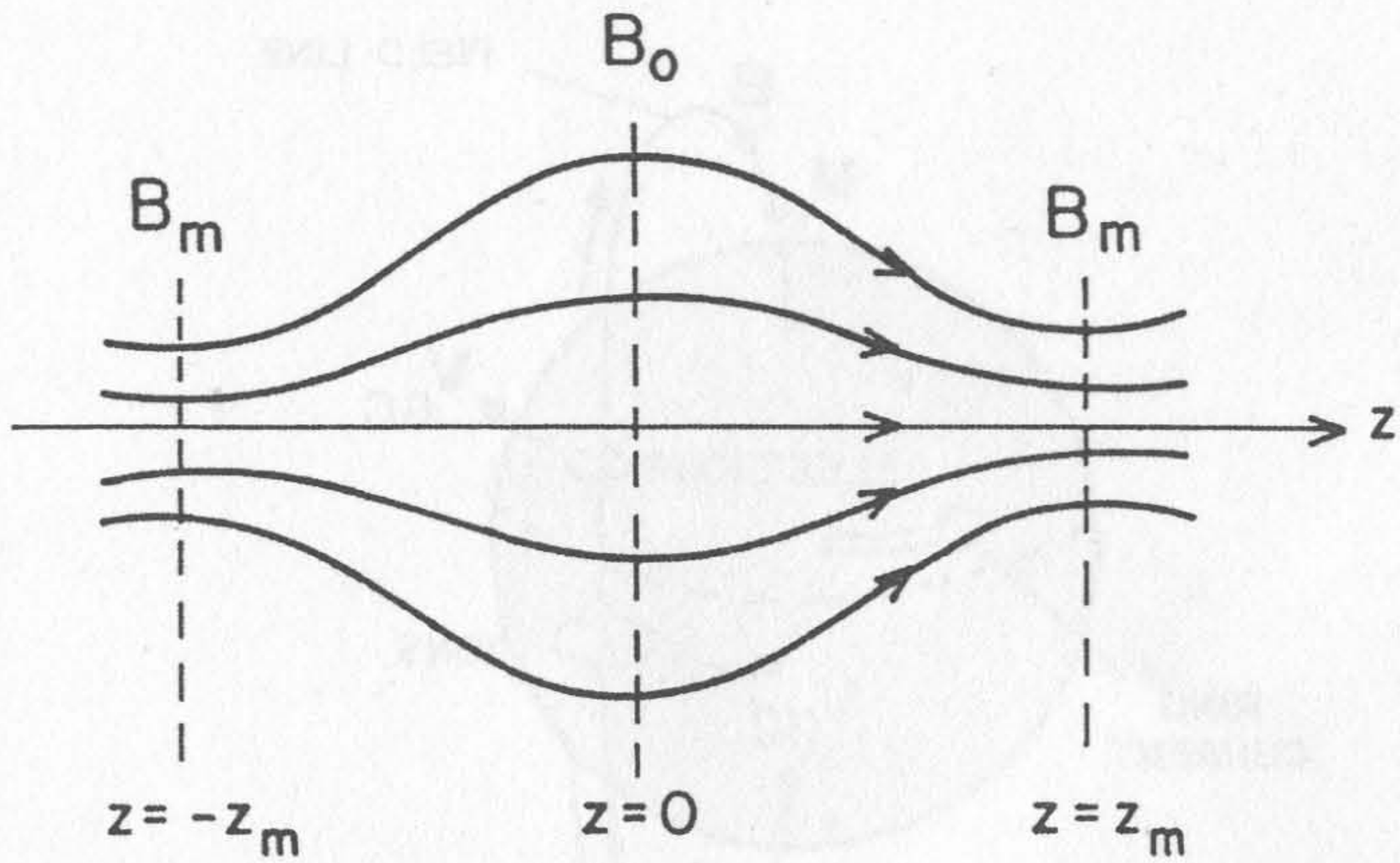


Fig. 20 *Magnetic field line geometry for a system of two coaxial magnetic mirrors whose axis coincides with the z axis, being symmetrical about the the plane $z = 0$.*

(b) The average force acting on the particle guiding center, along the z axis, is given by

$$< \mathbf{F}_{\parallel} > = -|\mathbf{m}| \left(\frac{\partial B}{\partial z}\right)\widehat{\mathbf{z}}$$

Show that the particle performs a simple harmonic motion between the mirroring planes, with a period given by

$$T = 2\pi a_0 \left(\frac{m}{2\,|\mathbf{m}|\,B_0}\right)^{1/2}$$

(c) If the motion of the particle is to be limited to the region $|z| < z_m$, what restriction must be imposed on the total energy and on the magnetic moment?

3.6 Consider a toroidal magnetic field, as shown in Fig. 21.
(a) Show that the magnetic flux density along the axis of the torus is given by

$$\mathbf{B} = B_a \left(\frac{a}{r}\right)\widehat{\mathbf{\Phi}}$$

where B_a denotes the magnitude of $\mathbf{B}$ at the radial distance $r = a$.
(b) In what direction is the gradient drift associated with the radial variation of B_ϕ? Examine qualitatively the type of charge separation that occurs. Neglect the effect of the magnetic field line curvature.

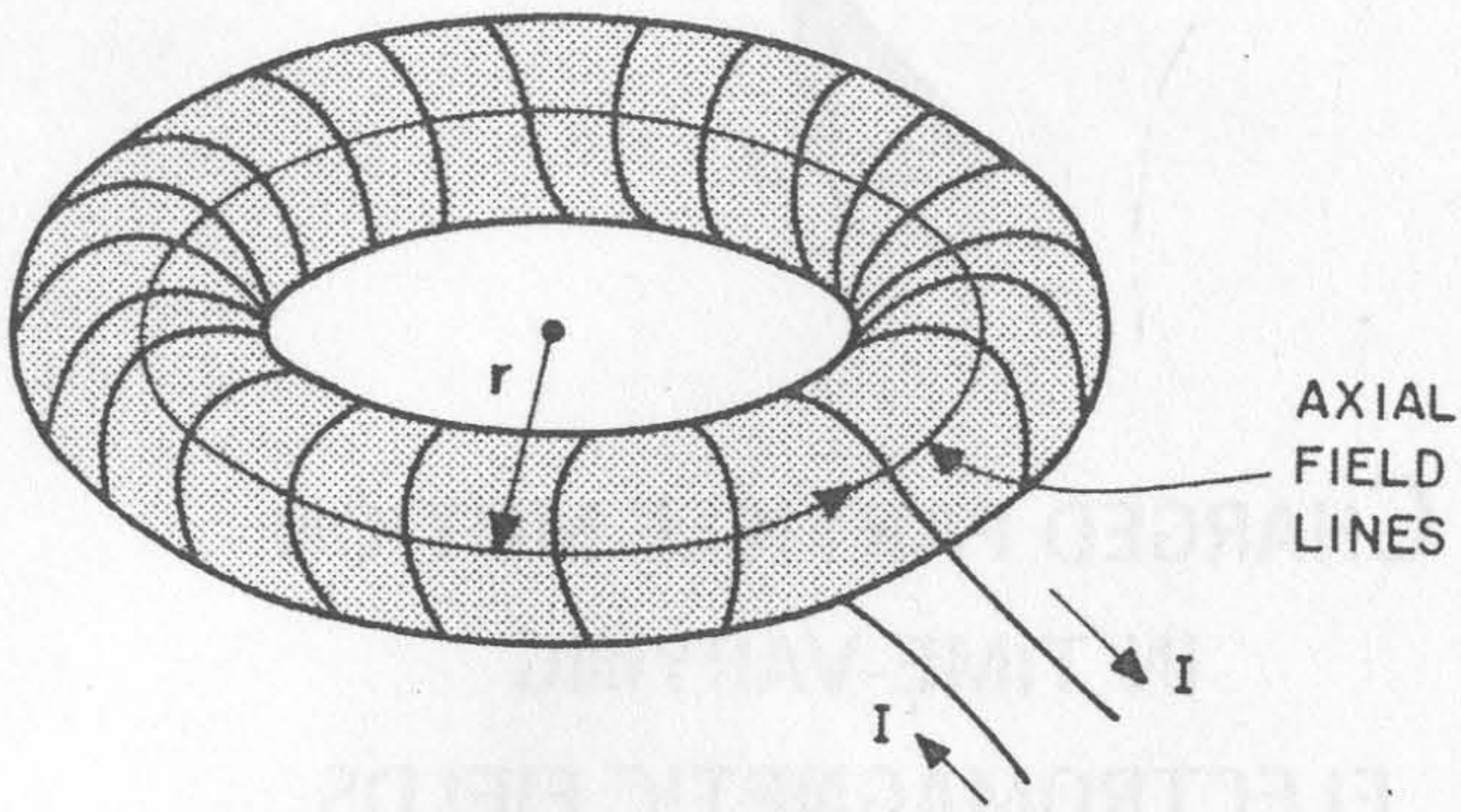


Fig. 21 *Magnetic field line with toroidal geometry.*

(c) If $\mathbf{E}$ denotes the induced electric field due to charge separation, in what direction is the $\mathbf{E} \times \mathbf{B}$ drift?
(d) Show that it is not possible to confine a plasma in a purely toroidal magnetic field, because of the gradient drift and the $\mathbf{E} \times \mathbf{B}$ drift.

3.7 Consider a spatially nonuniform magnetostatic field expressed in terms of a Cartesian coordinate system by

$$\mathbf{B}(x, z) = B_0[\alpha z\widehat{\mathbf{x}} + (1 + \alpha x)\widehat{\mathbf{z}}]$$

where B_0 and α are positive constants, $|\alpha x| \ll 1$ and $|\alpha z| \ll 1$.
(a) Show that this magnetic field is consistent with Maxwell equations, so that both gradient and curvature terms are present. Determine the equation of a magnetic flux line.
(b) Write down the Cartesian components of the equation of motion for an electron moving in the region near the origin under the action of this magnetic field.
(c) Consider the following initial conditions for the electron:

$$\mathbf{r}(0) = (x_0 + v_{\perp 0}/\Omega_c)\widehat{\mathbf{x}}$$

$$\mathbf{v}(0) = v_{\perp 0}\widehat{\mathbf{y}} + v_{z0}\widehat{\mathbf{z}}$$

Solve the equation of motion using a perturbation technique, retaining only terms up to the first order in the small parameter α. Show that

the leading terms in the velocity components, after eliminating the time periodic parts, are given by

$$\mathbf{v}_x = \alpha v_{z0}^2 \, t \, \widehat{\mathbf{x}}$$

$$\mathbf{v}_y = -(\alpha/\Omega_c)\left(\tfrac{1}{2}v_{\perp 0}^2 + v_{z0}^2\right)\widehat{\mathbf{y}}$$

$$\mathbf{v}_z = v_{z0}\widehat{\mathbf{z}}$$

(d) Show that the average position of the electron in the (x, z) plane follows the magnetic flux line that passes through its initial position.
(e) Show that the gradient and curvature drift velocities are given, respectively, by

$$\mathbf{v}_G = -(\alpha/\Omega_c)(\tfrac{1}{2}v_\perp^2)\widehat{\mathbf{y}}$$

$$\mathbf{v}_C = -(\alpha/\Omega_c)v_{z0}^2\widehat{\mathbf{y}}$$

so that the total drift velocity is precisely the nonperiodic part of $\mathbf{v}_y$.

3.8 The Earth's magnetic field can be represented, in a first approximation, by a magnetic dipole placed in the Earth's center, at least up to distances of a few Earth radii (R_E).
(a) Using the fact that, at one of the magnetic poles, the field has a magnitude of approximately 0.5 gauss near the surface, calculate the dipole magnetic moment.
(b) Consider the motion of an electron of energy E_0 at a radial distance r_0, where $r_0 > R_E$. Calculate its cyclotron frequency and gyroradius.
(c) Assuming that the electron is confined to move in the equatorial plane, calculate its gradient and curvature drift velocities, and determine the time it takes to drift once around the Earth, at the radial distance r_0.
(d) Calculate the period of the bounce motion of the electron, as it gets reflected back and forth between the magnetic mirrors near the poles. What is the altitude of the reflection points? Assume that $W_\parallel = W_\perp$ at the magnetic equatorial plane.
(e) Obtain numerical values for the results of items (b), (c) and (d), considering $E_0 = 1$ MeV and $r_0 = 4R_E$. Examine these results in terms of typical values for charged particles in the outer Van Allen radiation belt.
(f) Assuming that there is an isotropic population of 1 MeV protons and 100 keV electrons at about 4 R_E, each having a density $n_e = n_i = 10^7$ m^{-3} in the equatorial plane, calculate the ring current density in ampere/m^2.

3.9 Imagine an infinite straight wire carrying a current I and uniformly charged to a negative electrostatic potential ϕ. Analyse the motion of an

electron in the vicinity of this wire using first-order orbit theory. Sketch the path described by the electron, indicating the relative directions of the electromagnetic, gradient and curvature drift velocities.

3.10 The field of a *magnetic monopole* can be represented by

$$\mathbf{B}(\mathbf{r}) = \lambda \frac{\mathbf{r}}{r^3}$$

where λ is a constant. Solve the equation of motion to determine the trajectory of a charged particle in this field. (You may refer to the book by B. Rossi and S. Olbert, *Introduction to the Physics of Space*, Chapter 2, Mc Graw-Hill, 1970).

3.11 Analyze the motion of a charged particle in the field of a *magnetic dipole*. Determine the two constants of the motion and analyze their physical meaning. (For this problem, you may refer to the book by S. Stormer, *The Polar Aurora*, University Oxford Press, 1955, or to B. Rossi and S. Olbert, *Introduction to the Physics of Space*, Chapter 3, Mc Graw-Hill, 1970).

4

CHARGED PARTICLE MOTION IN TIME-VARYING ELECTROMAGNETIC FIELDS

1. INTRODUCTION

In this chapter we analyze the motion of charged particles in the presence of time-varying fields. Initially, in the two following sections, we consider a time-varying electric field and a constant magnetic field, both fields being spatially uniform. The assumption of a constant and spatially uniform **B** field is well justified if the externally applied magnetostatic field is much larger than the magnetic field associated with the time-varying **E** field. Also, the assumption of an electric field which is spatially uniform is valid if the charged particle cyclotron radius is much smaller than the scale length of the spatial variation of **E**. Both these requirements are assumed to hold in the analysis presented in sections 2 and 3. In section 4 we consider a time-varying magnetic field and the corresponding space-varying electric field.

2. SLOWLY TIME-VARYING ELECTRIC FIELD

2.1 Equation of Motion and Polarization Drift

For the moment we shall assume that the characteristic time scale for variation of the electric field is much larger than the particle cyclotron period. The component of the charged particle motion along the magnetic field lines is given by (2.5.4), from which we can write in general

$$\mathbf{v}_{\parallel}(t) - \mathbf{v}_{\parallel}(0) = \frac{q}{m}\int_0^t \mathbf{E}_{\parallel}(t')dt' \tag{2.1}$$

This result however does not lead to any new interesting information.

Since the $\mathbf{E}$ field is varying slowly in time, the component of the motion across the magnetic field lines is expected to be not very different from that for a constant $\mathbf{E}$ field. Therefore, it is reasonable to seek the solution for $\mathbf{v}_\perp$ in a form similar to (2.5.8). Hence, we take

$$\mathbf{v}_\perp = \mathbf{v}'_\perp + \mathbf{v}_E + \mathbf{v}_p \tag{2.2}$$

where $\mathbf{v}_E = \mathbf{E}\times\mathbf{B}/B^2$ is the electromagnetic plasma drift velocity (2.5.11). Note that $\mathbf{v}_E$ varies slowly in time, since $\mathbf{E}$ is slowly time-varying. The substitution of (2.2) into the perpendicular component equation of motion (2.5.5) yields

$$m\frac{d}{dt}(\mathbf{v}'_\perp + \mathbf{v}_E + \mathbf{v}_p) = q[\mathbf{E}_\perp + (\mathbf{v}'_\perp + \mathbf{v}_E + \mathbf{v}_p)\times\mathbf{B}] \tag{2.3}$$

From (2.5.11) $\mathbf{v}_E = (\mathbf{E}_\perp \times \mathbf{B})/B^2$, so that we can write (2.3) as

$$m\frac{d\mathbf{v}'_\perp}{dt} + m\frac{d}{dt}\left(\frac{\mathbf{E}_\perp\times\mathbf{B}}{B^2}\right) + m\frac{d\mathbf{v}_p}{dt} = q\mathbf{v}'_\perp\times\mathbf{B} + q\mathbf{v}_p\times\mathbf{B} \tag{2.4}$$

Thus, if we set

$$\mathbf{v}_p = \frac{m}{qB^2}\left(\frac{\partial\mathbf{E}_\perp}{\partial t}\right) \tag{2.5}$$

we can write (2.4) in the form

$$m\frac{d\mathbf{v}'_\perp}{dt} + m\frac{d\mathbf{v}_p}{dt} = q\mathbf{v}'_\perp\times\mathbf{B} \tag{2.6}$$

When the second term on the left-hand side can be neglected, this equation becomes identical to (2.5.12), which describes a circular motion about the magnetic field lines. Comparing the relative magnitudes of the second term on the left, with that on the right-hand side of (2.6), we find

$$\frac{|m\, d\mathbf{v}_p/dt|}{|q\mathbf{v}'_\perp\times\mathbf{B}|} = \frac{|(m^2/qB^2)(\partial^2 E_\perp/\partial t^2)|}{|qv'_\perp B|} =$$

$$= |(E_\perp/B)/v'_\perp|\,(\omega^2 m^2)/(q^2B^2) = |v_E/v'_\perp|\,(\omega/\Omega_c)^2 \tag{2.7}$$

where we have assumed that $\mathbf{E}_\perp$ has a harmonic time dependence with a characteristic angular frequency ω. Considering that this characteristic frequency is much smaller than the cyclotron frequency,

$$\omega \ll \Omega_c \tag{2.8}$$

and, further, if $|v_E/v'_\perp|$ is also small, then the term $m(d\mathbf{v}_p/dt)$ can be neglected in comparison to the other terms of (2.6), and we obtain

$$m\frac{d\mathbf{v}'_\perp}{dt} = q\mathbf{v}'_\perp \times \mathbf{B} \tag{2.9}$$

which is identical to (2.5.12). Therefore, $\mathbf{v}'_\perp$ corresponds to the usual circular motion of the charged particle about the magnetic field, and is independent of the variations of the electric field. Superposed upon this circular motion velocity are the drift velocities

$$\mathbf{v}_E = \frac{\mathbf{E}_\perp \times \mathbf{B}}{B^2} \tag{2.10}$$

and

$$\mathbf{v}_p = \frac{m}{qB^2}\left(\frac{\partial \mathbf{E}_\perp}{\partial t}\right) \tag{2.11}$$

Thus, the effect of a slowly varying electric field is the addition of the drift velocity $\mathbf{v}_p$ called the *polarization drift velocity.*

Since $\mathbf{v}_p$ is in opposite directions for charges of opposite sign, the time dependent electric field produces a net polarization current in a neutral plasma, so that the plasma medium behaves like a *dielectric.* The *polarization current density* $\mathbf{J}_p$ is the rate of flow of positive and negative charges across unit area, and is given by

$$\mathbf{J}_p = \frac{1}{\delta V}\sum_i q_i \mathbf{v}_{pi} = \left(\frac{1}{\delta V}\sum_i m_i\right)\frac{1}{B^2}\left(\frac{\partial \mathbf{E}_\perp}{\partial t}\right) = \frac{\rho_m}{B^2}\left(\frac{\partial \mathbf{E}_\perp}{\partial t}\right) \tag{2.12}$$

where the summation is over all positive and negative charges contained in the small volume element δV, and ρ_m is the mass density of the plasma.

2.2 Plasma Dielectric Constant

The polarization effect in a plasma is due to the time variation of the electric field. The application of a steady $\mathbf{E}$ field does not result in a polarization field, since the ions and electrons can move around to preserve quasineutrality. Since the plasma behaves like a dielectric, the polarization current density $\mathbf{J}_p$ can be taken into account through the introduction of the *dielectric constant* of the plasma.

For this purpose, we can separate the *total* current density $\mathbf{J}$ into the polarization current density $\mathbf{J}_p$ and the current density $\mathbf{J}_0$ due to other sources,

$$\mathbf{J} = \mathbf{J}_p + \mathbf{J}_0 \tag{2.13}$$

Thus, combining $\mathbf{J}_p$ with the term $\epsilon_0 \partial \mathbf{E}_\perp / \partial t$ which appears on the right-hand side of Maxwell $\nabla \times \mathbf{B}$ equation, we obtain

$$\epsilon_0 \frac{\partial \mathbf{E}_\perp}{\partial t} + \frac{\rho_m}{B^2} \frac{\partial \mathbf{E}_\perp}{\partial t} = \epsilon_0 \left(1 + \frac{\rho_m}{\epsilon_0 B^2}\right) \frac{\partial \mathbf{E}_\perp}{\partial t} = \epsilon \frac{\partial \mathbf{E}_\perp}{\partial t} \tag{2.14}$$

where

$$\epsilon = \epsilon_0 \epsilon_r = \epsilon_0 \left(1 + \frac{\rho_m}{\epsilon_0 B^2}\right) \tag{2.15}$$

is the *effective electric permittivity* perpendicular to the magnetic field. In some cases the *relative permittivity* ϵ_r of a plasma can be very high. As an example, if we consider a number density of 10^{20} *particles*$/m^3$ and $B = 1$ *tesla* we have $\epsilon_r = 10^4$.

The resulting charge density ρ_p, which accumulates as a result of the polarization current density $\mathbf{J}_p$, satisfies the charge continuity equation

$$\frac{\partial \rho_p}{\partial t} + \nabla \cdot \mathbf{J}_p = 0 \tag{2.16}$$

From (2.16) and (2.12) we have

$$\rho_p = -\frac{\rho_m}{B^2} \nabla \cdot \mathbf{E}_\perp \tag{2.17}$$

The total charge density ρ can be separated as

$$\rho = \rho_0 + \rho_p \tag{2.18}$$

where ρ_0 corresponds to $\mathbf{J}_0$. Assuming that the parallel component of the electric field vanishes, we see that

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}(\rho_0 + \rho_p) = \frac{\rho_0}{\epsilon_0} - \frac{\rho_m}{\epsilon_0 B^2} \nabla \cdot \mathbf{E} \tag{2.19}$$

from which we find, using (2.15),

$$\nabla \cdot \mathbf{E} = \frac{\rho_0}{\epsilon} \tag{2.20}$$

Thus, the resulting charge density ρ_p can also be correctly taken into account by the introduction of an effective electric permittivity ϵ.

We can further verify the correctness of introducing the effective electric permittivity of the plasma, by calculating the *total energy density*

associated with the **E** field, which for an ordinary dielectric medium of effective permittivity ϵ is given by $\epsilon E^2/2$. The *energy density* in the electric field is given by

$$W_E = \tfrac{1}{2}\epsilon_0 E^2 \tag{2.21}$$

To calculate the additional drift kinetic energy acquired by the particle as a result of the polarization drift, we note that for a change $\Delta\mathbf{E}_\perp$ in the electric field, in a time interval Δt, the displacement $\Delta\mathbf{r}$ of the guiding center, due to this change, is

$$\Delta\mathbf{r} = \mathbf{v}_p\Delta t = \frac{m}{qB^2}\Big(\frac{\partial\mathbf{E}_\perp}{\partial t}\Big)\Delta t = \frac{m}{qB^2}\Delta\mathbf{E}_\perp \tag{2.22}$$

The corresponding work done by the electric field is, using (2.22),

$$\Delta W = q\mathbf{E}_\perp\cdot(\Delta\mathbf{r}) = \frac{m}{B^2}\mathbf{E}_\perp\cdot(\Delta\mathbf{E}_\perp) = \Delta(\tfrac{1}{2}mE_\perp^2/B^2) \tag{2.23}$$

Hence, using (2.10), the change in the particle kinetic energy associated with the polarization drift is given by

$$\Delta W = \Delta(\tfrac{1}{2}mv_E^2) \tag{2.24}$$

This result shows that the work done by the electric field, during polarization, is equal to the change in the kinetic energy associated with the particle motion at the electromagnetic drift velocity $\mathbf{v}_E$. Note that $\mathbf{v}_E$ does not lead to any energy exchange between field and particle, since the displacement associated with $\mathbf{v}_E$ is perpendicular to the electric field. Summing (2.24) over all particles in a unit volume gives the change in the total kinetic energy density of the system

$$\Delta W_V = \Delta(\tfrac{1}{2}\rho_m v_E^2) = \Delta(\tfrac{1}{2}\rho_m E_\perp^2/B^2) \tag{2.25}$$

The kinetic energy density associated with the circular motion of the particles is not affected by changes in the electric field. Thus, the total energy density ($W_T = W_E + W_V$) associated with the electric field is

$$W_T = \tfrac{1}{2}\epsilon_0 E^2 + \tfrac{1}{2}\rho_m v_E^2 = \tfrac{1}{2}\epsilon_0 E^2\Big(1 + \frac{\rho_m}{\epsilon_0 B^2}\Big) = \tfrac{1}{2}\epsilon E^2 \tag{2.26}$$

assuming that there is no parallel component of the electric field. This result completes our discussion about the legitimacy of the introduction of an effective electric permittivity for the plasma.

3. ELECTRIC FIELD WITH ARBITRARY TIME VARIATION

3.1 Solution of the Equation of Motion

We consider now an arbitrary time variation of the electric field but, again, the field is spatially uniform. The applied magnetic field is static and uniform, as before. Without loss of generality, the time variation of $\mathbf{E}$ can be assumed to be harmonic with angular frequency ω,

$$\mathbf{E}(t) = \mathbf{E}_0 e^{-i\omega t} \tag{3.1}$$

where the complex amplitude $\mathbf{E}_0$ is independent of time. According to the usual convention, only the real part of this expression is to be taken for the physical interpretation of the results. An arbitrary time variation of $\mathbf{E}$ can be written as a superposition of terms similar to (3.1), corresponding to all possible values of ω, since the equation of motion (2.5.5) is linear. Using (3.1), the equation of motion becomes

$$m\frac{d\mathbf{v}}{dt} = q(\mathbf{E}_0 e^{-i\omega t} + \mathbf{v} \times \mathbf{B}) \tag{3.2}$$

It is natural to expect the forced oscillations of the charged particle to have the same frequency as that of the forcing electric field. Thus, the particle velocity vector may be conveniently decomposed into two parts

$$\mathbf{v} = \mathbf{v}_m + \mathbf{v}_e e^{-i\omega t} \tag{3.3}$$

where $\mathbf{v}_m$ is the velocity associated with the magnetic field alone and, thus, contains no time variation at the angular frequency ω, while $\mathbf{v}_e$ is due to the oscillating electric field. The substitution of (3.3) into (3.2) gives

$$m\frac{d\mathbf{v}_m}{dt} - i\omega m\mathbf{v}_e e^{-i\omega t} = q(\mathbf{E}_0 e^{-i\omega t} + \mathbf{v}_m \times \mathbf{B} + \mathbf{v}_e \times \mathbf{B}e^{-i\omega t}) \tag{3.4}$$

The terms containing the periodicity at the angular frequency ω are decoupled from those that do not, so that (3.4) separates into two equations, one involving only the velocity $\mathbf{v}_m$ associated with the magnetic field according to

$$m\frac{d\mathbf{v}_m}{dt} = q\mathbf{v}_m \times \mathbf{B} \tag{3.5}$$

and the other one which involves only the velocity $\mathbf{v}_e$,

$$-i\omega m\mathbf{v}_e = q(\mathbf{E}_0 + \mathbf{v}_e \times \mathbf{B}) \tag{3.6}$$

Equation (3.5) corresponds to the usual circular motion of the particle about the magnetic field lines at the cyclotron frequency $\mathbf{\Omega}_c$.

To solve (3.6) for $\mathbf{v}_e$ it is appropriate to separate this equation into components parallel and perpendicular to $\mathbf{B}$. The *parallel* velocity component is obtained immediately as

$$\mathbf{v}_{e\parallel} = \frac{iq}{\omega m}\mathbf{E}_{0\parallel} \tag{3.7}$$

while the *perpendicular* velocity component satisfies the equation

$$\left(-i\omega + \frac{q}{m}\mathbf{B}\times\right)\mathbf{v}_{e\perp} = \frac{q}{m}\mathbf{E}_{0\perp} \tag{3.8}$$

Introducing the cyclotron frequency vector, defined by

$$\mathbf{\Omega}_c = -\frac{q}{m}\mathbf{B} \tag{3.9}$$

we can rewrite (3.8) in the form

$$(i\omega + \mathbf{\Omega}_c\times)\mathbf{v}_{e\perp} = -\frac{q}{m}\mathbf{E}_{0\perp} \tag{3.10}$$

In order to solve this equation for $\mathbf{v}_{e\perp}$, we multiply both sides by the conjugate operator $-(i\omega - \mathbf{\Omega}_c\times)$. First, we note that

$$\begin{aligned}(i\omega - \mathbf{\Omega}_c\times)(i\omega + \mathbf{\Omega}_c\times)\mathbf{v}_{e\perp} &= (i\omega)^2\mathbf{v}_{e\perp} - \mathbf{\Omega}_c \times (\mathbf{\Omega}_c \times \mathbf{v}_{e\perp}) = \\ &= (\Omega_c^2 - \omega^2)\mathbf{v}_{e\perp}\end{aligned} \tag{3.11}$$

Therefore, (3.10) becomes

$$\mathbf{v}_{e\perp} = \frac{q}{m}\frac{(i\omega - \mathbf{\Omega}_c\times)}{(\omega^2 - \Omega_c^2)}\mathbf{E}_{0\perp} \tag{3.12}$$

Combining the results contained in (3.12), (3.7) and (3.5), we obtain the following expresion for the total velocity vector (3.3)

$$\mathbf{v} = \mathbf{v}_m + \frac{q}{m}\left[\frac{i}{\omega}\mathbf{E}_{0\parallel} + \frac{(i\omega - \mathbf{\Omega}_c\times)}{(\omega^2 - \Omega_c^2)}\mathbf{E}_{0\perp}\right]e^{-i\omega t} \tag{3.13}$$

3.2 Physical Interpretation

The result contained in (3.13) shows that *along* the magnetic field lines the particle oscillates with frequency ω and amplitude $\mathbf{v}_{e\parallel}$, as given by (3.7), so that the velocity oscillation *lags* 90 degrees behind the oscillation of the applied electric field. This result is easily seen by taking the *real part* of the vectors $\mathbf{v}_{e\parallel}e^{-i\omega t}$ and $\mathbf{E}_{0\parallel}e^{-i\omega t}$,

$$\Re\{\mathbf{v}_{e\parallel}e^{-i\omega t}\} = \frac{q}{\omega m}\mathbf{E}_{0\parallel}\,\sin(\omega t) \tag{3.14}$$

$$\Re\{\mathbf{E}_{0\parallel}e^{-i\omega t}\} = \mathbf{E}_{0\parallel}\,\cos(\omega t) \tag{3.15}$$

which clearly are 90 degrees out of phase.

In the plane *perpendicular* to $\mathbf{B}$ the particle motion is the superposition of the circular motion at the cyclotron frequency $\mathbf{\Omega}_c$, with an oscillation at the frequency ω and amplitude given by (3.12).

In order to analyze the *physical meaning* of the motion in the plane *perpendicular* to $\mathbf{B}$, it is convenient to decompose the oscillating electric field vector into two *circularly polarized* components, with opposite directions of rotation. The advantage of using the two circularly polarized components is that (3.12), for the perpendicular velocity component $\mathbf{v}_{e\perp}$, uncouples into two separate equations pertaining to the two circular polarizations rotating in opposite directions. Thus, we take

$$\mathbf{E}_\perp = \mathbf{E}_R + \mathbf{E}_L = (\mathbf{E}_{0R} + \mathbf{E}_{0L})e^{-i\omega t} \tag{3.16}$$

with

$$\mathbf{E}_{0R} = \tfrac{1}{2}(1 + i\widehat{\mathbf{B}}\times)\mathbf{E}_{0\perp} \tag{3.17}$$

$$\mathbf{E}_{0L} = \tfrac{1}{2}(1 - i\widehat{\mathbf{B}}\times)\mathbf{E}_{0\perp} \tag{3.18}$$

where $\widehat{\mathbf{B}} = \mathbf{B}/B$ is a unit vector pointing in the direction of the magnetic field. The component $\mathbf{E}_R$ represents a circularly polarized field with its electric vector rotating to the *right* (*clockwise* direction), and the component $\mathbf{E}_L$ represents a circularly polarized field with its electric vector rotating to the *left* (*counterclockwise* direction), as seen by an observer looking in the *direction* of the $\mathbf{B}$ field.

To understand the physical meaning of this decomposition, let us consider a Cartesian coordinate system with the z axis pointing along $\mathbf{B}$ and the x axis pointing along $\mathbf{E}_\perp$. Then, we have

$$\widehat{\mathbf{B}} \times \mathbf{E}_{0\perp} = \widehat{\mathbf{z}} \times (\widehat{\mathbf{x}}E_{0\perp}) = \widehat{\mathbf{y}}E_{0\perp} \tag{3.19}$$

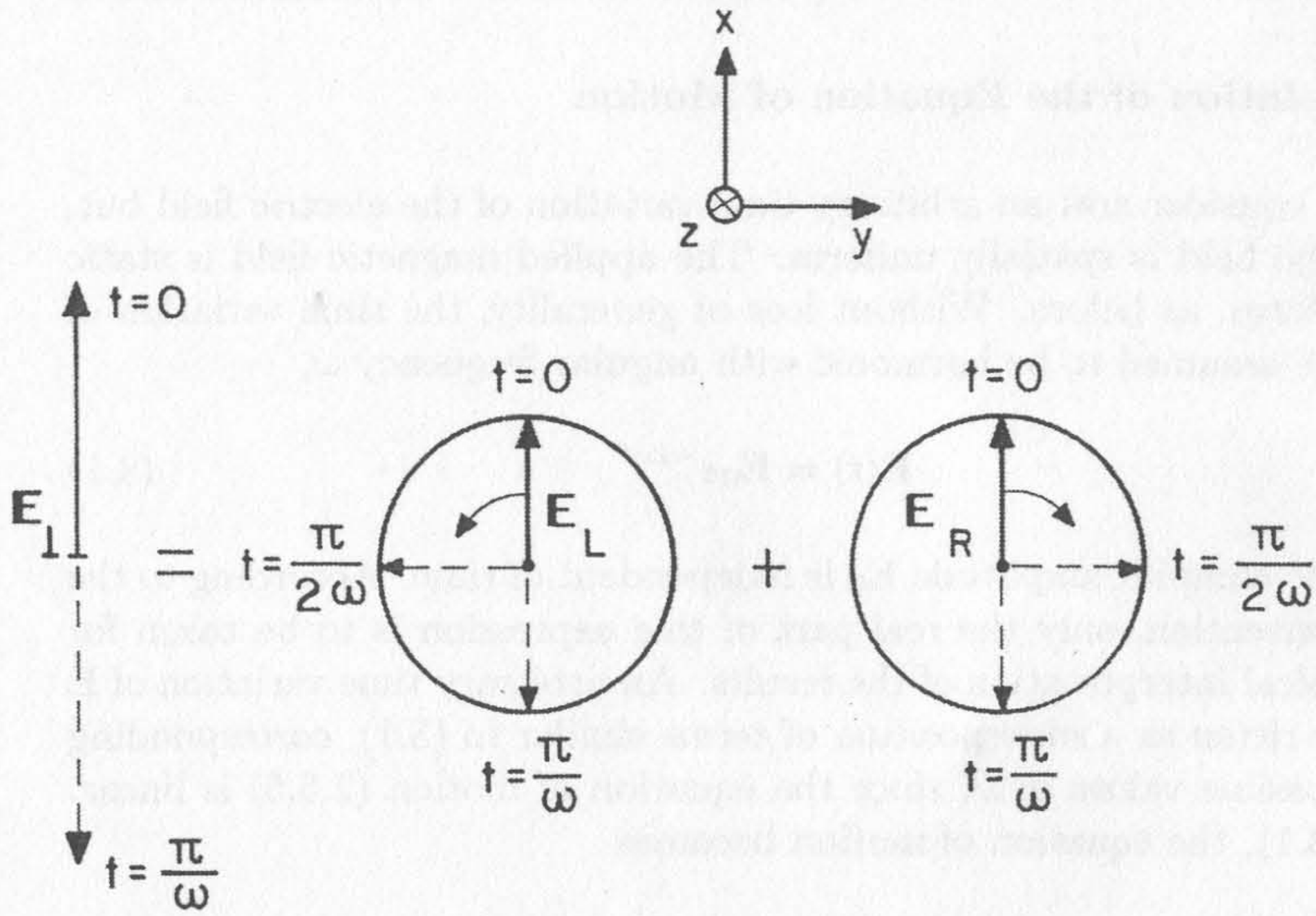


Fig. 1 *The electric field of a plane polarized wave, represented by the double-headed arrow* $\mathbf{E}_\perp$*, is equivalent to the vector sum of left and right circularly polarized waves* $\mathbf{E}_L$ *and* $\mathbf{E}_R$*.*

and from (3.17) and (3.18),

$$\mathbf{E}_R = \mathbf{E}_{0R}e^{-i\omega t} = \tfrac{1}{2}E_{0\perp}(\widehat{\mathbf{x}} + i\widehat{\mathbf{y}})e^{-i\omega t} \tag{3.20}$$

$$\mathbf{E}_L = \mathbf{E}_{0L}e^{-i\omega t} = \tfrac{1}{2}E_{0\perp}(\widehat{\mathbf{x}} - i\widehat{\mathbf{y}})e^{-i\omega t} \tag{3.21}$$

The components of the actual electric field are obtained by taking the *real part* of these two equations,

$$\Re\{\mathbf{E}_R\} = \tfrac{1}{2}E_{0\perp}(\widehat{\mathbf{x}}\cos\omega t + \widehat{\mathbf{y}}\sin\omega t) \tag{3.22}$$

$$\Re\{\mathbf{E}_L\} = \tfrac{1}{2}E_{0\perp}(\widehat{\mathbf{x}}\cos\omega t - \widehat{\mathbf{y}}\sin\omega t) \tag{3.23}$$

Thus, the fields $\mathbf{E}_R$ and $\mathbf{E}_L$ are constant in magnitude, but sweep around in a circle at the frequency ω. For $\mathbf{E}_R$ the rotation is *clockwise* looking in the direction of $\mathbf{B}$, and it is called a *right circularly polarized* (RCP) component, while for $\mathbf{E}_L$ the rotation is *counterclockwise* and it is called a *left circularly polarized* (LCP) component. This behavior is illustrated in Fig. 1.

Proceeding in the analysis of the particle motion in the perpendicular plane, we must now substitute the decomposed form (3.16) into (3.12). The operator $(i\omega - \mathbf{\Omega}_c\times)$ when applied on the right circularly polarized component $\mathbf{E}_{0R}$ gives

$$(i\omega - \mathbf{\Omega}_c\times)\mathbf{E}_{0R} = \tfrac{1}{2}(i\omega - \mathbf{\Omega}_c\times)(1 + i\widehat{\mathbf{B}}\times)\mathbf{E}_{0\perp} =$$

$$= \tfrac{1}{2}i\omega(\mathbf{E}_{0\perp} + i\widehat{\mathbf{B}}\times\mathbf{E}_{0\perp}) + \tfrac{1}{2}(qB/m)\widehat{\mathbf{B}}\times\mathbf{E}_{0\perp} - \tfrac{1}{2}i(qB/m)\mathbf{E}_{0\perp} =$$

$$= \tfrac{1}{2}i\omega(\mathbf{E}_{0\perp} + i\widehat{\mathbf{B}}\times\mathbf{E}_{0\perp}) - \tfrac{1}{2}i(qB/m)(\mathbf{E}_{0\perp} + i\widehat{\mathbf{B}}\times\mathbf{E}_{0\perp}) =$$

$$= i(\omega - qB/m)\mathbf{E}_{0R} \tag{3.24}$$

In a similar fashion we find for the left circularly polarized component,

$$(i\omega - \mathbf{\Omega}_c\times)\mathbf{E}_{0L} = i(\omega + qB/m)\mathbf{E}_{0L} \tag{3.25}$$

Therefore, both $\mathbf{E}_{0R}$ and $\mathbf{E}_{0L}$ are *eigenvectors* of the complex operator appearing in (3.12). Using (3.24) and (3.25), (3.12) becomes

$$\mathbf{v}_{e\perp} = i\left(\frac{q}{m}\right)\left[\frac{1}{(\omega \pm \Omega_c)}\mathbf{E}_{0R} + \frac{1}{(\omega \mp \Omega_c)}\mathbf{E}_{0L}\right] \tag{3.26}$$

In this equation the signs are coupled such that the *upper* signs apply to a *positively* charged particle and the *lower* signs to a *negatively* charged particle, since by definition $\Omega_c = |q|\ B/m$ is always positive. Thus, $\mathbf{v}_{e\perp}$ also separates into two vectors rotating in opposite directions,

$$\mathbf{v}_{e\perp} = \mathbf{v}_R + \mathbf{v}_L \tag{3.27}$$

where

$$\mathbf{v}_R = i\left(\frac{q}{m}\right)\frac{1}{(\omega \pm \Omega_c)}\mathbf{E}_{0R} \tag{3.28}$$

$$\mathbf{v}_L = i\left(\frac{q}{m}\right)\frac{1}{(\omega \mp \Omega_c)}\mathbf{E}_{0L} \tag{3.29}$$

For a *positive* ion we see that, as ω approaches the ion cyclotron frequency (Ω_{ci}), there is resonance between the ion and the *left* circularly polarized component of the electric field. For an *electron*, as ω approaches the electron cyclotron frequency (Ω_{ce}), resonance occurs with the *right* circularly polarized component.

3.3 Mobility Dyad

The expression for $\mathbf{v}_e = \mathbf{v}_{e\parallel} + \mathbf{v}_R + \mathbf{v}_L$ can be written in a compact form through the introduction of the *mobility dyad* $\mathcal{M}$ defined by

$$\mathbf{v}_e = \mathcal{M} \cdot \mathbf{E}_0 \tag{3.30}$$

Using (3.7), (3.28) and (3.29) we see that the mobility dyad $\mathcal{M}$ is diagonal in the rotating system, so that (3.30) becomes in matrix form,

$$\begin{pmatrix} v_R \\ v_L \\ v_{e\parallel} \end{pmatrix} = \frac{iq}{\omega m} \begin{pmatrix} \dfrac{\omega}{(\omega \pm \Omega_c)} & 0 & 0 \\ 0 & \dfrac{\omega}{(\omega \mp \Omega_c)} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E_{0R} \\ E_{0L} \\ E_{0\parallel} \end{pmatrix} \tag{3.31}$$

If, instead of the rotating system, we use a stationary Cartesian coordinate system with the z axis pointing along the magnetostatic field, we can write (3.7) and (3.12) in matrix form as

$$\begin{pmatrix} v_{ex} \\ v_{ey} \\ v_{ez} \end{pmatrix} = \frac{iq}{\omega m} \begin{pmatrix} \dfrac{\omega^2}{(\omega^2 - \Omega_c^2)} & \pm\dfrac{i\omega\Omega_c}{(\omega^2 - \Omega_c^2)} & 0 \\ \mp\dfrac{i\omega\Omega_c}{(\omega^2 - \Omega_c^2)} & \dfrac{\omega^2}{(\omega^2 - \Omega_c^2)} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E_{0x} \\ E_{0y} \\ E_{0z} \end{pmatrix} \tag{3.32}$$

3.4 Plasma Conductivity Dyad

Denoting by n_0 the number density of electrons (charge $-e$) and ions (charge $+e$) in a plasma, the electric current density can be expressed as

$$\mathbf{J} = -n_0 e \mathbf{v}_e + n_0 e \mathbf{v}_i = n_0 e(\mathcal{M}_i - \mathcal{M}_e) \cdot \mathbf{E} \tag{3.33}$$

where $\mathcal{M}_e$ and $\mathcal{M}_i$ are the mobility dyads for the electrons and ions, respectively. Introducing the conductivity dyad $\mathcal{S}$ by

$$\mathbf{J} = \mathcal{S} \cdot \mathbf{E} = (\mathcal{S}_e + \mathcal{S}_i) \cdot \mathbf{E} \tag{3.34}$$

we obtain from (3.33), for the electron and ion conductivities, respectively,

$$\mathcal{S}_e = -n_0 e \mathcal{M}_e \tag{3.35}$$

$$\mathcal{S}_i = n_0 e \mathcal{M}_i \tag{3.36}$$

With the help of (3.32), these conductivities can be expressed in matrix form, in a Cartesian coordinate system, as

$$\mathcal{S}_e = \frac{in_0e^2}{\omega m_e}\begin{pmatrix} \dfrac{\omega^2}{(\omega^2-\Omega_{ce}^2)} & -\dfrac{i\omega\Omega_{ce}}{(\omega^2-\Omega_{ce}^2)} & 0 \\ \dfrac{i\omega\Omega_{ce}}{(\omega^2-\Omega_{ce}^2)} & \dfrac{\omega^2}{(\omega^2-\Omega_{ce}^2)} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.37}$$

$$\mathcal{S}_i = \frac{in_0e^2}{\omega m_i}\begin{pmatrix} \dfrac{\omega^2}{(\omega^2-\Omega_{ci}^2)} & \dfrac{i\omega\Omega_{ci}}{(\omega^2-\Omega_{ci}^2)} & 0 \\ -\dfrac{i\omega\Omega_{ci}}{(\omega^2-\Omega_{ci}^2)} & \dfrac{\omega^2}{(\omega^2-\Omega_{ci}^2)} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.38}$$

The fact that the conductivity dyad $\mathcal{S}$ is *imaginary* implies that $\mathbf{J}$ and $\mathbf{E}$ are 90 degrees out of *phase*, since the actual physical expressions for $\mathbf{J}$ and $\mathbf{E}$ are obtained by taking the real part of (3.34) and (3.1), respectively.

3.5 Cyclotron Resonance

The particle velocities, given in (3.7), (3.28) and (3.29), do not represent correctly the motion of the particle when the frequency ω of the applied electric field is equal to the particle cyclotron frequency Ω_c. For example, when the forcing electric field rotates in the counterclockwise direction, looking along $\mathbf{B}$ (LCP component), a positive particle is able to absorb energy from the electric field, so that its speed increases continuouslly and indefinitely in time. The same holds for an electron, say, and the RCP component of the forcing electric field when $\omega = \Omega_{ce}$. This phenomenon is called *cyclotron resonance.*

To investigate the particle motion under resonance conditions, it is necessary to go back to the original equation of motion and solve the problem for the case when $\omega = \Omega_c$. For simplicity, let us assume that the component of the $\mathbf{E}$ field along $\mathbf{B}$ vanishes, that is, $\mathbf{E} = \mathbf{E}_\perp$. Hence, we take

$$\mathbf{E} = \mathbf{E}_{0\perp}e^{-i\Omega_c t} \tag{3.39}$$

where $\Omega_c = |q|\ B/m$. In view of this assumption, the particle velocity along $\mathbf{B}$ is constant and is equal to the initial parallel velocity. The component of the equation of motion in the plane *normal* to the magnetic field becomes

$$\frac{d\mathbf{v}_\perp(t)}{dt} = \frac{q}{m}[\mathbf{E}_{0\perp}e^{-i\Omega_c t} + \mathbf{v}_\perp(t)\times\mathbf{B}] \tag{3.40}$$

Taking the derivative of this equation with respect to time, yields

$$\frac{d^2\mathbf{v}_\perp}{dt^2} = \frac{q}{m}\left(-i\Omega_c\mathbf{E}_{0\perp}e^{-i\Omega_c t} + \frac{d\mathbf{v}_\perp}{dt}\times\mathbf{B}\right) \tag{3.41}$$

and using (3.40) to eliminate $(d\mathbf{v}_\perp/dt)$ in (3.41), we obtain, after rearranging the terms,

$$\frac{d^2\mathbf{v}_\perp}{dt^2} + \Omega_c^2\mathbf{v}_\perp = -i\Omega_c\frac{q}{m}\mathbf{E}_{0\perp}e^{-i\Omega_c t} - \frac{q^2}{m^2}(\mathbf{B}\times\mathbf{E}_{0\perp})e^{-i\Omega_c t} \tag{3.42}$$

The solution of this *inhomogeneous* differential equation is given by the sum of the solution of the *homogeneous* equation plus a *particular integral* of the inhomogeneous equation. The solution of the homogeneous equation

$$\frac{d^2\mathbf{v}_\perp}{dt^2} + \Omega_c^2\mathbf{v}_\perp = 0 \tag{3.43}$$

is just the cyclotron motion described previously by the velocity $\mathbf{v}_m$. A particular integral of (3.42) is provided by the function .

$$\mathbf{v}_\perp = \mathbf{A}\ t\ e^{-i\Omega_c t} \tag{3.44}$$

To determine the constant vector $\mathbf{A}$ we differentiate (3.44) twice with respect to time

$$\frac{d\mathbf{v}_\perp}{dt} = \mathbf{A}(1 - i\Omega_c t)e^{-i\Omega_c t} \tag{3.45}$$

$$\frac{d^2\mathbf{v}_\perp}{dt^2} = -2i\Omega_c\mathbf{A}e^{-i\Omega_c t} - \Omega_c^2\mathbf{v}_\perp \tag{3.46}$$

Comparing (3.46) with (3.42) we see that (3.44) satisfies (3.42) provided we take

$$\mathbf{A} = \frac{q}{2m}\mathbf{E}_{0\perp} - \frac{iq^2}{2\Omega_c m^2}(\mathbf{B}\times\mathbf{E}_{0\perp}) \tag{3.47}$$

Therefore, the complete solution of (3.42) is

$$\mathbf{v}_\perp = \mathbf{v}_m + \frac{q}{2m}(\mathbf{E}_{0\perp} \mp i\widehat{\mathbf{B}}\times\mathbf{E}_{0\perp})\ t\ e^{-i\Omega_c t} \tag{3.48}$$

where we have replaced qB/m by $\pm\Omega_c$, the upper and lower signs corresponding to positive and negative charges, respectively. Using (3.17) and (3.18), which define the right and left circularly polarized components of

the electric field, respectively, we can write (3.48) for a *positively* charged particle ($q > 0$) as

$$\mathbf{v}_\perp = \mathbf{v}_m + \frac{q}{m}\mathbf{E}_{0L}\, t\, e^{-i\Omega_c t} \tag{3.49}$$

and for a *negatively* charged particle ($q < 0$) as

$$\mathbf{v}_\perp = \mathbf{v}_m + \frac{q}{m}\mathbf{E}_{0R}\, t\, e^{-i\Omega_c t} \tag{3.50}$$

Hence, the particle velocity increases indefinitely with time. Note that the expressions $(q/m)\mathbf{E}_{0L}$ and $(q/m)\mathbf{E}_{0R}$ represent a constant acceleration. A positive charge resonates with $\mathbf{E}_L$ and a negative one with $\mathbf{E}_R$. The particle moves in circles of ever increasing radii, with its velocity increasing continuouslly during this spiral motion at the expense of the electric field energy. A typical resonant spiral for an electron is shown in Fig. 2.

This phenomenon can be used as a method of increasing the particle speed and hence the kinetic temperature of a plasma through particle collisions. This method is known as *radio frequency heating* of the plasma by *cyclotron resonance.*

4. TIME-VARYING MAGNETIC FIELD AND SPACE-VARYING ELECTRIC FIELD

From Maxwell equations it is seen that a time-varying magnetic field is also accompanied by a space-varying electric field. The $\mathbf{E}$ field associated with the time-dependent $\mathbf{B}$ field satisfies

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{4.1}$$

We shall assume that the fractional change in the magnetic field, in a time interval of the order of the cyclotron period, is very small.

4.1 Equation of Motion and Adiabatic Invariants

Consider the magnetic field in the z direction and suppose that it is spatially uniform and increases with time within the circular orbit of the particle. From Faraday's law, an electric field is induced along the path of the particle orbit (see Fig. 3) which accelerates the particle, with the result that the orbit is no longer a circle. However, since the time variation

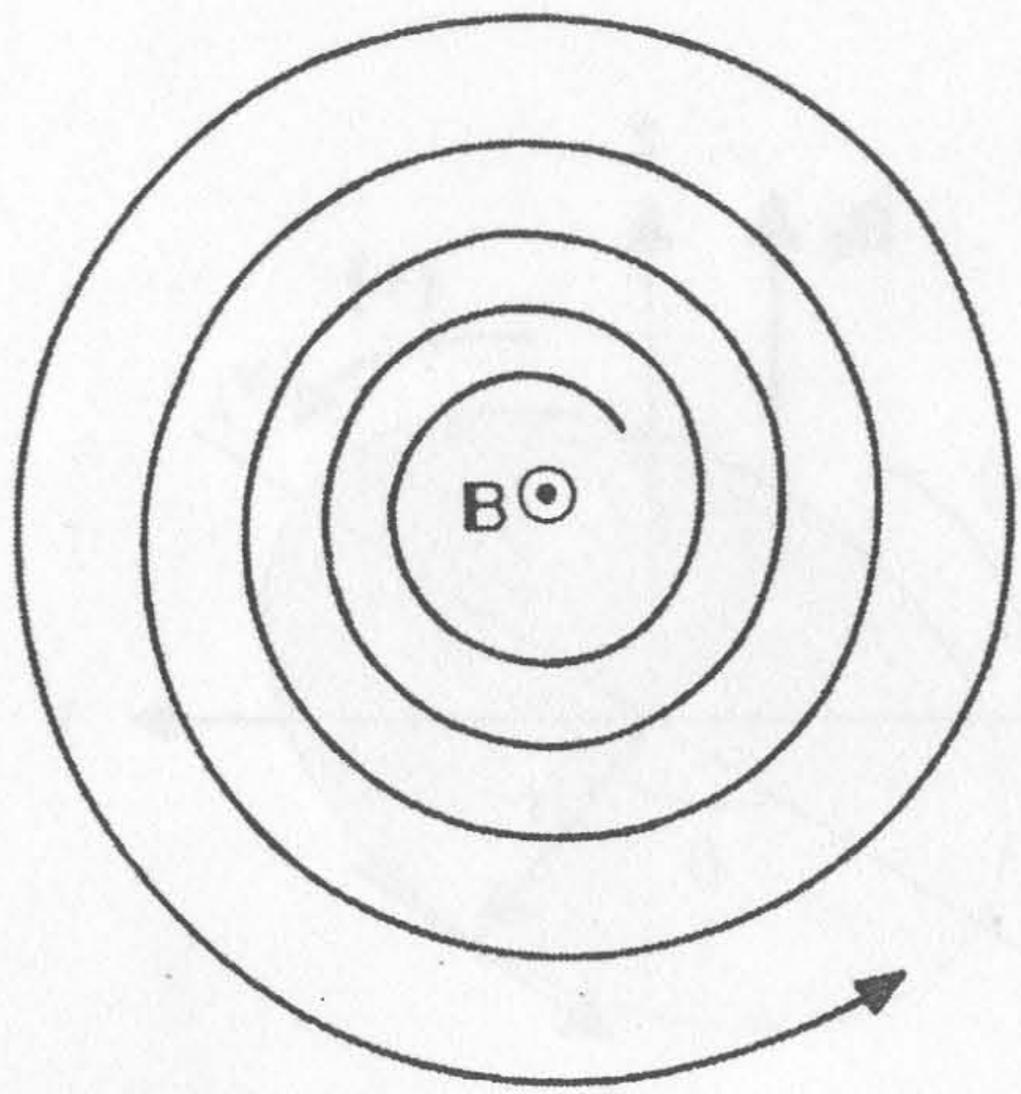


Fig. 2 *Outward spiral motion of an electron, in the plane normal to the magnetic field, under cyclotron resonance.*

of the **B** field is small, the azimuthal component of the electric field **E** is also small, and the orbit will be nearly a circle.

Expressing (4.1) in cylindrical coordinates, taking $\mathbf{B} = B\widehat{\mathbf{z}}$, and considering $\mathbf{E} = \widehat{\boldsymbol{\Theta}} E_\theta(r)$, we find

$$\frac{1}{r}\frac{\partial}{\partial r}(rE_\theta) = -\frac{\partial B}{\partial t} \tag{4.2}$$

Integrating with respect to r, noting that $(\partial B/\partial t)$ can be taken outside the integral sign since B is a slowly varying function, yields

$$\int_0^r \frac{\partial}{\partial r'}(r'E_\theta)\, dr' = -\frac{\partial B}{\partial t}\int_0^r r'\, dr' \tag{4.3}$$

from which we obtain the induced electric field as

$$E_\theta = -\tfrac{1}{2}\, r\, \frac{\partial B}{\partial t} \tag{4.4}$$

Further, since $\widehat{\mathbf{r}} \times \widehat{\mathbf{z}} = -\widehat{\boldsymbol{\Theta}}$, (4.4) can be written in vector form as

$$\mathbf{E}_\theta = \tfrac{1}{2}\, \mathbf{r} \times \frac{\partial \mathbf{B}}{\partial t} \tag{4.5}$$

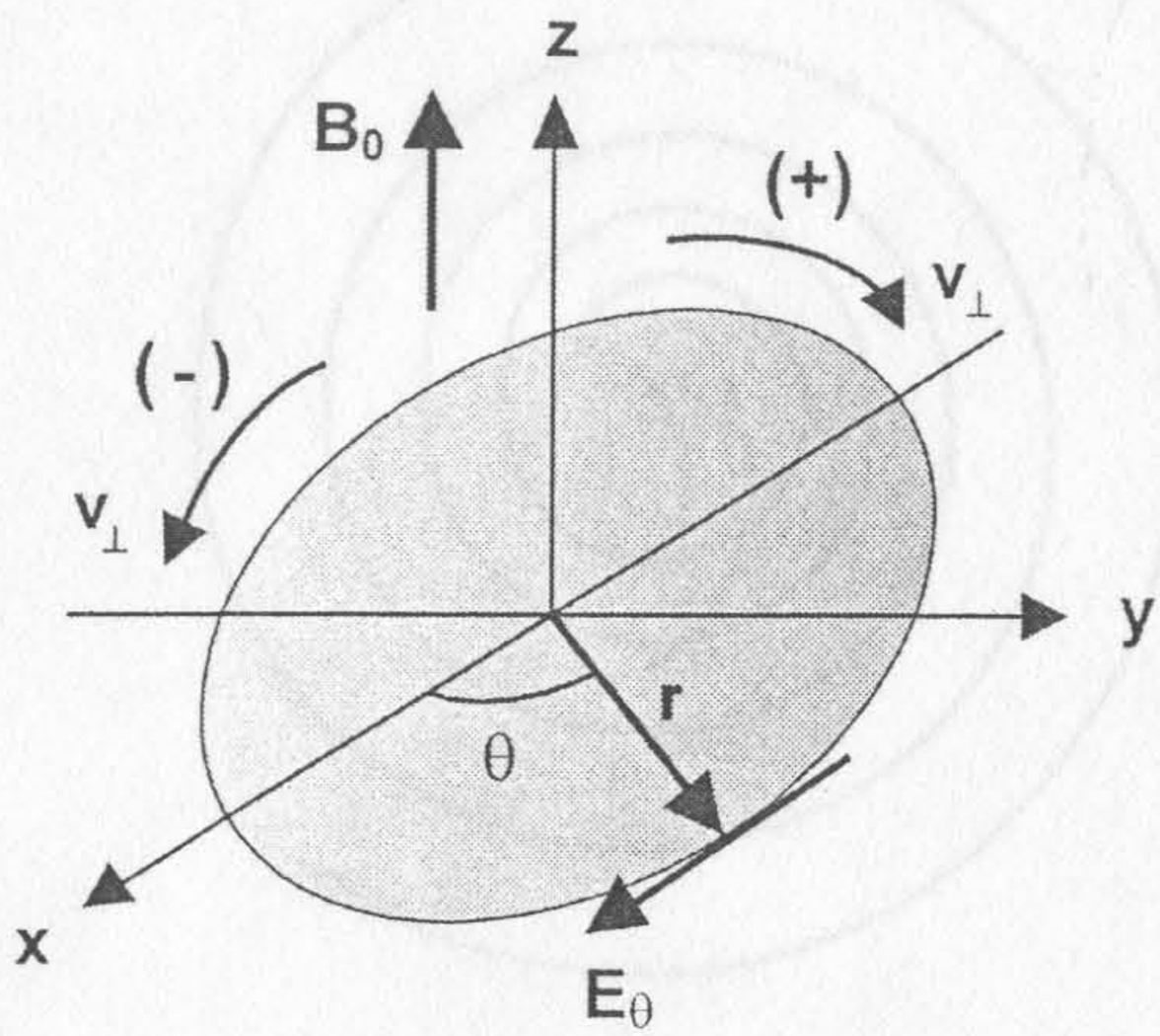


Fig. 3 *Azimuthal electric field* $\mathbf{E}_\theta$ *induced by a time-varying magnetic field. The magnetic field inside the orbit is uniform, parallel to the z axis, and increases slowly in time.*

Using this result in the Lorentz force equation, we obtain for the equation of motion, after some rearrangement,

$$\frac{d\mathbf{v}}{dt} = -\tfrac{1}{2}\,\mathbf{r} \times \frac{\partial \mathbf{\Omega}_c}{\partial t} + \mathbf{\Omega}_c \times \mathbf{v} \tag{4.6}$$

Instead of solving (4.6) directly, we shall determine a relation between the radius vector $\mathbf{r}$ and the time rate of change of $\mathbf{B}$, by calculating the change in the particle transverse kinetic energy over one gyration period, which results from the action of the induced electric field. Since the force acting on the particle due to the electric field is given by $q\mathbf{E}_\theta$, the increase in the transverse kinetic energy over one gyration period is given by

$$\delta(\tfrac{1}{2}mv_\perp^2) = q \oint \mathbf{E}_\theta \cdot d\mathbf{r} \tag{4.7}$$

where $d\mathbf{r}$ denotes an element of path along the particle trajectory, so that $\mathbf{v}_\perp = d\mathbf{r}/dt$. Since the field changes very slowly, we can calculate the line integral in (4.7) as if the orbit were closed. Using Stokes' theorem we replace the line integral by a surface integral over the unperturbed orbit,

$$\delta(\tfrac{1}{2}mv_\perp^2) = q \int_S (\nabla \times \mathbf{E}_\theta) \cdot d\mathbf{S} = -q \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} \tag{4.8}$$

where use was made of (4.1). Here $\mathbf{S}$ denotes the surface enclosed by the cyclotron orbit, its direction being such that $\mathbf{B} \cdot d\mathbf{S} < 0$ for ions, and $\mathbf{B} \cdot d\mathbf{S} > 0$ for electrons, in view of the diamagnetic character of a plasma. Thus (4.8) becomes

$$\delta(\tfrac{1}{2}mv_\perp^2) = |q| \, \frac{\partial B}{\partial t} \, \pi r_c^2 \tag{4.9}$$

Now, the change in the magnetic field over one gyration period $(2\pi/\Omega_c)$ is

$$\delta B = \left(\frac{\partial B}{\partial t}\right) \frac{2\pi}{\Omega_c} \tag{4.10}$$

and using the relations $r_c^2 = v_\perp^2/\Omega_c^2$ and $\Omega_c = |q| \, B/m$, (4.9) can be rewritten as

$$\delta(\tfrac{1}{2}mv_\perp^2) = (\tfrac{1}{2}mv_\perp^2/B) \, \delta B = |\mathbf{m}| \, \delta B \tag{4.11}$$

where the quantity $|\mathbf{m}| = (\frac{1}{2}mv_\perp^2)/B$ is the orbital magnetic moment of the charged particle. Now, since the left-hand side of (4.11) is $\delta(|\mathbf{m}|B)$, we obtain

$$\delta|\mathbf{m}| = 0 \tag{4.12}$$

This result shows that the *magnetic moment* is *invariant* in slowly varying magnetic fields for which $(\partial B/\partial t)(2\pi/\Omega_c) \ll B$.

From the constancy of the magnetic moment, we can easily verify that

$$B\pi r_c^2 = constant \tag{4.13}$$

Therefore, as the magnetic field increases, the radius of gyration decreases, as shown in Fig. 4. Further, since the magnetic flux Φ_m through a Larmor orbit is given by

$$\Phi_m = BS = B\pi r_c^2 \tag{4.14}$$

it is clear that the *magnetic flux* through the particle orbit is also an *adiabatic invariant*. Hence, as the magnetic field strength increases, the radius of the orbit decreases in such a way that the particle always encircles the same number of magnetic flux lines.

When the time variation of the magnetic field is not spatially uniform within the particle orbit, but if it occurs in an unsymmetrical way, then the induced electric field acting on the particle can considerably modify its orbit from the one shown in Fig. 4. In the most general cases the particle orbit can be extremely complicated. In order to obtain a general idea of the particle orbit, let us consider the simple case of a magnetic field varying in time with cylindrical symmetry over a region of radius R,

which is much larger that the cyclotron radius r_c, as shown in Fig. 5. The azimuthal component of the induced electric field $\mathbf{E}_\theta$, at the point P (see Fig. 5) is given, from (4.5), by

$$\mathbf{E}_\theta = \tfrac{1}{2}\mathbf{R} \times \frac{\partial \mathbf{B}}{\partial t} \tag{4.15}$$

A charged particle located at the point P is now acted upon by crossed electric and magnetic fields, resulting in a drift velocity given by

$$\mathbf{v}_E = \frac{\mathbf{E}_\theta \times \mathbf{B}}{B^2} = \tfrac{1}{2}\left(\mathbf{R} \times \frac{\partial \mathbf{B}}{\partial t}\right) \times \frac{\mathbf{B}}{B^2} \tag{4.16}$$

Since $\mathbf{B}$ is in the z direction (normal to the vector $\mathbf{R}$), (4.16) yields

$$\mathbf{v}_E = -\tfrac{1}{2}\left(\frac{\partial B}{\partial t}\right)\frac{\mathbf{R}}{B} \tag{4.17}$$

Therefore, the particle guiding center drifts radially inward with the drift velocity $\mathbf{v}_E$ given in (4.17). As the particle drifts radially inward, its radius of gyration decreases in such a way that the flux encircled by the gyrating particle remains constant (see Fig. 5). Since the density of the magnetic flux lines increases as the magnetic field strength increases, this radial particle drift can be pictured as a radially inward motion of the magnetic flux lines at the velocity $\mathbf{v}_E$, with the guiding center attached to a given flux line.

4.2 Magnetic Heating of a Plasma

The adiabatic invariance of the particle orbital magnetic moment ($|\mathbf{m}| = \frac{1}{2}mv_\perp^2/B = constant$) implies that when the magnetic field increases, the particle transverse kinetic energy ($W_\perp = \frac{1}{2}mv_\perp^2$) increases linearly with B. Further, since the magnetic flux encircled by the gyrating particle is also constant, as the magnetic flux density increases, the magnetic flux tube contracts and the particle guiding center moves radially inward, accompanying the radial displacement of the magnetic field lines, as if it was frozen in the field lines. Consequently, the increase in the magnetic flux density causes the charged particles in a plasma to approach each other, resulting in a *magnetic compression.* In the present case of two-dimensional compression, since the increase in the number density (n) of the particles is proportional to the cross-sectional area πr_c^2,

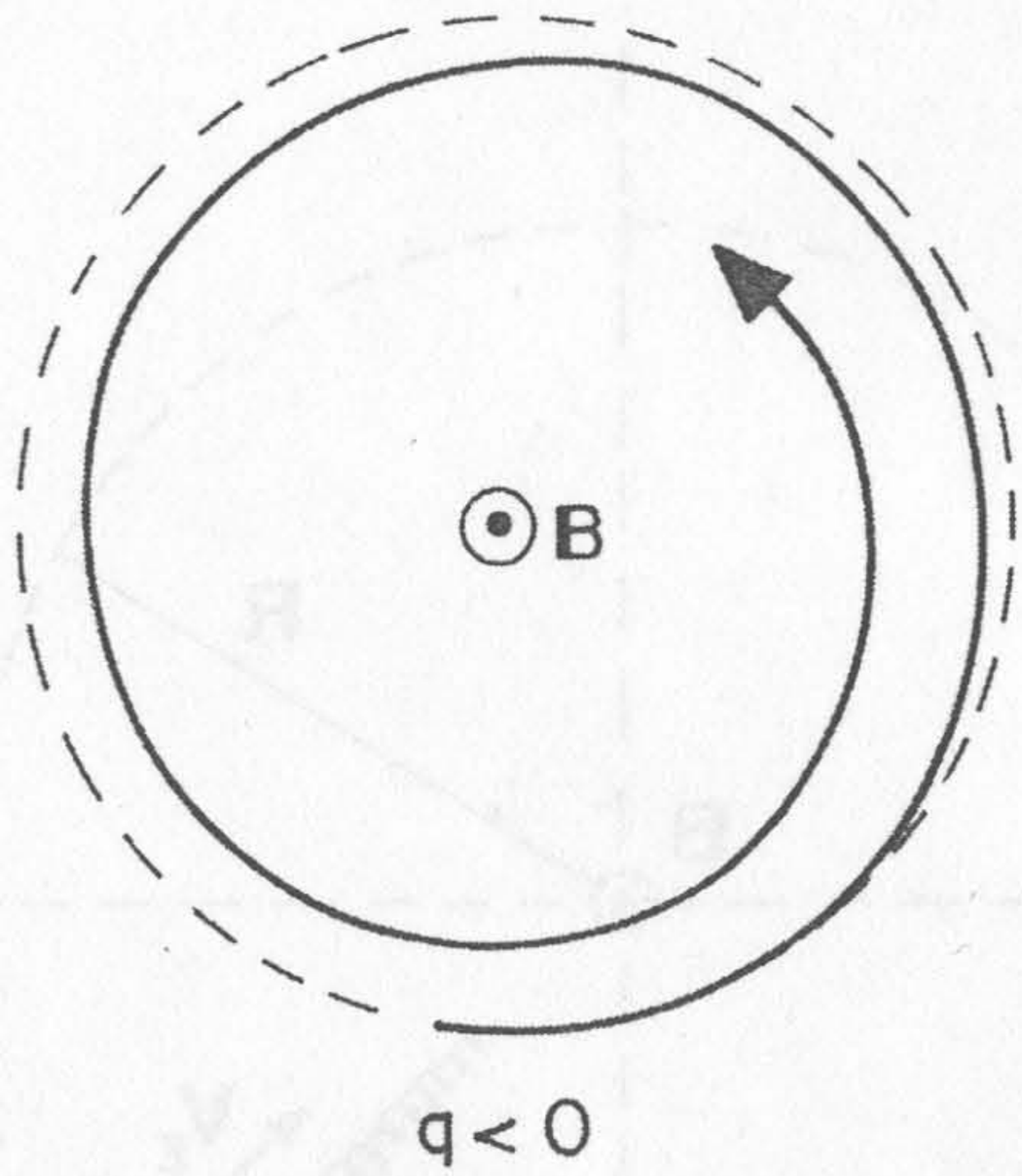


Fig. 4 *Motion of an electron in a time-varying magnetic field. The field is spatially uniform and increases in time.*

it follows that under magnetic compression in two dimensions n increases linearly with B. Similarly, when B decreases, n also decreases, resulting in a *magnetic decompression.* Thus, for two-dimensional compression we have $W_\perp \propto B \propto n$.

This property is used as a method of *plasma heating*, known as *magnetic pumping*, which consists in periodic magnetic compressions and decompressions of the plasma. The compression and the decompression must take place in a time interval very large compared to the Larmor period but, at the same time, very small compared to the relaxation time necessary for the achievement of thermal equilibrium. For the present case of two-dimensional compression-decompression, obtained by varying the axial magnetic field, only the velocity and the energy in the two directions normal to the magnetic field are changed.

In order to be able to heat the plasma in a compression-decompression cycle, it is necessary to transfer part of the energy increase in $W_\perp$, obtained by compression, to the energy $W_\parallel$ which is unaffected by decompression. This energy transfer is brought about by particle collisions, therefore increasing the particle thermal kinetic energy at the expense of

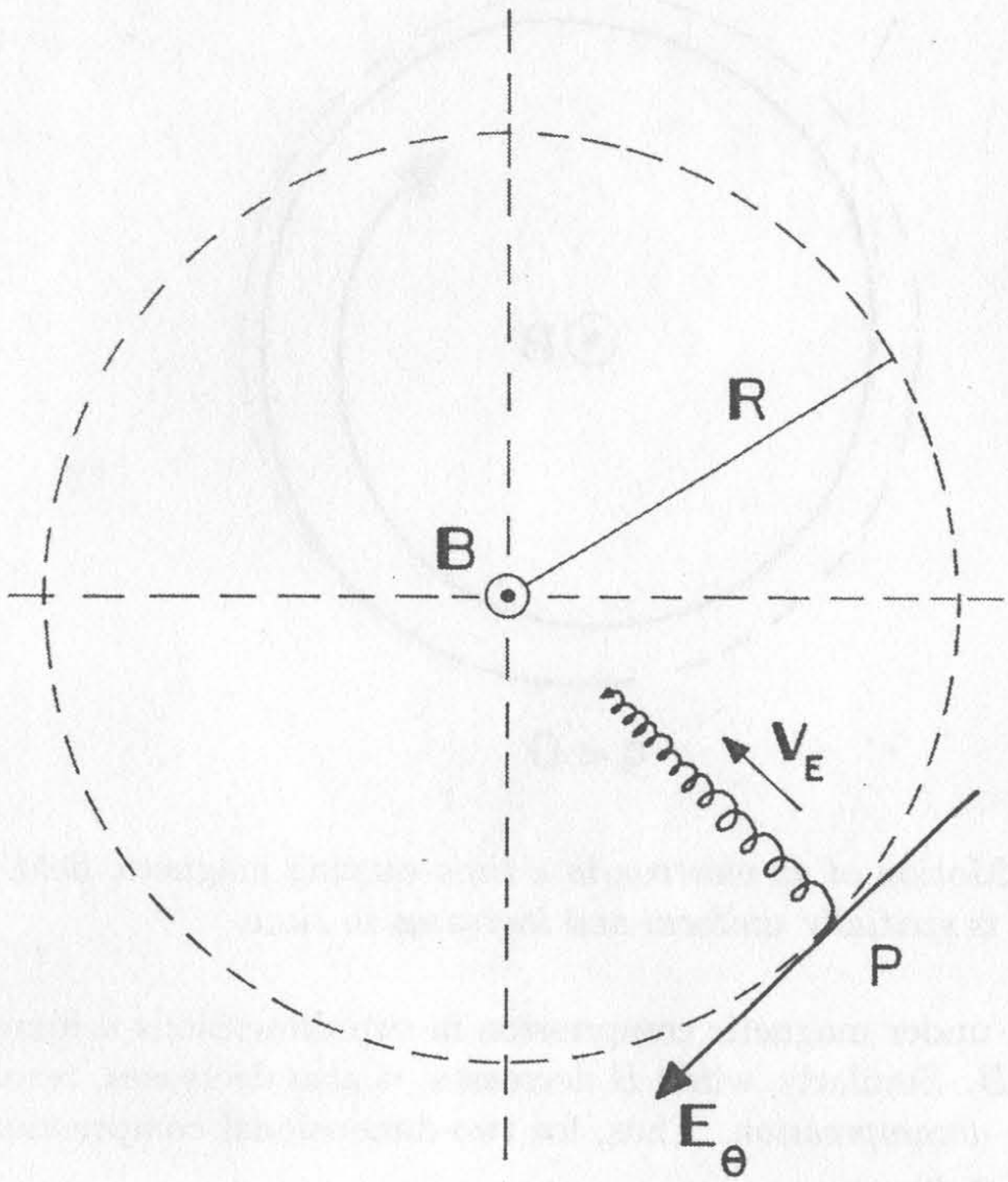


Fig. 5 *Motion of a negatively charged particle in an increasing magnetic field (directed out of the page) with cylindrical symmetry over a region of radius R much greater than the Larmor radius r_c.*

the electric field energy. Thus, in a complete cycle consisting of a compression, a relaxation time and a decompression, part of the energy increase in $W_\perp$ achieved during compression is transferred to the degree of freedom parallel to B, as a result of collisions, causing an increase in $W_\parallel$ which is not afected during the decompression, whereas $W_\perp$ is decreased correspondingly. Therefore, by a periodic repetition of these cycles of adiabatic compression-decompression, the plasma thermal energy and therefore the temperature is increased.

5. SUMMARY OF GUIDING CENTER DRIFTS AND CURRENT DENSITIES

5.1 Guiding Center Drifts

Electric field:

$$\mathbf{v}_E = \frac{\mathbf{E} \times \mathbf{B}}{B^2} \tag{2.5.15}$$

Gravitational field:

$$\mathbf{v}_g = \frac{m\mathbf{g} \times \mathbf{B}}{qB^2} \tag{2.6.3}$$

General force:

$$\mathbf{v}_F = \frac{\mathbf{F} \times \mathbf{B}}{qB^2} \tag{2.6.2}$$

Gradient of $\mathbf{B}$:

$$\mathbf{v}_G = -\frac{|\mathbf{m}|\,(\nabla B) \times \mathbf{B}}{qB^2} \tag{3.5.1}$$

Curvature of $\mathbf{B}$:

$$\mathbf{v}_C = -\frac{mv_\parallel^2\,[(\mathbf{B}\cdot\nabla)\mathbf{B}] \times \mathbf{B}}{qB^4} \tag{3.7.9}$$

Gradient-curvature of $\mathbf{B}$ (in vacuum field):

$$\mathbf{v}_{GC} = -\frac{m(v_\parallel^2 + \frac{1}{2}v_\perp^2)(\frac{1}{2}\nabla B^2) \times \mathbf{B}}{qB^4} \tag{3.8.2}$$

Polarization:

$$\mathbf{v}_p = \frac{m(\partial \mathbf{E}_\perp/\partial t)}{qB^2} \tag{2.11}$$

5.2 Current Densities

Magnetization:

$$\mathbf{J}_M = \nabla \times \mathbf{M} \tag{2.4.40}$$

Gravitational:

$$\mathbf{J}_g = \frac{\rho_m \mathbf{g} \times \mathbf{B}}{B^2} \tag{2.6.5}$$

Gradient of $\mathbf{B}$:

$$\mathbf{J}_G = -\Big(\frac{1}{\delta V}\sum_i |\mathbf{m}_i|\Big)\frac{(\nabla B) \times \mathbf{B}}{B^2} \tag{3.5.3}$$

Curvature of B:

$$\mathbf{J}_C = -2\Big(\frac{1}{\delta V}\sum_i W_{i\parallel}\Big)\frac{[(\mathbf{B}\cdot\nabla)\mathbf{B}]\times\mathbf{B}}{B^4} \tag{3.7.12}$$

Polarization:

$$\mathbf{J}_p = \frac{\rho_m(\partial\mathbf{E}_\perp/\partial t)}{B^2} \tag{2.12}$$

PROBLEMS

4.1 With reference to a magnetic field pointing along the z axis ($\mathbf{B} = B_0\widehat{\mathbf{z}}$), describe the type of polarization of the following electric field:

$$\mathbf{E} = E_0(\widehat{\mathbf{x}}\cos\omega t - \widehat{\mathbf{y}}\sin\omega t)$$

Make a drawing which shows the orientation of $\mathbf{E}$ for the instants $t = 0$, $t = \pi/(2\omega)$ and $t = \pi/\omega$. How can you represent this electric field in complex notation?

4.2 Describe, in a semiquantitative way, the motion of an electron in the presence of a constant magnetic field $\mathbf{B} = B_0\widehat{\mathbf{z}}$ and a time-varying electric field given by

$$\mathbf{E} = \tfrac{1}{2}E_0(\widehat{\mathbf{x}} + i\widehat{\mathbf{y}})\, e^{-i\Omega_c t}$$

where E_0 and B_0 are positive constants and $\Omega_c = eB_0/m_e$. What type of polarization has this electric field?

4.3 Solve the equation of motion to determine the transient response of a charged particle in the presence of a spatially uniform AC electric field $\mathbf{E}(t) = \widehat{\mathbf{x}}E\sin(\omega t)$, which is switched on at $t = 0$. Assume that initially, at $t = 0$, the particle is at rest at the origin. Make a plot of the particle trajectory and velocity as a function of time.

4.4 Consider an electron acted upon by a constant and uniform magnetic field $\mathbf{B} = B_0\widehat{\mathbf{z}}$, and a uniform but time-varying electric field $\mathbf{E} = \widehat{\mathbf{y}}E_{y0}\sin(\omega t)$. Assume that the initial conditions are such that the motion takes place in the (x, y) plane and that at $t = 0$ the electron is at rest

($\mathbf{v}_0 = 0$) at the origin.
(a) Show that the orbit of the electron is given by

$$x(t) = -\frac{eE_{y0}}{m(\omega^2 - \Omega_c^2)}\left[\frac{\omega}{\Omega_c}(\cos\Omega_c t - 1) - \frac{\Omega_c}{\omega}(\cos\omega t - 1)\right]$$

$$y(t) = -\frac{eE_{y0}}{m(\omega^2 - \Omega_c^2)}\left(\frac{\omega}{\Omega_c}\sin\Omega_c t - \sin\omega t\right)$$

(b) In the low-frequency limit, $\omega \ll \Omega_c$, show that the electron orbits at the angular frequency ω around an ellipse which has its major axis perpendicular to the electric field. Determine the ratio of the minor to the major axis of the ellipse.
(c) In the high-frequency limit, $\omega \gg \Omega_c$, show that the electron moves in a circle at the cyclotron frequency Ω_c.

4.5 Integrate (3.49) and (3.50) to determine the particle trajectory in the plane normal to $\mathbf{B}$ and sketch the path of the particle for $q > 0$ and $q < 0$.

4.6 Consider the motion of an electron in the presence of a uniform magnetostatic field $\mathbf{B} = B_0\hat{\mathbf{z}}$, and an electric field which oscillates in time at the electron cyclotron frequency Ω_c, according to

$$\mathbf{E}(t) = E_0(\hat{\mathbf{x}}\cos\Omega_c t + \hat{\mathbf{y}}\sin\Omega_c t)$$

(a) What type of polarization has this electric field?
(b) Obtain the following uncoupled differential equations satisfied by the velocity components $v_x(t)$ and $v_y(t)$:

$$\frac{d^2 v_x}{dt^2} + \Omega_c^2 v_x = 2\frac{eE_0}{m}\Omega_c \sin\Omega_c t$$

$$\frac{d^2 v_y}{dt^2} + \Omega_c^2 v_y = -2\frac{eE_0}{m}\Omega_c \cos\Omega_c t$$

(c) Assume that, at $t = 0$, the electron is located at the origin of the coordinate system, with zero velocity. Neglect the time-varying part of $\mathbf{B}$. Show that the electron velocity is given by

$$v_x(t) = -\frac{eE_0}{m}\, t\cos\Omega_c t$$

$$v_y(t) = -\frac{eE_0}{m}\, t \sin \Omega_c t$$

(d) Show that the electron trajectory is given by

$$x(t) = -\frac{eE_0}{m}\Big(\frac{1}{\Omega_c^2}\cos \Omega_c t + \frac{t}{\Omega_c}\sin \Omega_c t - \frac{1}{\Omega_c^2}\Big)$$

$$y(t) = -\frac{eE_0}{m}\Big(\frac{1}{\Omega_c^2}\sin \Omega_c t - \frac{t}{\Omega_c}\cos \Omega_c t\Big)$$

4.7 Solve the equation of motion to determine the velocity and the trajectory of an electron in the presence of a uniform magnetostatic field $\mathbf{B} = B_0\widehat{\mathbf{z}}$, and an oscillating electric field given by

$$\mathbf{E}(t) = \widehat{\mathbf{x}}E_x \sin \omega t + \widehat{\mathbf{z}}E_z \cos \omega t$$

Consider the same assumptions and initial conditions as in the previous problem.

4.8 Consider the motion of an electron in a spatially uniform magnetic field $\mathbf{B} = B_z\widehat{\mathbf{z}}$, such that B_z has a slow time variation given by

$$B_z(t) = B_0(1 - \alpha t)$$

where B_0 and α are positive constants, and $|\alpha t| \ll 1$. Assume the following initial conditions: $\mathbf{r}(0) = (r_c, 0, 0)$ and $\mathbf{v}(0) = (0, v_{\perp 0}, 0)$, where r_c is the Larmor radius, $v_{\perp 0} = \Omega_c r_c$ and $\Omega_c = |q|\, B_0/m$.
(a) Write the equation of motion, considering the Lorentz force, and solve it by a perturbation technique including only terms up to the first order in the small parameter α. Show that the particle velocity is given by

$$v_x(t) = -\Omega_c r_c \sin \Omega_c t + \tfrac{1}{2}\alpha\Omega_c r_c t(\sin \Omega_c t + \Omega_c t \cos \Omega_c t)$$

$$v_y(t) = \Omega_c r_c \cos \Omega_c t + \tfrac{1}{2}\alpha\Omega_c r_c t(-\cos \Omega_c t + \Omega_c t \sin \Omega_c t)$$

(b) Show that the particle orbit is given by

$$x(t) = r_c(1 + \tfrac{1}{2}\alpha t)\cos \Omega_c t + \frac{r_c\alpha}{2\Omega_c}(\Omega_c^2 t^2 - 1)\sin \Omega_c t$$

$$y(t) = r_c(1 + \tfrac{1}{2}\alpha t)\sin \Omega_c t - \frac{r_c\alpha}{2\Omega_c}(\Omega_c^2 t^2 - 1)\cos \Omega_c t - \frac{r_c\alpha}{2\Omega_c}$$

$$z(t) = v_{z0}t$$

(c) Determine the orbital magnetic moment and verify its adiabatic invariance, retaining only terms up to the first order in α.

4.9 Consider the motion of a charged particle in a spatially uniform magnetic field which varies slowly in time as compared to the particle cyclotron period.
(a) Show that the equation of motion can be written in vector form as

$$\frac{d\mathbf{v}(t)}{dt} = \mathbf{\Omega}_c(t) \times \mathbf{v}(t) - \tfrac{1}{2}\mathbf{r}(t) \times \frac{\partial \mathbf{\Omega}_c(t)}{\partial t}$$

where $\mathbf{\Omega}_c(t) = -q\mathbf{B}(t)/m$.
(b) Considering that $\mathbf{B}(t) = \hat{\mathbf{z}}B_0 f(t)$, where B_0 is constant, obtain the following equations for the motion of the particle in the plane normal to $\mathbf{B}$

$$\frac{d^2x(t)}{dt^2} - \Omega_c\left[f(t)\frac{dy(t)}{dt} + \tfrac{1}{2}y(t)\frac{df(t)}{dt}\right] = 0$$

$$\frac{d^2y(t)}{dt^2} + \Omega_c\left[f(t)\frac{dx(t)}{dt} + \tfrac{1}{2}x(t)\frac{df(t)}{dt}\right] = 0$$

where $\Omega_c = |q|\ B/m$.
(c) Define a complex variable $u(t) = x(t) + iy(t)$ and a function $\xi(t)$ by

$$\xi(t) = u(t)\ \exp\ [\tfrac{1}{2}i\Omega_c \int_0^t f(t')\ dt']$$

and show that the equation satisfied by $\xi(t)$ is

$$\frac{d^2\xi(t)}{dt^2} + \frac{\Omega_c^2}{4}f^2(t)\ \xi(t) = 0$$

(d) If $\xi_1(t)$ and $\xi_2(t)$ are two linearly independent solutions of this equation, subject to the initial conditions

$$\xi_1(0) = 0 \qquad (d\xi_1(t)/dt)_{t=0} = 1$$

$$\xi_2(0) = 1 \qquad (d\xi_2(t)/dt)_{t=0} = 0$$

show that the solution for $u(t)$ can be written as

$$u(t) = \{u_0\xi_2(t) + \xi_1(t)[du_0/dt + \tfrac{1}{2}i\Omega_c f(0)u_0]\}\ \exp\ [-\tfrac{1}{2}i\Omega_c \int_0^t f(t')\ dt']$$

where u_0 and du_0/dt represent the initial position and velocity, respectively.
(e) Considering now that the particle is initially ($t = 0$) at the origin and moving with velocity v_0 along the negative y axis, that is, $u_0 = 0$ and $du_0/dt = -iv_0$, show that

$$u(t) = -iv_0\, \xi_1(t) \exp\left[-\tfrac{1}{2}i\Omega_c \int_0^t f(t')\, dt'\right]$$

and, consequently,

$$x(t) = v_0\, \xi_1(t) \sin\left[\tfrac{1}{2}\Omega_c \int_0^t f(t')\, dt'\right]$$

$$y(t) = -v_0\, \xi_1(t) \cos\left[\tfrac{1}{2}\Omega_c \int_0^t f(t')\, dt'\right]$$

4.10 (a) Assume that $f(t)$, in Problem 4.9, is given by $\exp(-\alpha t)$. Show that, in this case, $\xi(t)$ satisfies the *Bessel equation* of zero order,

$$\frac{d^2\xi(\tau)}{d\tau^2} + \frac{1}{\tau}\frac{d\xi(\tau)}{d\tau} + \xi(\tau) = 0$$

where $\tau = (\Omega_c/2\alpha) \exp(-\alpha t)$. Determine the two solutions of this equation which satisfy the initial conditions stated in Problem 4.9 and interpret them physically.
(b) Considering now that $f(t) = (1 - \alpha t)$, solve the equation for $\xi(t)$ in Problem 4.9 in a power series in α, and determine the particle trajectory to order α. Show that the ratio $(v_x^2 + v_y^2)/B(t)$ has no terms of order α, thus verifying the *adiabatic invariance* of the magnetic moment. Compare these results with those of Problem 4.8.

4.11 For an electron with initial velocity $v_0\hat{\mathrm{x}}$ and initial position $x_0\hat{\mathrm{x}}$, acted upon by an electric field $\mathbf{E} = \hat{\mathrm{x}}E\cos(kx - \omega t)$, show that its velocity is given by

$$v(t) = v_0 - \frac{eE}{m_e}\int_0^t \cos(kx - \omega t')\, dt'$$

Using a perturbation approach, in which to lowest order $\mathbf{E} = 0$, show that

$$v(t) = v_0 - \frac{eE}{m_e(kv_0 - \omega)}\{\sin[kx_0 + (kv_0 - \omega)t] - \sin(kx_0)\}$$

Notice that the velocity perturbation will be large only when v_0 is close to the phase velocity ω/k.

4.12 Using the Maxwell equation (1.5.3) and the equation (3.34) which defines the plasma conductivity dyad $\mathcal{S}$, and considering the time variation indicated in (3.1), show that

$$\nabla \times \mathbf{B} = -i\omega\mu_0 \, \mathcal{E} \cdot \mathbf{E}$$

where $\mathcal{E}$ is the plasma electric permittivity dyad given by

$$\mathcal{E} = \epsilon_0 \left(\mathbf{1} + \frac{i}{\omega\epsilon_0} \mathcal{S} \right)$$

where $\mathbf{1}$ denotes the unit dyad, which in Cartesian coordinates can be written as

$$\mathbf{1} = \widehat{\mathbf{x}}\widehat{\mathbf{x}} + \widehat{\mathbf{y}}\widehat{\mathbf{y}} + \widehat{\mathbf{z}}\widehat{\mathbf{z}}$$

5

ELEMENTS OF PLASMA KINETIC THEORY

1. INTRODUCTION

A plasma is a system containing a very large number of interacting charged particles, so that for its analysis it is appropriate and convenient to use a *statistical* approach. In this chapter we present the basic elements of *kinetic theory*, introducing the concepts of *phase space* and of *distribution function* necessary for a statistical description.

All physically interesting information about the system is contained in the distribution function. From a knowlegde of the distribution function the *macroscopic variables* of physical interest, necessary for a macroscopic description of the plasma behavior, can be systematically deduced. These macroscopic variables are functions of position and time and are directly related to the *average values* of the various particle physical quantities of interest, which can be considered as functions of the particle velocities, with the distribution function used as a weighting function in phase space.

The differential kinetic equation satisfied by the distribution function, generally known as the *Boltzmann equation*, is deduced in Section 5. At this point, the effects due to particle collisions are incorporated into this kinetic equation only through a general, unspecified collision term. In Chapter 21 we shall deduce explicit expressions for the collision term, in particular for the *Boltzmann collision integral* and for the *Fokker-Planck collision term*. Only a simple approximate expression for the collision term is presented at this point, the so-called *relaxation model* or *Krook collision term*. The *Vlasov equation* for a plasma is introduced in the last section.

2. PHASE SPACE

At any instant of time each particle in the plasma can be localized by a position vector $\mathbf{r}$ drawn from the origin of a coordinate system to the center of mass of the particle. In a Cartesian frame of reference, as illustrated in Fig. 1, we have

$$\mathbf{r} = x\widehat{\mathbf{x}} + y\widehat{\mathbf{y}} + z\widehat{\mathbf{z}} \tag{2.1}$$

where $\widehat{\mathbf{x}}$, $\widehat{\mathbf{y}}$ and $\widehat{\mathbf{z}}$ denote unit vectors along the axes x, y and z, respectively. The linear velocity of the center of mass of the particle can be represented by the vector

$$\mathbf{v} = v_x\widehat{\mathbf{x}} + v_y\widehat{\mathbf{y}} + v_z\widehat{\mathbf{z}} \tag{2.2}$$

with $v_x = dx/dt$, $v_y = dy/dt$ and $v_z = dz/dt$.

In analogy with the *configuration space* defined by the position coordinates (x, y, z), it is convenient to introduce the *velocity space* defined by the velocity coordinates (v_x, v_y, v_z). In this space the velocity vector $\mathbf{v}$ can be viewed as a position vector drawn from the origin of the coordinate system (v_x, v_y, v_z) to the center of mass of the particle, as indicated schematically in Fig. 1.

2.1 Single-Particle Phase Space

From the point of view of classical mechanics the instantaneous *dynamical state* of each particle can be specified by its position and velocity vectors. It is convenient therefore to consider the *phase space* defined by the six coordinates (x, y, z, v_x, v_y, v_z).

In this six-dimensional space the dynamical state of each particle is appropriately represented by a single point. The coordinates $(\mathbf{r}, \mathbf{v})$ of the representative point give the position and velocity of the particle. When the particle moves, its representative point describes a trajectory in phase space. At each instant of time the dynamical state of a system of N particles is represented by N points in phase space.

2.2 Many-Particle Phase Space

The phase space just defined, often called *μ-space*, is the phase space for a single particle, in contrast with the many-particle phase space or

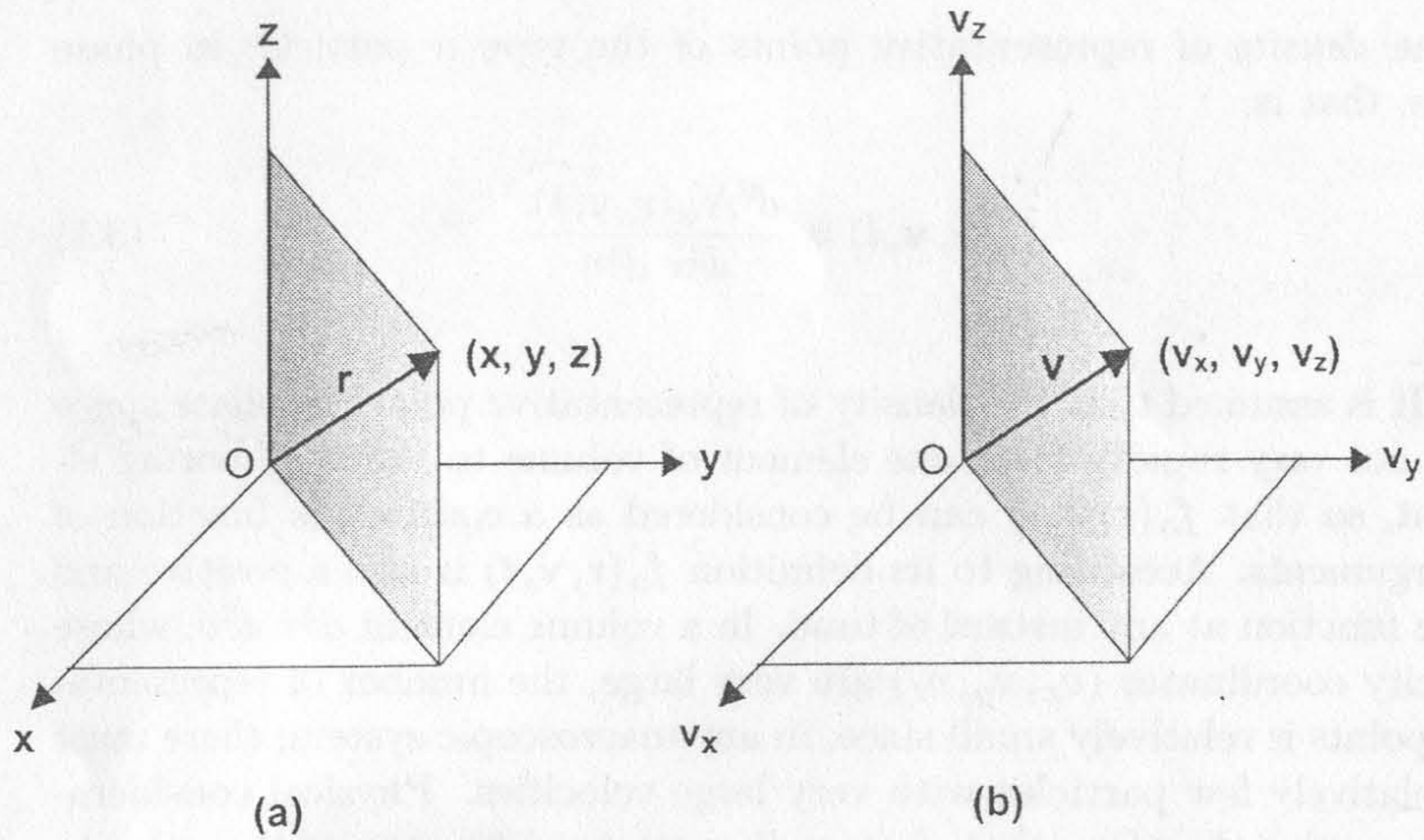


Fig. 1 *Position vectors (a) in configuration space and (b) in velocity space.*

Γ-*space* for the whole system of particles. In the latter, a system consisting of N particles, with no internal degrees of freedom, is represented by a single point in a *6N-dimensional* space defined by the *3N* position coordinates $(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_N)$ and the *3N* velocity coordinates $(\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_N)$. Thus, a point in Γ-*space* corresponds to a single microscopic state for the whole system of particles. This many-particle phase space is often used in statistical mechanics and advanced kinetic theory. The single-particle phase space is the one normally used in elementary kinetic theory and basic plasma physics, and is the space which will be considered in what follows.

2.3 Volume Elements

A small element of volume in configuration space is represented by $d^3r = dx\ dy\ dz$. This differential element of volume should not be taken literally as a mathematically infinitesimal quantity but as a finite element of volume, sufficiently large to contain a very large number of particles, yet sufficiently small in comparison with the characteristic lengths associated with the spatial variation of physical parameters of interest such as, for example, density and temperature. In a gas containing 10^{18} molecules/m^3,

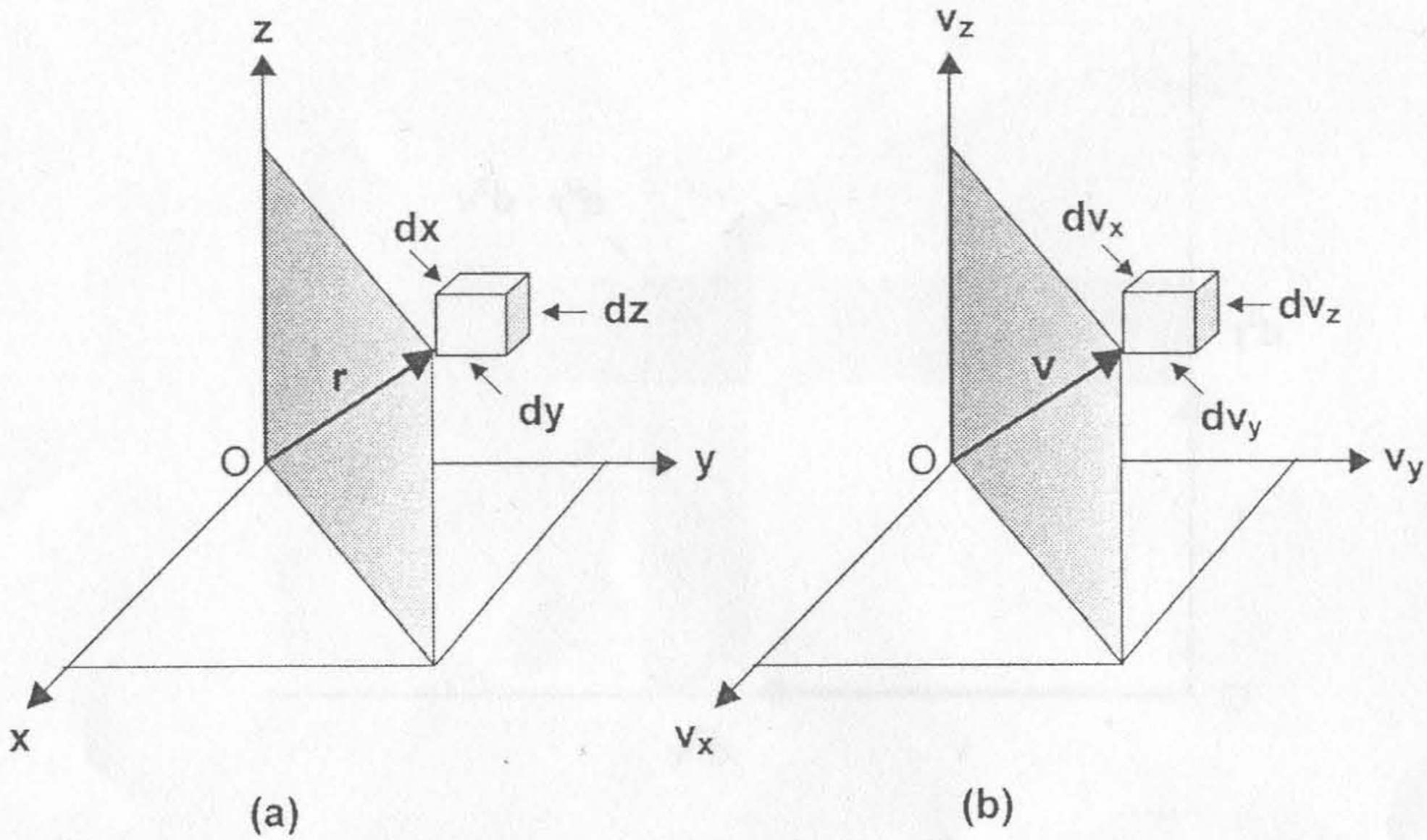


Fig. 2 *(a) The element of volume $d^3r = dx\ dy\ dz$ around the terminal point of* **r**, *in configuration space, and (b) the element of volume $d^3v = dv_x\ dv_y\ dv_z$, in velocity space, around the terminal point of* **v**.

for example, if we take $d^3r = 10^{-12}$ m^3, which in a macroscopic scale can be considered as a point, there are still 10^6 molecules inside d^3r. Plasmas that do not allow a choice of differential volume elements as indicated, cannot be analyzed statistically.

When we refer to a particle as being situated inside d^3r, at **r**, it is meant that the x coordinate of the particle lies between x and $x + dx$, the y coordinate lies between y and $y + dy$, and the z coordinate lies between z and $z + dz$, that is, inside the volume element $dx\ dy\ dz$ situated around the terminal point of the position vector $\mathbf{r} = x\widehat{\mathbf{x}} + y\widehat{\mathbf{y}} + z\widehat{\mathbf{z}}$. It is important to note that the particles localized inside d^3r, at **r**, may have completely arbitrary velocities which would be represented by scattered points in velocity space.

A small element of volume in velocity space is represented by $d^3v = dv_x\ dv_y\ dv_z$. For a particle to be included in d^3v, around the terminal point of the velocity vector **v**, its v_x velocity component must lie between v_x and $v_x + dv_x$, the v_y component between v_y and $v_y + dv_y$, and the v_z component between v_z and $v_z + dv_z$. The differential elements of volume d^3r and d^3v are schematically represented in Fig. 2.

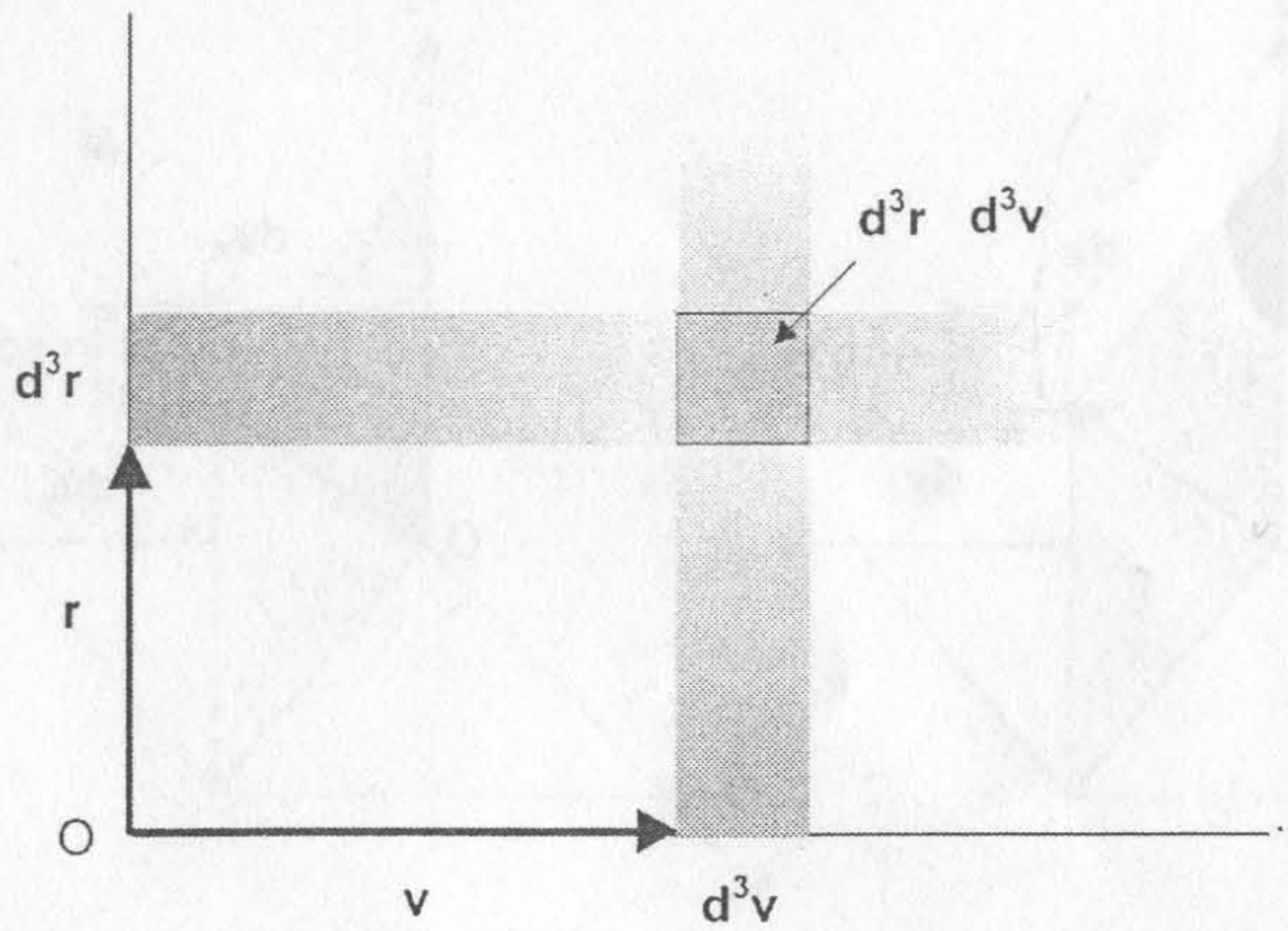


Fig. 3 *Schematic representation of the element of volume $d^3r\ d^3v$ in the six-dimensional phase space, around the representative point* $(\mathbf{r},\mathbf{v})$.

In phase space (*μ-space*) a differential element of volume may be imagined as a six-dimensional cube, represented by

$$d^3r\ d^3v = dx\ dy\ dz\ dv_x\ dv_y\ dv_z \tag{2.3}$$

as shown schematically in Fig. 3. Note that inside $d^3r\ d^3v$, at the position $(\mathbf{r},\mathbf{v})$ in phase space, there are only the particles inside d^3r around $\mathbf{r}$ whose velocities lie inside d^3v about $\mathbf{v}$. The number of representative points inside the volume element $d^3r\ d^3v$ is, in general, a function of time and of the position of this element in phase space. It is important to note that the coordinates $\mathbf{r}$ and $\mathbf{v}$ of phase space are considered to be *independent* variables, since they represent the position of individual volume elements (containing many particles) in phase space.

3. DISTRIBUTION FUNCTION

Let $d^6\mathcal{N}_\alpha(\mathbf{r},\mathbf{v},t)$ denote the number of particles of type α inside the volume element $d^3r\ d^3v$ around the phase space coordinates $(\mathbf{r},\mathbf{v})$, at the instant t. The *distribution function* in phase space, $f_\alpha(\mathbf{r},\mathbf{v},t)$, is defined

as the *density* of representative points of the type α particles in phase space, that is,

$$f_\alpha(\mathbf{r}, \mathbf{v}, t) = \frac{d^6\mathcal{N}_\alpha(\mathbf{r}, \mathbf{v}, t)}{d^3r\ d^3v} \tag{3.1}$$

It is assumed that the density of representative points in phase space does not vary rapidly from one element of volume to the neighboring element, so that $f_\alpha(\mathbf{r}, \mathbf{v}, t)$ can be considered as a continuous function of its arguments. According to its definition $f_\alpha(\mathbf{r}, \mathbf{v}, t)$ is also a positive and finite function at any instant of time. In a volume element $d^3r\ d^3v$, whose velocity coordinates (v_x, v_y, v_z) are very large, the number of representative points is relatively small since, in any macroscopic system, there must be relatively few particles with very large velocities. Physical considerations require therefore that $f_\alpha(\mathbf{r}, \mathbf{v}, t)$ must tend to zero as the velocity becomes infinitely large.

The distribution function is, in general, a function of the position vector $\mathbf{r}$. When this is the case the corresponding plasma is said to be *inhomogeneous*. In the absence of external forces, however, a plasma initially inhomogeneous reaches, in the course of time, an equilibrium state as a result of the mutual particle interactions. In this *homogeneous* state the distribution function does not depend on $\mathbf{r}$.

In velocity space the distribution function can be *anisotropic*, when it depends on the orientation of the velocity vector $\mathbf{v}$, or *isotropic*, when it does not depend on the orientation of $\mathbf{v}$ but only on its magnitude, i.e., on the particle speed $v = |\mathbf{v}|$.

The statistical description of different types of plasmas requires the use of inhomogeneous or homogeneous, as well as anisotropic or isotropic distribution functions. A plasma in thermal equilibrium, for example, is characterized by a homogeneous, isotropic and time-independent distribution function.

In a statistical sense the distribution function provides a complete description of the system under consideration. Knowing $f_\alpha(\mathbf{r}, \mathbf{v}, t)$ we can deduce all the macroscopic variables of physical interest for the type α species. One of the primary problems of kinetic theory consists in determining the distribution function for a given system. The differential equation that governs the temporal and spatial variation of the distribution function under given conditions, known generally as the Boltzmann equation, will be derived in Section 5.

4. NUMBER DENSITY AND AVERAGE VELOCITY

The *number density*, $n_\alpha(\mathbf{r}, t)$, is a macroscopic variable defined in configuration space as the number of particles of type α, per unit volume, irrespective of velocity. It can be obtained by integrating $d^6\mathcal{N}_\alpha(\mathbf{r}, \mathbf{v}, t)$ over all of velocity space and dividing the result by the volume element d^3r of configuration space,

$$n_\alpha(\mathbf{r}, t) = \frac{1}{d^3r} \int_v d^6\mathcal{N}_\alpha(\mathbf{r}, \mathbf{v}, t) \tag{4.1}$$

or, using the definition (3.1),

$$n_\alpha(\mathbf{r}, t) = \int_v f_\alpha(\mathbf{r}, \mathbf{v}, t)\, d^3v \tag{4.2}$$

The single integral sign indicated here represents in fact a triple integral extending over all velocity space, that is, over each one of the variables v_x, v_y and v_z from $-\infty$ to $+\infty$. For convenience and simplification of notation only a single integral sign will be indicated, being implicit the fact that the integral extends over all velocity space.

The *average velocity* $\mathbf{u}_\alpha(\mathbf{r}, t)$ is defined as the *macroscopic* flow velocity of the particles of type α in the neighborhood of the position vector $\mathbf{r}$ at the instant t. In order to relate $\mathbf{u}_\alpha(\mathbf{r}, t)$ to the distribution function, consider the particles of type α contained in the volume element $d^3r\ d^3v$ about $(\mathbf{r},\mathbf{v})$ at the instant t, which we have denoted by $d^6\mathcal{N}_\alpha(\mathbf{r}, \mathbf{v}, t)$. The average velocity of the particles of type α can be obtained as follows. First we multiply $d^6\mathcal{N}_\alpha(\mathbf{r}, \mathbf{v}, t)$ by the particle velocity $\mathbf{v}$, next we integrate over all possible velocities, and finally we divide the result by the total number of type α particles contained in d^3r, irrespective of velocity. Therefore,

$$\mathbf{u}_\alpha(\mathbf{r}, t) = \frac{1}{n_\alpha(\mathbf{r}, t)\ d^3r} \int_v \mathbf{v}\ d^6\mathcal{N}_\alpha(\mathbf{r}, \mathbf{v}, t) \tag{4.3}$$

The procedure just described is the usual statistical definition of average values. Using the definition of $f_\alpha(\mathbf{r}, \mathbf{v}, t)$, given in (3.1), we obtain

$$\mathbf{u}_\alpha(\mathbf{r}, t) = \frac{1}{n_\alpha(\mathbf{r}, t)} \int_v \mathbf{v}\ f_\alpha(\mathbf{r}, \mathbf{v}, t)\ d^3v \tag{4.4}$$

Note that both $n_\alpha(\mathbf{r}, t)$ and $\mathbf{u}_\alpha(\mathbf{r}, t)$ are *macroscopic* variables which depend only upon the coordinates $\mathbf{r}$ and t.

A systematic method for deducing the macroscopic variables (such as momentum flux, pressure, temperature, heat flux and so on), in terms of the distribution function, is formally presented in Chapter 6.

5. THE BOLTZMANN EQUATION

In order to calculate the average values of the particle physical properties (the macroscopic variables of interest), it is necessary to know the distribution function for the system under consideration. The dependence of the distribution function on the independent variables $\mathbf{r}$, $\mathbf{v}$ and t is governed by an equation known as the *Boltzmann equation.* We present in this section a derivation of the collisionless Boltzmann equation and the general form it takes when the effects of the particle interactions are taken into account, without explicity deriving any particular expression for the collision term.

5.1 Collisionless Boltzmann Equation

Recall that

$$d^6\mathcal{N}_\alpha(\mathbf{r},\mathbf{v},t) = f_\alpha(\mathbf{r},\mathbf{v},t)\ d^3r\ d^3v \tag{5.1}$$

represents the number of particles of type α which, at the instant t, are situated within the volume element $d^3r\ d^3v$ of phase space, about the coordinates ($\mathbf{r}$,$\mathbf{v}$). Suppose that each particle is subjected to an *external* force $\mathbf{F}$. In the absence of particle interactions, a particle of type α with coordinates about ($\mathbf{r}$,$\mathbf{v}$) in phase space, at the instant t, will be found after a time interval dt about the new coordinates ($\mathbf{r}$',$\mathbf{v}$') such that

$$\mathbf{r}'(t+dt) = \mathbf{r}(t) + \mathbf{v}\ dt \tag{5.2}$$

$$\mathbf{v}'(t+dt) = \mathbf{v}(t) + \mathbf{a}\ dt \tag{5.3}$$

where $\mathbf{a} = \mathbf{F}/m_\alpha$ is the particle acceleration and m_α its mass. Thus, all particles of type α inside the volume element $d^3r\ d^3v$ of phase space, about ($\mathbf{r}$,$\mathbf{v}$) at the instant t, will occupy a new volume element $d^3r'\ d^3v'$, about ($\mathbf{r}$',$\mathbf{v}$') after the interval dt (see Fig. 4). Since we are considering the same particles at t and at $t+dt$, we must have, in the absence of collisions,

$$f_\alpha(\mathbf{r}',\mathbf{v}',t+dt)\ d^3r'\ d^3v' = f_\alpha(\mathbf{r},\mathbf{v},t)\ d^3r\ d^3v \tag{5.4}$$

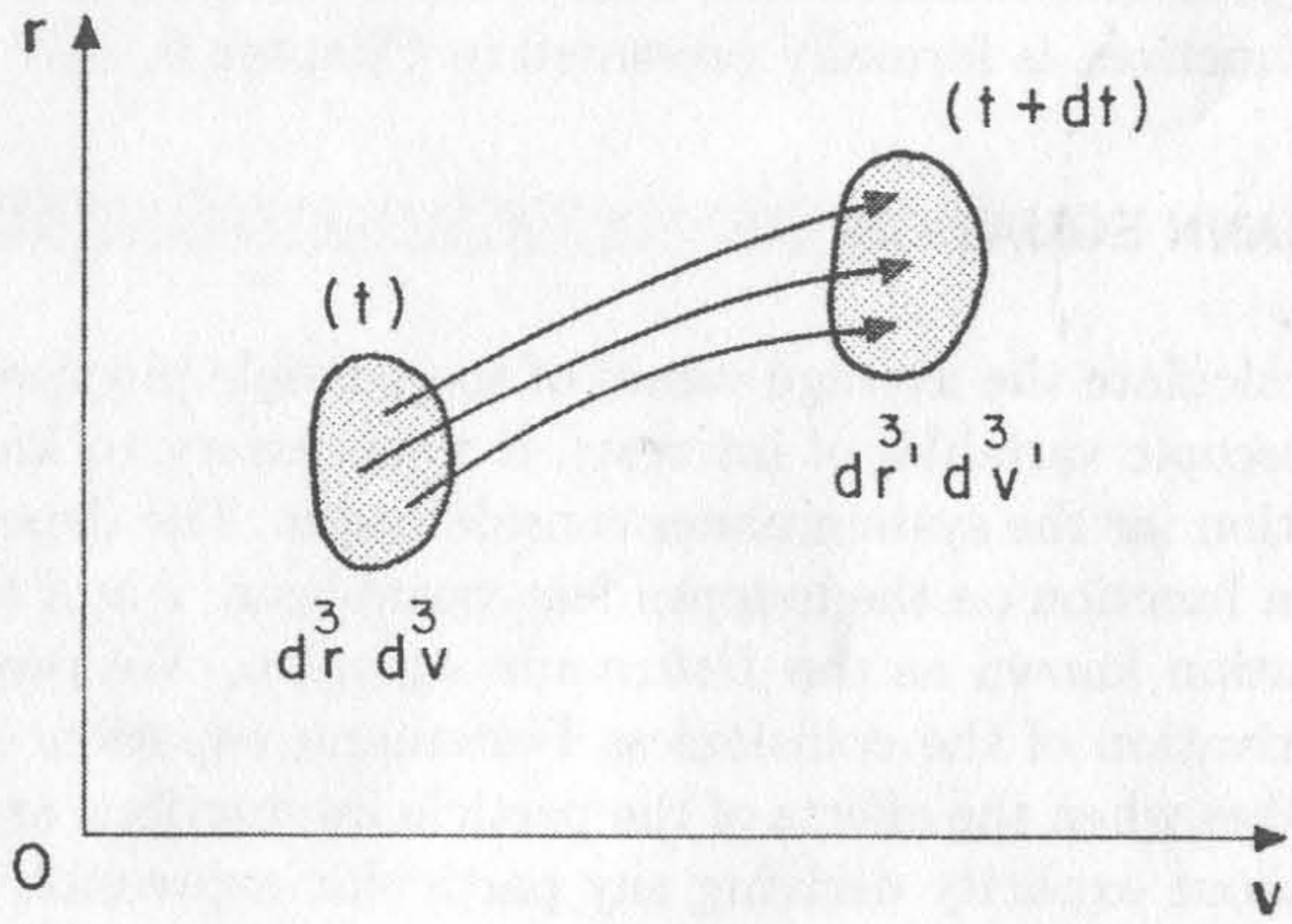


Fig. 4 *In the absence of collisions the particles within the volume element $d^3r\ d^3v$ about* (**r**,**v**), *at an instant t, will occupy after a time interval dt a new volume element $d^3r'\ d^3v'$, about* (**r**', **v**').

The element of volume $d^3r\ d^3v$ may become distorted in shape as a result of the particle motion. The relation between the new element of volume and the initial one is given by

$$d^3r'\ d^3v' = |J|\ d^3r\ d^3v \tag{5.5}$$

where J stands for the Jacobian of the transformation from the initial coordinates (**r**,**v**) to the final ones (**r**',**v**'). It will be shown in the next subsection that for the transformation defined by (5.2) and (5.3) we have $|J| = 1$, so that

$$d^3r'\ d^3v' = d^3r\ d^3v \tag{5.6}$$

and (5.4) becomes

$$[f_\alpha(\mathbf{r}', \mathbf{v}', t + dt) - f_\alpha(\mathbf{r}, \mathbf{v}, t)]\ d^3r\ d^3v = 0 \tag{5.7}$$

The first term on the left-hand side of (5.7) can be expanded in a Taylor series about $f_\alpha(\mathbf{r}, \mathbf{v}, t)$ as follows

$$f_\alpha(\mathbf{r} + \mathbf{v}\ dt, \mathbf{v} + \mathbf{a}\ dt, t + dt) = f_\alpha(\mathbf{r}, \mathbf{v}, t) + \left[\ \frac{\partial f_\alpha}{\partial t} + \right.$$

$$+\left(v_x\frac{\partial f_\alpha}{\partial x}+v_y\frac{\partial f_\alpha}{\partial y}+v_z\frac{\partial f_\alpha}{\partial z}\right)+\left(a_x\frac{\partial f_\alpha}{\partial v_x}+a_y\frac{\partial f_\alpha}{\partial v_y}+a_z\frac{\partial f_\alpha}{\partial v_z}\right)\Big]\,dt \quad (5.8)$$

neglecting terms of order $(dt)^2$ and higher. Using the del operator notation

$$\nabla=\widehat{\mathbf{x}}\frac{\partial}{\partial x}+\widehat{\mathbf{y}}\frac{\partial}{\partial y}+\widehat{\mathbf{z}}\frac{\partial}{\partial z} \quad (5.9)$$

and, in a similar way, defining a del operator in velocity space by

$$\nabla_v=\widehat{\mathbf{x}}\frac{\partial}{\partial v_x}+\widehat{\mathbf{y}}\frac{\partial}{\partial v_y}+\widehat{\mathbf{z}}\frac{\partial}{\partial v_z} \quad (5.10)$$

we obtain from (5.8)

$$f_\alpha(\mathbf{r}+\mathbf{v}\,dt,\mathbf{v}+\mathbf{a}\,dt,t+dt)=f_\alpha(\mathbf{r},\mathbf{v},t)+$$

$$\Big[\frac{\partial f_\alpha(\mathbf{r},\mathbf{v},t)}{\partial t}+\mathbf{v}\cdot\nabla f_\alpha(\mathbf{r},\mathbf{v},t)+\mathbf{a}\cdot\nabla_v f_\alpha(\mathbf{r},\mathbf{v},t)\Big]\,dt \quad (5.11)$$

Substituting this result into (5.7) gives

$$\boxed{\frac{\partial f_\alpha(\mathbf{r},\mathbf{v},t)}{\partial t}+\mathbf{v}\cdot\nabla f_\alpha(\mathbf{r},\mathbf{v},t)+\mathbf{a}\cdot\nabla_v f_\alpha(\mathbf{r},\mathbf{v},t)=0} \quad (5.12)$$

which is the Boltzmann equation in the absence of collisions.

This equation can be rewritten as

$$\frac{\mathcal{D}f_\alpha(\mathbf{r},\mathbf{v},t)}{\mathcal{D}t}=0 \quad (5.13)$$

where the operator

$$\frac{\mathcal{D}}{\mathcal{D}t}=\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla+\mathbf{a}\cdot\nabla_v \quad (5.14)$$

represents the *total derivative* with respect to time, in phase space. Eq. (5.13) is a statement of the conservation of the density of representative points in phase space. If we move along with a representative point in phase space and observe the density of representative points $f_\alpha(\mathbf{r},\mathbf{v},t)$ in its neighborhood, we find that this density remains constant in time. This result is known as *Liouville's theorem.* Note that this result applies

only to the special case in which collisions, as well as radiation losses and processes of production and loss of particles are unimportant.

5.2 Jacobian of the Transformation in Phase Space

To determine the Jacobian of the transformation defined by (5.2) and (5.3) recall that, from its definition, we have

$$J = \frac{\partial(\mathbf{r}', \mathbf{v}')}{\partial(\mathbf{r}, \mathbf{v})} = \frac{\partial(x', y', z', v'_x, v'_y, v'_z)}{\partial(x, y, z, v_x, v_y, v_z)} \tag{5.15}$$

which corresponds to the determinant of the 6×6 matrix

$$J = \begin{pmatrix} \partial x'/\partial x & \partial y'/\partial x & \cdots & \partial v'_z/\partial x \\ \partial x'/\partial y & \partial y'/\partial y & \cdots & \partial v'_z/\partial y \\ \cdots & \cdots & \cdots & \cdots \\ \partial x'/\partial v_z & \partial y'/\partial v_z & \cdots & \partial v'_z/\partial v_z \end{pmatrix} \tag{5.16}$$

We can separate the external force $\mathbf{F}$ into two parts,

$$\mathbf{F} = \mathbf{F}' + q_\alpha(\mathbf{v} \times \mathbf{B}) \tag{5.17}$$

where $\mathbf{F}$' represents a velocity-independent force and the second term is the velocity-dependent force due to an externally applied magnetic field $\mathbf{B}$, the only velocity-dependent force that may concern us in this treatment. The partial derivatives appearing in the matrix J are

$$\frac{\partial x'_i}{\partial x_j} = \delta_{ij} \quad , \qquad \frac{\partial v'_i}{\partial x_j} = \frac{1}{m_\alpha}\frac{\partial F'_i}{\partial x_j}\, dt$$

$$\frac{\partial x'_i}{\partial v_j} = \delta_{ij}\, dt \quad , \qquad \frac{\partial v'_i}{\partial v_j} = \delta_{ij} + \frac{q_\alpha}{m_\alpha}\frac{\partial(\mathbf{v} \times \mathbf{B})_i}{\partial v_j}\, dt \tag{5.18}$$

where (5.2), (5.3) and (5.17) have been used, and where $x_{i,j} = x, y, z$ and $v_{i,j} = v_x, v_y, v_z$. The symbol δ_{ij} is the *Kronecker delta.* The matrix (5.16) can be written in the form

$$J = \begin{pmatrix} (J)_1 & (J)_2 \\ (J)_3 & (J)_4 \end{pmatrix} \tag{5.19}$$

where the $(J)'_i s$, with $i = 1, 2, 3, 4$, represent the following 3×3 submatrices

$$(J)_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.20}$$

$$(J)_2 = \frac{dt}{m_\alpha} \begin{pmatrix} \partial F'_x/\partial x & \partial F'_y/\partial x & \partial F'_z/\partial x \\ \partial F'_x/\partial y & \partial F'_y/\partial y & \partial F'_z/\partial y \\ \partial F'_x/\partial z & \partial F'_y/\partial z & \partial F'_z/\partial z \end{pmatrix} \tag{5.21}$$

$$(J)_3 = \begin{pmatrix} dt & 0 & 0 \\ 0 & dt & 0 \\ 0 & 0 & dt \end{pmatrix} \tag{5.22}$$

$$(J)_4 = \begin{pmatrix} 1 & -aB_z & aB_y \\ aB_z & 1 & -aB_x \\ -aB_y & aB_x & 1 \end{pmatrix} \tag{5.23}$$

where the constant a stands for $(q_\alpha/m_\alpha)\ dt$. Neglecting terms of order $(dt)^2$ it can be easily verified that $|J| = 1$. Thus, up to and including the terms of first order in the infinitesimal dt, we have

$$d^3r'\ d^3v' = d^3r\ d^3v \tag{5.24}$$

which is the result (5.6) used in the previous subsection.

5.3 Effects of Particle Interactions

When the effects due to the particle interactions are taken into account, (5.12) needs to be modified. As a result of collisions during the time interval dt, some of the particles of type α which were initially within the volume element $d^3r\ d^3v$ of phase space may be removed from it, and particles of type α initially outside this volume element may end up inside it. This is indicated schematically in Fig. 5. Generally, the number of particles of type α inside $d^3r\ d^3v$ about the coordinates (**r**,**v**), at an instant t, will be different from the number of particles of type α inside this same volume element about the coordinates (**r**',**v**') at the instant $t + dt$. We shall denote this net gain or loss of particles of type α, as a result of collisions during the interval dt, in the volume element $d^3r\ d^3v$, by

$$\left[\frac{\delta f_\alpha(\mathbf{r}, \mathbf{v}, t)}{\delta t}\right]_{coll} d^3r\ d^3v\ dt \tag{5.25}$$

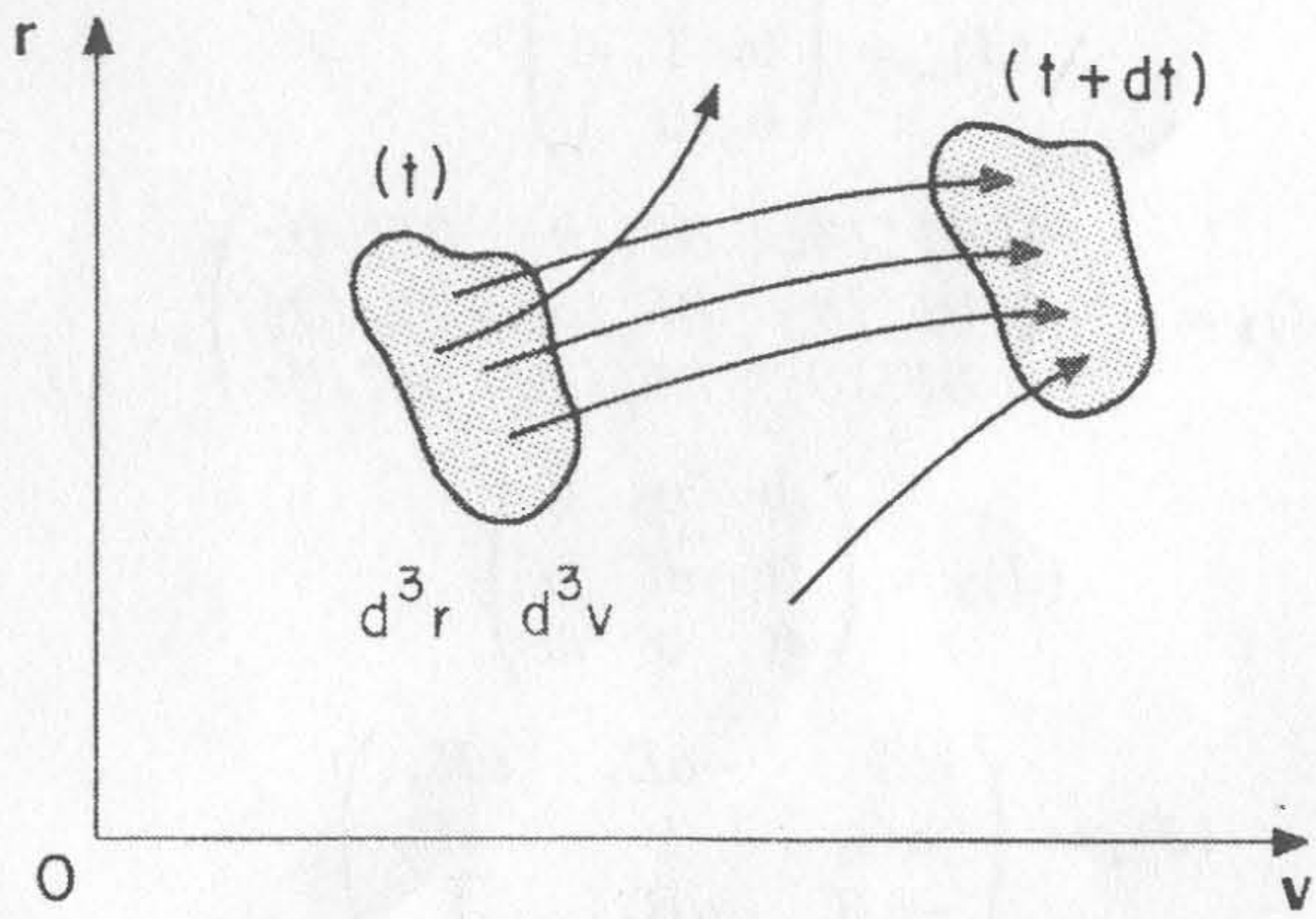


Fig. 5 *Schematic representation of the motion of the volume element $d^3r\ d^3v$ in phase space, showing particles entering and leaving this volume element, as a result of collisions during the time interval dt.*

where $(\delta f_\alpha/\delta t)_{coll}$ represents the rate of change of $f_\alpha(\mathbf{r}, \mathbf{v}, t)$ due to collisions. Thus, when collisions are considered, (5.7) becomes

$$[f_\alpha(\mathbf{r}', \mathbf{v}', t + dt) - f_\alpha(\mathbf{r}, \mathbf{v}, t)]\ d^3r\ d^3v = \left(\frac{\delta f_\alpha}{\delta t}\right)_{coll} d^3r\ d^3v\ dt \qquad (5.26)$$

and the following modified form of Eq. (5.12) results

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v} \cdot \nabla f_\alpha + \mathbf{a} \cdot \nabla_v f_\alpha = \left(\frac{\delta f_\alpha}{\delta t}\right)_{coll} \qquad (5.27)$$

Using the total time derivative operator, defined in (5.14), we can rewrite this equation in compact form as

$$\frac{\mathcal{D} f_\alpha}{\mathcal{D} t} = \left(\frac{\delta f_\alpha}{\delta t}\right)_{coll} \qquad (5.28)$$

This equation is obviously incomplete, since the precise form of the collision term is not known. In the following section we will consider a very simple expression for the collision term, known as the *Krook model* or *relaxation model.* More elaborate expressions, such as the *Boltzmann*

collision integral and the *Fokker-Planck collision term*, will be considered in Chapter 21.

6. RELAXATION MODEL FOR THE COLLISION TERM

A very simple method for taking into account collision effects is provided by the relaxation model. In this model it is assumed that the effect of collisions is to restore a situation of *local equilibrium*, characterized by a local equilibrium distribution function $f_{\alpha 0}(\mathbf{r}, \mathbf{v})$. In the absence of external forces, it assumes that a situation initially not in equilibrium, described by a distribution function $f_\alpha(\mathbf{r}, \mathbf{v}, t)$ different from $f_{\alpha 0}(\mathbf{r}, \mathbf{v})$, reaches a local equilibrium condition exponentially with time, as a result of collisions, with a *relaxation time* τ. This relaxation time is of the order of the time between collisions and may also be written as ν^{-1} where ν represents a *relaxation collision frequency*. This model was originally developed by *Krook* and can be expressed mathematically as

$$\left(\frac{\delta f_\alpha}{\delta t}\right)_{coll} = -\frac{(f_\alpha - f_{\alpha 0})}{\tau} \tag{6.1}$$

According to this expression for the collision term, when $f_\alpha = f_{\alpha 0}$ we have $(\delta f_\alpha / \delta t)_{coll} = 0$, so that in a state of local equilibrium the distribution function is not altered as a result of collisions.

In order to bring out the physical meaning of the relaxation model, let us consider the Boltzmann equation with this collision term, in the absence of external forces and spatial gradients, and when $f_{\alpha 0}$ and τ are time-independent,

$$\frac{\partial f_\alpha}{\partial t} = -\frac{(f_\alpha - f_{\alpha 0})}{\tau} \tag{6.2}$$

which can be rewritten as

$$\frac{\partial f_\alpha}{\partial t} + \frac{f_\alpha}{\tau} = \frac{f_{\alpha 0}}{\tau} \tag{6.3}$$

This simple inhomogeneous differential equation has $Ce^{-t/\tau}$ as the homogeneous solution, where C is a constant, and $f_{\alpha 0}$ as a particular integral. Therefore, the complete solution is

$$f_\alpha(\mathbf{v}, t) = f_{\alpha 0} + [f_\alpha(\mathbf{v}, 0) - f_{\alpha 0}]\, e^{-t/\tau} \tag{6.4}$$

Thus, the difference between f_α and $f_{\alpha 0}$ decreases exponentially in time at a rate governed by the relaxation collision frequency $\nu = 1/\tau$.

This collisional model has proved to be useful and, in some cases, leads to results almost identical to the ones obtained using the Boltzmann collision integral (deduced in Chapter 21). It is particularly applicable to a weakly ionized plasma in which only charge-neutral collisions are important. However, it oversimplifies the entire relaxation phenomena and does not predict correctly the different relaxation collision frequencies for the various physical quantities of interest, such as the macroscopic velocity, momentum and energy. According to the relaxation model, these macroscopic physical variables approach equilibrium at the *same* rate ν. A detailed analysis of the collision process, however, shows that this is not the case and the relaxation times for the various macroscopic variables differs to some extent. For nonrelativistic velocities, while the relaxation times for the average velocity and the momentum are found to be the same, approximately τ, that of the average thermal energy is approximately $(m_\beta/2m_\alpha)\tau$. Hence, for collisions between electrons and neutral particles, the relaxation time for the kinetic enrgy of the electrons is longer than that for the average velocity by a factor which is of the order of the ratio of the neutral particle mass to the electron mass. The relaxation model is therefore strictly applicable only to the cases of collisions between particles of the same mass. In spite of this limitation, the relaxation model is still useful partly because of its simplicity and partly because it usually gives a first approximation to the problem under consideration.

7. THE VLASOV EQUATION

A very useful approximate way to describe the dynamics of a plasma is to consider that the plasma particle motions are governed by the applied *external* fields *plus* the macroscopic *average internal* fields, smoothed in space and time, due to the presence and motion of all plasma particles. The problem of obtaining the macroscopic (smoothed) internal electromagnetic fields, however, is still a complex one and requires that a *self-consistent* solution be obtained.

The Vlasov equation is a partial differential equation that describes the time evolution of the distribution function in phase space and which directly incorporates the smoothed macroscopic internal electromagnetic fields. It may be obtained from the Boltzmann equation (5.27) with the collision term $(\delta f_\alpha/\delta t)_{coll}$ equal to zero, but including the *internal smoothed fields* in the force term,

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v}\cdot\nabla f_\alpha + \frac{1}{m_\alpha}[\mathbf{F}_{ext} + q_\alpha(\mathbf{E}_i + \mathbf{v}\times\mathbf{B}_i)]\cdot\nabla_v f_\alpha = 0 \qquad (7.1)$$

Here $\mathbf{F}_{ext}$ represents the external force, including the Lorentz force associated with any *externally* applied electric and magnetic fields, and $\mathbf{E}_i$ and $\mathbf{B}_i$ are *internal* smoothed electric and magnetic fields due to the presence and motion of all charged particles inside the plasma. In order that the internal macroscopic electromagnetic fields $\mathbf{E}_i$ and $\mathbf{B}_i$ be consistent with the macroscopic charge and current densities existing in the plasma itself, they must satisfy Maxwell equations

$$\nabla \cdot \mathbf{E}_i = \frac{\rho}{\epsilon_0} \tag{7.2}$$

$$\nabla \cdot \mathbf{B}_i = 0 \tag{7.3}$$

$$\nabla \times \mathbf{E}_i = -\frac{\partial \mathbf{B}_i}{\partial t} \tag{7.4}$$

$$\nabla \times \mathbf{B}_i = \mu_0 \left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}_i}{\partial t} \right) \tag{7.5}$$

with the plasma charge density ρ and the plasma current density $\mathbf{J}$ given by the expressions

$$\rho(\mathbf{r}, t) = \sum_\alpha q_\alpha \, n_\alpha(\mathbf{r}, t) = \sum_\alpha q_\alpha \int_v f_\alpha(\mathbf{r}, \mathbf{v}, t) \, d^3v \tag{7.6}$$

$$\mathbf{J}(\mathbf{r}, t) = \sum_\alpha q_\alpha \, n_\alpha(\mathbf{r}, t) \, \mathbf{u}_\alpha(\mathbf{r}, t) = \sum_\alpha q_\alpha \int_v \mathbf{v} \, f_\alpha(\mathbf{r}, \mathbf{v}, t) \, d^3v \tag{7.7}$$

the summations being over the different charged particle species in the plasma. Here $\mathbf{u}_\alpha(\mathbf{r}, t)$ denotes the macroscopic *average velocity* for the particles of type α, given in (4.4).

Eqs. (7.1) to (7.7) constitute a complete set of *self-consistent* equations to be solved simultaneously. For example, in an iterative procedure assuming starting approximate values for $\mathbf{E}_i(\mathbf{r}, t)$ and $\mathbf{B}_i(\mathbf{r}, t)$, Eq. (7.1) can be solved to yield $f_\alpha(\mathbf{r}, \mathbf{v}, t)$ for the various different species. Using the calculated $f'_\alpha s$ in (7.6) and (7.7), leads to values for the charge and current densities (ρ and $\mathbf{J}$) in the plasma, which can be substituted into Maxwell equations and solved for $\mathbf{E}_i(\mathbf{r}, t)$ and $\mathbf{B}_i(\mathbf{r}, t)$. These values are then plugged back into the Vlasov equation, and so on, in order to obtain a *self-consistent* solution for the single particle distribution function.

Although the Vlasov equation does not explicitly include a collision term in its right-hand side and, hence, does not take into account short-range collisions, it is not so restrictive as it may appear, since a significant

part of the effects of the particle interactions has already been included in the Lorentz force, through the internal self-consistent smoothed electromagnetic fields.

PROBLEMS

5.1 Consider a system of particles uniformly distributed in space, with a constant particle number density n_0, and characterized by a velocity distribution function $f(v)$ such that

$$f(v) = K_0 \quad for \quad |v_i| \leq v_0 \quad (i = x, y, z)$$

$$f(v) = 0 \quad otherwise,$$

where K_0 is a non-zero positive constant. Determine the value of K_0 in terms of n_0 and v_0.

5.2 Consider the following *two-dimensional* Maxwellian distribution function

$$f(v_x, v_y) = n_0 \left(\frac{m}{2\pi kT}\right) exp\left[-\frac{m(v_x^2 + v_y^2)}{2kT}\right]$$

(a) Verify that n_0 represents correctly the particle number density, that is, the number of particles per unit area.
(b) Sketch, in a three-dimensionsal perspective view, the surface for this distribution function, plotting $f(v_x, v_y)$ in terms of v_x and v_y. Draw, on this surface, curves of constant v_x, curves of constant v_y, and curves of constant f.

5.3 The electrons inside a system of two coaxial magnetic mirrors can be described by the so-called *loss-cone distribution function*

$$f(\mathbf{v}) = \frac{n_0}{\pi^{3/2}\alpha_\perp^2 \alpha_\parallel}\left(\frac{v_\perp}{\alpha_\perp}\right)^2 exp\left[-\left(\frac{v_\perp}{\alpha_\perp}\right)^2 - \left(\frac{v_\parallel}{\alpha_\parallel}\right)^2\right]$$

where $v_\parallel$ and $v_\perp$ denote the magnitudes of the electron velocities in the directions parallel and perpendicular to the magnetic bottle axis, respectively, and where $\alpha_\parallel^2 = 2kT_\parallel/m$ and $\alpha_\perp^2 = 2kT_\perp/m$.
(a) Verify that the number density of the electrons in the magnetic bottle is given by n_0.

(b) Justify the applicability of the loss-cone distribution function to a magnetic mirror bottle, by analyzing its dependence on $v_\parallel$ and $v_\perp$. Sketch, in a three-dimensionsal perspective view, the surface for $f(\mathbf{v})$ as a function of $v_\parallel$ and $v_\perp$.

5.4 Consider the motion of charged particles, in one dimension only, in the presence of an electric potential $V(x)$. Show, by direct substitution, that a function of the form

$$f = f(\tfrac{1}{2}mv^2 + qV)$$

is a solution of the Boltzmann equation under steady state conditions.

5.5 (a) Show that the Boltzmann equation, in cylindrical coordinates, can be written as

$$\frac{\partial f}{\partial t} + \dot{r}\frac{\partial f}{\partial r} + \dot{\phi}\frac{\partial f}{\partial \phi} - r\dot{\phi}^2\frac{\partial f}{\partial \dot{r}} + \frac{2\dot{r}\dot{\phi}}{r}\frac{\partial f}{\partial \dot{\phi}} + \dot{z}\frac{\partial f}{\partial z} +$$

$$+\frac{1}{m}\left(F_r\frac{\partial f}{\partial \dot{r}} + \frac{F_\phi}{r}\frac{\partial f}{\partial \dot{\phi}} + F_z\frac{\partial f}{\partial \dot{z}}\right) = \left(\frac{\delta f}{\delta t}\right)_{coll}$$

where the dot over the symbols stands for the time derivative operator d/dt and where $F_r = m(d^2r/dt^2)$, $F_\phi = mr(d^2\phi/dt^2)$ and $F_z = m(d^2z/dt^2)$.
(b) Show, by direct substitution, that in the presence of an azimuthally symmetric magnetic field (in the z direction) a function of the form

$$f = f(\tfrac{1}{2}mv^2, mr^2\dot{\phi} + qrA_\phi)$$

is a solution of the Boltzmann equation under steady conditions, where the constant canonical momentum is given by $p_\phi = mr^2\dot{\phi} + qrA_\phi$, and where A_ϕ denotes the ϕ component of the magnetic potential $\mathbf{A}$, defined such that $\mathbf{B} = \nabla \times \mathbf{A}$.

5.6 Show that the Vlasov equation for a homogeneous plasma under the influence of a uniform external magnetostatic field $\mathbf{B}_0$, in the equilibrium state, is satisfied by any homogeneous distribution function, $f(v_\parallel, v_\perp)$, which is cylindrically symmetric with respect to the magnetostatic field.

5.7 The entropy of a system can be expressed, in terms of the distribution function, as

$$S = -k\int_r\int_v f\,\ln(f)\;d^3r\;d^3v$$

Show that, for a system which obeys the collisionless Boltzmann equation, the total time derivative of the entropy vanishes.

5.8 Consider a one-dimensional harmonic oscillator whose total energy can be expressed by

$$E = \tfrac{1}{2}(mv^2 + cx^2)$$

where c is a constant and x its displacement coordinate. Show that the trajectory described by the representative point of the oscillator, in phase space, is an ellipse.

6

AVERAGE VALUES AND MACROSCOPIC VARIABLES

1. AVERAGE VALUE OF A PHYSICAL QUANTITY

A systematic method for obtaining the average values of functions of particle velocities is presented in this chapter. The *macroscopic variables*, such as number density, flow velocity, kinetic pressure, thermal energy flux, and so on, can be considered as average values of physical quantities, involving the collective behavior of a large number of particles. These macroscopic variables are related to the the various *moments* of the distribution function. A formal definition of the moments of the distribution function is presented in Section 10.

To each particle in the plasma we can associate some molecular property, $\chi(\mathbf{r}, \mathbf{v}, t)$, which in general may be a function of the position $\mathbf{r}$ of the particle, of its velocity $\mathbf{v}$ and of the time t. This property may be, for example, the mass, the velocity, the momentum, or the energy of the particle.

In order to calculate the average value of the property $\chi(\mathbf{r}, \mathbf{v}, t)$, recall that $d^6\mathcal{N}_\alpha(\mathbf{r}, \mathbf{v}, t)$ represents the number of particles of type α inside the phase space volume element $d^3r\ d^3v$ about $(\mathbf{r},\mathbf{v})$, at the instant t. Thus, the total value of $\chi(\mathbf{r}, \mathbf{v}, t)$ for all the particles of type α inside $d^3r\ d^3v$ is given by

$$\chi(\mathbf{r}, \mathbf{v}, t)\ d^6\mathcal{N}_\alpha(\mathbf{r}, \mathbf{v}, t) = \chi(\mathbf{r}, \mathbf{v}, t)\ f_\alpha(\mathbf{r}, \mathbf{v}, t)\ d^3r\ d^3v \tag{1.1}$$

The total value of $\chi(\mathbf{r}, \mathbf{v}, t)$ for all the particles of type α inside the volume element d^3r of configuration space, irrespective of velocity, is obtained by

integrating (1.1) over all possible velocities,

$$d^3r \int_v \chi(\mathbf{r}, \mathbf{v}, t)\ f_\alpha(\mathbf{r}, \mathbf{v}, t)\ d^3v \tag{1.2}$$

The average value of $\chi(\mathbf{r}, \mathbf{v}, t)$ can now be obtained by dividing (1.2) by the number of particles of type α inside d^3r about $\mathbf{r}$, at the instant t, i.e. by $n_\alpha(\mathbf{r}, t)\ d^3r$. We define, therefore, the average value of the property $\chi(\mathbf{r}, \mathbf{v}, t)$ for the particles of type α by

$$< \chi(\mathbf{r}, \mathbf{v}, t) >_\alpha = \frac{1}{n_\alpha(\mathbf{r}, t)} \int_v \chi(\mathbf{r}, \mathbf{v}, t)\ f_\alpha(\mathbf{r}, \mathbf{v}, t)\ d^3v \tag{1.3}$$

The symbol $< \quad >_\alpha$ denotes the average value with respect to velocity space for the particles of type α. Note that the average value is always independent of $\mathbf{v}$, being a function of only $\mathbf{r}$ and t.

If we take $\chi(\mathbf{r}, \mathbf{v}, t) = 1$ in (1.3), the expression for the number density $n_\alpha(\mathbf{r}, t)$, given in (5.4.2), is obtained.

2. AVERAGE VELOCITY AND PECULIAR VELOCITY

Consider now $\chi(\mathbf{r}, \mathbf{v}, t)$ as being the velocity $\mathbf{v}$ of the type α particles in the vicinity of the position $\mathbf{r}$, at the instant t. Replacing $\chi(\mathbf{r}, \mathbf{v}, t)$ in (1.3) by $\mathbf{v}$ gives the macroscopic *average velocity* or *flow velocity* $\mathbf{u}_\alpha(\mathbf{r}, t)$ for the particles of type α,

$$\mathbf{u}_\alpha(\mathbf{r}, t) = < \mathbf{v} >_\alpha = \frac{1}{n_\alpha(\mathbf{r}, t)} \int_v \mathbf{v}\ f_\alpha(\mathbf{r}, \mathbf{v}, t)\ d^3v \tag{2.1}$$

which is the same expression given earlier in (5.4.4).

Note that $\mathbf{r}$, $\mathbf{v}$ and t are taken as independent variables, whereas the average velocity $\mathbf{u}_\alpha(\mathbf{r}, t)$ depends on $\mathbf{r}$ and t. For the cases in which $\chi(\mathbf{r}, \mathbf{v}, t)$ is independent of the particle velocity, we have

$$< \chi(\mathbf{r}, t) >_\alpha = \chi_\alpha(\mathbf{r}, t) \tag{2.2}$$

so that, for example, $< \mathbf{u}_\alpha > = \mathbf{u}_\alpha$. In what follows, the index α after the average value symbol will be omitted whenever it is redundant, that is, $< \mathbf{u}_\alpha >_\alpha \equiv \mathbf{u}_\alpha$.

The *peculiar velocity* or *random velocity* $\mathbf{c}_\alpha$ is defined as the velocity of a type α particle relative to the average velocity $\mathbf{u}_\alpha(\mathbf{r}, t)$,

$$\mathbf{c}_\alpha = \mathbf{v} - \mathbf{u}_\alpha \tag{2.3}$$

Consequently, we always have $< \mathbf{c}_\alpha > = 0$, since $< \mathbf{v}_\alpha > = \mathbf{u}_\alpha(\mathbf{r}, t)$. The peculiar velocity $\mathbf{c}_\alpha$ is the one associated with the *random* or *thermal motions* of the particles of type α. When $\mathbf{u}_\alpha(\mathbf{r}, t)$ vanishes we have $\mathbf{c}_\alpha \equiv \mathbf{v}$.

3. FLUX

From the concept of distribution function many other macroscopic variables can be defined in terms of average values. Macroscopic variables such as the *particle current density* (or *particle flux*), the *pressure dyad* or *tensor*, and the *heat flow vector* (or *thermal energy flux*), involve the *flux* of some molecular property $\chi(\mathbf{r}, \mathbf{v}, t)$. The flux of $\chi(\mathbf{r}, \mathbf{v}, t)$ is defined as the amount of the quantity $\chi(\mathbf{r}, \mathbf{v}, t)$ transported across some given surface, per unit area and per unit time.

Consider a surface element $d\mathbf{S}$ inside the plasma. If the distribution of velocities is isotropic, the flux will be independent of the relative orientation in space of the surface element $d\mathbf{S}$. However, more generally, when the velocity distribution is anisotropic the flux will depend on the relative spatial orientation of $d\mathbf{S}$. Suppose, therefore, that the surface element of magnitude dS is oriented along some direction specified by the unit vector $\widehat{\mathbf{n}}$,

$$d\mathbf{S} = dS\ \widehat{\mathbf{n}} \tag{3.1}$$

$\widehat{\mathbf{n}}$ being normal to the surface element. In the case of an open surface there are two possible directions for the normal $\widehat{\mathbf{n}}$, one opposite to the other. The direction which is taken as positive is related to the positive sense of traversing the perimeter (bounding curve) of the open surface, according to the following convention: if the positive sense of traversal of the perimeter of a horizontal open surface is taken as counterclockwise, then the positive normal to the open surface is up; if the positive sense of traversal of the perimeter is clockwise, then the positive normal to the open surface is down, as shown in Fig. 1. For a closed surface the normal unit vector is conventionally chosen to point outward.

The particles inside the plasma, due to their velocities, will move across the surface element $d\mathbf{S}$ carrying the property $\chi(\mathbf{r}, \mathbf{v}, t)$ with them. We want to calculate the number of particles of type α that move across $d\mathbf{S}$ during the time interval dt.

The particles with velocity between $\mathbf{v}$ and $\mathbf{v} + d\mathbf{v}$ that will cross $d\mathbf{S}$ in the interval between t and $t + dt$ must lie initially in the volume of the prism of base dS and side $v\ dt$, as indicated in Fig. 2. The volume of this prism is

$$d^3r = d\mathbf{S} \cdot \mathbf{v}\ dt = \widehat{\mathbf{n}} \cdot \mathbf{v}\ dS\ dt \tag{3.2}$$

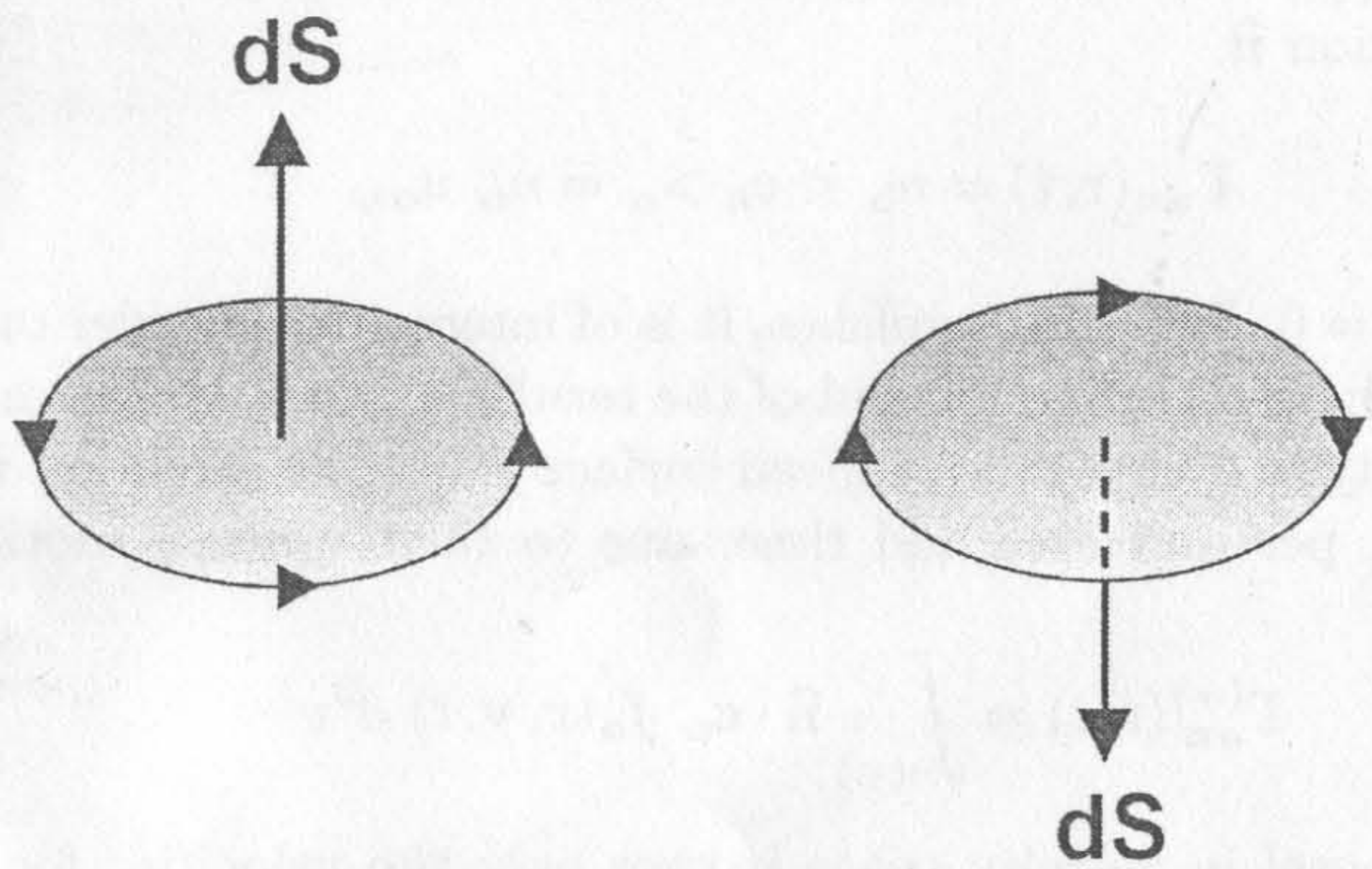


Fig. 1 *Direction of the positive normal to the surface element $d\mathbf{S}$ as related to the sense of traversing the perimeter of $d\mathbf{S}$.*

From the definition of $f_\alpha(\mathbf{r}, \mathbf{v}, t)$, the number of particles of type α in the volume of this prism, that have velocities between $\mathbf{v}$ and $\mathbf{v} + d\mathbf{v}$, is

$$f_\alpha(\mathbf{r}, \mathbf{v}, t)\ d^3r\ d^3v = f_\alpha(\mathbf{r}, \mathbf{v}, t)\ \widehat{\mathbf{n}} \cdot \mathbf{v}\ dS\ dt\ d^3v \tag{3.3}$$

so that the total amount of $\chi(\mathbf{r}, \mathbf{v}, t)$ transported across $d\mathbf{S}$, in the interval dt, is obtained by multiplying this number of particles by $\chi(\mathbf{r}, \mathbf{v}, t)$ and integrating the result over all possible velocities,

$$\int_v \chi(\mathbf{r}, \mathbf{v}, t)\ f_\alpha(\mathbf{r}, \mathbf{v}, t)\ \widehat{\mathbf{n}} \cdot \mathbf{v}\ d^3v\ dS\ dt \tag{3.4}$$

Note that the contributions corresponding to a rotation of the segment $v\ dt$ over all possible directions about $d\mathbf{S}$ are taken into account in the integration over velocity space. Particles that cross $d\mathbf{S}$ in a direction such that $\widehat{\mathbf{n}} \cdot \mathbf{v}$ is positive give a positive contribution to the flux in the direction of $\widehat{\mathbf{n}}$, while particles that cross $d\mathbf{S}$ in a direction such that $\widehat{\mathbf{n}} \cdot \mathbf{v}$ is negative give a negative contribution to the flux in the direction of $\widehat{\mathbf{n}}$, as illustrated in Fig. 3.

The net amount of the quantity $\chi(\mathbf{r}, \mathbf{v}, t)$ transported by the particles of type α, per unit area and per unit time, is obtained by dividing expression (3.4) by $dS\ dt$. The flux in the direction $\widehat{\mathbf{n}}$, $\Phi_{\alpha n}(\chi)$, is therefore given by

$$\Phi_{\alpha n}(\chi) = \int_v \chi(\mathbf{r}, \mathbf{v}, t)\ f_\alpha(\mathbf{r}, \mathbf{v}, t)\ \widehat{\mathbf{n}} \cdot \mathbf{v}\ d^3v \tag{3.5}$$

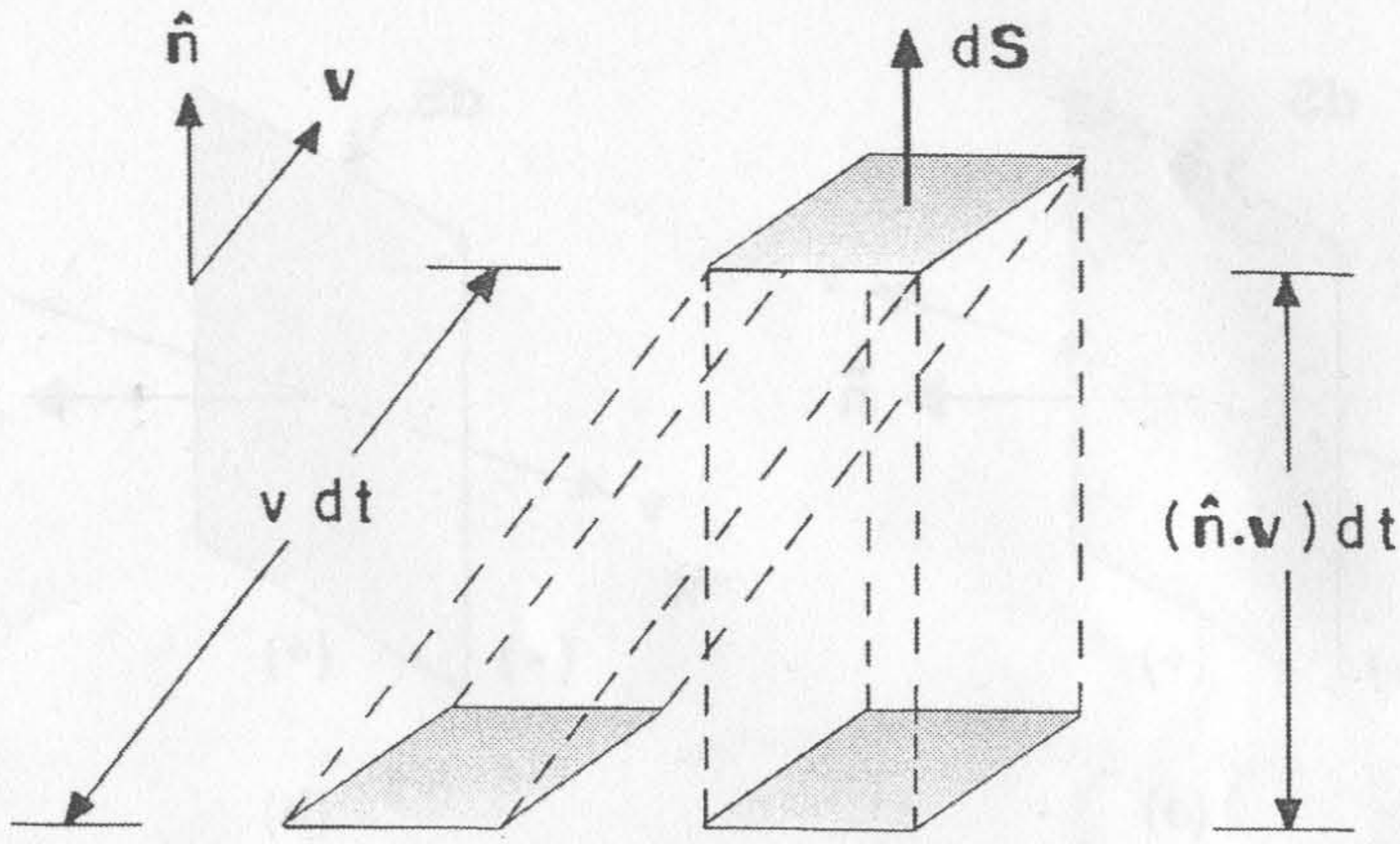


Fig. 2 *Prism of volume* $d^3r = d\mathbf{S}\cdot\mathbf{v}\ dt = \widehat{\mathbf{n}}\cdot\mathbf{v}\ dS\ dt$ *containing the particles of type* α *with velocities between* $\mathbf{v}$ *and* $\mathbf{v}+d\mathbf{v}$, *and which will cross* $d\mathbf{S}$ *in the time interval* dt.

or, using the average value symbol,

$$\Phi_{\alpha n}(\chi) = n_\alpha(\mathbf{r},t) < \chi(\mathbf{r},\mathbf{v},t)\ \widehat{\mathbf{n}}\cdot\mathbf{v} >_\alpha = n_\alpha < \chi v_n >_\alpha \tag{3.6}$$

where $v_n = \widehat{\mathbf{n}}\cdot\mathbf{v}$ denotes the component of $\mathbf{v}$ along the direction specified by the unit vector $\widehat{\mathbf{n}}$.

When $\chi(\mathbf{r},\mathbf{v},t)$ is a *scalar* quantity, $\Phi_{\alpha n}(\chi)$ can be considered as the component, along $\widehat{\mathbf{n}}$, of a *vector* flux $\mathbf{\Phi}_\alpha(\chi)$, that is,

$$\Phi_{\alpha n}(\chi) = \widehat{\mathbf{n}}\cdot\mathbf{\Phi}_\alpha(\chi) \tag{3.7}$$

with

$$\mathbf{\Phi}_\alpha(\chi) = n_\alpha < \chi\mathbf{v} >_\alpha \tag{3.8}$$

If $\chi(\mathbf{r},\mathbf{v},t)$ represents a *vector* quantity, which in this case we shall denote it by $\mathbf{X}(\mathbf{r},\mathbf{v},t)$, then we will have a flux *dyad* (or *tensor*),

$$\Phi_\alpha(\chi) = n_\alpha < \mathbf{X}\mathbf{v} >_\alpha \tag{3.9}$$

and if it represents a *dyad* quantity we will have a *flux triad* , and so on.

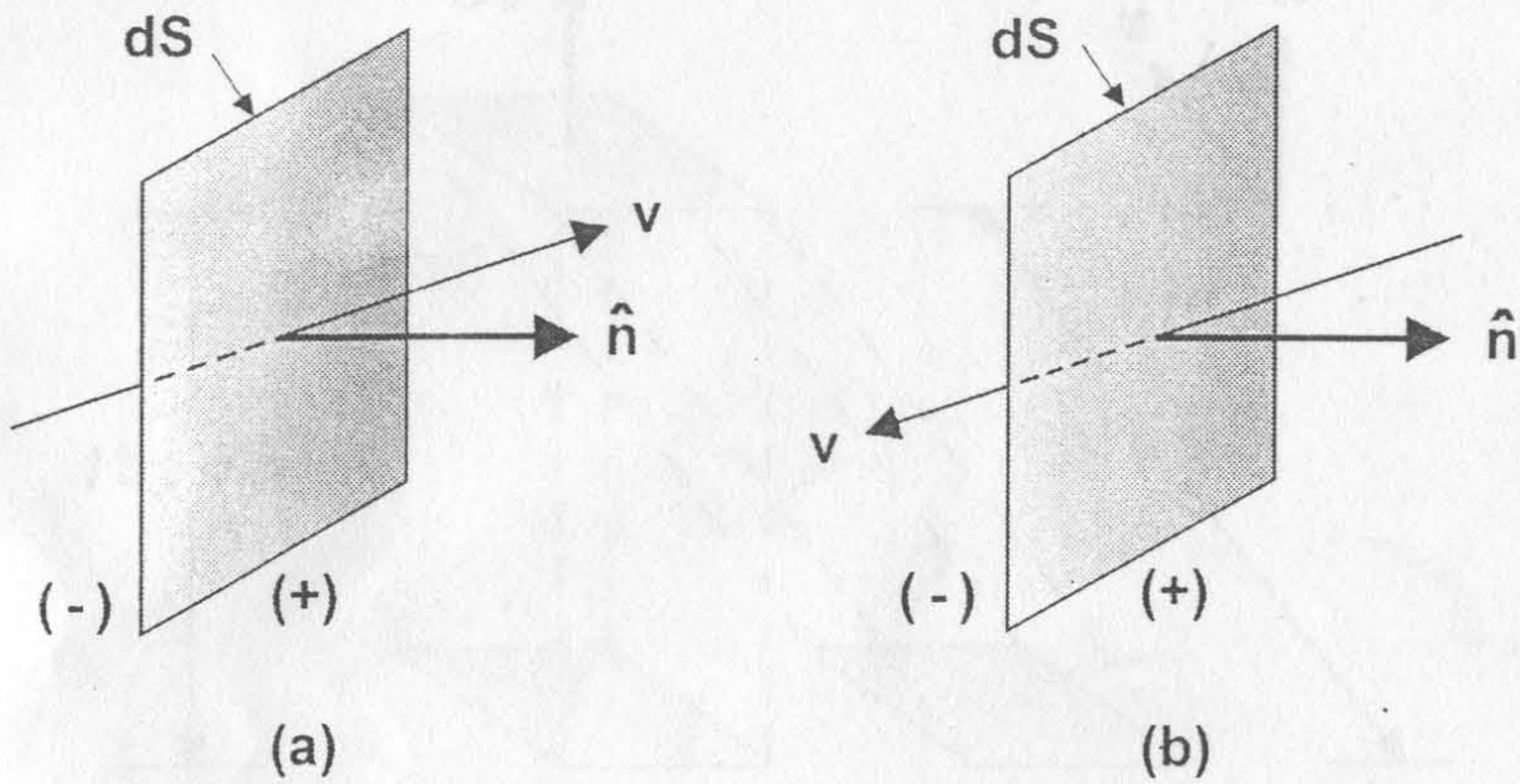


Fig. 3 *(a) Particles that cross $d\mathbf{S}$ from the (-) region to the (+) region contribute positively to the flux in the direction $\widehat{\mathbf{n}}$, while (b) particles that cross $d\mathbf{S}$ from the (+) region to the (-) region contribute negatively to the flux in the direction $\widehat{\mathbf{n}}$.*

In many situations of practical interest it is important to separately consider the contribution to the flux due to the *average* velocity $\mathbf{u}_\alpha(\mathbf{r},t)$, and that due to the *random* velocity $\mathbf{c}_\alpha$ of the particles of type α. Substituting $\mathbf{v} = \mathbf{c}_\alpha + \mathbf{u}_\alpha$ in (3.6), gives

$$\Phi_{\alpha n}(\chi) = n_\alpha < \chi\, c_{\alpha n} > + n_\alpha < \chi\, u_{\alpha n} > \tag{3.10}$$

where $c_{\alpha n} = \widehat{\mathbf{n}} \cdot \mathbf{c}_\alpha$ and $u_{\alpha n} = \widehat{\mathbf{n}} \cdot \mathbf{u}_\alpha$.

For the cases in which the flow velocity $\mathbf{u}_\alpha$ is zero or, equivalently, if we take $d\mathbf{S}$ to be in a frame of reference moving with the average velocity $\mathbf{u}_\alpha$, (3.10) becomes

$$\Phi_{\alpha n}(\chi) = n_\alpha < \chi\, c_{\alpha n} > \tag{3.11}$$

which is the flux of $\chi(\mathbf{r}, \mathbf{v}, t)$ along $\widehat{\mathbf{n}}$ due to the *random* motions of the particles of type α.

4. PARTICLE CURRENT DENSITY

The *particle current density* (or *particle flux*) is defined as the number of particles passing through a given surface, per unit area and per unit

time. Taking $\chi(\mathbf{r}, \mathbf{v}, t) = 1$ in (3.6), we obtain the flux of particles of type α in the direction $\widehat{\mathbf{n}}$,

$$\Gamma_{\alpha n}(\mathbf{r}, t) = n_\alpha < v_n >_\alpha = n_\alpha \, u_{\alpha n} \tag{4.1}$$

since $< c_{\alpha n} > = 0$. When $\mathbf{u}_\alpha$ vanishes, it is of interest to consider only the flux in the positive direction instead of the resulting net flux. The number of particles of type α that cross a given surface along the direction $\widehat{\mathbf{n}}$ from the *same* side, per unit area and time, due to their random motions, is given by

$$\Gamma^{(+)}_{\alpha n}(\mathbf{r}, t) = \int_{v(+)} \widehat{\mathbf{n}} \cdot \mathbf{c}_\alpha \, f_\alpha(\mathbf{r}, \mathbf{v}, t) \, d^3v \tag{4.2}$$

where the integral in velocity space is over only the velocities for which $\widehat{\mathbf{n}} \cdot \mathbf{c}_\alpha > 0$.

The *random mass flux* in the positive direction of $\widehat{\mathbf{n}}$ is consequently given by $m_\alpha \Gamma^{(+)}_{\alpha n}(\mathbf{r}, t)$, where m_α is the mass of the type α particle.

5. MOMENTUM FLOW DYAD OR TENSOR

This quantity is defined as the net momentum transported per unit area and time through some surface element $\widehat{\mathbf{n}} \, dS$. If we take, in (3.6), $\chi(\mathbf{r}, \mathbf{v}, t)$ as the component of momentum of the type α particles along some direction specified by the unit vector $\widehat{\mathbf{j}}$, that is,

$$\chi_j = m_\alpha \mathbf{v} \cdot \widehat{\mathbf{j}} = m_\alpha \, v_j \tag{5.1}$$

we obtain the element $\Pi_{\alpha jn}(\mathbf{r}, t)$ of the momentum flow tensor

$$\Pi_{\alpha jn}(\mathbf{r}, t) = n_\alpha < m_\alpha (\widehat{\mathbf{j}} \cdot \mathbf{v})(\mathbf{v} \cdot \widehat{\mathbf{n}}) >_\alpha = \rho_{m\alpha} < v_j \, v_n >_\alpha \tag{5.2}$$

where $\rho_{m\alpha} = n_\alpha m_\alpha$ denotes the mass density of the type α particles. Thus, the momentum flow element $\Pi_{\alpha jn}(\mathbf{r}, t)$ represents the flux of the jth component of the momentum of the type α particles through a surface element whose normal is oriented along $\widehat{\mathbf{n}}$. Since $\mathbf{v} = \mathbf{c}_\alpha + \mathbf{u}_\alpha$, we obtain

$$\Pi_{\alpha jn}(\mathbf{r}, t) = \rho_{m\alpha} < c_{\alpha j} \, c_{\alpha n} > \; + \rho_{m\alpha} \, u_{\alpha j} \, u_{\alpha n} \tag{5.3}$$

or, in dyadic form,

$$\mathit{\Pi}_\alpha(\mathbf{r}, t) = \rho_{m\alpha} < \mathbf{c}_\alpha \, \mathbf{c}_\alpha > \; + \rho_{m\alpha} \mathbf{u}_\alpha \, \mathbf{u}_\alpha \tag{5.4}$$

where we have used the result $< \mathbf{u}_\alpha \, \mathbf{c}_\alpha > = \mathbf{u}_\alpha < \mathbf{c}_\alpha > = 0$.

In a Cartesian coordinate system (x, y, z) the momentum flow dyad can be written in the following form, in terms of its components,

$$\begin{aligned} \mathit{\Pi}_\alpha = {} & \widehat{\mathrm{x}}\widehat{\mathrm{x}}\Pi_{\alpha xx} + \widehat{\mathrm{x}}\widehat{\mathrm{y}}\Pi_{\alpha xy} + \widehat{\mathrm{x}}\widehat{\mathrm{z}}\Pi_{\alpha xz} \\ & + \widehat{\mathrm{y}}\widehat{\mathrm{x}}\Pi_{\alpha yx} + \widehat{\mathrm{y}}\widehat{\mathrm{y}}\Pi_{\alpha yy} + \widehat{\mathrm{y}}\widehat{\mathrm{z}}\Pi_{\alpha yz} \\ & + \widehat{\mathrm{z}}\widehat{\mathrm{x}}\Pi_{\alpha zx} + \widehat{\mathrm{z}}\widehat{\mathrm{y}}\Pi_{\alpha zy} + \widehat{\mathrm{z}}\widehat{\mathrm{z}}\Pi_{\alpha zz} \end{aligned} \tag{5.5}$$

From the rules of matrix multiplication $\mathit{\Pi}_\alpha$ can be expressed as

$$\mathit{\Pi}_\alpha = (\widehat{\mathrm{x}} \;\; \widehat{\mathrm{y}} \;\; \widehat{\mathrm{z}}) \begin{pmatrix} \Pi_{\alpha xx} & \Pi_{\alpha xy} & \Pi_{\alpha xz} \\ \Pi_{\alpha yx} & \Pi_{\alpha yy} & \Pi_{\alpha yz} \\ \Pi_{\alpha zx} & \Pi_{\alpha zy} & \Pi_{\alpha zz} \end{pmatrix} \begin{pmatrix} \widehat{\mathrm{x}} \\ \widehat{\mathrm{y}} \\ \widehat{\mathrm{z}} \end{pmatrix} \tag{5.6}$$

It is usual, however, to omit the pre and post-multiplicative dyadic signs, such as $\widehat{\mathrm{x}}\widehat{\mathrm{x}}$, and so on, and denote the dyad only by the 3×3 matrix containing the elements $\Pi_{\alpha ij}$. Thus, $\Pi_{\alpha ij}$ corresponds to the element of the ith row and the jth column. From (5.3) it is clear that $\Pi_{\alpha ij} = \Pi_{\alpha ji}$ and, consequently, the 3×3 matrix in (5.6) is *symmetric*. Therefore, only six of the components of the momentum flow dyad are independent.

6. PRESSURE DYAD OR TENSOR

6.1 Concept of Pressure

The pressure of a gas is usually defined as the force per unit area exerted by the gas molecules through collisions with the walls of the containing vessel. This force is equal to the rate of transfer of molecular momentum to the walls of the container. This definition applies also to any surface immersed in the gas as, for example, the surface of a material body.

We may generalize this definition of pressure so that it can be applied to any point inside the gas. To this end, we will define pressure in terms of an imaginary surface element $d\mathbf{S} = \widehat{\mathbf{n}}\, dS$, inside the gas, moving with its average flow velocity. The pressure on $d\mathbf{S}$ is then defined as the *rate* of transport of molecular momentum per unit area, that is, the *flux* of momentum across $d\mathbf{S}$ due to the *random* particle motions.

When different species of particles are present, as in a plasma, it is useful to define a (*partial*) pressure due to the particles of type α, as the

flux of momentum transported by the type α particles as they move back and forth across the surface element $\widehat{\mathbf{n}}\, dS$ which is moving at the average velocity $\mathbf{u}_\alpha(\mathbf{r}, t)$. In the reference frame of $d\mathbf{S}$ (3.11) applies, and taking $\chi(\mathbf{r}, \mathbf{v}, t)$ as the jth component of momentum of the type α particles, $m_\alpha\, c_{\alpha j}$, we obtain the element $P_{\alpha jn}$ of the pressure tensor,

$$P_{\alpha jn} = \rho_{m\alpha} < c_{\alpha j}\, c_{\alpha n} > \tag{6.1}$$

The *pressure dyad* is therefore given by

$$\mathcal{P}_\alpha = \rho_{m\alpha} < \mathbf{c}_\alpha\, \mathbf{c}_\alpha > \tag{6.2}$$

From (5.4) we find the following relation between the pressure dyad $\mathcal{P}_\alpha$ and the momentum flow dyad Π_α,

$$\mathcal{P}_\alpha = \Pi_\alpha - \rho_{m\alpha}\, \mathbf{u}_\alpha\, \mathbf{u}_\alpha \tag{6.3}$$

They are equal only when the flow velocity $\mathbf{u}_\alpha(\mathbf{r}, t)$ vanishes.

6.2 Force per Unit Area

Consider now a small element of volume inside the plasma, bounded by the closed surface S, and let $d\mathbf{S} = \widehat{\mathbf{n}}\, dS$ be an element of area belonging to S, with the unit vector $\widehat{\mathbf{n}}$ normal to the surface element and pointing *outward* (see Fig. 4). The *force per unit area*, $\mathbf{f}_\alpha$, acting on the area element $\widehat{\mathbf{n}}\, dS$, as the result of the *random* particle motions, is given by

$$\mathbf{f}_\alpha = -\mathcal{P}_\alpha \cdot \widehat{\mathbf{n}} = -\rho_{m\alpha} < \mathbf{c}_\alpha\, (\mathbf{c}_\alpha \cdot \widehat{\mathbf{n}}) > \tag{6.4}$$

The reason for the minus sign can be seen as follows. Suppose, for the moment, that all type α particles have the same velocity $\mathbf{c}_\alpha$. If $\mathbf{c}_\alpha$ forms an angle of less than 90^0 with $\widehat{\mathbf{n}}$, then the quantity $n_\alpha(\mathbf{c}_\alpha \cdot \widehat{\mathbf{n}})\, dS$ is the number of type α particles *leaving*, per unit time, the volume enclosed by the closed surface S, through $d\mathbf{S}$. The corresponding change (*decrease*) in the momentum of the plasma enclosed by the surface S is given by the expression $-n_\alpha m_\alpha \mathbf{c}_\alpha(\mathbf{c}_\alpha \cdot \widehat{\mathbf{n}})\, dS$, since $(\mathbf{c}_\alpha \cdot \widehat{\mathbf{n}})$ is positive. On the other hand, if $\mathbf{c}_\alpha$ forms an angle greater than 90^0 with $\widehat{\mathbf{n}}$, then $-n_\alpha(\mathbf{c}_\alpha \cdot \widehat{\mathbf{n}})\, dS$ represents the number of particles *entering*, per unit time, the bounded volume through $d\mathbf{S}$, and the corresponding change (*increase*) in the momentum of the plasma within the closed surface S is again $-n_\alpha m_\alpha \mathbf{c}_\alpha(\mathbf{c}_\alpha \cdot \widehat{\mathbf{n}})\, dS$, since now $(\mathbf{c}_\alpha \cdot \widehat{\mathbf{n}})$ is *negative*.

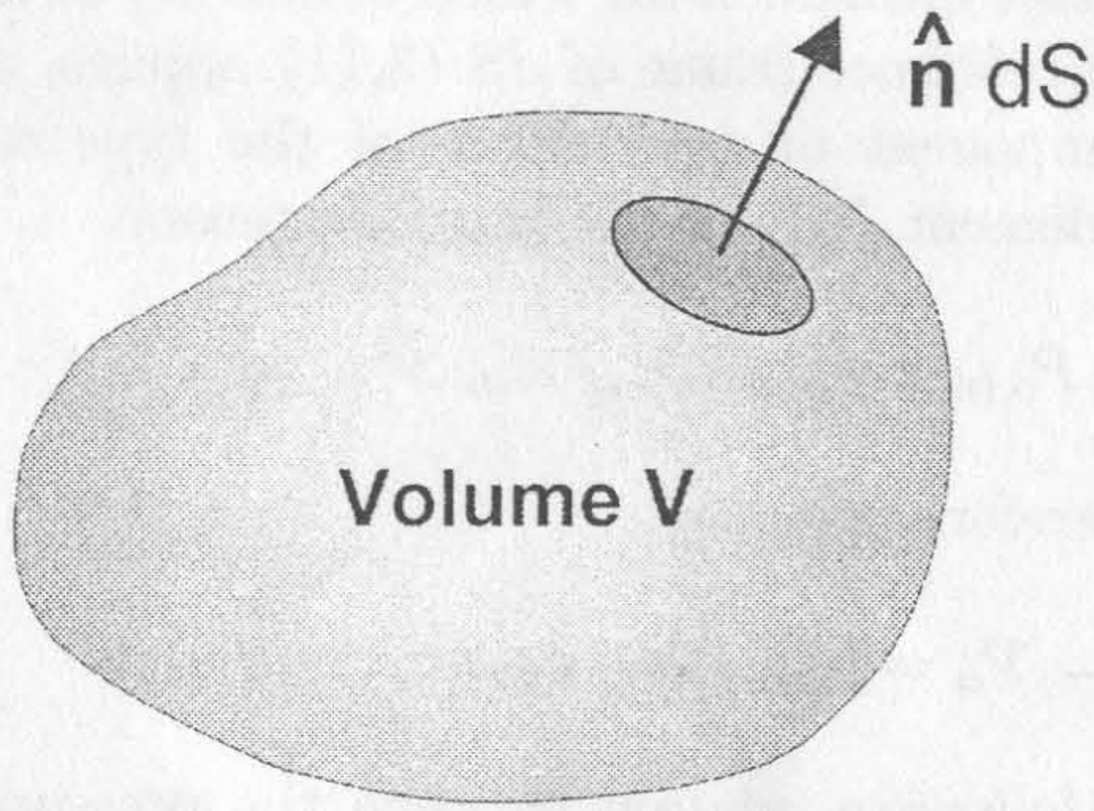


Fig. 4 *Element of volume V bounded by a closed surface S, with the surface element $\hat{\mathbf{n}}\, dS$ pointing outward.*

We conclude, by generalizing this result, that for any arbitrary distribution of individual velocities, the vector quantity

$$-n_\alpha m_\alpha < \mathbf{c}_\alpha(\mathbf{c}_\alpha \cdot \hat{\mathbf{n}}) > \; dS = -\mathcal{P}_\alpha \cdot \hat{\mathbf{n}}\, dS \tag{6.5}$$

represents the rate of change of the plasma momentum within the closed surface S, due to the exchange of type α particles through the surface element $\hat{\mathbf{n}}\, dS$. Therefore, the force per unit area exerted on an element of area oriented along the unit vector $\hat{\mathbf{n}}$ is $-\mathcal{P}_\alpha \cdot \hat{\mathbf{n}}$. If we take, for example, an element of area oriented along the x direction, that is, $\hat{\mathbf{n}} = \hat{\mathbf{x}}$, we have

$$-\mathcal{P}_\alpha \cdot \hat{\mathbf{n}} = -\hat{\mathbf{x}} P_{\alpha xx} - \hat{\mathbf{y}} P_{\alpha yx} - \hat{\mathbf{z}} P_{\alpha zx} \tag{6.6}$$

where $P_{\alpha xx}$ is *normal* to the surface and towards it, just like a *hydrostatic* pressure, whereas the components $P_{\alpha yx}$ and $P_{\alpha zx}$ are pressures due to *shear* forces which are *tangential* to the surface, as indicated in Fig. 5. All other components of $\mathcal{P}_\alpha$ are interpreted in an analogous way. Generally, the force per unit area $P_{\alpha jn}$ acts along the negative direction of the axis denoted by the first subscript (j) on a surface whose outward normal is parallel to the axis indicated by the second subscript (n). Alternatively, if the outward normal to the surface is in the negative direction of the axis indicated by the second subscript (n), then the force acts in the same direction as the axis denoted by the first subscript (j).

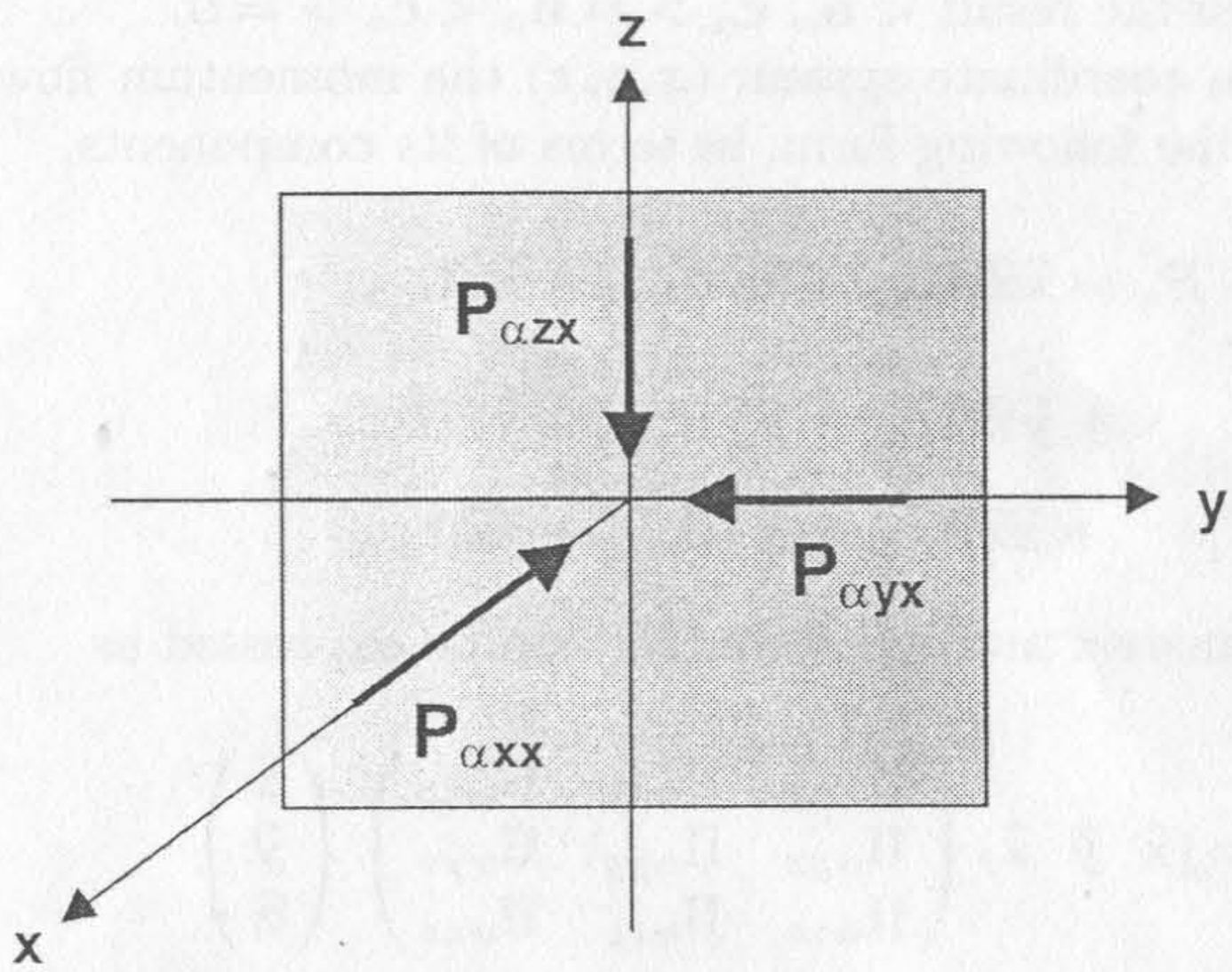


Fig. 5 *Components of the pressure tensor $\mathcal{P}$ corresponding to the tangential shear stresses $P_{\alpha yx}$ and $P_{\alpha zx}$, and to the normal stress $P_{\alpha xx}$, acting on a surface element whose normal is oriented along the unit vector $\widehat{\mathrm{x}}$.*

6.3 Force per Unit Volume

The force per unit volume inside the plasma, due to the random particle motions, can be obtained by integrating (6.5) over the closed surface S bounding the volume element V, dividing the result by V, and then taking the limit as V tends to zero. This procedure is just the definition of the divergence,

$$-\nabla \cdot \mathcal{P}_\alpha = -\lim_{V \to 0} \left[\frac{1}{V} \oint_S \mathcal{P}_\alpha \cdot \widehat{\mathbf{n}}\, dS \right] \tag{6.7}$$

and, from Gauss divergence theorem,

$$-\oint_S \mathcal{P}_\alpha \cdot \widehat{\mathbf{n}}\, dS = -\int_V \nabla \cdot \mathcal{P}_\alpha\, d^3r \tag{6.8}$$

We conclude, therefore, that the negative divergence of the kinetic pressure dyad $(-\nabla \cdot \mathcal{P}_\alpha)$ is the force exerted on a *unit volume* of the

plasma due to the random particle motions, and $\mathcal{P}_\alpha \cdot \hat{\mathbf{n}}$ is the force acting on a *unit area* of a surface normal to the unit vector $\hat{\mathbf{n}}$.

6.4 Scalar Pressure and Absolute Temperature

An important macroscopic variable is the *scalar pressure*, or *mean hydrostatic pressure*, p_α. It is defined as one third the trace of the pressure tensor,

$$p_\alpha = \tfrac{1}{3}\sum_{i,j} P_{\alpha ij}\ \delta_{ij} = \tfrac{1}{3}\sum_{i} P_{\alpha ii} = \tfrac{1}{3}(P_{\alpha xx} + P_{\alpha yy} + P_{\alpha zz}) \tag{6.9}$$

where δ_{ij} is the *Kronecker delta*, defined such that $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$. The pressure tensor elements $P_{\alpha ii}$, with $i = x, y, z$, are just the hydrostatic pressures *normal* to the surfaces described by $i = constant$. Using (6.1),

$$p_\alpha = \tfrac{1}{3}\rho_{m\alpha} < c_{\alpha x}^2 + c_{\alpha y}^2 + c_{\alpha z}^2 > \tag{6.10}$$

Since $c_\alpha^2 = c_{\alpha x}^2 + c_{\alpha y}^2 + c_{\alpha z}^2$, we have

$$p_\alpha = \tfrac{1}{3}\rho_{m\alpha} < c_\alpha^2 > \tag{6.11}$$

Another important parameter for a macroscopic description of a plasma is its temperature. The *absolute temperature* T_α, for the type α particles, is a measure of the *mean kinetic energy* of the *random* particle motions. According to the thermodynamic definition of absolute temperature, there is a mean thermal energy of $kT_{\alpha i}/2$ associated with each translational degree of freedom ($i = x, y, z$), so that

$$\tfrac{1}{2}kT_{\alpha i} = \tfrac{1}{2}m_\alpha < c_{\alpha i}^2 > \tag{6.12}$$

where k is Boltzmann's constant.

When the distribution of random velocities is *isotropic*, as is the case of the Maxwell-Boltzmann distribution function (to be considered in the next chapter) which characterizes the state of thermal equilibrium of a gas, we have $c_{\alpha x}^2 = c_{\alpha y}^2 = c_{\alpha z}^2 = c_\alpha^2/3$, and therefore,

$$p_\alpha = P_{\alpha xx} = P_{\alpha yy} = P_{\alpha zz} = \rho_{m\alpha} < c_{\alpha i}^2 > \tag{6.13}$$

Combining (6.13) and (6.12), gives

$$p_\alpha = n_\alpha k T_\alpha \tag{6.14}$$

which is the *equation of state* of an *ideal* gas. For the Maxwell-Boltzmann distribution function the nondiagonal elements of the kinetic pressure dyad are all zero and it reduces to

$$\mathcal{P}_\alpha = (\widehat{\mathbf{x}}\widehat{\mathbf{x}} + \widehat{\mathbf{y}}\widehat{\mathbf{y}} + \widehat{\mathbf{z}}\widehat{\mathbf{z}})\, p_\alpha = \mathbf{1}\, p_\alpha \tag{6.15}$$

where $\mathbf{1}$ stands for the unit dyad, which in matrix form can be written as

$$\mathbf{1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{6.16}$$

In this case the negative divergence of the pressure dyad becomes

$$-\nabla \cdot \mathcal{P}_\alpha = -\left(\widehat{\mathbf{x}}\frac{\partial}{\partial x}p_\alpha + \widehat{\mathbf{y}}\frac{\partial}{\partial y}p_\alpha + \widehat{\mathbf{z}}\frac{\partial}{\partial z}p_\alpha\right) = -\nabla p_\alpha \tag{6.17}$$

Thus, for an *isotropic* velocity distribution, the force per unit volume due to the random variations of the peculiar velocities is given by the negative gradient of the scalar pressure.

In some problems a simplification of practical interest for the general form of the kinetic pressure dyad consists in taking

$$\mathcal{P}_\alpha = \widehat{\mathbf{x}}\widehat{\mathbf{x}}P_{\alpha xx} + \widehat{\mathbf{y}}\widehat{\mathbf{y}}P_{\alpha yy} + \widehat{\mathbf{z}}\widehat{\mathbf{z}}P_{\alpha zz} \tag{6.18}$$

or, in matrix form,

$$\mathcal{P}_\alpha = \begin{pmatrix} P_{\alpha xx} & 0 & 0 \\ 0 & P_{\alpha yy} & 0 \\ 0 & 0 & P_{\alpha zz} \end{pmatrix} \tag{6.19}$$

where the diagonal elements are different from one another but all nondiagonal elements vanish. This expression corresponds to an *anisotropy* of the peculiar velocities and the *absence* of shear forces and viscous drag. The effects of viscosity and shear stresses are incorporated in the nondiagonal elements of the pressure dyad. Usually the effects of viscosity are relatively unimportant for most plasmas and the nondiagonal elements of $\mathcal{P}_\alpha$ can, in many cases, be neglected.

In this anisotropic case, different absolute temperatures $T_{\alpha i}$ can be defined for each direction in space, according to (6.12).

7. HEAT FLOW VECTOR

The component of the heat flow vector, $q_{\alpha n}$, is defined as the flux of *random* or *thermal energy* across a surface whose normal points in the direction of the unit vector $\widehat{\mathbf{n}}$. Taking $\chi(\mathbf{r}, \mathbf{v}, t)$, in (3.11), as the kinetic energy of random motion of the particles of type α, that is, $\chi = m_\alpha c_\alpha^2/2$, we obtain for the component of the heat flow vector along $\widehat{\mathbf{n}}$,

$$q_{\alpha n} = \mathbf{q}_\alpha \cdot \widehat{\mathbf{n}} = \tfrac{1}{2}\rho_{m\alpha} < c_\alpha^2 \; \mathbf{c}_\alpha \cdot \widehat{\mathbf{n}} > \tag{7.1}$$

The *heat flow vector* is therefore given by

$$\mathbf{q}_\alpha = \tfrac{1}{2}\rho_{m\alpha} < c_\alpha^2 \; \mathbf{c}_\alpha > \tag{7.2}$$

8. HEAT FLOW TRIAD

It is convenient, at this point, to introduce a *triad* of *thermal energy flux* defined by

$$\mathcal{Q}_\alpha = \rho_{m\alpha} < \mathbf{c}_\alpha \mathbf{c}_\alpha \mathbf{c}_\alpha > \tag{8.1}$$

Its components are, explicity,

$$Q_{\alpha ijk} = \rho_{m\alpha} < c_{\alpha i} c_{\alpha j} c_{\alpha k} > \tag{8.2}$$

Using Cartesian coordinates, the thermal energy flux triad can be written in the form

$$\mathcal{Q}_\alpha = \mathcal{Q}_{\alpha x}\widehat{\mathbf{x}} + \mathcal{Q}_{\alpha y}\widehat{\mathbf{y}} + \mathcal{Q}_{\alpha z}\widehat{\mathbf{z}} \tag{8.3}$$

where each of the *dyads* $\mathcal{Q}_{\alpha n}$, with $n = x, y, z$, can be expressed in matrix form as

$$\mathcal{Q}_{\alpha n} = \begin{pmatrix} Q_{\alpha xxn} & Q_{\alpha xyn} & Q_{\alpha xzn} \\ Q_{\alpha yxn} & Q_{\alpha yyn} & Q_{\alpha yzn} \\ Q_{\alpha zxn} & Q_{\alpha zyn} & Q_{\alpha zzn} \end{pmatrix} \tag{8.4}$$

To obtain a relation between the heat flow vector $\mathbf{q}_\alpha$ and the thermal energy flux triad $\mathcal{Q}_\alpha$, note that (7.1) can be written as

$$q_{\alpha n} = \tfrac{1}{2}\rho_{m\alpha} \left(< c_{\alpha x}^2 c_{\alpha n} > + < c_{\alpha y}^2 c_{\alpha n} > + < c_{\alpha z}^2 c_{\alpha n} > \right) \tag{8.5}$$

and comparing this equation with (8.2), we see that $q_{\alpha n}$ can be written as

$$q_{\alpha n} = \frac{1}{2}\left(Q_{\alpha xxn} + Q_{\alpha yyn} + Q_{\alpha zzn}\right) \tag{8.6}$$

9. TOTAL ENERGY FLUX TRIAD

In analogy with the definition of the heat flow triad $\mathcal{Q}_\alpha$ (with components denoted by $Q_{\alpha ijk}$), consider now the quantity

$$E_{\alpha ijk}(\mathbf{r}, t) = \rho_{m\alpha} < v_i\ v_j\ v_k >_\alpha \tag{9.1}$$

which represents one of the nine components of the *total energy flux triad* $\mathcal{E}_\alpha(\mathbf{r}, t)$. This quantity can be considered as the sum of three parts. Substituting $v_i = u_{\alpha i} + c_{\alpha i}$ in (9.1) and expanding,

$$\begin{aligned} < v_i\ v_j\ v_k >_\alpha = < & c_{\alpha i}c_{\alpha j}c_{\alpha k} + u_{\alpha i}c_{\alpha j}c_{\alpha k} + u_{\alpha j}c_{\alpha k}c_{\alpha i} \\ & + u_{\alpha k}c_{\alpha i}c_{\alpha j} + u_{\alpha i}u_{\alpha j}c_{\alpha k} + u_{\alpha j}u_{\alpha k}c_{\alpha i} \\ & + u_{\alpha k}u_{\alpha i}c_{\alpha j} + u_{\alpha i}u_{\alpha j}u_{\alpha k} > \end{aligned} \tag{9.2}$$

Noting that $< u_{\alpha i} >= u_{\alpha i}$ and $< c_{\alpha i} >= 0$, and using (8.2) and (6.1), we obtain

$$\rho_{m\alpha} < v_i\ v_j\ v_k >_\alpha = \rho_{m\alpha}u_{\alpha i}u_{\alpha j}u_{\alpha k} + (\mathbf{u}_\alpha, \mathcal{P}_\alpha)_{ijk} + Q_{\alpha ijk} \tag{9.3}$$

where the following notation was used

$$(\mathbf{u}_\alpha, \mathcal{P}_\alpha)_{ijk} = u_{\alpha i}P_{\alpha jk} + u_{\alpha j}P_{\alpha ki} + u_{\alpha k}P_{\alpha ij} \tag{9.4}$$

Therefore, we can write (9.1) in triadic form as

$$\mathcal{E}_\alpha(\mathbf{r}, t) = \rho_{m\alpha} < \mathbf{v}\ \mathbf{v}\ \mathbf{v} >_\alpha = \rho_{m\alpha}\mathbf{u}_\alpha\mathbf{u}_\alpha\mathbf{u}_\alpha + (\mathbf{u}_\alpha, \mathcal{P}_\alpha) + \mathcal{Q}_\alpha \tag{9.5}$$

The total energy flux triad can therefore be considered as the sum of the energy flux transported by the *convective* particle motions, represented by the first two terms in the right-hand side of (9.5), and the *thermal* energy flux $\mathcal{Q}_\alpha$ due to the random thermal motions of the particles of type α.

The physical interpretation of the heat flow triad $\mathcal{Q}_\alpha$ is, in some sense, analogous to the physical interpretation of the heat flow vector $\mathbf{q}_\alpha$. For this purpose, consider the quantity

$$\frac{1}{2}\rho_{m\alpha} < v^2\ \mathbf{v} >_\alpha \tag{9.6}$$

which represents the *average energy flux* transported by the particles of type α. This quantity can be written as the sum of three terms. Substituting $\mathbf{v} = \mathbf{c}_\alpha + \mathbf{u}_\alpha$ in expression (9.6) and expanding,

$$\tfrac{1}{2}\rho_{m\alpha} < v^2\ \mathbf{v} >_\alpha = \tfrac{1}{2}\rho_{m\alpha} < u_\alpha^2\ \mathbf{u}_\alpha + 2(\mathbf{u}_\alpha \cdot \mathbf{c}_\alpha)\mathbf{u}_\alpha$$

$$+ c_\alpha^2\ \mathbf{u}_\alpha + u_\alpha^2\ \mathbf{c}_\alpha + 2(\mathbf{u}_\alpha \cdot \mathbf{c}_\alpha)\mathbf{c}_\alpha + c_\alpha^2 \mathbf{c}_\alpha > \tag{9.7}$$

and since $< \mathbf{c}_\alpha >= 0$ and $< \mathbf{u}_\alpha >= \mathbf{u}_\alpha$, we obtain

$$\tfrac{1}{2}\rho_{m\alpha} < v^2\ \mathbf{v} >_\alpha = \tfrac{1}{2}\rho_{m\alpha}(u_\alpha^2 + < c_\alpha^2 >)\mathbf{u}_\alpha$$

$$+ \rho_{m\alpha}\mathbf{u}_\alpha \cdot < \mathbf{c}_\alpha\ \mathbf{c}_\alpha > + \tfrac{1}{2}\ \rho_{m\alpha} < c_\alpha^2\ \mathbf{c}_\alpha > \tag{9.8}$$

If we now use (6.2) and (7.2), which define $\mathcal{P}_\alpha$ and $\mathbf{q}_\alpha$, respectively, we obtain the identity

$$\tfrac{1}{2}\rho_{m\alpha} < v^2\ \mathbf{v} >_\alpha = W_\alpha \mathbf{u}_\alpha + \mathbf{u}_\alpha \cdot \mathcal{P}_\alpha + \mathbf{q}_\alpha \tag{9.9}$$

where W_α is the *mean kinetic energy density* of the type α particles,

$$W_\alpha = \tfrac{1}{2}\rho_{m\alpha} u_\alpha^2 + \tfrac{1}{2}\rho_{m\alpha} < c_\alpha^2 > \tag{9.10}$$

Eq. (9.9) is written in a form analogous to (9.5). It shows that the flux (rate of transport per unit area) of the average energy of the type α particles can be separated into three parts: the first term in the right-hand side of (9.9) represents the flux of the mean kinetic energy transported convectively, the second term is the rate of work per unit area done by the kinetic pressure dyad and the third one is the random thermal energy flux transported by the particles due to their random thermal motions. It is instructive to note that in a frame of reference moving with the average velocity $\mathbf{u}_\alpha(\mathbf{r}, t)$, the particle velocities become identical to their random velocities, that is, $\mathbf{v} = \mathbf{c}_\alpha$, so that (9.9) reduces to (7.2) which defines the thermal energy flux vector $\mathbf{q}_\alpha$.

When the thermal velocities $\mathbf{c}_\alpha$ are distributed uniformly in all directions, that is *isotropically*, it turns out that $\mathbf{q}_\alpha = 0$ (since the integrand is an odd function of $\mathbf{c}_\alpha$). Consequently, $\mathbf{q}_\alpha$ can be considered as a *partial* measure of the anisotropies in the distribution of the thermal velocities. The thermal energy flux triad $\mathcal{Q}_\alpha$ considerably extends the concept of the heat flux vector and in this sense can be considered as a *complete* measure of the anisotropies in the distribution of the thermal velocities of the particles.

10. HIGHER MOMENTS OF THE DISTRIBUTION FUNCTION

The first four *moments of the distribution function* $f_\alpha(\mathbf{r}, \mathbf{v}, t)$ are related, respectively, to the number density $n_\alpha(\mathbf{r}, t)$, to the average velocity $\mathbf{u}_\alpha(\mathbf{r}, t)$, to the momentum flow dyad $\Pi_\alpha(\mathbf{r}, t)$ and to the total energy flow triad $\mathcal{E}_\alpha(\mathbf{r}, t)$. For easy reference and convenience we gather here their mathematical expressions,

$$n_\alpha(\mathbf{r}, t) = \int_v f_\alpha(\mathbf{r}, \mathbf{v}, t)\, d^3v \tag{10.1}$$

$$u_{\alpha i}(\mathbf{r}, t) = < v_i >_\alpha = \frac{1}{n_\alpha(\mathbf{r}, t)} \int_v v_i\, f_\alpha(\mathbf{r}, \mathbf{v}, t)\, d^3v \tag{10.2}$$

$$\Pi_{\alpha ij}(\mathbf{r}, t) = \rho_{m\alpha} < v_i\, v_j >_\alpha = m_\alpha \int_v v_i\, v_j\, f_\alpha(\mathbf{r}, \mathbf{v}, t)\, d^3v \tag{10.3}$$

$$E_{\alpha ijk}(\mathbf{r}, t) = \rho_{m\alpha} < v_i\, v_j\, v_k >_\alpha = m_\alpha \int_v v_i\, v_j\, v_k\, f_\alpha(\mathbf{r}, \mathbf{v}, t)\, d^3v \tag{10.4}$$

When the average velocity $\mathbf{u}_\alpha(\mathbf{r}, t)$ vanishes, we have $\mathbf{v} = \mathbf{c}_\alpha$, the momentum flow dyad Π_α becomes the same as the pressure dyad $\mathcal{P}_\alpha$ and the total energy flux triad $\mathcal{E}_\alpha$ becomes the same as the thermal energy flux triad $\mathcal{Q}_\alpha$.

As a formal extension of these definitions we may, whenever necessary, consider higher moments of the distribution function. The moment of order N can be defined by the expression

$$M^{(N)}_{\alpha ij\,\ldots\,k}(\mathbf{r}, t) = \int_v v_i\, v_j\, \ldots\, v_k\, f_\alpha(\mathbf{r}, \mathbf{v}, t)\, d^3v \tag{10.5}$$

where the velocity components v_i appear N times in the integrand of expression (10.5).

PROBLEMS

6.1 Consider a system of particles characterized by the distribution function given in Problem 5.1.

(a) Show that the absolute temperature of the system is given by

$$T = \frac{m v_0^2}{3k}$$

where m is the mass of each particle and k is Boltzmann's constant.
(b) Obtain the following expression for the pressure dyad

$$\mathcal{P} = \tfrac{1}{3}\rho_m v_0^2 \, \mathbf{1}$$

where $\rho_m = nm$ and $\mathbf{1}$ is the unit dyad.
(c) Verify that the heat flow vector $\mathbf{q} = 0$.

6.2 Suppose that the peculiar (random) velocities of the electrons in a given plasma satisfy the following modified Maxwell-Boltzmann distribution function (considering $\mathbf{u} = 0$),

$$f(\mathbf{c}) = n_0 \left(\frac{m}{2\pi k T_\perp}\right)\left(\frac{m}{2\pi k T_\parallel}\right)^{1/2} exp\left[-\frac{m}{2k}\left(\frac{c_x^2 + c_y^2}{T_\perp} + \frac{c_z^2}{T_\parallel}\right)\right]$$

(a) Verify that the electron number density is given by n_0.
(b) Considering a Cartesian coordinate system with the z axis coincident with the parallel direction, show that the kinetic pressure dyad is given by

$$\mathcal{P} = n_0 k \left[T_\perp(\widehat{\mathbf{x}}\widehat{\mathbf{x}} + \widehat{\mathbf{y}}\widehat{\mathbf{y}}) + T_\parallel \widehat{\mathbf{z}}\widehat{\mathbf{z}}\right]$$

which indicates the presence of an *anisotropy* in the z direction.
(c) Calculate the heat flow vector $\mathbf{q}$.
(d) Show that

$$\tfrac{1}{2}m < v_\parallel^2 >= \tfrac{1}{2}kT_\parallel$$

$$\tfrac{1}{2}m < v_\perp^2 >= kT_\perp$$

6.3 For the loss-cone distribution function of Problem 5.3, show that

$$\tfrac{1}{2}m < v_\parallel^2 >= \tfrac{1}{4}m\alpha_\parallel^2$$

$$\tfrac{1}{2}m < v_\perp^2 >= m\alpha_\perp^2$$

Compare these results with those of Problem 6.2(d) and provide physical arguments to justify the difference in the perpendicular part of the average thermal energy.

6.4 Verify, by symmetry arguments, that there are only *ten independent elements* in the thermal energy flux triad $\mathcal{Q}$. Note that, according to its definition, $Q_{ijk} = nm < c_i \; c_j \; c_k >$ is symmetric under the interchange of any two of its three indices.

6.5 A plasma is made up of a mixture of various particle species, the type α species having mass m_α, number density n_α, average macroscopic velocity $\mathbf{u}_\alpha$, random velocity $\mathbf{c}_\alpha = \mathbf{v} - \mathbf{u}_\alpha$, scalar pressure $p_\alpha = n_\alpha k T_\alpha$, temperature $T_\alpha = (m_\alpha/3k) < c_\alpha^2 >$, pressure dyad $\mathcal{P}_\alpha = n_\alpha m_\alpha < \mathbf{c}_\alpha \mathbf{c}_\alpha >$, and heat flow vector $q_\alpha = (n_\alpha m_\alpha/2) < c_\alpha^2 \mathbf{c}_\alpha >$. Similar quantitites can be defined for the plasma *as a whole*, for example, we can define the *total* number density by

$$n_0 = \sum_\alpha n_\alpha$$

the *average* mass by

$$m_0 = \frac{1}{n_0} \sum_\alpha n_\alpha m_\alpha$$

and the *average* flow velocity by

$$\mathbf{u}_0 = \frac{1}{n_0 m_0} \sum_\alpha n_\alpha m_\alpha \mathbf{u}_\alpha$$

We can also define an *alternative* random velocity for the type α species, with reference to $\mathbf{u}_0$, as $\mathbf{c}_{\alpha 0} = \mathbf{v} - \mathbf{u}_0$, as well as an *alternative* absolute temperature by

$$T_{\alpha 0} = \frac{m_\alpha < c_{\alpha 0}^2 >}{3k}$$

a corresponding pressure dyad by

$$\mathcal{P}_{\alpha 0} = n_\alpha m_\alpha < \mathbf{c}_{\alpha 0} \mathbf{c}_{\alpha 0} >$$

and heat flow vector by

$$\mathbf{q}_{\alpha 0} = \tfrac{1}{2} n_\alpha m_\alpha < c_{\alpha 0}^2 \mathbf{c}_{\alpha 0} > .$$

(a) Show that, for the plasma as a whole , the *total* pressure dyad is given by

$$\mathcal{P}_0 = \sum_\alpha (\mathcal{P}_\alpha + n_\alpha m_\alpha \mathbf{w}_\alpha \mathbf{w}_\alpha)$$

and the *total* scalar pressure by

$$p_0 = \sum_\alpha (p_\alpha + \tfrac{1}{3} n_\alpha m_\alpha w_\alpha^2)$$

where $\mathbf{w}_\alpha = \mathbf{u}_\alpha - \mathbf{u}_0$ is the macroscopic *diffusion* velocity.
(b) If $\mathbf{c}_\alpha$ is isotropic, that is, $< c_{\alpha i}^2 >= (1/3) < c_\alpha^2 >$, for $i = x, y, z$, show that the *total* heat flow vector is given by

$$\mathbf{q}_0 = \sum_\alpha \left(\mathbf{q}_\alpha + \tfrac{5}{2} p_\alpha \mathbf{w}_\alpha + \tfrac{1}{2} n_\alpha m_\alpha w_\alpha^2 \mathbf{w}_\alpha\right)$$

(c) If an *average* temperature T_0, for the plasma as a whole, is defined by requiring that $p_0 = n_0 k T_0$, show that

$$T_0 = \frac{1}{n_0} \sum_\alpha n_\alpha \left(T_\alpha + \frac{m_\alpha w_\alpha^2}{3k}\right)$$

(d) Verify that

$$\tfrac{1}{2} \textstyle\sum_\alpha n_\alpha m_\alpha < c_{\alpha 0}^2 >= \tfrac{3}{2} n_0 k T_0$$

so that there is an average thermal energy of $kT_0/2$ per degree of freedom.

6.6 (a) Show that the time rate of increase of momentum in an infinitesimal volume element $d^3r = dx\ dy\ dz$ inside a gas of number density n, as a result of particles of mass m entering d^3r with average velocity $\mathbf{u}$, is given by $-\nabla \cdot (nm\mathbf{uu})\ d^3r$.
(b) If the infinitesimal volume element d^3r moves with the average particle velocity $\mathbf{u}$, show that, because of the work done by the kinetic pressure dyad $\mathcal{P}$, the particle energy inside d^3r increases at a time rate given by $-\nabla \cdot (\mathbf{u} \cdot \mathcal{P}) d^3r$.
(c) Verify, by expansion, that $(\mathcal{P} \cdot \hat{\mathbf{n}}) \cdot \mathbf{u} = (\mathbf{u} \cdot \mathcal{P}) \cdot \hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ denotes an outward unit vector, normal to the surface bounding the volume element.

6.7 Consider (5.6.4), which is the solution of the Boltzmann equation with the relaxation model for the collision term, in the absence of external forces and spatial gradients, and when $f_{\alpha 0}$ and the relaxation time τ are time-independent. Show that, according to this simplified equation, we have

$$G_\alpha(t) = G_{\alpha 0} + [G_\alpha(0) - G_{\alpha 0}] \exp\,(-t/\tau)$$

where

$$G_\alpha(t) = \int_v f_\alpha\, \chi\, d^3v = n_\alpha < \chi >_\alpha$$

$$G_{\alpha 0} = \int_v f_{\alpha 0}\, \chi\, d^3v = n_\alpha < \chi >_{\alpha 0}$$

Thus, according to the relaxation model for the collision term, every average value $< \chi >_\alpha$ approaches equilibrium with the same relaxation time.

7

THE EQUILIBRIUM STATE

1. THE EQUILIBRIUM STATE DISTRIBUTION FUNCTION

The equilibrium distribution function is the time-independent solution of the Boltzmann equation in the absence of external forces. In the equilibrium state the particle interactions do not cause any change in the distribution function with time and there are no spatial gradients in the particle number density. We deduce in this section an expression for the equilibrium distribution function, known as the *Maxwell-Boltzmann* or *Maxwellian velocity distribution function.*

For simplicity we will consider a gas consisting of only one particle species. The extension to a mixture will be indicated in a subsequent section of this chapter. We assume that there are no external forces acting on the system ($\mathbf{F}_{ext} = 0$) and that the particles are uniformly distributed in space. Under these conditions the distribution function is homogeneous ($\nabla f = 0$) and, since we are looking for a steady state solution of the Boltzmann equation, it is also time-independent ($\partial f/\partial t = 0$). Therefore, it can be denoted by $f(\mathbf{v})$. According to the Boltzmann equation (5.5.27), the equilibrium distribution function satisfies the following condition

$$\left(\frac{\delta f}{\delta t}\right)_{coll} = 0 \tag{1.1}$$

Hence, under equilibrium conditions, there are no changes in the distribution function as a result of collisions between the particles. In Chapter 21 we shall derive the expression for the equilibrium distribution function using the Boltzmann collision integral. For the moment, however, in order

to simplify matters, it is appropriate to consider a simple derivation based on the *general principle of detailed balance* of statistical mechanics.

1.1 The General Principle of Detailed Balance and Binary Collisions

In general, this principle asserts that, under equilibrium conditions, the probability of occurrence of any physical process is equal to the probability of occurrence of the inverse process. Hence, for the case of a system of interacting particles in the state of equilibrium, the principle of detailed balance asserts that the effect of each type of collision is exactly compensated by the effect of the corresponding inverse collision.

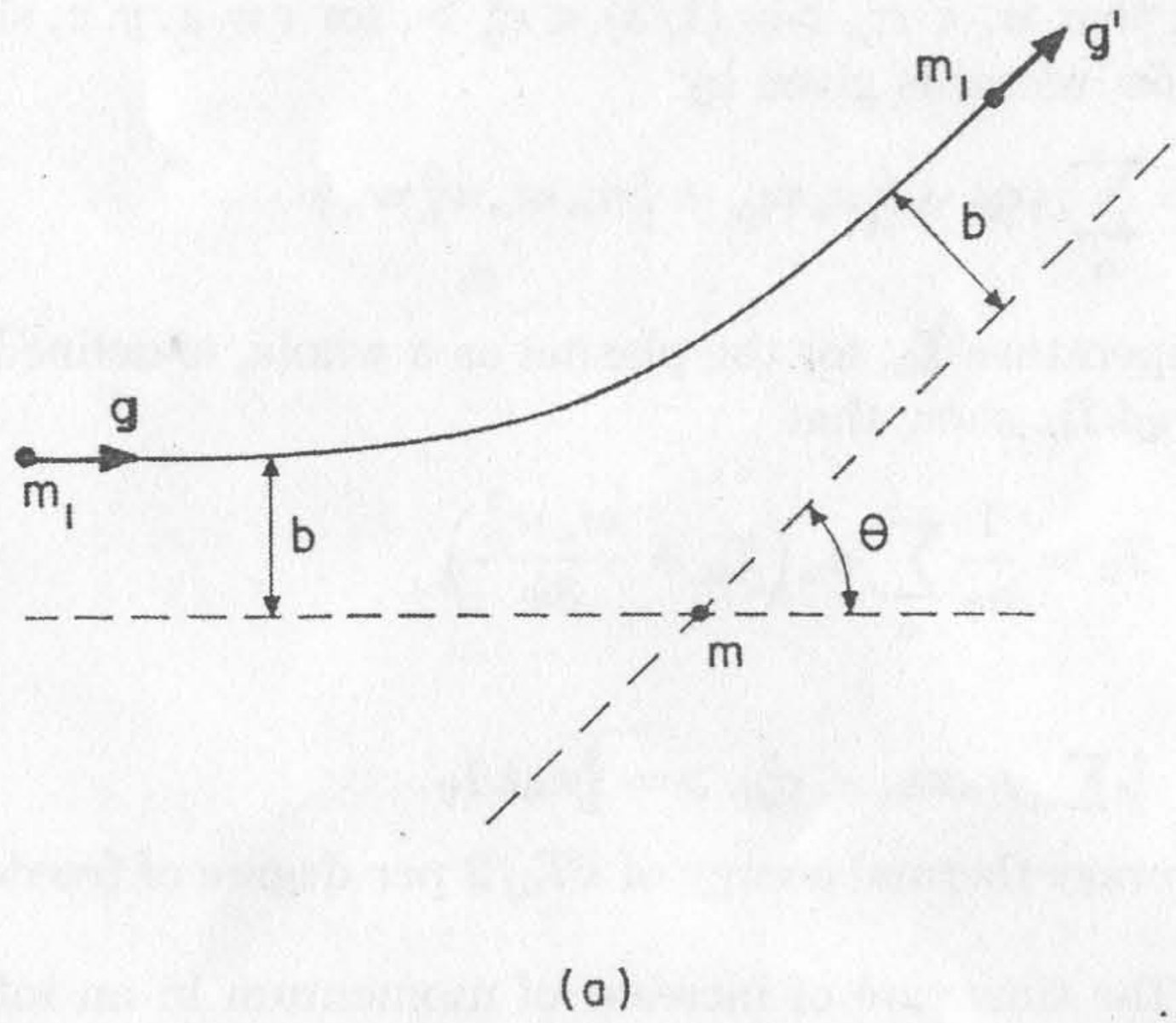
Consider an *elastic* collision between two particles having velocities $\mathbf{v}$ and $\mathbf{v}_1$ *before* collision, and $\mathbf{v}'$ and $\mathbf{v}_1'$ *after* collision. The corresponding *inverse* collision refers to an elastic collision in which a particle with *initial* velocity $\mathbf{v}'$ collides with another particle with velocity $\mathbf{v}_1'$, the velocities *after* collision being $\mathbf{v}$ and $\mathbf{v}_1$, respectively. Such events are illustrated schematically in Fig. 1, in a reference frame in which one of the particles is at rest.

Assuming that the velocities of the particles before collision are uncorrelated, the number of binary collisions occurring per unit time in a given volume d^3r, about the position $\mathbf{r}$ in configuration space, between particles having velocities within the velocity space element d^3v, about $\mathbf{v}$, and particles with velocities within d^3v_1, about $\mathbf{v}_1$, in the same configuration space element d^3r (see Fig. 2), is proportional to the product of the respective number of particles, that is, to $(f\ d^3r\ d^3v)(f_1\ d^3r\ d^3v_1)$, where f_1 represents $f(\mathbf{v}_1)$.

In a similar way, assuming the particle velocities to be uncorrelated, the number of corresponding inverse binary collisions occurring per unit time in the same volume element d^3r, about $\mathbf{r}$, in configuration space, between particles having velocities within d^3v', about $\mathbf{v}'$, and particles with velocities within d^3v_1', about $\mathbf{v}_1'$, is proportional to the product $(f'\ d^3r\ d^3v')(f_1'\ d^3r\ d^3v_1')$, where $f' = f(\mathbf{v}')$ and $f_1' = f(\mathbf{v}_1')$.

According to the principle of detailed balance, in the equilibrium state the effect of each direct collision is compensated by the effect of the corresponding inverse collision, so that

$$f\ f_1\ d^3v\ d^3v_1 = f'\ f_1'\ d^3v'\ d^3v_1' \tag{1.2}$$



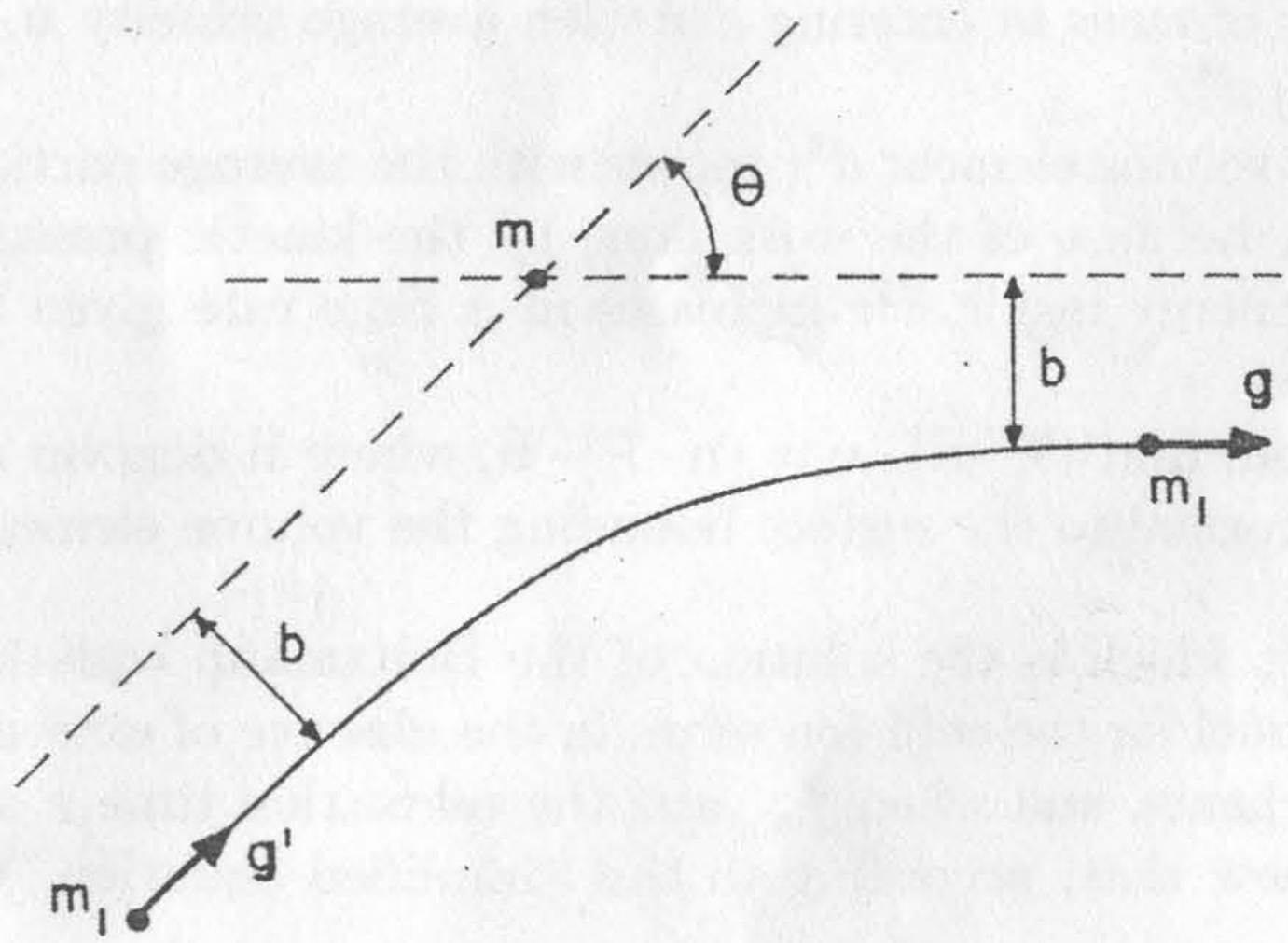


Fig. 1 *(a) Direct binary collision and (b) the corresponding inverse binary collision. Here* $\mathbf{g} = \mathbf{v}_1 - \mathbf{v}$ *and* $\mathbf{g}' = \mathbf{v}_1' - \mathbf{v}'$.

Since it can be shown that we have $d^3v\ d^3v_1 = d^3v'\ d^3v_1'$ (see Section 2 in Chapter 21), equation (1.2) yields

$$f(\mathbf{v})\ f(\mathbf{v}_1) = f(\mathbf{v}')\ f(\mathbf{v}_1') \tag{1.3}$$

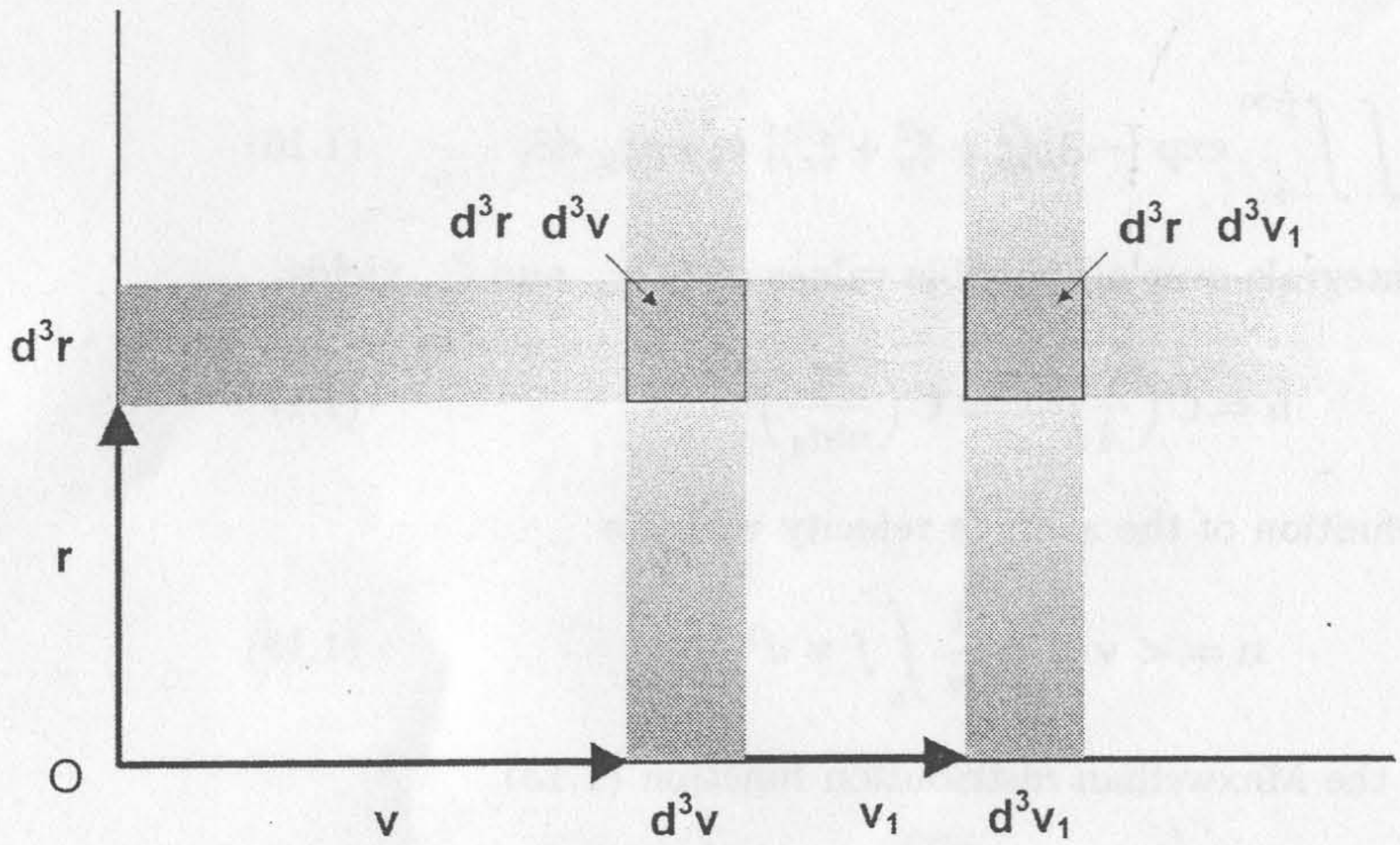


Fig. 2 *Schematic representation of the volume elements $d^3r\ d^3v$ and $d^3r\ d^3v_1$ in phase space.*

The assumption that the particle velocities are uncorrelated is known as the *molecular chaos* assumption. It is well justified when the density of the gas is sufficiently small so that the mean free path is larger than the characteristic range of the interparticle forces. Although this is certainly not a general situation for a plasma, the validity of the Maxwell-Boltzmann distribution function is very well justified experimentally.

1.2 Summation Invariants

It is convenient to introduce at this moment the concept of *summation invariants*. Consider a collision between two particles and let $\chi(\mathbf{v})$ be a physical quantity (scalar or vector) associated with each particle, which in general may be a function of the particle velocity. If the sum of the quantity $\chi(\mathbf{v})$ for the two particles is conserved during the collision process, then $\chi(\mathbf{v})$ is called a summation invariant. For a collision between two particles having initial velocities $\mathbf{v}$ and $\mathbf{v}_1$, and velocities after collision $\mathbf{v}'$ and $\mathbf{v}_1'$, repectively, we have for a summation invariant quantity

$$\chi(\mathbf{v}) + \chi(\mathbf{v}_1) = \chi(\mathbf{v}') + \chi(\mathbf{v}_1') \tag{1.4}$$

From the laws of conservation of mass, of momentum and of energy, it is clear that these physical quantities are summation invariants in binary elastic collisions. Denoting the masses of the two colliding particles by m and m_1, we can express the laws of conservation of mass, of momentum and of energy as

$$m + m_1 = m + m_1 \tag{1.5}$$

$$m\mathbf{v} + m_1\mathbf{v}_1 = m\mathbf{v}' + m_1\mathbf{v}_1' \tag{1.6}$$

$$\tfrac{1}{2}mv^2 + \tfrac{1}{2}m_1v_1^2 = \tfrac{1}{2}m(\mathbf{v}')^2 + \tfrac{1}{2}m_1(\mathbf{v}_1')^2 \tag{1.7}$$

Eq. (1.5) is a trivial one and does not lead to any new information. It only indicates that a numerical constant is a summation invariant. Eq. (1.6) represents three equations, one for each component of the momentum. The four equations (1.6) and (1.7), together with the equations involving the impact parameter b and the angle of the collision plane ϵ (for more details refer to the analysis of the dynamics of binary collisions, presented in Chapter 20), constitute six equations to be solved for the determination of the six unknown quantities, which are the components of the after-collision velocities $\mathbf{v}'$ and $\mathbf{v}_1'$, in terms of the initial velocities $\mathbf{v}$ and $\mathbf{v}_1$. The binary collision problem is therefore uniquely determined by these summation invariants. Any other summation invariant in the collision process gives no additional information and cannot be an independent one and may be expressed as a linear combination of the summation invariants defined by (1.5), (1.6) and (1.7).

1.3 Maxwell-Boltzmann Distribution Function

We proceed now to derive the equilibrium velocity distribution function starting from (1.3) and the concept of summation invariants. Taking the natural logarithm of both sides of (1.3) gives

$$\ln f + \ln f_1 = \ln f' + \ln f_1' \tag{1.8}$$

which shows that $\ln f$ is a summation invariant in the collision process. Therefore, it can be written as a linear combination of the summation invariants m, $m\mathbf{v}$, and $mv^2/2$,

$$\ln f = m\left(a_0 + \mathbf{a}_1 \cdot \mathbf{v} - \tfrac{1}{2}a_2v^2\right) \tag{1.9}$$

where a_0, $\mathbf{a}_1 = a_{1x}\widehat{\mathbf{x}} + a_{1y}\widehat{\mathbf{y}} + a_{1z}\widehat{\mathbf{z}}$ and a_2 are constants. The negative sign in front of a_2 is chosen for convenience in the equations that follow. Completing the squares in the right-hand side of (1.9), using Cartesian coordinates, gives

$$\begin{aligned}\ln f &= m\,[a_0 + (a_{1x}^2 + a_{1y}^2 + a_{1z}^2)/(2a_2)] \\ &\quad -\tfrac{1}{2}ma_2[(v_x - a_{1x}/a_2)^2 + (v_y - a_{1y}/a_2)^2 \\ &\quad + (v_z - a_{1z}/a_2)^2] \\ &= m\,[a_0 + a_1^2/(2a_2)] - \tfrac{1}{2}ma_2(\mathbf{v} - \mathbf{a}_1/a_2)^2 \end{aligned} \tag{1.10}$$

Defining new constants by

$$\ln C = m\,[a_0 + a_1^2/(2a_2)] \tag{1.11}$$

$$\mathbf{v}_0 = \mathbf{a}_1/a_2 \tag{1.12}$$

we can write (1.10) in the form

$$f = C\,\exp\,[-\tfrac{1}{2}ma_2(\mathbf{v} - \mathbf{v}_0)^2] \tag{1.13}$$

This expression is known as the *Maxwell-Boltzmann*, or *Maxwellian equilibrium distribution function.*

1.4 Determination of the Constant Coefficients

The Maxwellian distribution function (1.13) contains five constant coefficients to be determined, namely C, a_2, v_{0x}, v_{0y} and v_{0z}. Note that this is exactly the same number of coefficients in the original equation (1.9). These constants can be expressed in terms of observable physical properties of the system, such as the number density n, the average velocity $\mathbf{u}$ and the kinetic temperature T (or the scalar pressure p, since from the equation of state we have $p = nkT$). To relate the observables n, $\mathbf{u}$ and T with the constant coefficients C, a_2 and $\mathbf{v}_0$, we proceed as follows.

From the definition of the number density we must have

$$n = \int_v f\,d^3v \tag{1.14}$$

Substituting the Maxwellian distribution function (1.13) into (1.14), results in

$$n = C\int_v \exp\,[-\tfrac{1}{2}ma_2(\mathbf{v} - \mathbf{v}_0)^2]\,d^3v \tag{1.15}$$

If we use the notation $A = ma_2/2$ and $\xi_i = (v_i - v_{0i})$, with $i = x, y, z$, (1.15) becomes

$$n = C \int\int\int_{-\infty}^{+\infty} \exp\left[-A(\xi_x^2 + \xi_y^2 + \xi_z^2)\right] d\xi_x \, d\xi_y \, d\xi_z \tag{1.16}$$

Performing the integrals over all possible values of ξ_x, ξ_y and ξ_z, yields

$$n = C\left(\frac{\pi}{A}\right)^{3/2} = C\left(\frac{2\pi}{ma_2}\right)^{3/2} \tag{1.17}$$

From the definition of the average velocity we have

$$\mathbf{u} = <\mathbf{v}> = \frac{1}{n}\int_v f \, \mathbf{v} \, d^3v \tag{1.18}$$

and substituting the Maxwellian distribution function (1.13),

$$\mathbf{u} = \frac{C}{n}\int_v \mathbf{v} \exp\left[-\tfrac{1}{2}ma_2(\mathbf{v} - \mathbf{v}_0)^2\right] d^3v \tag{1.19}$$

Using the same notation as in (1.16), we can write

$$\mathbf{u} = \frac{C}{n}\int\int\int_{-\infty}^{+\infty} (\xi_x\widehat{\mathbf{x}} + \xi_y\widehat{\mathbf{y}} + \xi_z\widehat{\mathbf{z}}) \exp\left[-A(\xi_x^2 + \xi_y^2 + \xi_z^2)\right] d\xi_x \, d\xi_y \, d\xi_z$$

$$+ \frac{C}{n}\mathbf{v}_0 \int\int\int_{-\infty}^{+\infty} \exp\left[-A(\xi_x^2 + \xi_y^2 + \xi_z^2)\right] d\xi_x \, d\xi_y \, d\xi_z \tag{1.20}$$

The first triple integral in the right-hand side of (1.20), over all possible values of ξ_x, ξ_y and ξ_z, vanishes, since the integrand is an odd function of ξ_i. According to (1.16) the second triple integral is equal to n/C. Thus, we obtain

$$\mathbf{u} = \mathbf{v}_0 \tag{1.21}$$

which shows that the constant $\mathbf{v}_0$ represents the average (flow) velocity of the particles. Recall that the particle velocity $\mathbf{v}$ can be written as the sum of the peculiar (random) velocity $\mathbf{c}$ and the average velocity $\mathbf{u}$, that is, $\mathbf{v} = \mathbf{c} + \mathbf{u}$. If the system has no translational motion as a whole, then $\mathbf{v}_0 = \mathbf{u} = 0$.

Consider now the thermodynamic definition of the kinetic temperature T,

$$\tfrac{3}{2}nkT = \tfrac{1}{2}nm < c^2 > = \tfrac{1}{2}m\int_v f \, c^2 \, d^3v \tag{1.22}$$

where k is Boltzmann's constant. Substituting the Maxwellian distribution function (1.13), noting that $\mathbf{c} = \mathbf{v} - \mathbf{u}$ and $d^3v = d^3c$, we obtain

$$\tfrac{3}{2}nkT = \tfrac{1}{2}mC\int_c c^2 \exp\left(-Ac^2\right) d^3c \tag{1.23}$$

Performing the triple integral over all possible values of c_x, c_y and c_z, gives

$$kT = \left(\frac{C}{na_2}\right)\left(\frac{2\pi}{ma_2}\right)^{3/2} \tag{1.24}$$

We can now solve (1.17) and (1.24) for C and a_2, to obtain

$$C = n\left(\frac{m}{2\pi kT}\right)^{3/2} \tag{1.25}$$

$$a_2 = \frac{1}{kT} \tag{1.26}$$

Substituting the results just obtained into (1.13), the Maxwellian distribution of random velocities becomes

$$f(c) = n\left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left(-\frac{mc^2}{2kT}\right) \tag{1.27}$$

This is the equilibrium distribution function for a system of particles uniformly distributed in space and free from the action of external forces. Note that the number density n and the temperature T are constants, independent of $\mathbf{r}$ and t. This distribution function represents the only permanent mode for the distribution of the particle velocities in the gas, for specified values of n and T. Whatever may be the velocity distribution function of a gas, initially not in equilibrium, it tends to the distribution function (1.27) in the course of time, if the gas is maintained isolated from the action of external forces.

When the system has no translational motion as a whole (if it is maintained inside a container, for example), the average (flow) velocity $\mathbf{u}$ is zero and consequently we have $\mathbf{c} = \mathbf{v}$ in (1.27). The equilibrium distribution function depends only on the *magnitude* of the random velocity $\mathbf{c}$ so that, when a perfectly reflecting surface is immersed in the gas, $f(c)$ remains unchanged since the magnitude of the random velocity does not change when the particles are reflected at the surface.

For a plasma under equilibrium conditions, in which the various particle species such as electrons, ions and neutrals have the same temperature, the random velocities of each species are separately described by a

Maxwell-Boltzmann distribution function with the corresponding number density.

1.5 Local Maxwell-Boltzmann Distribution Function

In many situations of interest we are dealing with a gas that, although not in equilibrium, is not very far from it. It is then a good approximation to consider that, in the neighborhood of any point in the gas, there is an equilibrium situation described by a *local* Maxwell-Boltzmann distribution function of the form

$$f(\mathbf{r},\mathbf{v},t) = n(\mathbf{r},t)\left[\frac{m}{2\pi kT(\mathbf{r},t)}\right]^{3/2} \exp\left\{-\frac{m[\mathbf{v}-\mathbf{u}(\mathbf{r},t)]^2}{2kT(\mathbf{r},t)}\right\} \tag{1.28}$$

where the number density n, the temperature T and the average velocity $\mathbf{u}$ are slowly varying functions of $\mathbf{r}$ and t.

2. THE MOST PROBABLE DISTRIBUTION

We have seen that the Maxwell-Boltzmann distribution function is the solution of the Boltzmann equation representing the equilibrium state of a gas, in the absence of external forces. One of the important conclusions obtained from the derivation of this distribution function is that it is independent of the cross section for the particle collisions. This means that the Maxwell-Boltzmann function is, in a certain way, universal in the description of the equilibrium state, and it should be possible to derive it without explicitly considering the particle interactions. A derivation in these terms is in fact presented in statistical mechanics, where it is shown that the Maxwell-Boltzmann distribution function represents the *most probable* distribution satisfying the macroscopic conditions (or constraints) imposed on the system.

In statistical mechanics, to a given macroscopic system there corresponds a very large number of possible microscopic states that lead to the same macroscopic parameters specifying the system, such as number density n, average velocity $\mathbf{u}$ and absolute temperature T. Each microscopic state is considered to be equiprobable. If we choose, at random, any particular microscopic state for the system, amongst all the possible microscopic states consistent with the specified macroscopic parameters (such as n, $\mathbf{u}$, T), the probability of choosing a Maxwellian distribution is overwhelmingly larger than any other distribution. It is also shown that

the entropy is proportional to the probability of having a given distribution. Consequently, the state having maximum entropy is the most probable state consistent with the macroscopic constraints imposed on the system.

The meaning of the Maxwell-Boltzmann distribution function can be further illustrated by the following example. If a dilute gas is prepared in an arbitrary nonequilibrium initial state, and if there are interactions between the particles so as to allow the gas to pass from the initial state to other states, as time passes the gas will certainly reach the Maxwellian state, since essentially almost all possible microscopic states, consistent with the macroscopic constraints, have a Maxwellian distribution.

The statistical mechanics derivation of the most probable distribution function provides information only on the equilibrium state, and cannot possibly tell, for example, how long (which depends on the collision cross section) a given distribution function, initially not the equilibrium one, takes to become Maxwellian. The Boltzmann equation, on the other hand, is much more general and provides information also for nonequilibrium situations.

3. MIXTURE OF VARIOUS PARTICLE SPECIES

For the case of a mixture containing different species of particles, each species having their own number density n_α, average velocity $\mathbf{u}_\alpha$ and temperature T_α, we can still perform a calculation to determine the most probable distribution subjected to these macroscopic constraints. This requires only that we set $f_\alpha\, f_{\alpha 1} = f'_\alpha\, f'_{\alpha 1}$ for each particle species, but not necessarily $f_\alpha\, f_{\beta 1} = f'_\alpha\, f'_{\beta 1}$ for $\alpha \neq \beta$. This condition, therefore, does not describe an equilibrium situation, unless the temperatures and mean velocities of all species are the same. To determine the most probable distribution function for this nonequilibrium gas mixture (each species having their own number density, mean velocity and temperature), we independently apply (1.3) to each species, in order to maximize the entropy for each species. This also maximizes the entropy for the gas mixture under the specified macroscopic constraints. The problem is completely analogous to the one just solved for a one-component gas and leads, in identical fashion, to

$$f_\alpha(v) = n_\alpha \left(\frac{m_\alpha}{2\pi k T_\alpha}\right)^{3/2} \exp\left[-\frac{m_\alpha(\mathbf{v}-\mathbf{u}_\alpha)^2}{2kT_\alpha}\right] \tag{3.1}$$

Thus, each species have a Maxwellian distribution of velocities, but with their *own* density, average velocity and temperature. Although this is *not*

an equilibrium distribution for the system, since the equilibrium condition $f_\alpha\, f_{\beta 1} = f'_\alpha\, f'_{\beta 1}$ for all α and β is not satisfied, it is, nevertheless, the *most probable distribution under the specified constraints.* Only if the temperatures and average velocities of all species are equal will this be an equilibrium situation. Indeed, if two systems with different species and at different temperatures are brought together, then, as time passes, there will be a transfer of energy through collisions between the different species, until equilibrium is reached with the various species at the same temperature and mean velocity.

4. PROPERTIES OF THE MAXWELL-BOLTZMANN DISTRIBUTION FUNCTION

Due to the importance of the equilibrium distribution function, we present in this section some of its basic properties. We consider a gas in thermal equilibrium having no average (flow) velocity, $\mathbf{u} = 0$. If, however, this average velocity is not zero, we suppose that the observer is moving with the average velocity of the gas. Thus, in either case, $\mathbf{v} = \mathbf{c}$. According to the definition of the distribution function, the number of particles per unit volume having velocities between $\mathbf{v}$ and $\mathbf{v} + d\mathbf{v}$ is given by

$$f(v)\, d^3v = n\left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left(-\frac{mv^2}{2kT}\right) d^3v \tag{4.1}$$

4.1 Distribution of a Velocity Component

The distribution function for *one component* of the velocity, $g(v_i)$, is defined such that $g(v_i)\, dv_i$ represents the number of particles per unit volume which have the i component of the velocity between v_i and $v_i + dv_i$, irrespective of the values of the other two velocity components.

For the x component, for example, $g(v_x)\, dv_x$ is obtained by integrating $f(v)$ over all possible values of the velocity components v_y and v_z,

$$g(v_x)\, dv_x = \int_{v_y}\int_{v_z} f(v)\, dv_x\, dv_y\, dv_z \tag{4.2}$$

Substituting the Maxwell-Boltzmann distribution function, yields

$$g(v_x)\, dv_x = n\left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left(-\frac{mv_x^2}{2kT}\right) dv_x \int_{-\infty}^{+\infty} \exp\left(-\frac{mv_y^2}{2kT}\right) dv_y \ .$$

$$\cdot \int_{-\infty}^{+\infty} \exp\left(-\frac{mv_z^2}{2kT}\right) dv_z \tag{4.3}$$

Each integral in (4.3) is equal to $(2\pi kT/m)^{1/2}$. Therefore,

$$g(v_x)\, dv_x = n\left(\frac{m}{2\pi kT}\right)^{1/2} \exp\left(-\frac{mv_x^2}{2kT}\right) dv_x \tag{4.4}$$

Obviously this expression applies to any of the velocity components. It shows that *each* of the velocity *components* has a *Gaussian* distribution, which is symmetric about the average value $< v_i > = 0$, for $i = x, y, z$. The distribution function given in (4.4) is plotted in Fig. 3. Note that it is properly normalized so that

$$\int_{-\infty}^{+\infty} g(v_x)\, dv_x = n \tag{4.5}$$

The fact that the average value $< v_i >$ vanishes is physically evident by symmetry, since each component of the velocity can be equally positive or negative. Mathematically, we have

$$< v_i > = \frac{1}{n}\int_{-\infty}^{+\infty} g(v_i)\, v_i\, dv_i$$

$$= \left(\frac{m}{2\pi kT}\right)^{1/2} \int_{-\infty}^{+\infty} \exp\left(-\frac{mv_i^2}{2kT}\right) v_i\, dv_i = 0 \tag{4.6}$$

since the integrand is an odd function of v_i. Consequently, if l represents any odd integer number,

$$< v_i^l > = 0 \qquad\qquad l = 1, 3, 5, ... \tag{4.7}$$

On the other hand, $< v_i^2 >$ is intrinsically positive and represents the *dispersion* or *variance* of v_i,

$$< v_i^2 > = \frac{1}{n}\int_{-\infty}^{+\infty} g(v_i)\, v_i^2\, dv_i = \frac{kT}{m} \tag{4.8}$$

This result is consistent with the theorem of equipartition of energy, according to which

$$\tfrac{1}{2}m < v_i^2 > = \tfrac{1}{2}kT \tag{4.9}$$

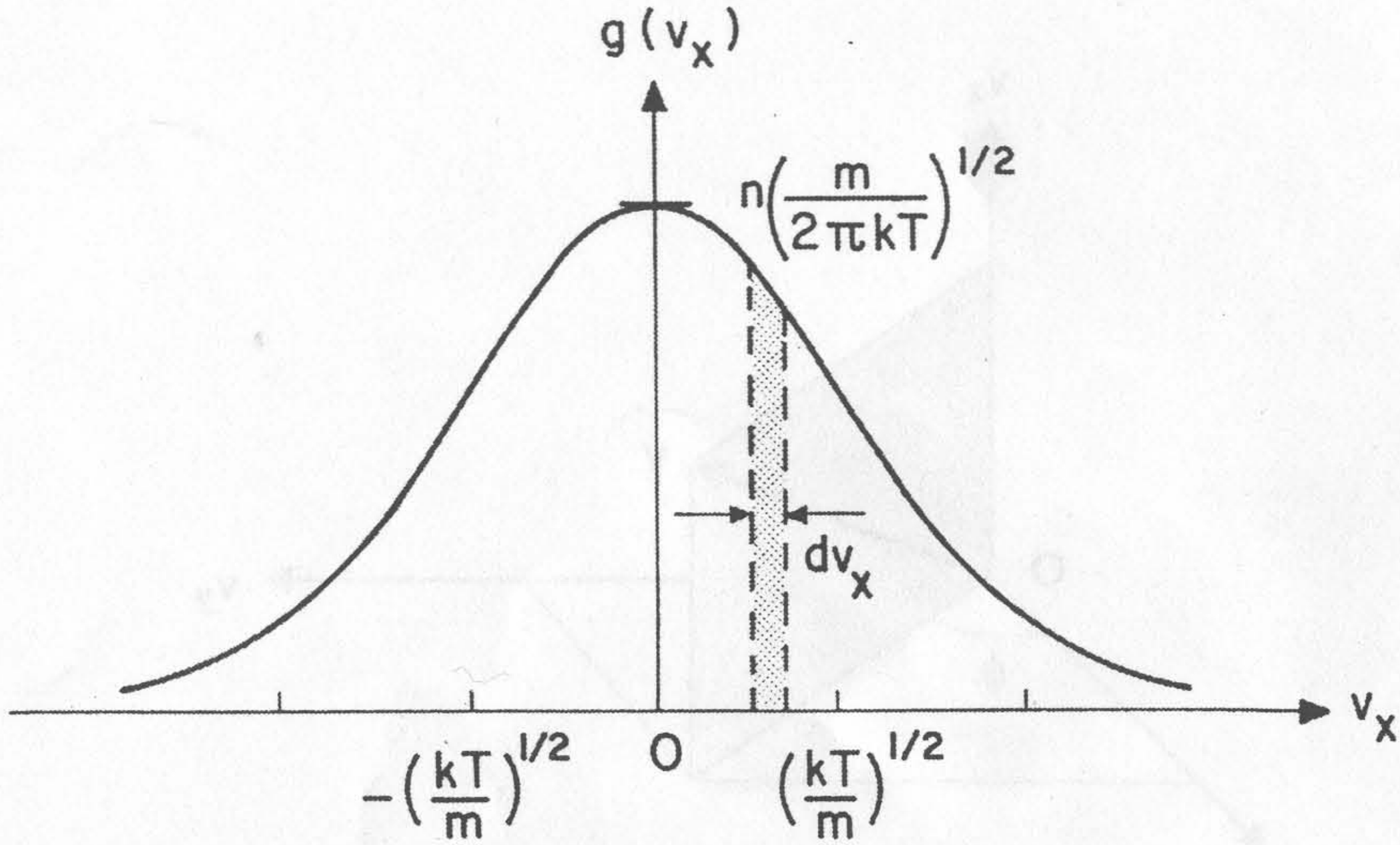


Fig. 3 *The equilibrium Maxwellian distribution function for each velocity component is a Gaussian distribution having zero expectation* ($< v_x > = 0$) *and root-mean-square width given by* $< v_x^2 >^{1/2} = (kT/m)^{1/2}$.

for $i = x, y, z$. The root-mean-square width of the Gaussian distribution $g(v_i)$ is therefore given by

$$< v_i^2 >^{1/2} = \left(\frac{kT}{m}\right)^{1/2} \tag{4.10}$$

which shows that the higher the temperature, the larger will be the width of the distribution function $g(v_i)$.

The velocity components behave, individually, like statistically independent quantities. Since $v^2 = v_x^2 + v_y^2 + v_z^2$, the probability that the particle velocity lies between $\mathbf{v}$ and $\mathbf{v} + d\mathbf{v}$ is equal to the product of the probabilities that the velocity components lie between v_i and $v_i + dv_i$, for $i = x, y, z$, that is

$$\frac{f(v)\ d^3v}{n} = \frac{g(v_x)\ dv_x}{n} \cdot \frac{g(v_y)\ dv_y}{n} \cdot \frac{g(v_z)\ dv_z}{n} \tag{4.11}$$

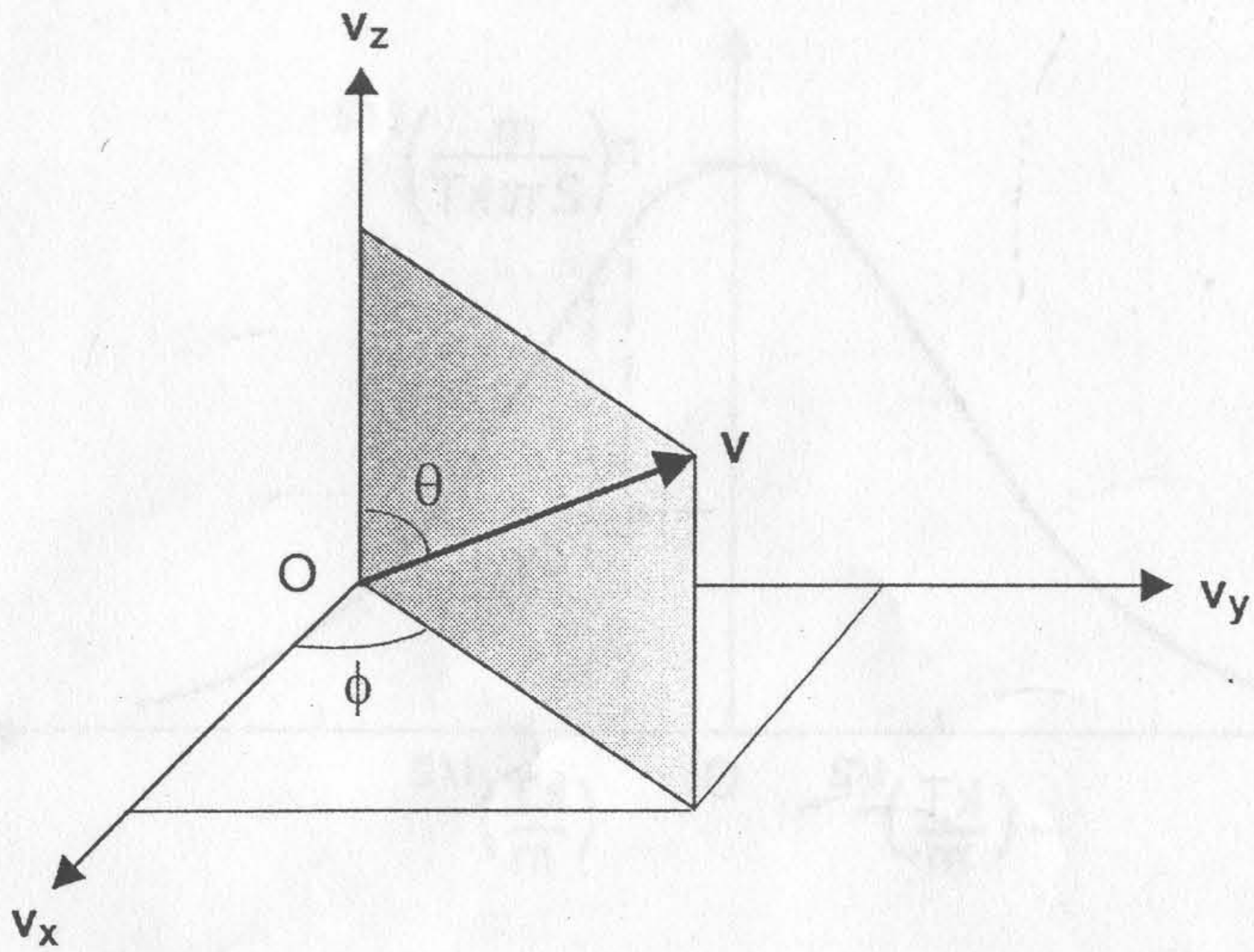


Fig. 4 *Spherical coordinate system (v, θ, ϕ) in velocity space.*

4.2 Distribution of Speeds

Since the Maxwell-Boltzmann velocity distribution function is isotropic, it is of interest to define a distribution function of *speeds* $v = |\mathbf{v}|$. For this purpose, consider a spherical polar coordinate system in velocity space (v, θ, ϕ), as shown in Fig. 4. The element of volume d^3v, in velocity space, between the coordinates (v, θ, ϕ) and $(v + dv, \theta + d\theta, \phi + d\phi)$, is given by

$$d^3v = v^2 \sin\theta \, d\theta \, d\phi \tag{4.12}$$

The distribution function of speeds $F(v)$ is defined such that $F(v)\, dv$ is the number of particles per unit volume having *speed* between v and $v+dv$, irrespective of the direction in space of the velocity vector $\mathbf{v}$. Hence, to determine $F(v)$ we integrate $f(v)$ over all velocities whose *magnitude* lies between v and $v + dv$, irrespective of the values of θ and ϕ, that is, whose velocity vector ends in a spherical shell in velocity space of internal radius v and external radius $v + dv$, as shown in Fig. 5. Therefore,

$$F(v)\, dv = \int_\theta \int_\phi f(v)\, v^2 \sin\theta \, d\theta \, d\phi \, dv \tag{4.13}$$

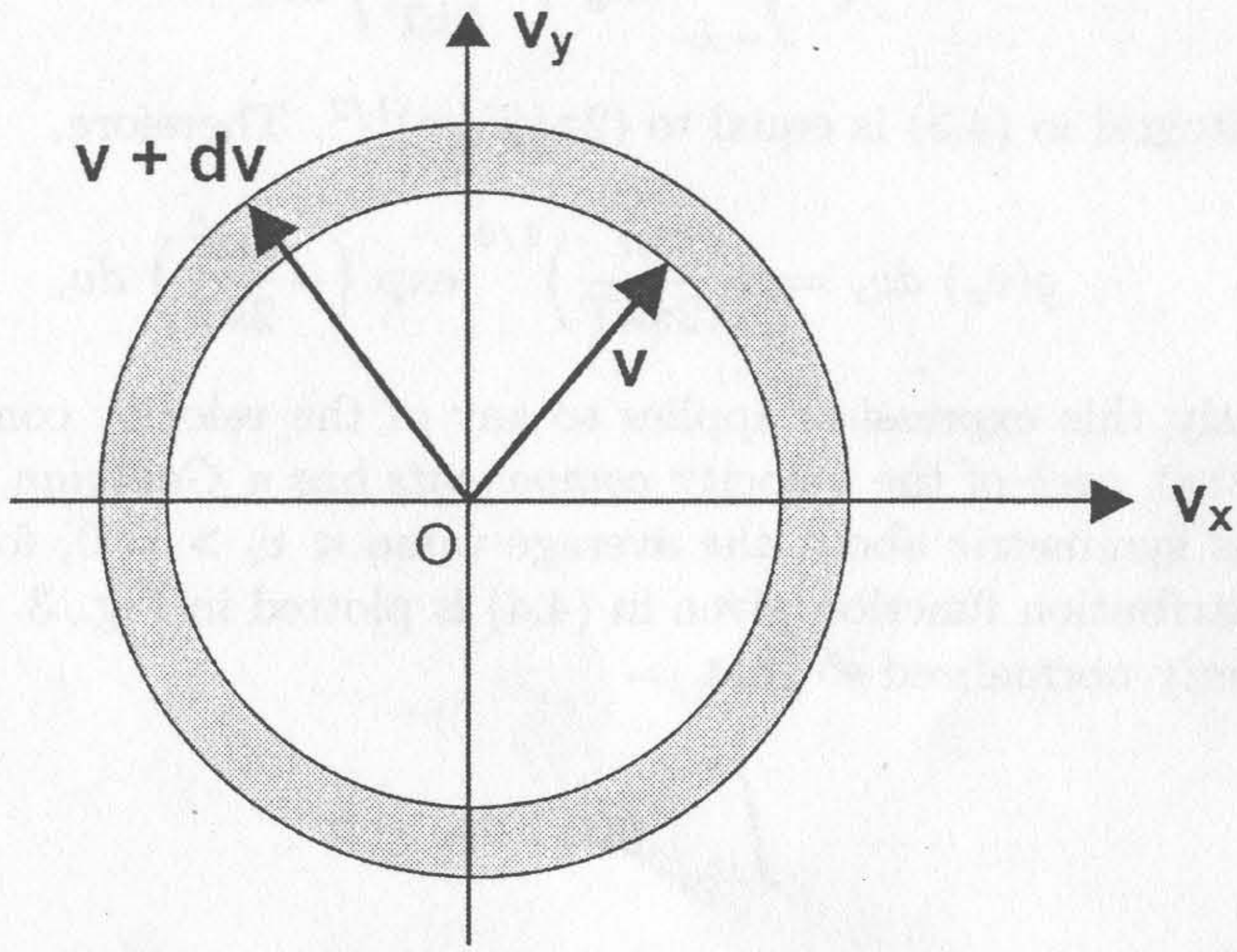


Fig. 5 *Schematic representation, in two dimensions, of a spherical shell in velocity space containing all particles with velocity having magnitude between v and $v + dv$.*

Since $f(v)$ depends only on the magnitude of **v**, but not on its direction,

$$F(v)\ dv = f(v)\ v^2\ dv \int_0^{\pi} \sin\theta\ d\theta \int_0^{2\pi} d\phi = 4\pi v^2\ f(v)\ dv \qquad (4.14)$$

Note that $4\pi\ v^2\ dv$ is the volume of the spherical shell in velocity space shown in Fig. 5. Substituting the Maxwell-Boltzmann distribution function for $f(v)$, we obtain the distribution of *speeds*,

$$F(v) = 4\pi n\left(\frac{m}{2\pi kT}\right)^{3/2} v^2 \exp\left(-\frac{mv^2}{kT}\right) \qquad (4.15)$$

This expression is properly normalized, so that

$$\int_0^{\infty} F(v)\ dv = n \qquad (4.16)$$

From the expression for $F(v)$ we see that, as v increases, the exponential factor decreases faster than v^2 increases, resulting in a maximum in $F(v)$

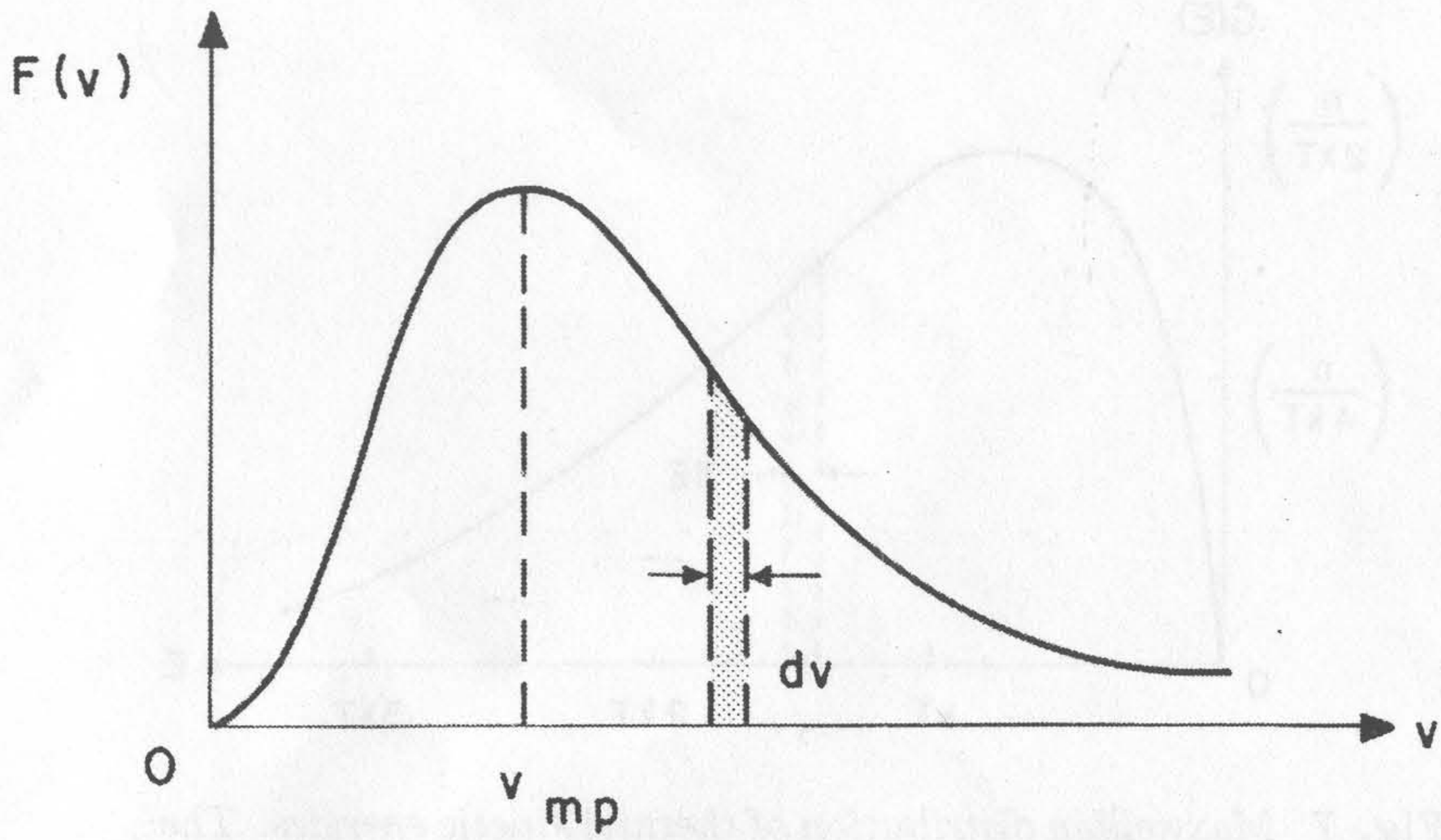


Fig. 6 *Maxwellian distribution of speeds, showing the most probable speed v_{mp}.*

for a given value of v which is called the *most probable speed.* The curve for $F(v)$ is shown in Fig. 6.

4.3 Mean Values Related to the Molecular Speeds

The *average value* of the speed is given by

$$< v > = \frac{1}{n} \int_v f \, v \, d^3v = \frac{1}{n} \int \int \int_{-\infty}^{+\infty} f \, v \, dv_x \, dv_y \, dv_z \tag{4.17}$$

or, equivalently, by

$$< v > = \frac{1}{n} \int_0^{\infty} F(v) \, v \, dv \tag{4.18}$$

It is intrinsically a positive quantity, since $v = |\mathbf{v}|$ is always positive. Using expression (4.15) for $F(v)$, we get

$$< v > = 4\pi \left(\frac{m}{2\pi kT} \right)^{3/2} \int_0^{\infty} v^3 \exp\left(-\frac{mv^2}{2kT} \right) dv$$

$$= 4\pi \left(\frac{m}{2\pi kT} \right)^{3/2} \frac{1}{2} \left(\frac{2kT}{m} \right)^2 \tag{4.19}$$

Therefore,

$$< v > = (8/\pi)^{1/2}(kT/m)^{1/2} \tag{4.20}$$

Integrals of the type

$$I(j) = \int_0^\infty x^j \exp\left(-\alpha x^2\right) dx \tag{4.21}$$

where j represents a positive integer number, occur frequentely in the computation of average values using the Maxwellian distribution of speeds. For future reference we present here the results for some integrals of the type (4.21),

$$\begin{aligned} I(0) &= \tfrac{1}{2}\pi^{1/2}\alpha^{-1/2} & I(1) &= \tfrac{1}{2}\alpha^{-1} \\ I(2) &= \tfrac{1}{4}\pi^{1/2}\alpha^{-3/2} & I(3) &= \tfrac{1}{2}\alpha^{-2} \\ I(4) &= \tfrac{3}{8}\pi^{1/2}\alpha^{-5/2} & I(5) &= \alpha^{-3} \end{aligned} \tag{4.22}$$

The average of the square of the speed is given by

$$< v^2 > = \frac{1}{n}\int\int\int_{-\infty}^{+\infty} f\, v^2\, dv_x\, dv_y\, dv_z = \frac{4\pi}{n}\int_0^\infty v^4\, f(v)\, dv \tag{4.23}$$

Substituting the Maxwellian distribution function for $f(v)$,

$$< v^2 > = 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2}\int_0^\infty v^4 \exp\left(-\frac{mv^2}{2kT}\right) dv \tag{4.24}$$

which gives

$$< v^2 > = 3kT/m \tag{4.25}$$

This result can also be obtained from (4.8), noting that $v^2 = v_x^2 + v_y^2 + v_z^2$ and that $< v_x^2 > = < v_y^2 > = < v_z^2 >$. The *root-mean-square speed* is given by

$$v_{rms} = < v^2 >^{1/2} = (3kT/m)^{1/2} \tag{4.26}$$

The *most probable speed* v_{mp} corresponds to the speed for which $F(v)$ is maximum, and can be obtained by the condition

$$\left(\frac{dF(v)}{dv}\right)_{v=v_{mp}} = 0 \tag{4.27}$$

Differentiating (4.15) with respect to v, yields

$$\frac{dF(v)}{dv} = 2v \exp\left(-\frac{mv^2}{2kT}\right) + v^2\left(-\frac{mv}{kT}\right) \exp\left(-\frac{mv^2}{2kT}\right) \tag{4.28}$$

which, for the condition of maximum expressed in (4.27), gives

$$v_{mp} = (2kT/m)^{1/2} \tag{4.29}$$

Note that the mean speeds $< v >$, v_{rms} and v_{mp} are all proportional to $(kT/m)^{1/2}$ and are such that $v_{mp} < (< v >) < v_{rms}$. Therefore, they increase with the temperature and, for a given temperature, particles having a larger mass will move with a smaller speed. We have also seen that the average kinetic energy of the random particle motions satisfies the relation

$$\tfrac{1}{2}m < v^2 > = \tfrac{3}{2}kT \tag{4.30}$$

4.4 Distribution of Thermal Kinetic Energy

The distribution of *thermal kinetic energy* $G(E)$, where $E = mv^2/2$, is defined such that $G(E)\ dE$ is the number of particles per unit volume having random kinetic energy between E and $E + dE$. It can be obtained from (4.15) substituting v by $(2E/m)^{1/2}$ and dv by $dE/(2mE)^{1/2}$. Hence,

$$G(E)\ dE = 4\pi n\left(\frac{m}{2\pi kT}\right)^{3/2}\left(\frac{2E}{m}\right)\exp\left(-\frac{E}{kT}\right)\frac{dE}{(2mE)^{1/2}} \tag{4.31}$$

Simplifying this expression,

$$G(E)\ dE = \frac{2nE^{1/2}}{\pi^{1/2}(kT)^{3/2}}\exp\left(-\frac{E}{kT}\right)dE \tag{4.32}$$

The function $G(E)$ is displayed in Fig. 7.

4.5 Random Particle Flux

We have seen in Chapter 6 that the particle flux, in a given direction specified by the unit vector $\hat{\mathbf{n}}$, is given by

$$\Gamma_n = n < v_n > = \int_v f\ \mathbf{v}\cdot\hat{\mathbf{n}}\ d^3v \tag{4.33}$$

Let us consider a surface element inside the gas. We are interested in determining the number of particles that reach this surface element, per

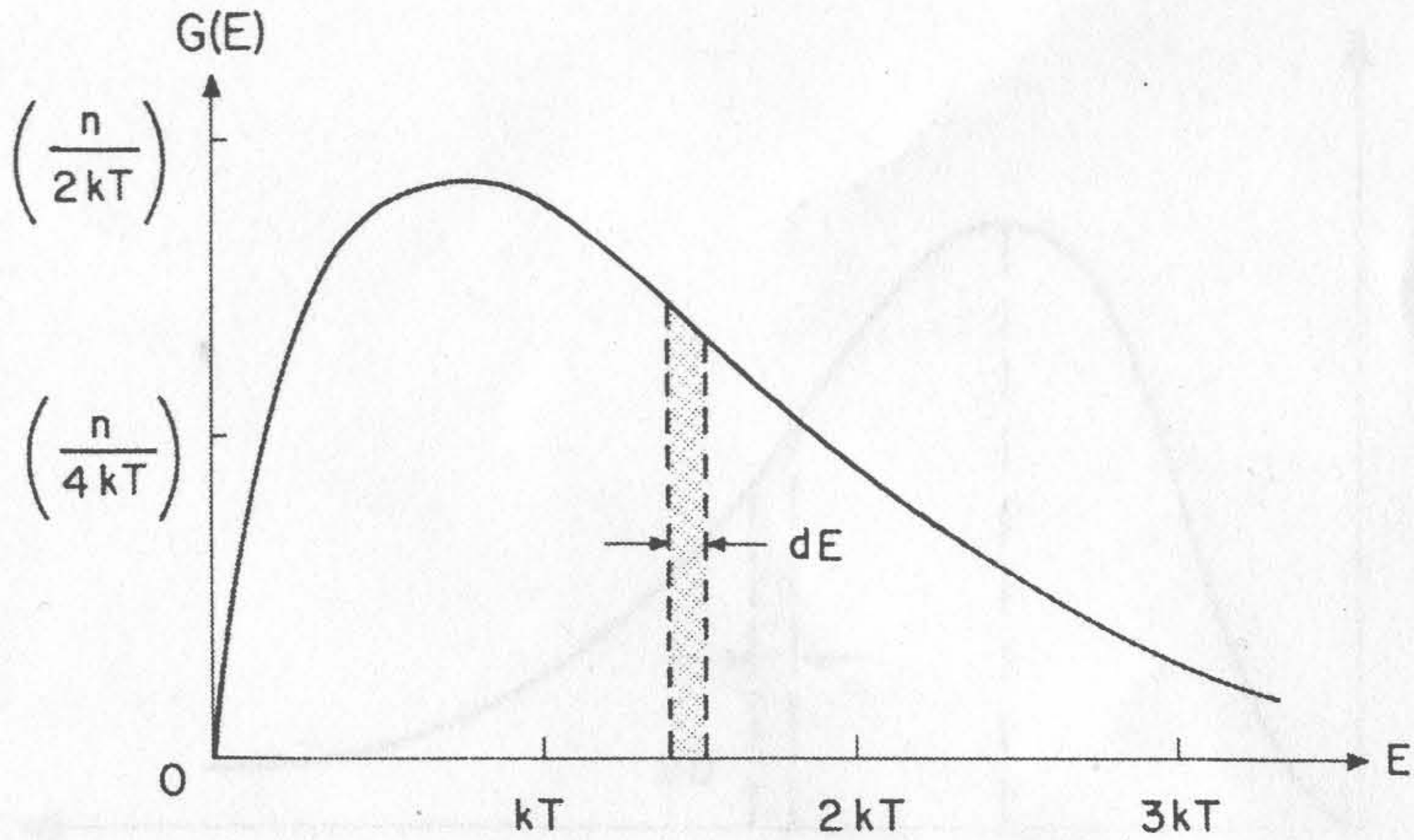


Fig. 7 *Maxwellian distribution of thermal kinetic energies. The shadowed area represents the number of particles having random kinetic energy between E and $E + dE$.*

unit area and time, due to the random particle motions. Eq. (4.33) takes into account particles that reach the surface element, oriented along the direction of $\widehat{\mathbf{n}}$, coming from all possible directions. Since we are assuming that the average velocity $\mathbf{u}$ is zero, the flux given by (4.33) is obviously zero, since $< \mathbf{c} > = 0$. In this case, it is of interest to consider only the flux of particles that cross the surface element from the *same side* (such that $\mathbf{v} \cdot \widehat{\mathbf{n}}$ is positive, say), due to their random motions.

Let $d\mathbf{S}$ be a surface element situated at the origin of a Cartesian coordinate system (x, y, z) and oriented along the z axis, that is, $d\mathbf{S} = \widehat{\mathbf{z}}\, dS$, as shown in Fig. 8. Consider the particles that cross $\widehat{\mathbf{z}}\, dS$ coming from the region $z < 0$, having velocities between $\mathbf{v}$ and $\mathbf{v} + d\mathbf{v}$, making an angle θ with the z axis, so that $\mathbf{v} \cdot \widehat{\mathbf{z}} = v\, cos(\theta)$. Expressing d^3v in terms of spherical coordinates (v, θ, ϕ),

$$d^3v = v^2 \sin\theta\, d\theta\, d\phi\, dv \tag{4.34}$$

the random particle current density, crossing $\widehat{\mathbf{z}}\, dS$ from the region $z < 0$, is given by

$$\Gamma_z = \int_0^\infty f\, v^3\, dv \int_0^{\pi/2} \sin\theta \cos\theta\, d\theta \int_0^{2\pi} d\phi = \pi \int_0^\infty f\, v^3\, dv \tag{4.35}$$

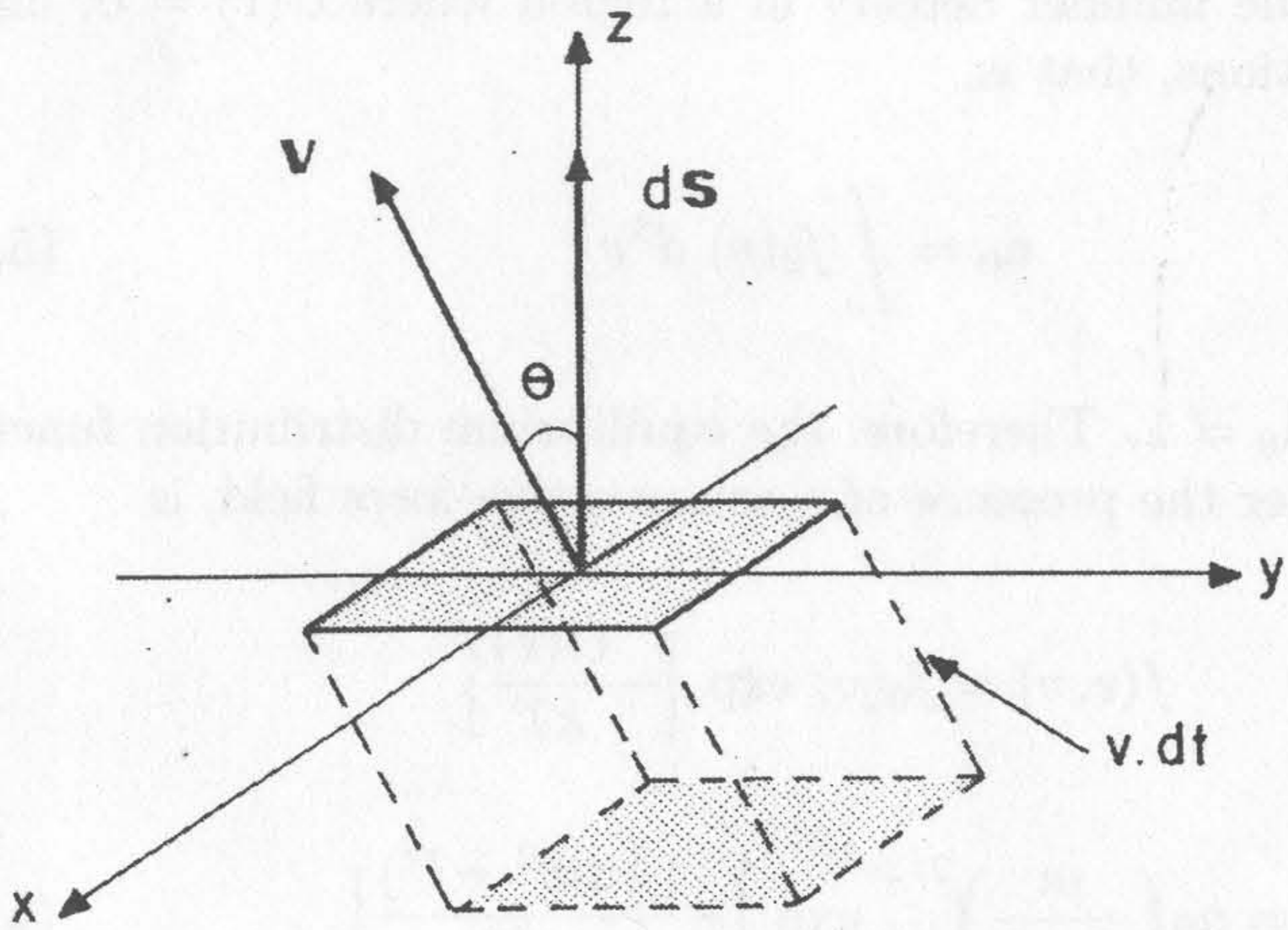


Fig. 8 *Prism of base* $d\mathbf{S} = \hat{\mathbf{z}}dS$ *containing the particles having velocity between* $\mathbf{v}$ *and* $\mathbf{v} + d\mathbf{v}$ *that will cross* $d\mathbf{S}$ *during the interval* dt.

Substituting the Maxwellian distribution for $f(v)$, we find

$$\Gamma_z = \pi n \left(\frac{m}{2\pi kT}\right)^{3/2} \int_0^\infty \exp\left(-\frac{mv^2}{2kT}\right) v^3 \, dv \tag{4.36}$$

and solving the integral, we obtain

$$\Gamma = n\left(\frac{kT}{2\pi m}\right)^{1/2} = \frac{1}{4} n < v > \tag{4.37}$$

In this result we have eliminated the index z from Γ, since the Maxwellian distribution function is isotropic, so that (4.37) applies to *any* direction inside the gas.

It is important to note that the random particle flux is inversely proportional to the square root of the particle mass. In a plasma the particle current density for the electrons is therefore much larger than that for the ions (the ratio of the electron mass to the proton mass, for example, is 1/1836). This difference in the thermal particle flux between electrons and ions plays a very important role in the interaction of a plasma with a material body immersed in it (see Chapter 11).

4.6 Kinetic Pressure and Heat Flux

From the definitions of the kinetic pressure dyad

$$\mathcal{P} = \rho_m < \mathbf{c}\,\mathbf{c} > = m \int_v \mathbf{c}\,\mathbf{c}\, f\, d^3v \tag{4.38}$$

and of the heat flux vector

$$\mathbf{q} = \tfrac{1}{2}\rho_m < c^2\,\mathbf{c} > = \tfrac{1}{2} m \int_v c^2\,\mathbf{c}\, f\, d^3v \tag{4.39}$$

we obtain, using the Maxwellian distribution function,

$$\begin{aligned} \mathcal{P} &= \rho_m\,(< c_x^2 > \widehat{\mathbf{x}}\widehat{\mathbf{x}} + < c_y^2 > \widehat{\mathbf{y}}\widehat{\mathbf{y}} + < c_z^2 > \widehat{\mathbf{z}}\widehat{\mathbf{z}}) \\ &= nkT\,(\widehat{\mathbf{x}}\widehat{\mathbf{x}} + \widehat{\mathbf{y}}\widehat{\mathbf{y}} + \widehat{\mathbf{z}}\widehat{\mathbf{z}}) \end{aligned} \tag{4.40}$$

and

$$\mathbf{q} = 0 \tag{4.41}$$

since the integrals having an odd integrand vanish. The scalar pressure is therefore,

$$p = nkT \tag{4.42}$$

5. EQUILIBRIUM IN THE PRESENCE OF AN EXTERNAL FORCE

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A gas under steady state conditions and immersed in a conservative force field is characterized by a distribution function that differs from the Maxwell-Boltzmann distribution by an exponential factor, known as the *Boltzmann factor*. The conservative force field can be specified in terms of a potential energy $U(\mathbf{r})$, such that

$$\mathbf{F}(\mathbf{r}) = -\nabla U(\mathbf{r}) \tag{5.1}$$

Since the conservative force field is a function only of the position vector $\mathbf{r}$, we expect the steady state solution of the Boltzmann equation for this case to be of the form

$$f(\mathbf{r}, v) = f_0(v)\,\psi(\mathbf{r}) \tag{5.2}$$

where $f_0(v)$ represents the Maxwell-Boltzmann equilibrium distribution function and $\psi(\mathbf{r})$ is a scalar function of $\mathbf{r}$ only, still to be determined. The function $\psi(\mathbf{r})$ can be determined by requiring (5.2) to satisfy the Boltzmann equation under equilibrium conditions in the presence of the conservative field,

p. 131 (5.12)

$$\mathbf{v} \cdot \nabla[f_0(v)\ \psi(\mathbf{r})] - \frac{1}{m}[\nabla U(\mathbf{r})] \cdot \nabla_v[f_0(v)\ \psi(\mathbf{r})] = 0 \tag{5.3}$$

From the expression for $f_0(v)$ it can be easily verified that

$$\nabla_v f_0(v) = -\frac{m\mathbf{v}}{kT}\ f_0(v) \tag{5.4}$$

Therefore, (5.3) simplifies to

$$f_0(v)\ \mathbf{v} \cdot [\nabla\psi(\mathbf{r}) + \frac{1}{kT}\psi(\mathbf{r})\ \nabla U(\mathbf{r})] = 0 \tag{5.5}$$

from which we can write

$$\frac{\nabla\psi(\mathbf{r})}{\psi(\mathbf{r})} = -\frac{1}{kT}\nabla U(\mathbf{r}) \tag{5.6}$$

Since $d\psi = \nabla\psi \cdot d\mathbf{r}$, (5.6) may also be written as

$$\frac{d\psi(\mathbf{r})}{\psi(\mathbf{r})} = -\frac{1}{kT}dU(\mathbf{r}) \tag{5.7}$$

The solution of this differential equation is

$$\psi(\mathbf{r}) = A_0\ \exp\left[-\frac{U(\mathbf{r})}{kT}\right] \tag{5.8}$$

where A_0 is a constant that can be determined by requiring that

$$\int_v f(\mathbf{r}, v)\ d^3v = n(\mathbf{r}) \tag{5.9}$$

from which we get

$$n(\mathbf{r}) = A_0\ \exp\left[-\frac{U(\mathbf{r})}{kT}\right] \cdot \int_v f_0(v)\ d^3v \tag{5.10}$$

Denoting by n_0 the number density in a region where $U(\mathbf{r}) = 0$, under equilibrium conditions, that is,

$$n_0 = \int_v f_0(v)\, d^3v \tag{5.11}$$

we must choose $A_0 = 1$. Therefore, the equilibrium distribution function (with $\mathbf{u} = 0$), under the presence of a conservative force field, is

$$f(\mathbf{r}, v) = f_0(v) \exp\left[-\frac{U(\mathbf{r})}{kT}\right]$$

$$= n_0 \left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left[-\frac{(\frac{1}{2}mv^2 + U)}{kT}\right] \tag{5.12}$$

The number density, for a system described by this velocity distribution function, is therefore given by

$$n(\mathbf{r}) = n_0 \exp\left[-\frac{U(\mathbf{r})}{kT}\right] \tag{5.13}$$

The factor $\exp\,[-U(\mathbf{r})/kT]$, responsible for the inhomogeneity of $f(\mathbf{r}, v)$ in (5.12), is known as the *Boltzmann factor.*

An important example is provided by a plasma in the presence of a conservative force due to an electrostatic field

$$\mathbf{E} = -\nabla\phi(\mathbf{r}) \tag{5.14}$$

where $\phi(\mathbf{r})$ is the electrostatic scalar potential. The potential energy, in this case, is

$$U(\mathbf{r}) = q\ \phi(\mathbf{r}) \tag{5.15}$$

The number density for particles of charge q in equilibrium under the action of an electrostatic field is therefore

$$n(\mathbf{r}) = n_0 \exp\left[-\frac{q\ \phi(\mathbf{r})}{kT}\right] \tag{5.16}$$

This expression is very useful for the analysis of electrostatic shielding in a plasma (see Chapter 11).

6. DEGREE OF IONIZATION IN EQUILIBRIUM AND THE SAHA EQUATION

From the methods of statistical mechanics we can determine the degree of ionization in a gas in thermal equilibrium at some temperature T, without considering the details of the ionization process. In order to ionize an atom or molecule, it is necessary to provide a certain amount of energy. This ionization energy is conveniently expressed in *electron volts*, and is normally called the *ionization potential.* Values for the first ionization potential of some atoms are given in Table 1. Note that to provide a mean thermal energy kT of 1 eV requires a temperature of 11,600 K. Hence, it is apparent that only at very high temperatures does the mean thermal kinetic energy $3kT/2$ of a particle exceed the ionization energy. However, we will show that a considerable degree of ionization can be achieved even when the mean thermal energy of the particles is far below the ionization energy, since some of the particles, the ones with the largest velocities (in the tail of the Maxwellian distribution function), have enough energy to produce ionization by collisions. The degree of ionization, under thermal equilibrium conditions, is then determined by a balance between the rate of ionization by collisions and the rate of recombination.

To calculate the relative numbers of ionized and neutral atoms in a plasma, at a given temperature, it is appropriate to use a particle distribution function similar to that given in (5.13). However, the physical situation is somewhat different because of the necessary quantum mechanical aspects of the problem. Denoting by n_a and n_b the number density of the particles having energies U_a and U_b, respectively, the ratio n_a/n_b is given, from statistical mechanics, by

$$\frac{n_a}{n_b} = \frac{g_a}{g_b} \exp\left[-\frac{(U_a - U_b)}{kT}\right] \tag{6.1}$$

where g_a and g_b are the *statistical weights* associated with the energies U_a and U_b, that is, the *degeneracy factors* giving the number of states having the energies U_a and U_b, respectively. For the particular case of a system having only two energy levels, U_a and U_b, the fraction α of all the particles that are in the higher energy state U_a is given by

$$\alpha = \frac{n_a}{(n_a + n_b)} = \frac{n_a}{n_b}\left(\frac{n_a}{n_b} + 1\right)^{-1} \tag{6.2}$$

or, using (6.1) with $U = U_a - U_b$,

$$\alpha = \frac{(g_a/g_b)\exp(-U/kT)}{(g_a/g_b)\exp(-U/kT) + 1} \tag{6.3}$$

TABLE 1
IONIZATION POTENTIAL ENERGY U
OF SOME ATOMS FOR THE FIRST ELECTRON

Element	U(eV)
Helium (He)	24.59
Argon (A)	15.76
Nitrogen (N)	14.53
Oxygen (O)	13.62
Hydrogen (H)	13.60
Mercury (Hg)	10.44
Iron (Fe)	7.87
Sodium (Na)	5.14
Potassium (K)	4.34
Cesium (Cs)	3.89

For the ionization problem, state a is taken as that of the ion-electron pair, state b is that of the neutral atom and $U = U_a - U_b$ is the ionization energy. The temperature T for which $\alpha = 0.5$, that is, for which fifty per cent of all the atoms are in the ionized state ($n_a = n_b$), can be determined by taking

$$\frac{g_a}{g_b} \exp\left(-\frac{U}{k\,T_{1/2}}\right) = 1 \tag{6.4}$$

which gives

$$T_{1/2} = \frac{U}{k \ln\,(g_a/g_b)} \tag{6.5}$$

Fig. 9 shows the plot of α as a function of T, according to (6.3).

The fraction of particles in the ionized state changes from nearly zero to nearly one over a small temperature range. An *estimate* of this temperature range can be obtained from the temperature difference ΔT that would exist between $\alpha = 0$ and $\alpha = 1$, if the curve of $\alpha(T)$ were a straight line with the slope of the true $\alpha(T)$ curve at $T_{1/2}$. Hence, we take

$$\left(\frac{d\alpha(T)}{dT}\right)_{T_{1/2}} = \frac{1}{\Delta T} \tag{6.6}$$

From (6.3) we obtain, assuming $d(g_a/g_b)/dT = 0$,

$$\left(\frac{d\alpha(T)}{dT}\right)_{T_{1/2}} = \left[\frac{U\alpha^2}{T^2(g_a/g_b)\exp\,(-U/kT)}\right]_{T_{1/2}} = \frac{U}{4\,T_{1/2}^2} \tag{6.7}$$

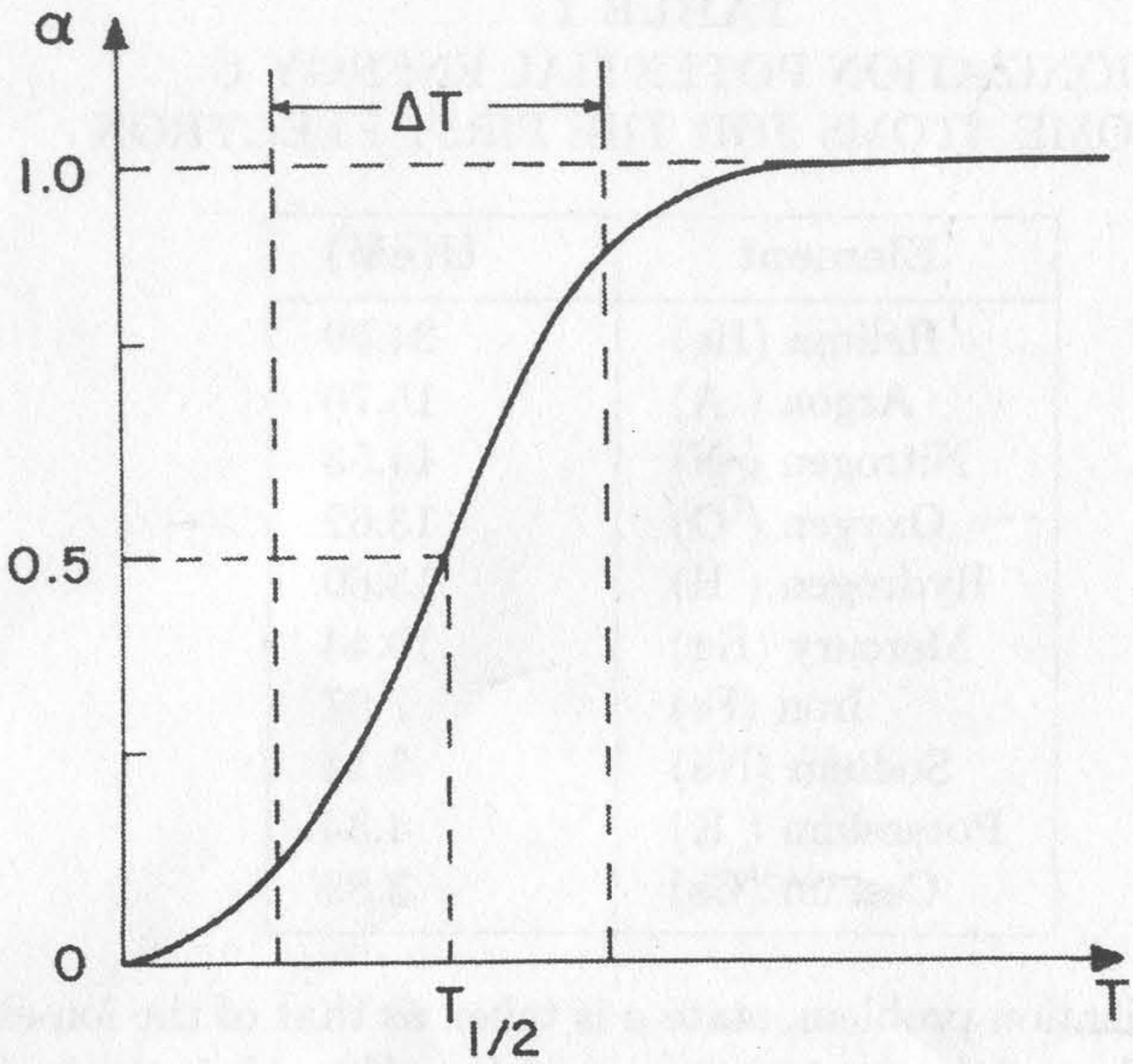

Fig. 9 *The function $\alpha(T)$, which gives the fraction of particles in the ionized state as a function of temperature T.*

so that

$$\Delta T = \frac{4\,T_{1/2}}{k\,\ln\,(g_a/g_b)} = \frac{4\,U}{[k\,\ln\,(g_a/g_b)]^2} \tag{6.8}$$

From this result we can see that, the larger is g_a/g_b, the smaller is ΔT.

Since the ionized state is much more degenerate than the neutral state ($g_a \gg g_b$), the curve of $\alpha(T)$ presents a very steep inclination near $T_{1/2}$, with most of the transitions from the neutral state to the ionized state occurring near $T_{1/2}$, given by (6.5). Thus, for $g_a \gg g_b$, the curve of $\alpha(T)$ will look approximately like a step function with the ionization occurring near $T_{1/2}$.

The degeneracy factors g_a and g_b can be obtained from a quantum mechanical calculation. If we neglect the small interaction potential between the ion and the free electron, and also the internal degrees of freedom of all the particles, it turns out that

$$\frac{g_a}{g_b} = \left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} \frac{1}{n_i} \tag{6.9}$$

where h is Planck's constant and n_i is the ion number density. For T expressed in *degrees Kelvin* and n_i in m^{-3},

$$\frac{g_a}{g_b} = 2.405 \times 10^{21}\, T^{3/2}\, \frac{1}{n_i} \tag{6.10}$$

Using this result in (6.1), we obtain the following equation

$$\frac{n_i}{n_n} = 2.405 \times 10^{21}\, T^{3/2}\, \frac{1}{n_i} \exp\left(-\frac{U}{kT}\right) \tag{6.11}$$

which is known as the *Saha equation.* Since 1 eV $= kT$ for $T = 11,600$ K, we can also write the Saha equation as

$$\frac{n_i}{n_n} = 3.00 \times 10^{27}\, T^{3/2}\, \frac{1}{n_i} \exp\left(-\frac{U}{T}\right) \tag{6.12}$$

with T in eV and n_i in m^{-3}. Thus, when the total number density $n_t = n_i + n_n$ is sufficiently low, a considerable degree of ionization can be achieved for temperatures that are well below the ionization energy. This point is illustrated in Fig. 10, which shows the degree of ionization of hydrogen as a function of temperature, for values of the total number density of 10^{16}, 10^{19}, 10^{22} and 10^{25} m^{-3}. It is clear that, as the number density decreases, the values of ΔT and $T_{1/2}$ decrease significantly, and a significant degree of ionization can be obtained at temperatures far below the ionization energy of atomic hydrogen (13.60 eV). In a gas like cesium, whose ionization energy is only 3.89 eV, a high degree of ionization can be obtained even at relatively low temperatures of the order of 1000 K.

PROBLEMS

7.1 A *two-dimensional* gas, consisting of only one species and whose particles are restricted to move in a plane (the $z = 0$ plane), is characterized by a homogeneous, isotropic, two-dimensional Maxwell-Boltzmann distribution function (with $\mathbf{u} = 0$),

$$f(v) = n_0 \left(\frac{m}{2\pi kT}\right) \exp\left[-\frac{m(v_x^2 + v_y^2)}{2kT}\right]$$

where n_0 represents the number of particles per unit area.

(a) Show that the most probable speed of the particles is

$$v_{mp} = (kT/m)^{1/2}$$

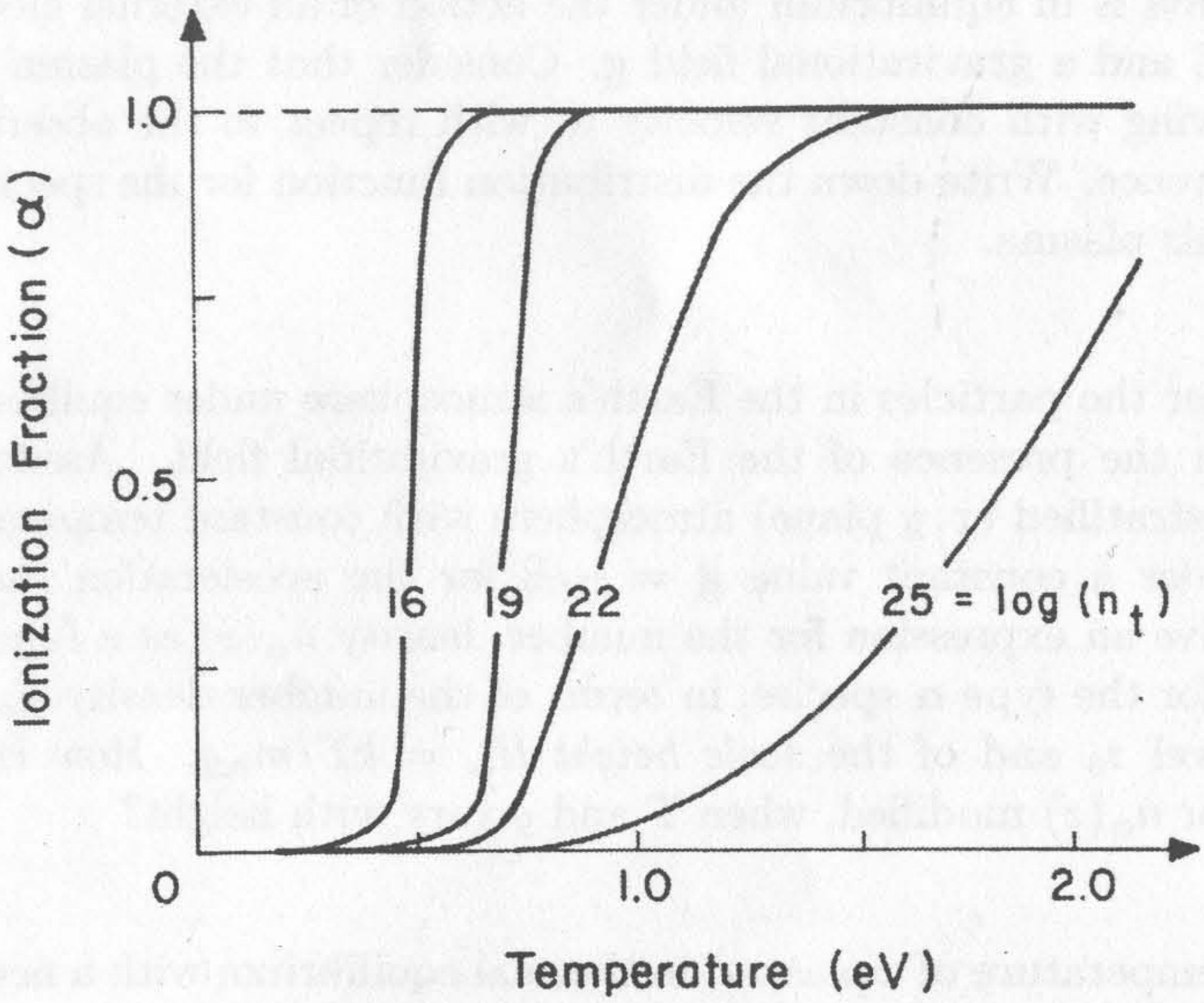

Fig. 10 *Degree of ionization $\alpha = n_i/(n_i + n_n)$ as a function of temperature for atomic hydrogen ($U = 13.60$ eV). The various curves refer to different number densities $n_t = n_i + n_n$ in m^{-3}.*

(b) Show that the fraction of the number of particles per unit area, which have speeds greater than the most probable speed, is given by $(1/e)^{1/2}$, where e is the base of natural logarithms.

(c) Show that the number of particles crossing a unit length per unit time (flux), from one side only, is given by

$$\Gamma = \frac{n_0}{\pi} < v > = n_0 \left(\frac{kT}{2\pi m} \right)^{1/2}$$

(d) Show that the kinetic pressure dyad is given by

$$\mathcal{P} = n_0 kT(\hat{\mathbf{x}}\hat{\mathbf{x}} + \hat{\mathbf{y}}\hat{\mathbf{y}})$$

7.2 Consider a gas of particles consisting of only one species and characterized by the Maxwell-Boltzmann equilibrium distribution function (with $\mathbf{u} = 0$)

$$f(v) = n_0 \left(\frac{m}{2\pi kT} \right)^{3/2} \exp\left(-\frac{mv^2}{2kT} \right)$$

(a) Show that the total number of particles crossing a unit area per unit time, lying within an element $d\Omega$ of solid angle, is given by

$$\frac{n_0}{\pi}\left(\frac{kT}{2\pi m}\right)^{1/2} \cos\theta \, d\Omega$$

where θ denotes the angle between the solid angle orientation and the direction of the normal to the area considered.

(b) Show that the fraction of particles that cross a unit area perpendicular to the x axis per unit time, from the same side, having the velocity components in the range $d^3v = dv_x \, dv_y \, dv_z$, about $\mathbf{v}$, is given by

$$\frac{1}{2\pi}\left(\frac{m}{kT}\right)^2 v_x \exp\left(-\frac{mv^2}{2kT}\right) d^3v$$

(c) Calculate the thermal energy flux triad for the Maxwellian gas.

7.3 The distribution of thermal kinetic energies E, for a gas in the Maxwellian state, is given by

$$G(E) = \frac{2nE^{1/2}}{\pi^{1/2}(kT)^{3/2}} \exp\left(-\frac{E}{kT}\right)$$

Calculate the *most probable energy* and show that the velocity of the particles, which have this energy, is equal to $(kT/m)^{1/2}$.

7.4 The entropy of a system can be expressed in terms of the distribution function as

$$S = -k \int_r \int_v f \ln f \, d^3v \, d^3r$$

Prove that, for a Maxwellian distribution function, the entropy satisfies the following thermodynamic relations

$$\left(\frac{\partial S}{\partial E}\right)_{V,N} = \frac{1}{T}$$

$$\left(\frac{\partial S}{\partial V}\right)_{E,N} = \frac{p}{T}$$

where N is the total number of particles in the system, V is the total volume and $E = 3NkT/2$ is the total energy.

7.5 Derive an expression for the *Doppler* intensity profile (thermal broadening) of a spectral lime emitted near the central frequency ν_0, assuming that the emitting atoms have a Maxwellian velocity distribution. Ignore all other factors that contribute to the shape of the line.
Hints: (1) The change in frequency due to the Doppler effect associated with the relative (nonrelativistic) motion of the emitting atoms, with respect to the direction of observation (x direction, e.g.), is given by

$$\nu - \nu_0 = -\nu_0 \frac{v_x}{c}$$

where in this expression c denotes the speed of light in vacuum. (2) The observed intensity in the frequency range between ν and $\nu + d\nu$, that is, $I(\nu)\ d\nu$, is proportional to the number of emitting atoms per unit volume, which have velocities along the direction of observation (x direction) between v_x and $v_x + dv_x$.

7.6 Consider a gas mixture containing n_e electrons and n_i oxygen ions per unit volume, all in thermal equilibrium at a temperature T and having no drift velocity.
(a) Resolve the motion of the particle species into the motion in space of the *center of mass* plus the *relative motion* of one species with respect to the other, but using the *reduced mass.* Calculate the Jacobian J of this velocity transformation and show that $|J| = 1$.
(b) Show that the center of mass velocities have a Maxwellian distribution and that the relative velocities also have a Maxwellian distribution, but with the reduced mass.
(c) What must be the magnitude of T such that 1/5 of the electrons have a relative kinetic energy greater than 2 eV? The following integral will be useful:

$$\int_{x_0}^{\infty} x^2 \exp\left(-\alpha x^2\right) dx = \frac{x_0 \exp\left(-\alpha x_0^2\right)}{2\alpha} + \frac{\pi^{1/2}}{4\alpha^{3/2}} \operatorname{erfc}\left(\alpha^{1/2} x_0\right)$$

where erfc $(\alpha^{1/2} x_0)$ denotes the *complementary error function.*

7.7 A gas of O_2 molecules is in the equilibrium state with number density n and absolute temperature T. Calculate the average value of the reciprocal of the particle velocity, that is, $< 1/v >$.

7.8 A plasma is in equilibrium under the action of an external electrostatic field $\mathbf{E}$ and a gravitational field $\mathbf{g}$. Consider that the plasma as a whole is moving with constant velocity $\mathbf{u}$, with repect to the observer's frame of reference. Write down the distribution function for the species of type α for this plasma.

7.9 Consider the particles in the Earth's atmosphere under equilibrium conditions in the presence of the Earth's gravitationl field. Assume a horizontally stratified (x, y plane) atmosphere with constant temperature T and consider a constant value $\mathbf{g} = -g\hat{\mathbf{z}}$ for the acceleration due to gravity. Derive an expression for the number density $n_\alpha(z)$ as a function of height z, for the type α species, in terms of the number density $n_\alpha(z_0)$ at a base level z_0 and of the *scale height* $H_\alpha = kT/m_\alpha g$. How is the expression for $n_\alpha(z)$ modified, when T and g vary with height?

7.10 The temperature of a plasma, in thermal equilibrium with a neutral gas, can be determined experimentally by measuring the electron density n_e with a microwave transmission experiment, for example, and the neutrals number density in a particular excited state through the rate of transitions to a lower state. Determine the temperature of a plasma which has only one type of ions, with the electron number density equal to 10^{20} m^{-3} and which is in equilibrium with a state of ionization potential equal to 2 eV whose population is 10^{15} m^{-3}.

7.11 Consider two large chambers which communicate with each other only through a small aperture of area A in a very thin wall, as indicated in Fig. 11. The chambers contain an ideal gas at a *very low* pressure, such that the particle mean free path is much larger than the dimensions of A. The temperatures of the chambers are T_1 and T_2. Determine the ratio p_1/p_2 of pressure in the two chambers assuming that, under equilibrium conditions, the flux of particles through the aperture A from one chamber must equal that from the other. What would be the result in the case of *normal* conditions of pressure? Give a physical explanation for the two different results.

7.12 Use the laws of conservation of momentum and of energy in a collision to show that the Maxwell-Boltzmann distribution function

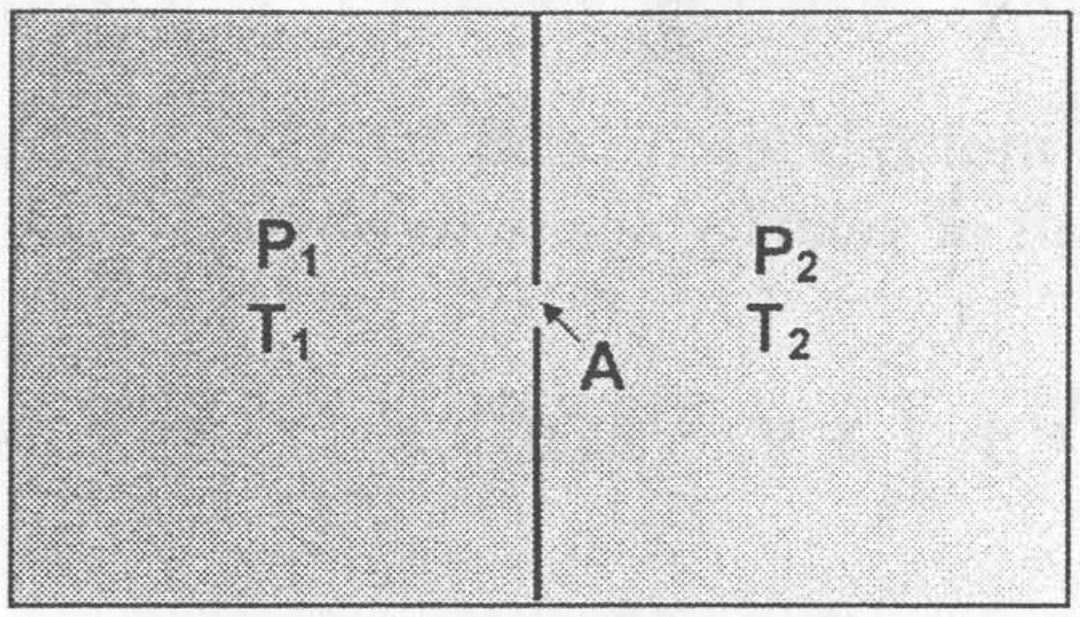

Fig. 11 *Two chambers connected through a small aperture of area A, at very low pressure, as illustration for Problem 7.11.*

$$f(\mathbf{v}) = n\left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left[-\frac{m(\mathbf{v}-\mathbf{u})^2}{2kT}\right]$$

satisfies the following equation of detailed balance

$$f'\, f_1' = f\, f_1$$

7.13 Show that the average thermal energy per particle, for a gas in thermodynamic equilibrium, is equal to 1.292×10^{-4} eV/K.

8

MACROSCOPIC TRANSPORT EQUATIONS

1. MOMENTS OF THE BOLTZMANN EQUATION

In the previous chapters we have seen that the macroscopic variables of physical interest for a plasma, such as number density n_α, mean velocity $\mathbf{u}_\alpha$, temperature T_α, and so on, can be calculated if we know the distribution function for the system under consideration. For the case of a system in thermal equilibrium we have calculated, in Chapter 7, several of these macroscopic parameters using the Maxwell-Boltzmann distribution function. In principle, the distribution function for a system not in equilibrium can be obtained by solving the Boltzmann equation. However, the solution of the Boltzmann equation is generally a matter of great difficulty. We will see, in this chapter, that it is not necessary to solve the Boltzmann equation for the distribution function in order to determine the macroscopic variables of physical interest. The differential equations governing the temporal and spatial variations of these macroscopic variables can be derived directly from the Boltzmann equation without solving it. These differential equations are known as the *macroscopic transport equations*, and their solutions, under certain assumptions, give us directly the macroscopic variables.

The macroscopic variables are related to the *moments of the distribution function* and the transport equations satisfied by these variables can be obtained by taking the various *moments of the Boltzmann equation.* The first three moments of the Boltzmann equation, obtained by multiplying it by m_α , $m_\alpha \mathbf{v}$ and $m_\alpha v^2/2$, respectively, and integrating over all of velocity space, give us the equation of conservation of mass,

the equation of conservation of momentum and the equation of conservation of energy. However, at each step of the hierarchy of moments of the Boltzmann equation, the resulting set of transport equations is not complete in the sense that the number of equations is not sufficient to determine all the macroscopic variables that appear in them. Each time a higher moment of the Boltzmann equation is calculated in an attempt to obtain a complete set of transport equations, a new macroscopic variable appears. It is necessary, therefore, to truncate the system of transport equations at some point of the hierarchy and to introduce a simplifying assumption concerning the highest moment of the distribution function that appears in the system. Thus, with such simpliflying approximation, we can obtain a complete set of transport equations sufficient to determine all the macroscopic variables appearing in the system. Since a plasma is composed of more than one particle species (electrons, ions and neutral particles), there is, consequentely, a system of transport equations for each species.

There are several different complete sets of transport equations (or hydrodynamic equations) that can be formed, depending on the assumptions considered. Amongst the possible complete systems of macroscopic equations, there are two which are widely used and which characterize the so called *cold* and *warm plasma models*. The equations that describe these two simple models and the corresponding approximations will be discussed in Sections 6 and 7 of this chapter.

2. GENERAL TRANSPORT EQUATION

We derive now a general partial differential equation that describes the temporal and spatial variation of the physically relevant *macroscopic* parameters. Let $\chi(\mathbf{v})$ represent some physical property of the particles in the plasma, which may be, in general, a function of the particle velocity. Since the average value of $\chi(\mathbf{v})$ is obtained by multiplying the distribution function by the property $\chi(\mathbf{v})$, integrating the product over all of velocity space and dividing the result by the particle number density, the differential equation governing the temporal and spatial variation of the average value of $\chi(\mathbf{v})$ can be obtained in a similar way by multiplying the Boltzmann equation by the function $\chi(\mathbf{v})$ and integrating the resulting equation over all of velocity space.

Consider the Boltzmann equation for the type α particles in the general form

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v} \cdot \nabla f_\alpha + \mathbf{a} \cdot \nabla_v f_\alpha = \left(\frac{\delta f_\alpha}{\delta t}\right)_{coll} \tag{2.1}$$

As indicated, we now multiply each term by $\chi(\mathbf{v})$ and integrate the resulting equation over all of velocity space to obtain

$$\int_v \chi \, \frac{\partial f_\alpha}{\partial t} \, d^3v + \int_v \chi \, \mathbf{v} \cdot \nabla f_\alpha \, d^3v + \int_v \chi \, \mathbf{a} \cdot \nabla_v f_\alpha \, d^3v$$

$$= \int_v \chi \left(\frac{\delta f_\alpha}{\delta t}\right)_{coll} d^3v \tag{2.2}$$

We proceed next to evaluate separately each of the terms in (2.2).

The *first term* of (2.2) may be rewritten as

$$\int_v \chi \, \frac{\partial f_\alpha}{\partial t} \, d^3v = \frac{\partial}{\partial t}\Big(\int_v \chi \, f_\alpha \, d^3v\Big) - \int_v f_\alpha \, \frac{\partial \chi}{\partial t} \, d^3v \tag{2.3}$$

since the limits of integration do not depend upon the space and time variables, and therefore the partial time derivative can be taken inside or outside the integral sign. The last integral in (2.3) vanishes since $\chi(\mathbf{v})$ does not depend upon t. Using the definition of average values, as presented in Chapter 6, we obtain

$$\int_v \chi \, \frac{\partial f_\alpha}{\partial t} \, d^3v = \frac{\partial}{\partial t}(n_\alpha < \chi >_\alpha) \tag{2.4}$$

Similarly, for the *second term* of (2.2) we can write

$$\int_v \chi \, \mathbf{v} \cdot \nabla f_\alpha \, d^3v = \nabla \cdot \Big(\int_v \mathbf{v} \, \chi \, f_\alpha \, d^3v \Big) - \int_v f_\alpha \, \mathbf{v} \cdot \nabla \chi \, d^3v$$

$$- \int_v f_\alpha \, \chi \, \nabla \cdot \mathbf{v} \, d^3v \tag{2.5}$$

The term involving $\nabla \cdot \mathbf{v}$ is zero, since $\mathbf{r}$, $\mathbf{v}$ and t are independent variables, as well as the term involving $\nabla\chi$, since $\chi(\mathbf{v})$ does not depend upon the space variables. Thus, the second term in (2.2) becomes

$$\int_v \chi \, \mathbf{v} \cdot \nabla f_\alpha \, d^3v = \nabla \cdot (n_\alpha < \chi \, \mathbf{v} >_\alpha) \tag{2.6}$$

For the *third term* of (2.2) we have, in a similar way,

$$\int_v \chi\, \mathbf{a}\cdot\nabla_v f_\alpha\, d^3v = \int_v \nabla_v\cdot(\mathbf{a}\,\chi\, f_\alpha)\, d^3v - \int_v f_\alpha\, \mathbf{a}\cdot\nabla_v\chi\, d^3v - \\ - \int_v f_\alpha\, \chi\, \nabla_v\cdot\mathbf{a}\, d^3v \tag{2.7}$$

The last integral in (2.7) vanishes if we assume that

$$\nabla_v\cdot\mathbf{a} = \frac{1}{m_\alpha}\nabla_v\cdot\mathbf{F} = 0 \tag{2.8}$$

that is, if the force component F_i is independent of the corresponding velocity component v_i, for $i = x, y, z$. Note that this restriction dos not exclude the force due to a magnetic field, $\mathbf{F} = q_\alpha\, \mathbf{v}\times\mathbf{B}$, since in this case F_i is still independent of v_i. For the x component, for example, we have

$$F_x = q_\alpha\, (v_y B_z\ - v_z B_y) \tag{2.9}$$

which is independent of v_x, and the same holds true for the other two components. The first integral in the right-hand side of (2.7) consists of a sum of three triple integrals,

$$\int_v \nabla_v\cdot(\mathbf{a}\,\chi\, f_\alpha)\, d^3v = \sum_i \int\int\int_{-\infty}^{+\infty} \frac{\partial}{\partial v_i}(a_i\,\chi\, f_\alpha)\, dv_x\, dv_y\, dv_z \tag{2.10}$$

For each one of these triple integrals ($i = x, y, z$) we have the result

$$\int\int\int_{-\infty}^{+\infty} \frac{\partial}{\partial v_x}(a_x\,\chi\, f_\alpha)\, dv_x\, dv_y\, dv_z = \\ \int\int_{-\infty}^{+\infty} dv_y\, dv_z (a_x\,\chi\, f_\alpha\,|_{-\infty}^{+\infty}) = 0 \tag{2.11}$$

since $f_\alpha(\mathbf{r}, \mathbf{v}, t)$ must be zero when v_i becomes infinitely large, as there are no particles with infinite velocity. Consequently, the first integral in the right-hand side of (2.7) vanishes. Therefore,

$$\int_v \chi\, \mathbf{a}\cdot\nabla_v f_\alpha\, d^3v = -n_\alpha < \mathbf{a}\cdot\nabla_v\chi >_\alpha \tag{2.12}$$

Combining the results contained in Eqs. (2.4), (2.6) and (2.12), we obtain the *general transport equation*,

$$\frac{\partial}{\partial t}(n_\alpha < \chi >_\alpha) + \nabla \cdot (n_\alpha < \chi \mathbf{v} >_\alpha) - n_\alpha < \mathbf{a} \cdot \nabla_v \chi >_\alpha =$$

$$= \left[\frac{\delta}{\delta t}(n_\alpha < \chi >_\alpha)\right]_{coll} \tag{2.13}$$

where the term in the right denotes the time rate of change of the quantity χ per unit volume, for the particles of type α, due to collisions,

$$\left[\frac{\delta}{\delta t}(n_\alpha < \chi >_\alpha)\right]_{coll} = \int_v \chi \left(\frac{\delta f_\alpha}{\delta t}\right)_{coll} d^3v \tag{2.14}$$

The equations to be derived in the subsequent sections of this chapter are very general and are not specifically dependent on any particular form of the collision term. A derivation of the general transport equation for the case when the property χ is a function of $\mathbf{r}$, $\mathbf{v}$ and t is included in Problem 8.6.

3. CONSERVATION OF MASS

3.1 Derivation of the Continuity Equation

The transport equation (2.13) is a general expression and it applies to any arbitrary function $\chi(\mathbf{v})$. The equation of continuity, or of conservation of mass, can be obtained by taking $\chi = m_\alpha$ in (2.13). Hence, consider

$$< \chi >_\alpha = m_\alpha \tag{3.1a}$$

$$< \chi \mathbf{v} >_\alpha = m_\alpha < \mathbf{v} >_\alpha = m_\alpha \mathbf{u}_\alpha \tag{3.1b}$$

$$\nabla_v \chi = \nabla_v m_\alpha = 0 \tag{3.1c}$$

The substitution of these results into the general transport equation gives the *continuity equation*,

$$\frac{\partial \rho_{m\alpha}}{\partial t} + \nabla \cdot (\rho_{m\alpha} \mathbf{u}_\alpha) = S_\alpha \tag{3.2}$$

where $\rho_{m\alpha} = n_\alpha m_\alpha$ represents the mass density and where the collision term S_α, defined by

$$S_\alpha = m_\alpha \int_v \left(\frac{\delta f_\alpha}{\delta t}\right)_{coll} d^3v = \left(\frac{\delta \rho_{m\alpha}}{\delta t}\right)_{coll} \tag{3.3}$$

represents the rate per unit volume at which particles of type α (with mass m_α) are produced or lost as a result of collisions. Contributions to

this term are due to processes of type α particle production or destruction such as ionization, recombination, attachment, charge transfer, and so on. In the absence of interactions leading to production or loss of particles of type α, the collision term (3.3) is equal to zero, since in this case the mass is conserved in the collision process. When $S_\alpha = 0$ the continuity equation reduces to

$$\frac{\partial \rho_{m\alpha}}{\partial t} + \nabla \cdot (\rho_{m\alpha} \mathbf{u}_\alpha) = 0 \tag{3.4}$$

Dividing each term in (3.4) by m_α, the continuity equation can be written in terms of the number density n_α, as

$$\frac{\partial n_\alpha}{\partial t} + \nabla \cdot (n_\alpha \mathbf{u}_\alpha) = 0 \tag{3.5}$$

The equation of *conservation of electric charge* follows from (3.5) by multiplying it by the particle charge q_α,

$$\frac{\partial \rho_\alpha}{\partial t} + \nabla \cdot \mathbf{J}_\alpha = 0 \tag{3.6}$$

where $\rho_\alpha = n_\alpha q_\alpha$ is the charge density and $\mathbf{J}_\alpha = \rho_\alpha \mathbf{u}_\alpha$ is the charge current density.

3.2 Derivation by the Method of Fluid Dynamics

The continuity equation can also be derived using the method of fluid dynamics, since $n_\alpha(\mathbf{r}, t)$ and $\mathbf{u}_\alpha(\mathbf{r}, t)$ are macroscopic variables. Consider a volume V in the fluid, limited by the closed surface S and let $d\mathbf{S} = \hat{\mathbf{n}}\, dS$ be an element of area on this surface, such that the unit normal vector $\hat{\mathbf{n}}$ points outward, as shown in Fig. 1. The average number of particles of type α that leave the volume V through the element of area $d\mathbf{S}$ per unit time is given by

$$n_\alpha \mathbf{u}_\alpha \cdot d\mathbf{S} \tag{3.7}$$

Therefore, the number of particles of type α that leave the volume V through the whole closed surface S per unit time is obtained by integrating expression (3.7) over the whole surface,

$$\oint_S n_\alpha \mathbf{u}_\alpha \cdot d\mathbf{S} \tag{3.8}$$

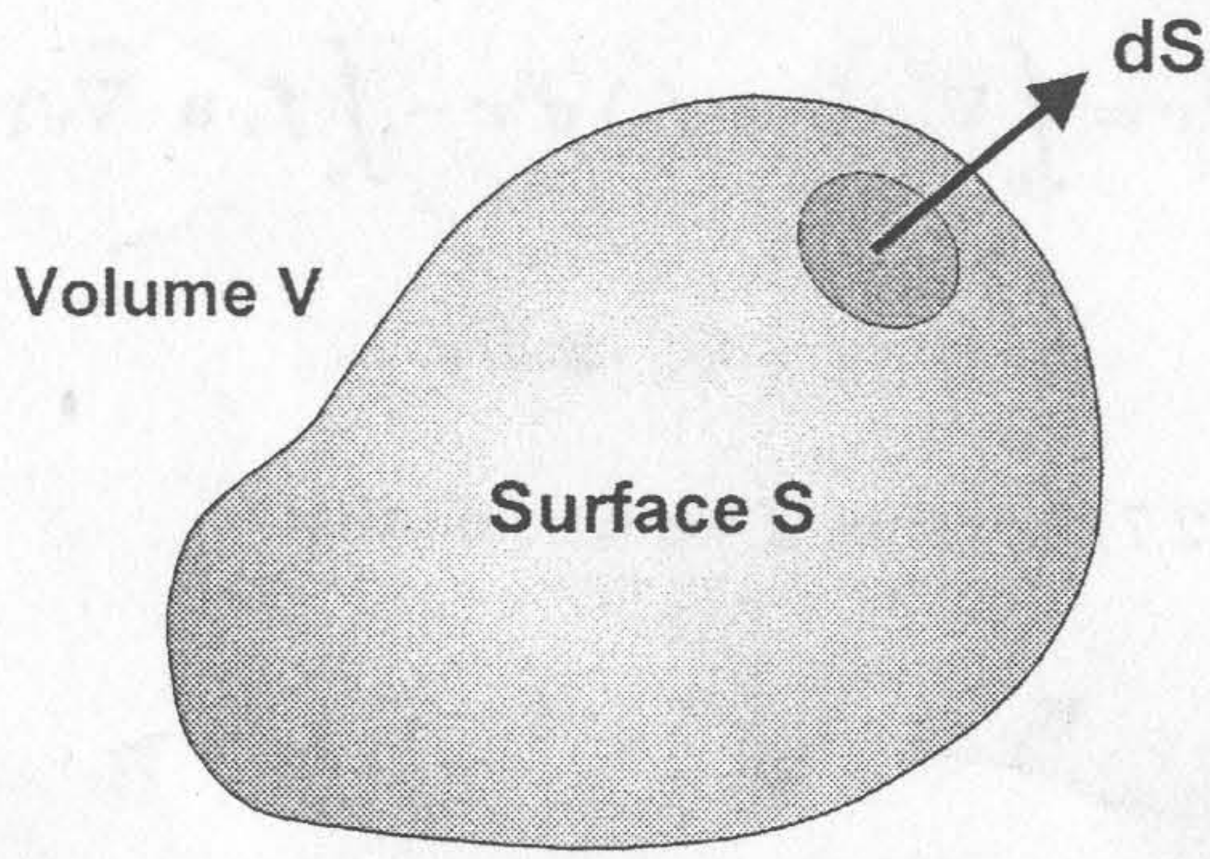

Fig. 1 *Closed surface S surrounding the arbitrary volume V inside the fluid, and the element of area $d\mathbf{S} = \widehat{\mathbf{n}}\, dS$ pointing outwards.*

On the other hand, the total number of particles of type α contained in V, at any time, is given by

$$\int_V n_\alpha \, d^3r \tag{3.9}$$

If we consider that there are no production or loss of particles inside the volume V, then the number of type α particles leaving V, must equal the time rate of decrease of the number of type α particles inside V. We must have therefore

$$\oint_S n_\alpha \mathbf{u}_\alpha \cdot d\mathbf{S} = -\frac{\partial}{\partial t} \int_V n_\alpha \, d^3r \tag{3.10}$$

Using Gauss' divergence theorem we can write

$$\oint_S n_\alpha \mathbf{u}_\alpha \cdot d\mathbf{S} = \int_V \nabla \cdot (n_\alpha \mathbf{u}_\alpha) \, d^3r \tag{3.11}$$

and (3.10) becomes

$$\int_V \left[\frac{\partial n_\alpha}{\partial t} + \nabla \cdot (n_\alpha \mathbf{u}_\alpha) \right] d^3r = 0 \tag{3.12}$$

This result must be valid for any arbitrary volume V, which implies that the integrand of (3.12) vanishes identically. Hence, we obtain the expression (3.5) for the continuity equation.

3.3 The Collision Term

Let us consider now the form of the collision term S_α for some mechanisms of production and loss of particles in plasmas. The processes leading to production and loss of particles are usually related to inelastic collisions, such as those involving ionization, recombination or electron attachment, for example.

The effect of *ionization* can be included in the continuity equation through a rate coefficient for ionization, k_i, defined such that the number of electrons produced per unit time is given by $k_i\ n_e$.

An important process leading to the loss of electrons and ions in a plasma is ion-electron *recombination.* Let k_r denote the recombination coefficient, which can be determined experimentally. The rate of electron recombination is proportional to the product of the electron and ion number densities. Assuming that there is only one ion species present, we have $n_i = n_e$, and the electron loss term, due to recombination, can be written as $k_r\ n_e^2$.

Another important mechanism for electron loss is the process of electron *attachment.* In this case, the electron loss rate is proportional to the product of the electron number density and the neutral particle number density. In a weakly ionized plasma the neutral particle number density can be considered to be approximately constant, and the loss term for the electrons due to attachment can be written as $k_a\ n_e$, where k_a is the attachment collision frequency, which can be determined experimentally.

For these inelastic collision mechanisms just described, the collision term S_α for the electrons can be expressed as

$$S_e = m_e(k_i n_e - k_r n_e^2 - k_a n_e) \tag{3.13}$$

4. CONSERVATION OF MOMENTUM

4.1 Derivation of the Equation of Motion

In order to derive the momentum transport equation, we replace $\chi(\mathbf{v})$ by $m_\alpha \mathbf{v}$ in the general transport equation (2.13). Taking $\mathbf{v} = \mathbf{c}_\alpha + \mathbf{u}_\alpha$ and noting that $< \mathbf{c}_\alpha >= 0$, the terms of the general transport equation (2.13) become

$$\frac{\partial}{\partial t}(\rho_{m\alpha} < \mathbf{v} >_\alpha) = \rho_{m\alpha}\frac{\partial \mathbf{u}_\alpha}{\partial t} + \mathbf{u}_\alpha \frac{\partial \rho_{m\alpha}}{\partial t} \tag{4.1a}$$

$$\nabla \cdot (\rho_{m\alpha} < \mathbf{vv} >_\alpha) = \nabla \cdot [\rho_{m\alpha}(\mathbf{u}_\alpha \mathbf{u}_\alpha + \mathbf{u}_\alpha < \mathbf{c}_\alpha > + < \mathbf{c}_\alpha > \mathbf{u}_\alpha +$$

$$+ < \mathbf{c}_\alpha \mathbf{c}_\alpha >)] = \nabla \cdot (\rho_{m\alpha} \mathbf{u}_\alpha \mathbf{u}_\alpha + \rho_{m\alpha} < \mathbf{c}_\alpha \mathbf{c}_\alpha >) \tag{4.1b}$$

$$-n_\alpha < \mathbf{F} \cdot \nabla_v \mathbf{v} >_\alpha = -n_\alpha < \left(F_x \frac{\partial}{\partial v_x} + F_y \frac{\partial}{\partial v_y} + F_z \frac{\partial}{\partial v_z} \right) \mathbf{v} >_\alpha$$

$$= -n_\alpha < F_x \widehat{\mathbf{x}} + F_y \widehat{\mathbf{y}} + F_z \widehat{\mathbf{z}} >_\alpha = -n_\alpha < \mathbf{F} >_\alpha \tag{4.1c}$$

Substituting these expressions into (2.13), results in the momentum conservation equation

$$\rho_{m\alpha} \frac{\partial \mathbf{u}_\alpha}{\partial t} + \mathbf{u}_\alpha \frac{\partial \rho_{m\alpha}}{\partial t} + \nabla \cdot (\rho_{m\alpha} \mathbf{u}_\alpha \mathbf{u}_\alpha) + \nabla \cdot (\rho_{m\alpha} < \mathbf{c}_\alpha \mathbf{c}_\alpha >) -$$

$$-n_\alpha < \mathbf{F} >_\alpha = \mathbf{A}_\alpha \tag{4.2}$$

where $\mathbf{A}_\alpha$ denotes the collision term

$$\mathbf{A}_\alpha = m_\alpha \int_v \mathbf{v} \left(\frac{\delta f_\alpha}{\delta t} \right)_{coll} d^3 v = \left[\frac{\delta(\rho_{m\alpha} \mathbf{u}_\alpha)}{\delta t} \right]_{coll} \tag{4.3}$$

The expression $\rho_{m\alpha} < \mathbf{c}_\alpha \mathbf{c}_\alpha >$ is the *kinetic pressure dyad* $\mathcal{P}_\alpha$ defined in (6.6.2). Therefore,

$$\nabla \cdot (\rho_{m\alpha} < \mathbf{c}_\alpha \mathbf{c}_\alpha >) = \nabla \cdot \mathcal{P}_\alpha \tag{4.4}$$

The third term in the left-hand side of (4.2) can be expanded as follows

$$\nabla \cdot (\rho_{m\alpha} \mathbf{u}_\alpha \mathbf{u}_\alpha) = \frac{\partial}{\partial x}(\rho_{m\alpha} u_{\alpha x} \mathbf{u}_\alpha) + \frac{\partial}{\partial y}(\rho_{m\alpha} u_{\alpha y} \mathbf{u}_\alpha) + \frac{\partial}{\partial z}(\rho_{m\alpha} u_{\alpha z} \mathbf{u}_\alpha)$$

$$= \rho_{m\alpha} \left(u_{\alpha x} \frac{\partial \mathbf{u}_\alpha}{\partial x} + u_{\alpha y} \frac{\partial \mathbf{u}_\alpha}{\partial y} + u_{\alpha z} \frac{\partial \mathbf{u}_\alpha}{\partial z} \right) +$$

$$+ \mathbf{u}_\alpha \left[\frac{\partial(\rho_{m\alpha} u_{\alpha x})}{\partial x} + \frac{\partial(\rho_{m\alpha} u_{\alpha y})}{\partial y} + \frac{\partial(\rho_{m\alpha} u_{\alpha z})}{\partial z} \right]$$

$$= \rho_{m\alpha} (\mathbf{u}_\alpha \cdot \nabla) \mathbf{u}_\alpha + \mathbf{u}_\alpha [\nabla \cdot (\rho_{m\alpha} \mathbf{u}_\alpha)] \tag{4.5}$$

Substituting (4.4) and (4.5) into (4.2), and using the continuity equation (3.2), we obtain

$$\rho_{m\alpha} \left[\frac{\partial \mathbf{u}_\alpha}{\partial t} + (\mathbf{u}_\alpha \cdot \nabla) \mathbf{u}_\alpha \right] + \nabla \cdot \mathcal{P}_\alpha - n_\alpha < \mathbf{F} >_\alpha = \mathbf{A}_\alpha - \mathbf{u}_\alpha S_\alpha \tag{4.6}$$

For the terms within brackets in this last equation we can use the total (or substantial) time derivative operator

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u}_\alpha \cdot \nabla \tag{4.7}$$

which corresponds to the time variation observed in a reference frame moving with the mean velocity $\mathbf{u}_\alpha$. If the electromagnetic Lorentz force and the gravitational force are considered, the last term in the left-hand side of (4.6) becomes

$$-n_\alpha < \mathbf{F} >_\alpha = -n_\alpha q_\alpha(\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}) - n_\alpha m_\alpha \mathbf{g} \tag{4.8}$$

where the fields $\mathbf{E}$ and $\mathbf{B}$, in this equation, represent smoothed macroscopic fields. The equation of motion can therefore be written as

$$\rho_{m\alpha}\frac{D\mathbf{u}_\alpha}{Dt} = n_\alpha q_\alpha(\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}) + \rho_{m\alpha}\mathbf{g} - \nabla \cdot \mathcal{P}_\alpha + \mathbf{A}_\alpha - \mathbf{u}_\alpha S_\alpha \tag{4.9}$$

Physically, this equation states that the time rate of change of the mean momentum, in each fluid element, is due to the external forces applied in the fluid, plus the shear (viscosity) and pressure forces of the fluid itself, and plus the internal forces associated with the collisional interactions. Thus, the equation of motion establishes the condition necessary to guarantee conservation of momentum, just as the continuity equation establishes the condition necessary to guarantee conservation of mass (or number of particles).

In Chapter 6 we have seen that the term $-\nabla \cdot \mathcal{P}_\alpha$ represents the force exerted in a unit volume of the plasma, due to the random variations in the particle peculiar velocities. This force per unit volume includes forces associated with the scalar pressure and tangential shear forces (viscous forces). In many cases, the effect of viscosity is relatively unimportant in plasmas and the nondiagonal terms of $\mathcal{P}_\alpha$ can be neglected. Furthermore, in the special case when the distribution of peculiar velocities is isotropic, the diagonal terms of $\mathcal{P}_\alpha$ are all equal and correspond to a scalar kinetic pressure p_α. Thus, neglecting viscosity effects and considering an isotropic velocity distribution, we have $\mathcal{P}_\alpha = p_\alpha \mathbf{1}$, and the force per unit volume becomes $-\nabla \cdot \mathcal{P}_\alpha = -\nabla p_\alpha$, according to (6.6.18).

With these simplifying approximations, and neglecting collisions leading to production or loss of particles ($S_\alpha = 0$), the momentum equation becomes

$$\rho_{m\alpha}\frac{D\mathbf{u}_\alpha}{Dt} = n_\alpha q_\alpha(\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}) + \rho_{m\alpha}\mathbf{g} - \nabla p_\alpha + \mathbf{A}_\alpha \tag{4.10}$$

The momentum conservation equation can also be derived using a fluid-dynamics approach, in a way similar to the derivation of the mass conservation equation presented in section 3.2, which we shall not discuss here.

4.2 The Collison Term

The symbol $\mathbf{A}_\alpha$ denotes the rate of change of the mean momentum per unit volume, due to collisons. As a consequence of conservation of the *total* momentum in an elastic collision, the change in the momentum of one of the particles must be equal and opposite to the change in momentum of the other particle participating in the collision event. This means that, for collisions involving particles of the same species, there is no variation in the total momentum per unit volume and therefore, in this case, $\mathbf{A}_\alpha = 0$. However, for a fluid composed of particles of different species, as in a plasma, the collision term $\mathbf{A}_\alpha$ is not zero in general. For collisons between electrons and neutral particles there is a net momentum transfer from the electron gas to the neutral gas. Collisions between electrons and ions also modify the total momentum of the electron gas. Therefore, for the case of collisions between particles of different species, a collision term must be included in the equation of momentum conservation.

An expression often used for the term of momentum transfer by collisions is

$$\mathbf{A}_\alpha = -\rho_{m\alpha} \sum_\beta \nu_{\alpha\beta}(\mathbf{u}_\alpha - \mathbf{u}_\beta) \tag{4.11}$$

which assumes that the force per unit volume exerted on the particles of type α due to collisions with particles of some other type (β) is proportional to the difference between the mean velocities of these particles. The proportionality constant $\nu_{\alpha\beta}$ (which has dimensions of sec^{-1}) is called the *collision frequency for momentum transfer* between the particles of type α and those of type β. Since the total momentum must be conserved during a collision, we must have

$$\rho_{m\alpha}\nu_{\alpha\beta}(\mathbf{u}_\alpha - \mathbf{u}_\beta) + \rho_{m\beta}\nu_{\beta\alpha}(\mathbf{u}_\beta - \mathbf{u}_\alpha) = 0 \tag{4.12}$$

The collision frequencies $\nu_{\alpha\beta}$ and $\nu_{\beta\alpha}$ satisfy, therefore, the following important relation

$$\rho_{m\alpha}\nu_{\alpha\beta} = \rho_{m\beta}\nu_{\beta\alpha} \tag{4.13}$$

The collision term $\mathbf{A}_\alpha$, defined in (4.3), will be considered in more detail in Chapter 21. We will see, then, that the expression (4.11) is not

generally valid, although this result is obtained when the difference between the mean velocities of the various particle species in the plasma is relatively small and when each particle species has a Maxwellian velocity distribution.

5. CONSERVATION OF ENERGY

5.1 Derivation of the Energy Transport Equation

To derive the energy transport equation, we substitute $\chi(\mathbf{v})$ by the particle kinetic energy $m_\alpha v^2/2$ in the general transport equation (2.13). In this case, we have

$$n_\alpha < \chi >_\alpha = \tfrac{1}{2}\rho_{m\alpha} < c_\alpha^2 > + \tfrac{1}{2}\rho_{m\alpha} u_\alpha^2 =$$

$$= \tfrac{1}{2}(3p_\alpha + \rho_{m\alpha} u_\alpha^2) \tag{5.1}$$

$$\nabla_v \chi = \tfrac{1}{2} m_\alpha \nabla_v (\mathbf{v}\cdot\mathbf{v}) = m_\alpha (\mathbf{v}\cdot\nabla_v)\mathbf{v} = m_\alpha \mathbf{v} \tag{5.2}$$

Therefore, the terms in the left-hand side of the general transport equation (2.13) become

$$\frac{\partial}{\partial t}(n_\alpha < \chi >_\alpha) = \frac{3}{2}\frac{\partial p_\alpha}{\partial t} + \frac{\partial}{\partial t}(\tfrac{1}{2}\rho_{m\alpha} u_\alpha^2) \tag{5.3a}$$

$$\nabla\cdot(n_\alpha < \chi\mathbf{v} >_\alpha) = \nabla\cdot[\tfrac{1}{2}\rho_{m\alpha} < (\mathbf{v}\cdot\mathbf{v})\mathbf{v} >_\alpha] \tag{5.3b}$$

$$-n_\alpha < (\mathbf{F}/m_\alpha)\cdot\nabla_v\chi >_\alpha = -n_\alpha < \mathbf{F}\cdot\mathbf{v} >_\alpha \tag{5.3c}$$

Adding these terms, results in the following *energy conservation equation*

$$\frac{3}{2}\frac{\partial p_\alpha}{\partial t} + \frac{\partial}{\partial t}(\tfrac{1}{2}\rho_{m\alpha} u_\alpha^2) + \nabla\cdot[\tfrac{1}{2}\rho_{m\alpha} < (\mathbf{v}\cdot\mathbf{v})\mathbf{v} >_\alpha] -$$

$$-n_\alpha < \mathbf{F}\cdot\mathbf{v} >_\alpha = M_\alpha \tag{5.4}$$

where M_α represents the rate of energy density change due to collisions,

$$M_\alpha = \tfrac{1}{2} m_\alpha \int_v v^2 \left(\frac{\delta f_\alpha}{\delta t}\right)_{coll} d^3v = \left[\frac{\delta(\frac{1}{2}\rho_{m\alpha} < v^2 >_\alpha)}{\delta t}\right]_{coll} \tag{5.5}$$

The energy conservation equation (5.4) can be written in an *alternative form* as follows. Consider, initially, the third term in the left-hand

side of (5.4). Taking $\mathbf{v} = \mathbf{c}_\alpha + \mathbf{u}_\alpha$, the quantity $< (\mathbf{v}\cdot\mathbf{v})\mathbf{v} >_\alpha$ can be expanded as

$$< [(\mathbf{u}_\alpha + \mathbf{c}_\alpha)\cdot(\mathbf{u}_\alpha + \mathbf{c}_\alpha)](\mathbf{u}_\alpha + \mathbf{c}_\alpha) >=< (u_\alpha^2 + 2\mathbf{u}_\alpha \cdot \mathbf{c}_\alpha + c_\alpha^2)(\mathbf{u}_\alpha + \mathbf{c}_\alpha) >$$

$$= u_\alpha^2 \mathbf{u}_\alpha + < c_\alpha^2 > \mathbf{u}_\alpha + 2 < \mathbf{c}_\alpha \mathbf{c}_\alpha > \cdot\, \mathbf{u}_\alpha + < c_\alpha^2 \mathbf{c}_\alpha > \tag{5.6}$$

The term $\rho_{m\alpha} < \mathbf{c}_\alpha \mathbf{c}_\alpha >$ represents the *kinetic pressure dyad* $\mathcal{P}_\alpha$ and $\frac{1}{2}\rho_{m\alpha} < c_\alpha^2 \mathbf{c}_\alpha >$ is the *heat flux vector* $\mathbf{q}_\alpha$, defined in Chapter 6. We have also seen that $\frac{1}{2}\rho_{m\alpha} < c_\alpha^2 >= 3p_\alpha/2$. Therefore,

$$\nabla\cdot[\tfrac{1}{2}\rho_{m\alpha} < (\mathbf{v}\cdot\mathbf{v})\mathbf{v} >_\alpha] = \nabla\cdot[\tfrac{1}{2}\rho_{m\alpha}u_\alpha^2\mathbf{u}_\alpha + \tfrac{1}{2}(3p_\alpha)\mathbf{u}_\alpha + \mathcal{P}_\alpha\cdot\mathbf{u}_\alpha + \mathbf{q}_\alpha] =$$

$$= \nabla\cdot(\tfrac{1}{2}\rho_{m\alpha}u_\alpha^2\mathbf{u}_\alpha) + \tfrac{1}{2}(3p_\alpha)(\nabla\cdot\mathbf{u}_\alpha) +$$

$$+ \tfrac{1}{2}(\mathbf{u}_\alpha\cdot\nabla)(3p_\alpha) + \nabla\cdot(\mathcal{P}_\alpha\cdot\mathbf{u}_\alpha) + \nabla\cdot\mathbf{q}_\alpha \tag{5.7}$$

Substituting this result into (5.4) and using the notation D/Dt for the total time derivative (4.7), we obtain

$$\frac{D}{Dt}\Big(\frac{3p_\alpha}{2}\Big) + \Big(\frac{3p_\alpha}{2}\Big)\nabla\cdot\mathbf{u}_\alpha + \frac{\partial}{\partial t}(\tfrac{1}{2}\rho_{m\alpha}u_\alpha^2) + \nabla\cdot(\tfrac{1}{2}\rho_{m\alpha}u_\alpha^2\mathbf{u}_\alpha) +$$

$$+ \nabla\cdot(\mathcal{P}_\alpha\cdot\mathbf{u}_\alpha) + \nabla\cdot\mathbf{q}_\alpha - n_\alpha < \mathbf{F}\cdot\mathbf{v} >_\alpha = M_\alpha \tag{5.8}$$

The third and fourth terms in the left-hand side can be written as

$$\frac{\partial}{\partial t}(\tfrac{1}{2}\rho_{m\alpha}\mathbf{u}_\alpha\cdot\mathbf{u}_\alpha) + \nabla\cdot[\tfrac{1}{2}\rho_{m\alpha}(\mathbf{u}_\alpha\cdot\mathbf{u}_\alpha)\mathbf{u}_\alpha] =$$

$$= \tfrac{1}{2}u_\alpha^2\frac{\partial\rho_{m\alpha}}{\partial t} + \rho_{m\alpha}\mathbf{u}_\alpha\cdot\frac{\partial\mathbf{u}_\alpha}{\partial t} +$$

$$+ \tfrac{1}{2}u_\alpha^2\nabla\cdot(\rho_{m\alpha}\mathbf{u}_\alpha) + \rho_{m\alpha}\mathbf{u}_\alpha\cdot[(\mathbf{u}_\alpha\cdot\nabla)\mathbf{u}_\alpha] =$$

$$= \tfrac{1}{2}u_\alpha^2\Big[\frac{\partial\rho_{m\alpha}}{\partial t} + \nabla\cdot(\rho_{m\alpha}\mathbf{u}_\alpha)\Big] + \rho_{m\alpha}\mathbf{u}_\alpha\cdot\frac{D\mathbf{u}_\alpha}{Dt} \tag{5.9}$$

Using the continuity equation (3.2) and the equation of motion (4.6), this last equation becomes

$$\tfrac{1}{2}u_\alpha^2 S_\alpha + n_\alpha\mathbf{u}_\alpha\cdot < \mathbf{F} >_\alpha - \mathbf{u}_\alpha\cdot(\nabla\cdot\mathcal{P}_\alpha) + \mathbf{u}_\alpha\cdot\mathbf{A}_\alpha - u_\alpha^2 S_\alpha \tag{5.10}$$

Taking this result back into (5.8) we obtain

$$\frac{D}{Dt}\Big(\frac{3p_\alpha}{2}\Big) + \frac{3p_\alpha}{2}\nabla\cdot\mathbf{u}_\alpha + \nabla\cdot(\mathcal{P}_\alpha\cdot\mathbf{u}_\alpha) - \mathbf{u}_\alpha\cdot(\nabla\cdot\mathcal{P}_\alpha) -$$

$$-n_\alpha < \mathbf{F}\cdot\mathbf{v} >_\alpha + n_\alpha \mathbf{u}_\alpha \cdot < \mathbf{F} >_\alpha + \nabla\cdot\mathbf{q}_\alpha =$$

$$= M_\alpha - \mathbf{u}_\alpha\cdot\mathbf{A}_\alpha + \tfrac{1}{2}u_\alpha^2 S_\alpha \tag{5.11}$$

The third and fourth terms in the left-hand side of this equation can be combined into one single term,

$$\nabla\cdot(\mathcal{P}_\alpha\cdot\mathbf{u}_\alpha) - \mathbf{u}_\alpha\cdot(\nabla\cdot\mathcal{P}_\alpha) = (\mathcal{P}_\alpha\cdot\nabla)\cdot\mathbf{u}_\alpha \tag{5.12}$$

as well as the fifth and the sixth terms, which give

$$-n_\alpha < \mathbf{F}\cdot\mathbf{v} >_\alpha + n_\alpha\mathbf{u}_\alpha\cdot < \mathbf{F} >_\alpha = -n_\alpha < \mathbf{F}\cdot\mathbf{c}_\alpha > \tag{5.13}$$

since

$$< \mathbf{F}\cdot\mathbf{v} >_\alpha = < \mathbf{F}\cdot(\mathbf{u}_\alpha + \mathbf{c}_\alpha) > = < \mathbf{F} >_\alpha \cdot \mathbf{u}_\alpha + < \mathbf{F}\cdot\mathbf{c}_\alpha > \tag{5.14}$$

For a velocity-independent force (5.13) vanishes, since, in this case,

$$< \mathbf{F}\cdot\mathbf{c}_\alpha > = \mathbf{F}\cdot < \mathbf{c}_\alpha > = 0 \tag{5.15}$$

For the force due to a magnetic field $\mathbf{B}$, the only velocity-dependent force that we are interested here, (5.13) also vanishes,

$$< \mathbf{F}\cdot\mathbf{c}_\alpha > = q_\alpha < (\mathbf{v}\times\mathbf{B})\cdot\mathbf{c}_\alpha > =$$

$$= q_\alpha(\mathbf{u}_\alpha\times\mathbf{B})\cdot < \mathbf{c}_\alpha > + q_\alpha < (\mathbf{c}_\alpha\times\mathbf{B})\cdot\mathbf{c}_\alpha > = 0 \tag{5.16}$$

where both terms vanish since $< \mathbf{c}_\alpha > = 0$ and $(\mathbf{c}_\alpha \times \mathbf{B})$ is normal to $\mathbf{c}_\alpha$. We obtain, finally, the following *alternative form* for the equation of *conservation of energy*

$$\frac{D}{Dt}\left(\frac{3p_\alpha}{2}\right) + \frac{3p_\alpha}{2}\nabla\cdot\mathbf{u}_\alpha + (\mathcal{P}_\alpha\cdot\nabla)\cdot\mathbf{u}_\alpha + \nabla\cdot\mathbf{q}_\alpha =$$

$$= M_\alpha - \mathbf{u}_\alpha\cdot\mathbf{A}_\alpha + \tfrac{1}{2}u_\alpha^2 S_\alpha \tag{5.17}$$

5.2 Physical Interpretation

The physical interpretation of this equation is as follows. The first term in the left-hand side represents the total rate of change of the particle *thermal energy density* in a volume element moving with the mean fluid velocity $\mathbf{u}_\alpha$. Note that the thermal energy density is given by $3p_\alpha/2 = \rho_{m\alpha} < c_\alpha^2 > /2$. The other terms of (5.17) contribute to some extent to this total rate of change of the thermal energy density. The second term in the left-hand side of (5.17) can be interpreted as the change in the thermal energy density due to particles entering the volume element with the mean velocity $\mathbf{u}_\alpha$. The third term is related to the work done on the unit volume by the kinetic pressure dyad acting on its surface, whereas the fourth term represents the change in the thermal energy density due to the heat flux. Finally, the terms in the right-hand side of (5.17) represent the rate of change in the thermal energy density as a consequence of collisions. In the case of a fluid containing only one type of particles, the collision terms vanish, as indicated previously.

The first two terms in the energy equation may also be combined, making use of the continuity equation (3.2). Expanding $\nabla \cdot (\rho_{m\alpha}\mathbf{u}_\alpha)$, (3.2) becomes

$$\Big(\frac{\partial}{\partial t} + \mathbf{u}_\alpha \cdot \nabla\Big)\rho_{m\alpha} + \rho_{m\alpha}\nabla \cdot \mathbf{u}_\alpha = S_\alpha \tag{5.18}$$

which gives

$$\nabla \cdot \mathbf{u}_\alpha = -\frac{1}{\rho_{m\alpha}}\Big(\frac{D\rho_{m\alpha}}{Dt} - S_\alpha\Big) \tag{5.19}$$

Substituting this result into (5.17), taking $\rho_{m\alpha} = n_\alpha m_\alpha$ and $p_\alpha = n_\alpha k T_\alpha$, yields the following alternative form for the energy equation in terms of the temperature T_α,

$$\frac{3}{2}n_\alpha k\frac{DT_\alpha}{Dt} + (\mathcal{P}_\alpha \cdot \nabla)\cdot \mathbf{u}_\alpha + \nabla \cdot \mathbf{q}_\alpha = M_\alpha - \mathbf{u}_\alpha \cdot \mathbf{A}_\alpha + \Big(\frac{1}{2}u_\alpha^2 - \frac{3}{2}\frac{kT_\alpha}{m_\alpha}\Big)S_\alpha \tag{5.20}$$

5.3 Simplifying Approximations

Several simplifying approximations can be considered for the energy equation, depending on the situation of interest.

(a) When the collision terms vanish, or are negligible, and when the mean

fluid velocity $\mathbf{u}_\alpha$ is equal to zero, (5.20) reduces to a *diffusion* equation for T_α, if we take the heat flux vector as

$$\mathbf{q}_\alpha = -K\nabla T_\alpha \tag{5.21}$$

where K denotes the thermal conductivity. Thus, in this case, (5.20) reduces to

$$\frac{3}{2}n_\alpha k\frac{DT_\alpha}{Dt} = \nabla\cdot(K\nabla T_\alpha) \tag{5.22}$$

The thermal conductivity coefficient K is related to the fluid viscosity coefficient.

(b) Consider now a non-viscous fluid, in which the pressure dyad reduces to a scalar pressure without thermal conductivity ($\mathbf{q}_\alpha = 0$). If we consider also that the collision terms vanish, the energy conservation equation (5.17) becomes

$$\frac{D}{Dt}\left(\frac{3p_\alpha}{2}\right) + \frac{3p_\alpha}{2}(\nabla\cdot\mathbf{u}_\alpha) + p_\alpha(\nabla\cdot\mathbf{u}_\alpha) = 0 \tag{5.23}$$

Substituting (5.19) for $(\nabla\cdot\mathbf{u}_\alpha)$, with $S_\alpha = 0$, yields

$$\frac{D}{Dt}\left(\frac{3p_\alpha}{2}\right) - \frac{5p_\alpha}{2\rho_{m\alpha}}\frac{D\rho_{m\alpha}}{Dt} = 0 \tag{5.24}$$

from which results

$$\frac{Dp_\alpha}{p_\alpha} = \frac{5}{3}\frac{D\rho_{m\alpha}}{\rho_{m\alpha}} \tag{5.25}$$

Integrating this equation gives

$$\frac{p_\alpha}{p_0} = \left(\frac{\rho_{m\alpha}}{\rho_{m0}}\right)^{5/3} \tag{5.26}$$

where p_0 and ρ_{m0} are constants, that is,

$$p_\alpha\rho_{m\alpha}^{-5/3} = constant \tag{5.27}$$

This is the *adiabatic energy equation* for a gas in which the ratio of the specific heats at constant pressure and at constant volume, γ, is equal to 5/3. We emphasize here that the energy equation reduces to this adiabatic equation only when the effects of viscosity, thermal conductivity and energy transfer due to collisions are neglected.

The parameter γ is related to the number of degrees of freedom, N, of a gas by the condition

$$\gamma = (2 + N)/N \tag{5.28}$$

For particles that have no internal degrees of freedom, as for example in a monoatomic gas, where the only degrees of freedom are those associated with the three possible directions of translational motion, we have $N = 3$ and therefore $\gamma = 5/3$. Other degrees of freedom exist in the case of diatomic or poliatomic molecules. The *adiabatic energy equation* often used in thermodynamics is

$$p\rho_m^{-\gamma} = constant \tag{5.29}$$

Differentiating this equation, yields

$$\rho_m^{-\gamma}\, dp - \gamma p \rho_m^{-(\gamma+1)}\, d\rho_m = 0 \tag{5.30}$$

or, equivalently,

$$dp = \left(\frac{\gamma p}{\rho_m}\right) d\rho_m = V_s^2\, d\rho_m \tag{5.31}$$

where we have defined

$$V_s = (\gamma p/\rho_m)^{1/2} = (\gamma kT/m)^{1/2} \tag{5.32}$$

which is the *adiabatic speed of sound* for the fluid.
(c) An equation which is also used in thermodynamics when the temperature is constant inside the fluid, is the *isothermal energy equation.* It can be easily obtained from the equation of state for an ideal gas, $p = nkT$. For an isothermal process ($T = constant$), we have

$$dp = kT\, dn = (p/\rho_m)\, d\rho_m = V_T^2\, d\rho_m \tag{5.33}$$

where the *isothermal speed of sound* is

$$V_T = (p/\rho_m)^{1/2} = (kT/m)^{1/2} \tag{5.34}$$

6. THE COLD PLASMA MODEL

In the previous sections we have seen that the differential equations governing the temporal and spatial variation of the macroscopic variables can be obtained by taking the various moments of the *Boltzmann equation.* The macroscopic parameters are all related to the moments of the *distribution function* $f_\alpha(\mathbf{r}, \mathbf{v}, t)$. The first four moments of the distribution function give us the *number density* n_α, the *mean velocity* $\mathbf{u}_\alpha$, the *momentum flow dyad* Π_α and the *energy flow triad* $\mathcal{E}_\alpha$.

The first moment of the Boltzmann equation gives us the *continuity equation* which relates the number density n_α (or the mass density $\rho_{m\alpha}$) with the mean velocity $\mathbf{u}_\alpha$ for the particles of type α. In order to determine these two macroscopic variables we need two independent macroscopic transport equations. Thus, we need to consider the second moment of the Boltzmann equation, which gives us the *equation of motion* (or momentum conservation equation), and which relates the mean velocity $\mathbf{u}_\alpha$, with the number density n_α and the kinetic pressure dyad $\mathcal{P}_\alpha$. This gives us now two transport equations involving three independent variables. We find, therefore, that the set of transport equations derived from the moments of the Boltzmann equation always includes more variables than independent equations. This situation is clearly evident in the three transport equations derived in this chapter. The *energy equation*, besides the variables n_α, $\mathbf{u}_\alpha$ and $\mathcal{P}_\alpha$, also includes the heat flow vector $\mathbf{q}_\alpha$. A more general energy conservation equation would include the energy flow triad $\mathcal{E}_\alpha$.

Any finite set of transport equations is, therefore, insufficient to form a *closed* system of equations. Consequentely, it is necessary to introduce a scheme of approximation to eliminate some of the independent variables, or to express some of these variables in terms of the others. It is common, therefore, to arbitrarily truncate the system of transport equations at some point in the hierarchy of moments of the Boltzmann equation, and consider some simplifying approximation for the highest moment of the distribution function appearing in the system.

The simplest closed system of macroscopic transport equations which can be formed is know as the *cold plasma model.* This simple model encompasses only the equations of conservation of mass and of momentum. The highest moment of the distribution function, appearing in the momentum equation, is the kinetic pressure dyad which, in this model, is taken equal to zero. This means that the effects due to the thermal motion of the particles and the force due to the divergence of the kinetic pressure dyad are neglected. For convenience, we collect here the two transport

equations pertinent to the cold plasma model,

$$\frac{\partial \rho_{m\alpha}}{\partial t} + \nabla \cdot (\rho_{m\alpha} \mathbf{u}_\alpha) = S_\alpha \tag{6.1}$$

$$\rho_{m\alpha} \frac{D\mathbf{u}_\alpha}{Dt} = n_\alpha q_\alpha (\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}) + \rho_{m\alpha} \mathbf{g} + \mathbf{A}_\alpha - \mathbf{u}_\alpha S_\alpha \tag{6.2}$$

In the absence of processes leading to production and loss of particles of type α (such as ionization and recombination), we have $S_\alpha = 0$. The expression normally used for the collision term for momentum transfer $\mathbf{A}_\alpha$ is the one indicated in (4.11). The cold plasma model assumes, in fact, a zero plasma temperature, so that the corresponding distribution function is a *Dirac delta function* centered at the macroscopic flow velocity, $f_\alpha(\mathbf{r}, \mathbf{v}, t) = \delta[\mathbf{v} - \mathbf{u}(\mathbf{r}, t)]$.

This model has been successfully applied, e.g., in the investigation of the properties of small amplitude electromagnetic waves propagating in plasmas, with phase velocities much larger than the thermal velocity of the particles. The theory of high frequency waves propagating in cold magnetized plasmas is commonly known as the *magnetoionic theory*.

7. THE WARM PLASMA MODEL

In this closed system of transport equations, the simplifying approximation is introduced in the equation of conservation of energy, in which we neglect the term involving the heat flux vector. Thus, the approximation consists in taking $\nabla \cdot \mathbf{q}_\alpha = 0$, which means that the processes occurring in the plasma are such that there is no thermal energy flux. This approximation is also called the *adiabatic approximation*. Since the thermal conductivity is zero in this case, it follows that the plasma is nonviscous and, consequently, the nondiagonal terms of the kinetic pressure dyad are all equal to zero. Further, the diagonal terms of $\mathcal{P}_\alpha$ are assumed to be equal and the kinetic pressure dyad is replaced by a scalar pressure p_α. Thus, the term $\nabla \cdot \mathcal{P}_\alpha$ in the momentum equation degenerates to ∇p_α.

The three macroscopic variables appearing in this case are the number density n_α, the mean velocity $\mathbf{u}_\alpha$ and the scalar pressure p_α. The three transport equations pertaining to the *warm plasma model* are, therefore,

$$\frac{\partial \rho_{m\alpha}}{\partial t} + \nabla \cdot (\rho_{m\alpha} \mathbf{u}_\alpha) = S_\alpha \tag{7.1}$$

$$\rho_{m\alpha} \frac{D\mathbf{u}_\alpha}{Dt} = n_\alpha q_\alpha (\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}) + \rho_{m\alpha} \mathbf{g} - \nabla p_\alpha + \mathbf{A}_\alpha - \mathbf{u}_\alpha S_\alpha \tag{7.2}$$

$$\frac{D}{Dt}\left(\frac{3p_\alpha}{2}\right) + \frac{5p_\alpha}{2}(\nabla \cdot \mathbf{u}_\alpha) = M_\alpha - \mathbf{u}_\alpha \cdot \mathbf{A}_\alpha + \tfrac{1}{2}u_\alpha^2 S_\alpha \tag{7.3}$$

Considering the additional approximation that the change in energy, as a result of collisions, is negligible, the energy equation (7.3) reduces to the following *adiabatic equation* (as shown in section 5),

$$p_\alpha \rho_{m\alpha}^{-\gamma} = constant \tag{7.4}$$

Generally, the warm plasma model gives a more precise description of the behavior of plasma phenomena as compared to the cold plasma model.

In the most general cases, in which the plasma is not in a state of local equilibrium, and when heat flow and viscosity need to be taken into account, it is more convenient and simple to work directly with the phase space distribution function. In this case, the plasma is usually said to be *hot*. After determining the distribution function f_α by solving the differential kinetic equation which governs the evolution of f_α in phase space, for the specific problem under consideration, the macroscopic variables can be obtained from f_α according to the systematic method presented in Chapter 6.

PROBLEMS

8.1 - Consider the following simplified steady state equation of motion, for each species in a fluid plasma,

$$nq(\mathbf{E} + \mathbf{u} \times \mathbf{B}) - \nabla p = 0$$

where the electric ($\mathbf{E}$) and magnetic ($\mathbf{B}$) fields are uniform, but the number density (n) and the kinetic pressure (p) have a spatial gradient. Taking the cross product of this equation with $\mathbf{B}$ show that, besides the $\mathbf{E} \times \mathbf{B}$ drift, there is also a *diamagnetic drift* given by

$$\mathbf{v}_D = -\frac{1}{nqB^2}(\nabla p) \times \mathbf{B}$$

Provide physical arguments to justify the physical reason for this fluid drift. Explain if there is any motion of the particle guiding centers, associated with this fluid drift, and why it does not appear in single particle orbit theory.

8.2 - (a) From Maxwell equations,

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \cdot \mathbf{H} = 0$$

$$\nabla \times \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

where $\mathbf{E}$ and $\mathbf{H}$ denote the electric and magnetic fields in a plasma, ρ denotes the electric charge density nq, and $\mathbf{J}$ the electric current density $nq\mathbf{u}$, show that

$$\epsilon_0 \mu_0 \frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{H}) = \nabla \cdot \mathcal{T} - nq(\mathbf{E} + \mathbf{u} \times \mathbf{B})$$

where $\epsilon_0 \mu_0 (\mathbf{E} \times \mathbf{H})$ is the electromagnetic momentum density, $\mathcal{T}$ is the electromagnetic stress dyad whose components are given by

$$T_{ij} = \epsilon_0 E_i E_j + \mu_0 H_i H_j - \tfrac{1}{2}(\epsilon_0 E^2 + \mu_0 H^2)\delta_{ij}$$

where δ_{ij} is the Kronecker delta.
(b) From this equation, which expresses conservation of the electromagnetic momentum density, and the continuity equation

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{u}) = 0$$

show that the momentum transport equation

$$nm \frac{D\mathbf{u}}{Dt} = nq(\mathbf{E} + \mathbf{u} \times \mathbf{B}) - \nabla \cdot \mathcal{P}$$

can be written in the form

$$\frac{\partial \mathbf{G}}{\partial t} + \nabla \cdot \mathbf{\Pi} = 0$$

where $\mathbf{G}$ denotes the total momentum density

$$\mathbf{G} = nm\mathbf{u} + \epsilon_0 \mu_0 (\mathbf{E} \times \mathbf{H})$$

and $\mathbf{\Pi}$ is the total momentum flux dyad (rate of transport of momentum through unit area)

$$\mathbf{\Pi} = nm\mathbf{uu} + \mathcal{P} - \mathcal{T}$$

(c) Using Maxwell curl equations, show that the energy transport equation

$$\frac{\partial}{\partial t}(\tfrac{1}{2}nm < c^2 >) + \frac{\partial}{\partial t}(\tfrac{1}{2}nmu^2) = -\nabla \cdot (\tfrac{1}{2}nm < c^2 > \mathbf{u}) -$$

$$-\nabla \cdot (\tfrac{1}{2}nmu^2\mathbf{u}) - \nabla \cdot \mathbf{q} - \nabla \cdot (\mathcal{P} \cdot \mathbf{u}) + nq\mathbf{u} \cdot \mathbf{E}$$

[note that $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{B}) = 0$] can be written in the form

$$\frac{\partial W}{\partial t} + \nabla \cdot \mathbf{S} = 0$$

where W denotes the total energy density

$$W = \tfrac{1}{2}(\epsilon_0 E^2 + \mu_0 H^2 + nmu^2 + nm < c^2 >)$$

and $\mathbf{S}$ is the total energy flux (power per unit area)

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} + \mathcal{P} \cdot \mathbf{u} + \tfrac{1}{2}nm\mathbf{u}(u^2 + < c^2 >) + \mathbf{q}$$

8.3 - In order to investigate the effect of the collision term (4.11) in the macroscopic fluid motion, consider a uniform mixture of different fluids (all spatial derivatives vanish), with no external forces, so that the equation of motion for the α species reduces to

$$\frac{d\mathbf{u}_\alpha}{dt} = -\sum_\beta \nu_{\alpha\beta}(\mathbf{u}_\alpha - \mathbf{u}_\beta)$$

Solve this equation to determine $\mathbf{u}(t)$, for a two-fluid mixture and for a three-fluid mixture (in the case of the three-fluid mixture it is convenient to use Laplace transforms). Notice that, at equilibrium (when $d\mathbf{u}_\alpha/dt = 0$), the velocities of all species must be the same.

8.4 - Consider a uniform mixture of different fluids (all spatial derivatives vanish), with no external forces, such that the equation of motion for the α species becomes

$$\frac{d\mathbf{u}_\alpha}{dt} = -\sum_\beta \nu_{\alpha\beta}(\mathbf{u}_\alpha - \mathbf{u}_\beta)$$

(a) Show that the time rate of change of the total fluid kinetic energy density, W_k, is given by

$$\frac{dW_k}{dt} = -\sum_{\alpha,\beta} \tfrac{1}{2}\rho_{m\alpha}\nu_{\alpha\beta}(\mathbf{u}_\alpha - \mathbf{u}_\beta)^2$$

where

$$W_k = \sum_\alpha \tfrac{1}{2}\rho_{m\alpha}u_\alpha^2$$

(b) Consider now the total fluid thermal energy density,

$$W_T = \sum_\alpha \frac{3}{2}n_\alpha k T_\alpha$$

If the energy equation for a homogeneous fluid mixture, with no external forces, is

$$\frac{dT_\alpha}{dt} = -\sum_\beta \frac{2m_\alpha\nu_{\alpha\beta}}{(m_\alpha + m_\beta)}\left[(T_\alpha - T_\beta) - \frac{m_\beta}{3k}(\mathbf{u}_\alpha - \mathbf{u}_\beta)^2\right]$$

then, show that the time rate of change of W_T is given by

$$\frac{dW_T}{dt} = \sum_{\alpha,\beta} \tfrac{1}{2}\rho_{m\alpha}\nu_{\alpha\beta}(\mathbf{u}_\alpha - \mathbf{u}_\beta)^2$$

Thus, the total thermal energy density W_T increases at exactly the same rate as the total kinetic energy density W_k decreases. As a hint for this problem, notice that for any function which is summed over two indices, the result is unchanged if we interchange the indices, so that

$$\sum_{\alpha,\beta} f_{\alpha\beta} = \sum_{\alpha,\beta} f_{\beta\alpha}$$

or

$$\sum_{\alpha,\beta} f_{\alpha\beta} = \sum_{\alpha,\beta} \tfrac{1}{2}(f_{\alpha\beta} + f_{\beta\alpha})$$

8.5 - Explain the reason why there is no term containing the magnetic flux density **B** in the energy equation (5.17).

8.6 - Derive the following *general transport equation*, similar to (2.13), for the case when the quantity χ depends on **r**, **v** and t,

$$\frac{\partial}{\partial t}(n_\alpha < \chi >_\alpha) - n_\alpha < \frac{\partial \chi}{\partial t} >_\alpha + \nabla \cdot (n_\alpha < \chi \mathbf{v} >_\alpha) -$$

$$-n_\alpha < (\mathbf{v} \cdot \nabla)\chi >_\alpha - n_\alpha < (\mathbf{a} \cdot \nabla_v)\chi >_\alpha = \left[\frac{\delta(n_\alpha < \chi >_\alpha)}{\delta t}\right]_{coll}$$

8.7 - Consider the general transport equation of the previous problem and let the property $\chi(\mathbf{r}, \mathbf{v}, t)$ be the random flux of thermal kinetic energy, that is, $\frac{1}{2} m_\alpha c_\alpha^2 \mathbf{c}_\alpha$, where $\mathbf{c}_\alpha = \mathbf{v} - \mathbf{u}_\alpha(\mathbf{r}, t)$. Show that (considering the Lorentz force for **F**)

$$\frac{\partial}{\partial t}(n_\alpha < \chi >_\alpha) = \frac{\partial \mathbf{q}_\alpha}{\partial t}$$

$$n_\alpha < \frac{\partial \chi}{\partial t} >_\alpha = -\left(\frac{\partial \mathbf{u}_\alpha}{\partial t}\right) \cdot \left(\mathcal{P}_\alpha + \frac{3}{2} p_\alpha \mathbf{1}\right)$$

$$\nabla \cdot (n_\alpha < \chi \mathbf{v} >_\alpha) = \nabla \cdot (\tfrac{1}{2} \rho_{m\alpha} < c_\alpha^2 \mathbf{c}_\alpha \mathbf{c}_\alpha > + \mathbf{u}_\alpha \mathbf{q}_\alpha)$$

$$n_\alpha < (\mathbf{v} \cdot \nabla)\chi >_\alpha = -(\mathcal{Q}_\alpha \cdot \nabla) \cdot \mathbf{u}_\alpha - (\mathbf{q}_\alpha \cdot \nabla)\mathbf{u}_\alpha -$$

$$-[(\mathbf{u}_\alpha \cdot \nabla)\mathbf{u}_\alpha] \cdot \left(\mathcal{P}_\alpha + \frac{3}{2} p_\alpha \mathbf{1}\right)$$

$$n_\alpha < (\mathbf{a} \cdot \nabla_v)\chi >_\alpha = \rho_{m\alpha} < \mathbf{a} \cdot (\mathbf{c}_\alpha \mathbf{c}_\alpha + \tfrac{1}{2} c_\alpha^2 \mathbf{1}) > =$$

$$= \frac{q_\alpha}{m_\alpha}\left[(\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}) \cdot \left(\mathcal{P}_\alpha + \frac{3}{2} p_\alpha \mathbf{1}\right) + \mathbf{q}_\alpha \times \mathbf{B}\right]$$

Using these results in the general transport equation, derive the following equation, known as the *heat flow equation*,

$$\frac{\partial \mathbf{q}_\alpha}{\partial t} - \frac{1}{\rho_{m\alpha}}(\nabla \cdot \mathcal{P}_\alpha) \cdot \left(\mathcal{P}_\alpha + \frac{3}{2} p_\alpha \mathbf{1}\right) +$$

$$+\nabla \cdot (\tfrac{1}{2} \rho_{m\alpha} < c_\alpha^2 \mathbf{c}_\alpha \mathbf{c}_\alpha > + \mathbf{u}_\alpha \mathbf{q}_\alpha) + (\mathcal{Q}_\alpha \cdot \nabla) \cdot \mathbf{u}_\alpha +$$

$$+(\mathbf{q}_\alpha \cdot \nabla)\mathbf{u}_\alpha - \frac{q_\alpha}{m_\alpha}(\mathbf{q}_\alpha \times \mathbf{B}) = \left(\frac{\delta \mathbf{q}_\alpha}{\delta t}\right)_{coll}$$

8.8 - In the general transport equation of Problem 8.6, consider that the property $\chi(\mathbf{r}, \mathbf{v}, t)$ is the random momentum flux, that is, $m_\alpha c_{\alpha j} c_{\alpha k}$. Show that, in this case,

$$\frac{\partial}{\partial t}(n_\alpha < \chi >_\alpha) = \frac{\partial P_{\alpha jk}}{\partial t}$$

$$n_\alpha < \frac{\partial \chi}{\partial t} >_\alpha = 0$$

$$\frac{\partial}{\partial x_i}(n_\alpha < \chi v_i >_\alpha) = \frac{\partial}{\partial x_i}(Q_{\alpha ijk} + u_{\alpha i} P_{\alpha jk})$$

$$n_\alpha < v_i \frac{\partial \chi}{\partial x_i} >_\alpha = -P_{\alpha ij} \frac{\partial u_{\alpha k}}{\partial x_i} - P_{\alpha ik} \frac{\partial u_{\alpha j}}{\partial x_i}$$

$$n_\alpha < a_i \frac{\partial \chi}{\partial v_i} >_\alpha = \rho_{m\alpha} < a_j c_{\alpha k} + a_k c_{\alpha j} >$$

where the summation convention on repeated indices is being used. Plug these results in the general transport equation to derive the following equation, known as the *viscous stress equation*,

$$\frac{\partial P_{\alpha jk}}{\partial t} + \frac{\partial}{\partial x_i}(Q_{\alpha ijk} + u_{\alpha i} P_{\alpha jk}) + P_{\alpha ij} \frac{\partial u_{\alpha k}}{\partial x_i} + P_{\alpha ik} \frac{\partial u_{\alpha j}}{\partial x_i} -$$

$$-\rho_{m\alpha} < a_j c_{\alpha k} + a_k c_{\alpha j} > = \left(\frac{\delta P_{\alpha jk}}{\delta t}\right)_{coll}$$

8.9 - Verify that the energy conservation equation, for the random kinetic energy $\frac{1}{2} m_\alpha c_\alpha^2$, can be obtained from the viscous stress equation (see Problem 8.8) by letting $j = k$, and summing over k.

8.10 - From the heat flow equation, derived in Problem 8.7, obtain the following simplified equation for *heat flow* in a stationary ($\mathbf{u} = 0$) electron gas

$$\frac{5p_e}{2} \nabla \left(\frac{p_e}{\rho_{me}}\right) + \Omega_{ce}(\mathbf{q}_e \times \mathbf{B}) = \left(\frac{\delta \mathbf{q}_e}{\delta t}\right)_{coll}$$

State all the assumptions necessary to obtain this result.

8.11 - Using the relaxation model (or Krook collision model) for the collision term,

$$\left(\frac{\delta f_\alpha}{\delta t}\right)_{coll} = -\nu(f_\alpha - f_{\alpha 0})$$

and the ideal gas law $p_e = n_e k T_e$, show that the heat flow equation of Problem 8.10 becomes

$$\mathbf{q}_e + \frac{\Omega_{ce}}{\nu}(\mathbf{q}_e \times \mathbf{B}) = -K_0 \nabla T_e$$

where

$$K_0 = \frac{5kp_e}{2m_e\nu}$$

is the thermal conductivity.

9

MACROSCOPIC EQUATIONS FOR A CONDUCTING FLUID

1. MACROSCOPIC VARIABLES FOR A PLASMA AS A CONDUCTING FLUID

A plasma can also be considered as a conducting fluid, without specifying its various individual species. The macroscopic transport equations, derived in the previous chapter, describe the macroscopic behavior of each individual plasma species (electrons, ions, neutral particles). We will determine now the set of transport equations which describe the macroscopic behavior of the plasma *as a whole*, without considering the individual species present. Each macroscopic variable is combined, by adding the contributions of the various particle species in the plasma. This procedure yields the *total* macroscopic parameters of interest, such as the total mass and charge densities, the total mass and charge current densities (or flux), the total kinetic pressure dyad and the total heat flux vector.

The *mass density* is the mass per unit volume of fluid and can be expressed as

$$\rho_m = \sum_\alpha \rho_{m\alpha} = \sum_\alpha n_\alpha m_\alpha \tag{1.1}$$

The *electric charge density* is the electric charge per unit volume of fluid,

$$\rho = \sum_\alpha n_\alpha q_\alpha \tag{1.2}$$

The *mean fluid velocity*, $\mathbf{u}$, is defined such that the total momentum density is the same as the sum of the momentum density of each species,

according to

$$\rho_m \mathbf{u} = \sum_\alpha \rho_{m\alpha} \mathbf{u}_\alpha \tag{1.3}$$

The mean velocity of the plasma, $\mathbf{u}$, is therefore a weighted mean value, where the velocity of each species is weighted proportionally to its mass density.

The mean velocity of each particle species, when considered in a reference frame moving with the global mean velocity $\mathbf{u}$ of the plasma, is called the *diffusion velocity* $\mathbf{w}_\alpha$,

$$\mathbf{w}_\alpha = \mathbf{u}_\alpha - \mathbf{u} = \mathbf{u}_\alpha - \frac{1}{\rho_m} \sum_\alpha \rho_{m\alpha} \mathbf{u}_\alpha \tag{1.4}$$

The *mass current density* or *mass flux* is given by

$$\mathbf{J}_m = \sum_\alpha n_\alpha m_\alpha \mathbf{u}_\alpha = \rho_m \mathbf{u} \tag{1.5}$$

and the *electric current density* or *charge flux* is expressed as

$$\mathbf{J} = \sum_\alpha n_\alpha q_\alpha \mathbf{u}_\alpha = \rho \mathbf{u} + \sum_\alpha n_\alpha q_\alpha \mathbf{w}_\alpha \tag{1.6}$$

Note that in (1.5) we have $\sum_\alpha \rho_{m\alpha} \mathbf{w}_\alpha = 0$, in virtue of (1.4), which defines the diffusion velocity $\mathbf{w}_\alpha$.

The *kinetic pressure dyad* for each particle species in the plasma is defined in (6.6.2) as

$$\mathcal{P}_\alpha = \rho_{m\alpha} < \mathbf{c}_\alpha \mathbf{c}_\alpha > \tag{1.7}$$

where $\mathbf{c}_\alpha = \mathbf{v} - \mathbf{u}_\alpha$ is the *peculiar* or *random* velocity of the type α particles. Note that the pressure is defined as the time rate in which momentum is transported by the particles through a surface element moving with the particle mean velocity. For the plasma as a whole it is necessary to define an alternative peculiar velocity $\mathbf{c}_{\alpha 0}$, for the particles of type α, relative to the global plasma mean velocity $\mathbf{u}$,

$$\mathbf{c}_{\alpha 0} = \mathbf{v} - \mathbf{u} \tag{1.8}$$

Thus, the total pressure is defined as the rate of momentum transfer due to all particles in the plasma, through a surface element moving with the

global mean velocity $\mathbf{u}$. The *total kinetic pressure dyad* $\mathcal{P}$ is therefore given by

$$\mathcal{P} = \sum_{\alpha} \rho_{m\alpha} < \mathbf{c}_{\alpha 0}\mathbf{c}_{\alpha 0} > \tag{1.9}$$

To relate $\mathcal{P}$, given in (1.9), with $\mathcal{P}_\alpha$, given in (1.7), we substitute $\mathbf{u}$ by $\mathbf{u}_\alpha - \mathbf{w}_\alpha$ and $\mathbf{v}$ by $\mathbf{c}_\alpha + \mathbf{u}_\alpha$ in (1.8), to obtain

$$\mathbf{c}_{\alpha 0} = \mathbf{c}_\alpha + \mathbf{w}_\alpha \tag{1.10}$$

Consequently,

$$\mathcal{P} = \sum_{\alpha} \rho_{m\alpha} < (\mathbf{c}_\alpha + \mathbf{w}_\alpha)(\mathbf{c}_\alpha + \mathbf{w}_\alpha) > \tag{1.11}$$

and expanding this expression,

$$\mathcal{P} = \sum_{\alpha} \rho_{m\alpha}(< \mathbf{c}_\alpha\mathbf{c}_\alpha > + < \mathbf{c}_\alpha\mathbf{w}_\alpha > + < \mathbf{w}_\alpha\mathbf{c}_\alpha > + < \mathbf{w}_\alpha\mathbf{w}_\alpha >) \tag{1.12}$$

From the definition of $\mathbf{w}_\alpha$ we see that $< \mathbf{w}_\alpha > = \mathbf{w}_\alpha$, since it is a macroscopic variable and therefore $< \mathbf{c}_\alpha\mathbf{w}_\alpha > = < \mathbf{c}_\alpha > \mathbf{w}_\alpha = 0$. Thus, (1.12) becomes

$$\mathcal{P} = \sum_{\alpha} \mathcal{P}_\alpha + \sum_{\alpha} \rho_{m\alpha}\mathbf{w}_\alpha\mathbf{w}_\alpha \tag{1.13}$$

Note that $\mathcal{P}_\alpha$ is a pressure relative to $\mathbf{u}_\alpha$, whereas $\mathcal{P}$ is relative to the global mean velocity $\mathbf{u}$.

The *total scalar pressure* p is defined as one third the trace of $\mathcal{P}$,

$$p = \frac{1}{3}\sum_i P_{ii} = \frac{1}{3}\sum_i\sum_\alpha \rho_{m\alpha} < c_{\alpha 0 i}c_{\alpha 0 i} > = \frac{1}{3}\sum_\alpha \rho_{m\alpha} < c_{\alpha 0}^2 > \tag{1.14}$$

Using (1.13) we can write

$$p = \sum_\alpha p_\alpha + \frac{1}{3}\sum_\alpha \rho_{m\alpha} w_\alpha^2 \tag{1.15}$$

Finally, we define the *total heat flux vector* $\mathbf{q}$ as

$$\mathbf{q} = \tfrac{1}{2}\sum_\alpha \rho_{m\alpha} < c_{\alpha 0}^2 \mathbf{c}_{\alpha 0} > \tag{1.16}$$

and the *thermal energy density* of the plasma as a whole as

$$\frac{3p}{2} = \tfrac{1}{2}\sum_{\alpha} \rho_{m\alpha} < c_{\alpha 0}^2 > \tag{1.17}$$

It is useful to relate $\mathbf{q}$, defined in (1.16), with the heat flux vector $\mathbf{q}_\alpha$ for the particles of type α,

$$\mathbf{q}_\alpha = \tfrac{1}{2}\rho_{m\alpha} < c_\alpha^2 \mathbf{c}_\alpha > \tag{1.18}$$

For this purpose, we substitute $\mathbf{c}_{\alpha 0}$ by $\mathbf{c}_\alpha + \mathbf{w}_\alpha$ in (1.16) and expand the resulting expression, obtaining

$$\mathbf{q} = \tfrac{1}{2}\sum_{\alpha} \rho_{m\alpha}[< c_\alpha^2 \mathbf{c}_\alpha > + w_\alpha^2 < \mathbf{c}_\alpha > + 2 < (\mathbf{w}_\alpha \cdot \mathbf{c}_\alpha)\mathbf{c}_\alpha > +$$

$$+ < c_\alpha^2 > \mathbf{w}_\alpha + w_\alpha^2 \mathbf{w}_\alpha + 2(< \mathbf{c}_\alpha > \cdot \mathbf{w}_\alpha)\mathbf{w}_\alpha] \tag{1.19}$$

The second and sixth terms in the right-hand side of this equation are equal to zero, since $< \mathbf{c}_\alpha >= 0$. Therefore,

$$\mathbf{q} = \tfrac{1}{2}\sum_{\alpha} \rho_{m\alpha}[< c_\alpha^2 \mathbf{c}_\alpha > + 2\mathbf{w}_\alpha \cdot < \mathbf{c}_\alpha \mathbf{c}_\alpha > + < c_\alpha^2 > \mathbf{w}_\alpha + w_\alpha^2 \mathbf{w}_\alpha] \tag{1.20}$$

Using (1.18), (1.7) and the relation $p_\alpha = \rho_{m\alpha} < c_\alpha^2 > /3$, we can write (1.20) as

$$\mathbf{q} = \sum_{\alpha}\left(\mathbf{q}_\alpha + \mathbf{w}_\alpha \cdot \mathcal{P}_\alpha + \frac{3}{2} p_\alpha \mathbf{w}_\alpha + \tfrac{1}{2}\rho_{m\alpha} w_\alpha^2 \mathbf{w}_\alpha\right) \tag{1.21}$$

In particular, for the isotropic case in which $\mathcal{P}_\alpha = p_\alpha \mathbf{1}$, we have $\mathbf{w}_\alpha \cdot \mathcal{P}_\alpha = \mathbf{w}_\alpha p_\alpha$, so that (1.21) becomes

$$\mathbf{q} = \sum_{\alpha}\left(\mathbf{q}_\alpha + \frac{5}{2} p_\alpha \mathbf{w}_\alpha + \tfrac{1}{2}\rho_{m\alpha} w_\alpha^2 \mathbf{w}_\alpha\right) \tag{1.22}$$

2. CONTINUITY EQUATION

To obtain the continuity equation for the plasma as a whole, we add (8.3.2) over all particle species in the plasma,

$$\sum_{\alpha} \frac{\partial \rho_{m\alpha}}{\partial t} + \sum_{\alpha} \nabla \cdot (\rho_{m\alpha} \mathbf{u}_\alpha) = \sum_{\alpha} S_\alpha \tag{2.1}$$

which gives

$$\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \mathbf{u}) = 0 \tag{2.2}$$

with ρ_m and $\mathbf{u}$ given by (1.1) and (1.3), respectively. The collision term S_α, when summed over all particle species must certainly vanish, as a consequence of conservation of the total mass of the system. It is of interest to note that, using the total time derivative operator $D/Dt = \partial/\partial t + \mathbf{u}\cdot\nabla$, (2.2) can also be written in the form

$$\frac{D\rho_m}{Dt} + \rho_m \nabla \cdot \mathbf{u} = 0 \tag{2.3}$$

3. EQUATION OF MOTION

Similarly, adding the equation of conservation of momentum (8.4.9) over all particle species in the plasma, yields

$$\sum_\alpha \rho_{m\alpha}\left[\frac{\partial \mathbf{u}_\alpha}{\partial t} + (\mathbf{u}_\alpha \cdot \nabla)\mathbf{u}_\alpha\right] = \sum_\alpha n_\alpha q_\alpha \mathbf{E} + \sum_\alpha n_\alpha q_\alpha (\mathbf{u}_\alpha \times \mathbf{B}) +$$

$$+ \sum_\alpha \rho_{m\alpha}\mathbf{g} - \sum_\alpha \nabla \cdot \mathcal{P}_\alpha + \sum_\alpha \mathbf{A}_\alpha - \sum_\alpha \mathbf{u}_\alpha S_\alpha \tag{3.1}$$

Since the total momentum of the particles in the plasma is conserved, the collision term for momentum transfer vanishes when summed over all species. Using the definitions (1.1), (1.2) and (1.6), and the relation (1.13), we can write (3.1) as

$$\sum_\alpha \rho_{m\alpha}\left[\frac{\partial \mathbf{u}_\alpha}{\partial t} + (\mathbf{u}_\alpha \cdot \nabla)\mathbf{u}_\alpha\right] = \rho\mathbf{E} + \mathbf{J} \times \mathbf{B} + \rho_m \mathbf{g} - \nabla \cdot \mathcal{P} +$$

$$+ \sum_\alpha \nabla \cdot (\rho_{m\alpha}\mathbf{w}_\alpha \mathbf{w}_\alpha) - \sum_\alpha \mathbf{u}_\alpha S_\alpha \tag{3.2}$$

The term involving S_α can be eliminated using the equation of continuity,

$$\sum_\alpha \mathbf{u}_\alpha S_\alpha = \sum_\alpha \mathbf{u}_\alpha \left[\frac{\partial \rho_{m\alpha}}{\partial t} + \nabla \cdot (\rho_{m\alpha}\mathbf{u}_\alpha)\right] \tag{3.3}$$

Combining this expression with the terms in the left-hand side of (3.2), results in the expression

$$\sum_\alpha \Big[\frac{\partial(\rho_{m\alpha}\mathbf{u}_\alpha)}{\partial t} + \nabla\cdot(\rho_{m\alpha}\mathbf{u}_\alpha\mathbf{u}_\alpha)\Big] \tag{3.4}$$

We can now substitute the mean velocity $\mathbf{u}_\alpha$ by $\mathbf{w}_\alpha + \mathbf{u}$ and expand the result. Noting that

$$\sum_\alpha \rho_{m\alpha}\mathbf{w}_\alpha = \sum_\alpha \rho_{m\alpha}(\mathbf{u}_\alpha - \mathbf{u}) = \rho_m\mathbf{u} - \rho_m\mathbf{u} = 0 \tag{3.5}$$

we can express (3.4) as

$$\sum_\alpha \Big[\frac{\partial(\rho_{m\alpha}\mathbf{u}_\alpha)}{\partial t} + \nabla\cdot(\rho_{m\alpha}\mathbf{u}_\alpha\mathbf{u}_\alpha)\Big] = \frac{\partial(\rho_m\mathbf{u})}{\partial t} + \nabla\cdot(\rho_m\mathbf{u}\mathbf{u}) +$$

$$+\sum_\alpha \nabla\cdot(\rho_{m\alpha}\mathbf{w}_\alpha\mathbf{w}_\alpha) = \rho_m\Big[\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\Big] + \mathbf{u}\Big[\frac{\partial\rho_m}{\partial t} + \nabla\cdot(\rho_m\mathbf{u})\Big] +$$

$$+\sum_\alpha \nabla\cdot(\rho_{m\alpha}\mathbf{w}_\alpha\mathbf{w}_\alpha) = \rho_m\frac{D\mathbf{u}}{Dt} + \sum_\alpha \nabla\cdot(\rho_{m\alpha}\mathbf{w}_\alpha\mathbf{w}_\alpha) \tag{3.6}$$

where we have used the continuity equation (2.2) and the total time derivative operator D/Dt. Taking this result back into the equation of motion (3.2), we obtain the following momentum equation for the plasma as a whole,

$$\rho_m\frac{D\mathbf{u}}{Dt} = \rho\mathbf{E} + \mathbf{J}\times\mathbf{B} + \rho_m\mathbf{g} - \nabla\cdot\mathcal{P} \tag{3.7}$$

This equation is an expression of Newton's second law of motion.

4. ENERGY EQUATION

To obtain the equation of conservation of energy for the plasma as a conducting fluid, we start from the energy equation (8.5.4) for the particles of type α, and add this equation over all plasma species,

$$\sum_\alpha \frac{\partial}{\partial t}(\tfrac{1}{2}\rho_{m\alpha} < v^2 >_\alpha) + \sum_\alpha \nabla\cdot\Big(\tfrac{1}{2}\rho_{m\alpha} < v^2\mathbf{v} >_\alpha\Big) -$$

$$-\sum_\alpha n_\alpha < \mathbf{F}\cdot\mathbf{v} >_\alpha = 0 \tag{4.1}$$

where the collision term M_α vanishes when summed over all species of particles. We substitute now $\mathbf{v}$ by $\mathbf{c}_{\alpha 0} + \mathbf{u}$ and expand each term of (4.1).

For the *first term* we have

$$\frac{\partial}{\partial t}\Big(\sum_\alpha \tfrac{1}{2}\rho_{m\alpha} < \mathbf{v}\cdot\mathbf{v} >_\alpha\Big) = \frac{\partial}{\partial t}\Big[\sum_\alpha \tfrac{1}{2}\rho_{m\alpha}(< c_{\alpha 0}^2 > + u^2 + 2\mathbf{w}_\alpha\cdot\mathbf{u})\Big]$$

$$= \frac{\partial}{\partial t}\Big(\sum_\alpha \tfrac{1}{2}\rho_{m\alpha} < c_{\alpha 0}^2 >\Big) + \frac{\partial}{\partial t}\Big(\tfrac{1}{2}\rho_m u^2\Big)$$

$$= \frac{\partial}{\partial t}\Big(\frac{3p}{2}\Big) + \frac{\partial}{\partial t}\Big(\tfrac{1}{2}\rho_m u^2\Big) \tag{4.2}$$

where we have used the definition (1.17) and the fact that $\sum_\alpha \rho_{m\alpha}\mathbf{w}_\alpha = 0$.

For the *second term* we note initially that

$$< v^2\mathbf{v} >_\alpha =< (c_{\alpha 0}^2 + u^2 + 2\mathbf{c}_{\alpha 0}\cdot\mathbf{u})(\mathbf{c}_{\alpha 0} + \mathbf{u}) >$$

$$= < c_{\alpha 0}^2\mathbf{c}_{\alpha 0} > + u^2\mathbf{w}_\alpha + 2 < \mathbf{c}_{\alpha 0}\mathbf{c}_{\alpha 0} > \cdot\,\mathbf{u} +$$

$$+ < c_{\alpha 0}^2 > \mathbf{u} + u^2\mathbf{u} + 2(\mathbf{w}_\alpha\cdot\mathbf{u})\mathbf{u} \tag{4.3}$$

since $\mathbf{c}_{\alpha 0} = \mathbf{c}_\alpha + \mathbf{w}_\alpha$ and $< \mathbf{c}_\alpha > = 0$. Therefore,

$$\nabla\cdot\Big(\sum_\alpha \tfrac{1}{2}\rho_{m\alpha} < v^2\mathbf{v} >_\alpha\Big) = \nabla\cdot\Big(\sum_\alpha \tfrac{1}{2}\rho_{m\alpha} < c_{\alpha 0}^2\mathbf{c}_{\alpha 0} >\Big) +$$

$$+\nabla\cdot\Big(\sum_\alpha \rho_{m\alpha} < \mathbf{c}_{\alpha 0}\mathbf{c}_{\alpha 0} > \cdot\,\mathbf{u}\Big) + \nabla\cdot\Big(\sum_\alpha \tfrac{1}{2}\rho_{m\alpha} < c_{\alpha 0}^2 > \mathbf{u}\Big) +$$

$$+\nabla\cdot\Big(\sum_\alpha \tfrac{1}{2}\rho_{m\alpha}u^2\mathbf{u}\Big) \tag{4.4}$$

Using the definitions of the total heat flux vector $\mathbf{q}$ and of the total kinetic pressure dyad $\mathcal{P}$, we can write (4.4) as

$$\nabla\cdot\Big(\sum_\alpha \tfrac{1}{2}\rho_{m\alpha} < v^2\mathbf{v} >_\alpha\Big) = \nabla\cdot\mathbf{q} + \nabla\cdot(\mathcal{P}\cdot\mathbf{u}) +$$

$$+\nabla\cdot\Big(\frac{3p}{2}\mathbf{u}\Big) + \nabla\cdot\Big(\tfrac{1}{2}\rho_m u^2\mathbf{u}\Big) \tag{4.5}$$

For the *third term* of (4.1) we have

$$\sum_\alpha n_\alpha < \mathbf{F}\cdot\mathbf{v} >_\alpha = \sum_\alpha n_\alpha[q_\alpha < \mathbf{E}\cdot\mathbf{v} >_\alpha + q_\alpha < (\mathbf{v}\times\mathbf{B})\cdot\mathbf{v} >_\alpha \; +$$

$$+ m_\alpha < \mathbf{g} \cdot \mathbf{v} >_\alpha] \tag{4.6}$$

where we have considered external forces due to electromagnetic and gravitational fields. Since $< \mathbf{v} >_\alpha = \mathbf{u}_\alpha$ and since, for any vector $\mathbf{v}$, we have $(\mathbf{v} \times \mathbf{B}) \cdot \mathbf{v} = 0$, we obtain

$$\sum_\alpha n_\alpha < \mathbf{F} \cdot \mathbf{v} >_\alpha = \mathbf{J} \cdot \mathbf{E} + \mathbf{J}_m \cdot \mathbf{g} \tag{4.7}$$

where we have used the definitions (1.5) and (1.6), and where $\mathbf{E}$ and $\mathbf{g}$ are smoothed macroscopic fields.

Combining the results contained in (4.2), (4.5) and (4.7), the energy equation becomes

$$\frac{\partial}{\partial t}\left(\frac{3p}{2}\right) + \nabla \cdot \left(\frac{3p}{2}\mathbf{u}\right) + \frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho_m u^2\right) + \nabla \cdot \left(\tfrac{1}{2}\rho_m u^2 \mathbf{u}\right) +$$

$$+\nabla \cdot \mathbf{q} + \nabla \cdot (\mathcal{P} \cdot \mathbf{u}) - \mathbf{J} \cdot \mathbf{E} - \mathbf{J}_m \cdot \mathbf{g} = 0 \tag{4.8}$$

This equation can be further simplified as follows. The third and fourth terms of (4.8) can be combined as

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho_m u^2\right) + \nabla \cdot \left(\tfrac{1}{2}\rho_m u^2 \mathbf{u}\right) = \tfrac{1}{2}u^2\left[\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \mathbf{u})\right] +$$

$$+\mathbf{u} \cdot \left(\rho_m \frac{D\mathbf{u}}{Dt}\right) \tag{4.9}$$

and using the continuity equation (2.2) and the equation of motion (3.7), we can express (4.9) as

$$\rho \mathbf{u} \cdot \mathbf{E} + \mathbf{u} \cdot (\mathbf{J} \times \mathbf{B}) + \mathbf{J}_m \cdot \mathbf{g} - \mathbf{u} \cdot (\nabla \cdot \mathcal{P}) \tag{4.10}$$

Taking this result back into the energy equation (4.8), yields

$$\frac{D}{Dt}\left(\frac{3p}{2}\right) + \frac{3p}{2}\nabla \cdot \mathbf{u} + \nabla \cdot \mathbf{q} + (\mathcal{P} \cdot \nabla) \cdot \mathbf{u} =$$

$$= \mathbf{J} \cdot \mathbf{E} - \mathbf{u} \cdot (\mathbf{J} \times \mathbf{B}) - \rho \mathbf{u} \cdot \mathbf{E} \tag{4.11}$$

The first term in the left-hand side of (4.11) represents the time rate of change of the total thermal energy density of the plasma $(3p/2)$ in a frame of reference moving with the global mean velocity $\mathbf{u}$. The second term contributes to this rate of change through the thermal energy transfered to this volume element, as a consequence of the particle motions. The

third term represents the heat flux and the fourth one the work done on the volume element by the pressure forces (normal and tangential). The terms in the right-hand side of (4.11) represent the work done on the volume element by the electric field existing in the frame of reference moving with the global mean velocity $\mathbf{u}$. These last terms can be combined as follows. We note, initially, that the charge current density consists of two parts

$$\mathbf{J} = \sum_\alpha n_\alpha q_\alpha \mathbf{u}_\alpha = \sum_\alpha n_\alpha q_\alpha \mathbf{w}_\alpha + \sum_\alpha n_\alpha q_\alpha \mathbf{u} = \mathbf{J}' + \rho\mathbf{u} \tag{4.12}$$

where $\rho\mathbf{u}$ is the *convection* charge current density, which represents the flux of the space charge with velocity $\mathbf{u}$, and $\mathbf{J}'$ is the *conduction* charge current density, which represents the charge current density in the frame of reference moving with the global mean velocity $\mathbf{u}$. On the other hand, we can write

$$\mathbf{u} \cdot (\mathbf{J} \times \mathbf{B}) = -\mathbf{J} \cdot (\mathbf{u} \times \mathbf{B}) = -\mathbf{J}' \cdot (\mathbf{u} \times \mathbf{B}) \tag{4.13}$$

Substituting (4.13) and (4.12) into the energy equation (4.11), we obtain

$$\frac{D}{Dt}\left(\frac{3p}{2}\right) + \frac{3p}{2}\nabla \cdot \mathbf{u} + \nabla \cdot \mathbf{q} + (\mathcal{P} \cdot \nabla) \cdot \mathbf{u} = \mathbf{J}' \cdot \mathbf{E}' \tag{4.14}$$

where $\mathbf{E}' = \mathbf{E} + \mathbf{u} \times \mathbf{B}$ is the electric field existing in the reference frame moving with the global mean velocity $\mathbf{u}$. The term $\mathbf{J}' \cdot \mathbf{E}'$ represents therefore the rate of change in the energy density due to Joule heating.

5. ELECTRODYNAMIC EQUATIONS FOR A CONDUCTING FLUID

In the previous sections we have derived the macroscopic transport equations for conservation of mass, of momentum and of energy in a conducting fluid. As mentioned before, this set of equations does not constitute a complete system, and it is necessary to truncate the hierarchy of macroscopic equations at some stage and make some simplifying assumptions. The continuity equation relates the mass density ρ_m with the global mean velocity $\mathbf{u}$. The equation of motion, which specifies the variation of $\mathbf{u}$, involves also the total kinetic pressure dyad $\mathcal{P}$. The energy equation, specifying the rate of change of the total thermal energy density $(3p/2)$,

includes also the heat flux vector $\mathbf{q}$. A more general energy equation would give us the variation of the total kinetic pressure dyad $\mathcal{P}$, which would include also the total heat flow triad $\mathcal{Q}$. We can continue taking higher order moments of the Boltzmann equation and obtain, for example, the transport equation governing the variation of the heat flow triad $\mathcal{Q}$. To obtain a complete system it is essential therefore to truncate the hierarchy of transport equations at some point. However, even after this truncation, the remaining equations include also the following electrodynamic variables: electric field $\mathbf{E}$, magnetic induction $\mathbf{B}$, charge current density $\mathbf{J}$, and charge density ρ. Besides the hydrodynamic transport equations, we need therefore ten electrodynamic equations which must relate the variations of $\mathbf{E}$, $\mathbf{B}$, $\mathbf{J}$ and ρ. These equations are considered next.

5.1 Maxwell Curl Equations

The following Maxwell equations

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{5.1}$$

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \tag{5.2}$$

provide six component equations, which can be considered as the equations governing the variations of the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$.

5.2 Conservation of Electric Charge

The electric charge conservation equation can be obtained by multiplying the mass conservation equation (8.3.2) by q_α/m_α, and adding over all species,

$$\frac{\partial}{\partial t}\Big(\sum_\alpha n_\alpha q_\alpha\Big) + \nabla \cdot \Big(\sum_\alpha n_\alpha q_\alpha \mathbf{u}_\alpha\Big) = \sum_\alpha \Big(\frac{q_\alpha}{m_\alpha}\Big) S_\alpha \tag{5.3}$$

Using the definitions of ρ and $\mathbf{J}$, and noting that the *total* electric charge does not change as a result of collisions, we obtain

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \tag{5.4}$$

It is worth noting here that (5.4) can also be derived, in an independent way, from the Maxwell curl equation (5.2) and the Maxwell divergence equation

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{5.5}$$

Taking the divergence of (5.2), yields

$$\nabla \cdot \mathbf{J} + \epsilon_0 \frac{\partial}{\partial t}(\nabla \cdot \mathbf{E}) = 0 \tag{5.6}$$

since the divergence of the curl of a vector field vanishes identically. This last equation, combined with (5.5), yields the charge conservation equation (5.4). The equations (5.4) and (5.5) cannot therefore be considered as independent. As we have just shown, the Maxwell equations (5.2) and (5.5) already imply in conservation of electric charge.

Another important aspect of Maxwell equations can be seen by taking the divergence of (5.1), which gives

$$\frac{\partial}{\partial t}(\nabla \cdot \mathbf{B}) = 0 \tag{5.7}$$

or

$$\nabla \cdot \mathbf{B} = constant \tag{5.8}$$

Therefore, the Maxwell equation

$$\nabla \cdot \mathbf{B} = 0 \tag{5.9}$$

can be considered as an *initial condition* for (5.1), since if initially we take $\nabla \cdot \mathbf{B} = 0$, (5.1) implies that this condition will remain satisfied for all subsequent times.

5.3 Generalized Ohm's Law

To obtain a differential equation governing the variation of the charge current density, $\mathbf{J}$, we proceed in a way analogous to the derivation of (5.4). Thus, we multiply the momentum conservation equation (8.4.9) by q_α/m_α and add over all particle species. This procedure leads to

$$\sum_\alpha n_\alpha q_\alpha \left(\frac{\partial \mathbf{u}_\alpha}{\partial t}\right) + \sum_\alpha n_\alpha q_\alpha (\mathbf{u}_\alpha \cdot \nabla)\mathbf{u}_\alpha = \sum_\alpha n_\alpha \left(\frac{q_\alpha}{m_\alpha}\right) < \mathbf{F} >_\alpha \; -$$

$$-\nabla\cdot\Big[\sum_\alpha\Big(\frac{q_\alpha}{m_\alpha}\Big)\mathcal{P}_\alpha\Big]+\sum_\alpha\Big(\frac{q_\alpha}{m_\alpha}\Big)\mathbf{A}_\alpha-\sum_\alpha\Big(\frac{q_\alpha}{m_\alpha}\Big)\mathbf{u}_\alpha S_\alpha \tag{5.10}$$

We define now the *electrokinetic pressure dyad* $\mathcal{P}_\alpha^E$ for the particles of type α, by

$$\mathcal{P}_\alpha^E=\Big(\frac{q_\alpha}{m_\alpha}\Big)\mathcal{P}_\alpha=n_\alpha q_\alpha<\mathbf{c}_\alpha\mathbf{c}_\alpha> \tag{5.11}$$

Consequently, for the plasma as a conducting fluid, we have the following relation analogous to (1.13)

$$\mathcal{P}^E=\sum_\alpha\mathcal{P}_\alpha^E+\sum_\alpha n_\alpha q_\alpha\mathbf{w}_\alpha\mathbf{w}_\alpha \tag{5.12}$$

The *second term* in the right-hand side of (5.10) becomes therefore

$$-\nabla\cdot\Big[\sum_\alpha\Big(\frac{q_\alpha}{m_\alpha}\Big)\mathcal{P}_\alpha\Big]=-\nabla\cdot\mathcal{P}^E+\nabla\cdot(\sum_\alpha n_\alpha q_\alpha\mathbf{w}_\alpha\mathbf{w}_\alpha) \tag{5.13}$$

Using the continuity equation (8.3.2) and substituting $\mathbf{u}_\alpha$ by $\mathbf{w}_\alpha+\mathbf{u}$, the *last term* in the right-hand side of (5.10) can be written as

$$-\sum_\alpha\Big(\frac{q_\alpha}{m_\alpha}\Big)\mathbf{u}_\alpha S_\alpha=-\sum_\alpha\mathbf{w}_\alpha\frac{\partial}{\partial t}(n_\alpha q_\alpha)-\sum_\alpha\mathbf{w}_\alpha[\nabla\cdot(n_\alpha q_\alpha\mathbf{w}_\alpha)]-$$

$$-\sum_\alpha\mathbf{w}_\alpha[\nabla\cdot(n_\alpha q_\alpha\mathbf{u})]-\mathbf{u}\frac{\partial\rho}{\partial t}-\mathbf{u}(\nabla\cdot\mathbf{J}) \tag{5.14}$$

Similarly, the *first* and *second terms* in the left-hand side of (5.10) can be combined in the form

$$\sum_\alpha n_\alpha q_\alpha\frac{\partial\mathbf{w}_\alpha}{\partial t}+\sum_\alpha(n_\alpha q_\alpha\mathbf{w}_\alpha\cdot\nabla)\mathbf{w}_\alpha+\sum_\alpha(n_\alpha q_\alpha\mathbf{u}\cdot\nabla)\mathbf{w}_\alpha+$$

$$+\rho\frac{\partial\mathbf{u}}{\partial t}+(\mathbf{J}\cdot\nabla)\mathbf{u} \tag{5.15}$$

We can now substitute expressions (5.13), (5.14) and (5.15) into (5.10) and simplify the result. Making use of the following identity for two vectors $\mathbf{a}$ and $\mathbf{b}$,

$$\nabla\cdot(\mathbf{ab})=\mathbf{b}(\nabla\cdot\mathbf{a})+(\mathbf{a}\cdot\nabla)\mathbf{b} \tag{5.16}$$

and the relation (4.12), we obtain

$$\frac{\partial\mathbf{J}}{\partial t}+\nabla\cdot(\mathbf{uJ}'+\mathbf{Ju})+\nabla\cdot\mathcal{P}^E=$$

$$= \sum_\alpha n_\alpha \Big(\frac{q_\alpha}{m_\alpha}\Big) < \mathbf{F} >_\alpha + \sum_\alpha \Big(\frac{q_\alpha}{m_\alpha}\Big) \mathbf{A}_\alpha \tag{5.17}$$

Equations (5.1), (5.2), (5.4) and (5.17) constitute ten component equations which complement the equations of conservation of mass, of momentum and of energy for a conducting fluid. Equation (5.17), however, is still in a very general form of little practical value. A very useful and simple expression can be obtained for the case of a completely ionized plasma consisting of electrons and only one type of ions. In what follows, we simplify (5.17) for this case.

The electric charge current density $\mathbf{J}$ and the electric charge density ρ for a completely ionized plasma containing only electrons and one type of ions of charge e are given, respectively, by

$$\mathbf{J} = \sum_\alpha n_\alpha q_\alpha \mathbf{u}_\alpha = e(n_i \mathbf{u}_i - n_e \mathbf{u}_e) \tag{5.18}$$

$$\rho = \sum_\alpha n_\alpha q_\alpha = e(n_i - n_e) \tag{5.19}$$

The global mean velocity $\mathbf{u}$, defined in (1.3), becomes

$$\mathbf{u} = \frac{1}{\rho_m}(\rho_{me}\mathbf{u}_e + \rho_{mi}\mathbf{u}_i) \tag{5.20}$$

where $\rho_m = \rho_{me} + \rho_{mi}$. Combining this last equation with (5.18) gives

$$\mathbf{u}_i = \frac{\mu}{\rho_{mi}}\Big(\frac{\rho_m \mathbf{u}}{m_e} + \frac{\mathbf{J}}{e}\Big) \tag{5.21}$$

$$\mathbf{u}_e = \frac{\mu}{\rho_{me}}\Big(\frac{\rho_m \mathbf{u}}{m_i} - \frac{\mathbf{J}}{e}\Big) \tag{5.22}$$

where $\mu = m_e m_i/(m_e + m_i)$ denotes the reduced mass.

We assume now that the mean velocity of the electrons and ions, relative to the global mean velocity $\mathbf{u}$, (that is, the diffusion velocities $\mathbf{w}_e$ and $\mathbf{w}_i$) are small compared to the thermal velocities. This condition being satisfied, (5.12) becomes

$$\mathcal{P}^E = \mathcal{P}_i^E + \mathcal{P}_e^E = e\Big(\frac{\mathcal{P}_i}{m_i} - \frac{\mathcal{P}_e}{m_e}\Big) \tag{5.23}$$

Considering that the conducting fluid is immersed in an electromagnetic field, the term containing the external force in (5.17) becomes

$$\sum_\alpha n_\alpha \Big(\frac{q_\alpha}{m_\alpha}\Big) < \mathbf{F} >_\alpha = \sum_\alpha n_\alpha \Big(\frac{q_\alpha}{m_\alpha}\Big)[q_\alpha(\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B})]$$

$$= e^2\Big(\frac{n_i}{m_i} + \frac{n_e}{m_e}\Big)\mathbf{E} + e^2\Big(\frac{n_i}{m_i}\mathbf{u}_i + \frac{n_e}{m_e}\mathbf{u}_e\Big) \times \mathbf{B} \tag{5.24}$$

Substituting the relations (5.21) and (5.22) in this last equation and simplifying, yields

$$\sum_\alpha n_\alpha\Big(\frac{q_\alpha}{m_\alpha}\Big) < \mathbf{F} >_\alpha = e^2\Big(\frac{n_i}{m_i} + \frac{n_e}{m_e}\Big)\mathbf{E} +$$

$$+ e^2\Big(\frac{n_i}{m_e} + \frac{n_e}{m_i}\Big)\mathbf{u}\times\mathbf{B} + e\Big(\frac{1}{m_i} - \frac{1}{m_e}\Big)\mathbf{J}\times\mathbf{B} \tag{5.25}$$

It is convenient at this moment to simplify this equation by making one additional approximation. Since the ion mass m_i is much larger than the electron mass m_e (for protons and electrons, for example, $m_i/m_e \simeq 1836$) and assuming macroscopic charge neutrality, $n_e = n_i = n$, we can take

$$\frac{1}{m_i} - \frac{1}{m_e} \simeq -\frac{1}{m_e} \tag{5.26}$$

$$\frac{n_i}{m_i} + \frac{n_e}{m_e} \simeq \frac{n}{m_e} \tag{5.27}$$

$$\frac{n_i}{m_e} + \frac{n_e}{m_i} \simeq \frac{n}{m_e} \tag{5.28}$$

Consequently, from (5.23) we have $\mathcal{P}^E = -(e/m_e)\mathcal{P}_e$, and from (5.25)

$$\sum_\alpha n_\alpha\Big(\frac{q_\alpha}{m_\alpha}\Big) < \mathbf{F} >_\alpha = \frac{ne^2}{m_e}(\mathbf{E} + \mathbf{u}\times\mathbf{B}) - \frac{e}{m_e}\mathbf{J}\times\mathbf{B} \tag{5.29}$$

For the collision term in (5.17) we make use of expression (8.4.11), that is,

$$\mathbf{A}_e = -\rho_{me}\nu_{ei}(\mathbf{u}_e - \mathbf{u}_i) \tag{5.30}$$

$$\mathbf{A}_i = -\rho_{mi}\nu_{ie}(\mathbf{u}_i - \mathbf{u}_e) \tag{5.31}$$

From (8.4.13) we have $\rho_{mi}\nu_{ie} = \rho_{me}\nu_{ei}$, so that

$$\sum_\alpha\Big(\frac{q_\alpha}{m_\alpha}\Big)\mathbf{A}_\alpha = e\rho_{me}\nu_{ei}(\mathbf{u}_e - \mathbf{u}_i)\Big(\frac{1}{m_i} + \frac{1}{m_e}\Big) = -\nu_{ei}\mathbf{J} \tag{5.32}$$

where we have used (5.18) for $\mathbf{J}$, and the approximations $m_i \gg m_e$ and $n_e = n_i = n$.

We can now substitute the results contained in (5.23), (5.29) and (5.32) into (5.17), to obtain

$$\frac{\partial\mathbf{J}}{\partial t} + \nabla\cdot(\mathbf{u}\mathbf{J}' + \mathbf{J}\mathbf{u}) - \frac{e}{m_e}\nabla\cdot\mathcal{P}_e =$$

$$= \frac{ne^2}{m_e}(\mathbf{E} + \mathbf{u} \times \mathbf{B}) - \frac{e}{m_e}\mathbf{J} \times \mathbf{B} - \nu_{ei}\mathbf{J} \tag{5.33}$$

Note that, since we assumed $n_e = n_i$, we must have $\rho = 0$ and $\mathbf{J}' = \mathbf{J}$. In some situations in which $\mathbf{J}$ and $\mathbf{u}$ can be considered as small perturbations, the nonlinear terms involving their product may be neglected compared to the other terms. With this simplifying approximation and using the notation

$$\sigma_0 = \frac{ne^2}{m_e \nu_{ei}} \tag{5.34}$$

which represents the *longitudinal electrical conductivity*, we obtain for (5.33)

$$\frac{m_e}{ne^2}\frac{\partial \mathbf{J}}{\partial t} - \frac{1}{ne}\nabla \cdot \mathcal{P}_e = \mathbf{E} + \mathbf{u} \times \mathbf{B} - \frac{1}{ne}\mathbf{J} \times \mathbf{B} - \frac{1}{\sigma_0}\mathbf{J} \tag{5.35}$$

This equation is known as the *generalized Ohm's law.* The terms in the right-hand side are the ones normally retained in magnetohydrodynamics, while all the others are neglected. The omission of the terms in the left-hand side of (5.35) is, generally, not always justifiable.

For cases in which $\mathbf{J}$ does not vary with time, that is, under steady state conditions, we have $\partial \mathbf{J}/\partial t = 0$. If we consider also that the pressure term in (5.35) is negligible, that is, $\nabla \cdot \mathcal{P}_e = 0$, then (5.35) simplifies to

$$\mathbf{J} = \sigma_0(\mathbf{E} + \mathbf{u} \times \mathbf{B}) - \frac{\sigma_0}{ne}\mathbf{J} \times \mathbf{B} \tag{5.36}$$

The last term in this equation is related to a phenomenon called the *Hall effect* in magnetodydrodynamic flow problems and, for this reason, it is normally called the Hall effect term. This term is small when $\sigma_0|\mathbf{B}|$ is much less than ne, that is, when $\Omega_{ce} \ll \nu_{ei}$. Thus, when the collision frequency is much larger than the magnetic gyrofrequency, the Hall effect term can be neglected and (5.36) reduces to

$$\mathbf{J} = \sigma_0(\mathbf{E} + \mathbf{u} \times \mathbf{B}) \tag{5.37}$$

In the absence of an external magnetic field, (5.37) reduces further to

$$\mathbf{J} = \sigma_0 \mathbf{E} \tag{5.38}$$

which is the expression commonly known as *Ohm's law.*

6. SIMPLIFIED MAGNETOHYDRODYNAMIC EQUATIONS

In the last two sections we have shown that the set of macroscopic transport equations for each individual species in a plasma can be substituted by transport equations for the whole plasma as a conducting fluid, complemented by the electrodynamic equations. These total macroscopic equations for a conducting fluid are usually known as the *magnetohydrodynamic (MHD) equations.* In their most general form they are essentially equivalent to the set of equations for each individual particle species, except for the loss of information regarding the variables for each individual species.

In practice, however, the MHD equations are seldom used in their general form. Several simplifying approximations are normally considered, based on physical arguments which allow the elimination of some of the terms in the equations. For steady state situations, or slowly-varying problems, the MHD equations are very convenient and, in many cases, lead to important results which would not be easily obtained from the individual equations for each particle species.

One of the approximations normally used in MHD consists in neglecting the term $\epsilon_0 \partial \mathbf{E}/\partial t$ in the Maxwell equation (5.2). In order to investigate the validity of this approximation it is conveninet to use dimensional analysis, as follows. We can express, in general, the charge current density as $\mathbf{J} = \mathcal{S} \cdot \mathbf{E}$, where $\mathcal{S}$ denotes the conductivity dyad, so that, dimensionally, we have

$$J \sim \sigma E \tag{6.1}$$

$$\epsilon_0 |\partial \mathbf{E}/\partial t| \sim \epsilon_0 E/\tau \tag{6.2}$$

where τ represents a characteristic time for changes in the electric field and σ represents a characteristic conductivity. The ratio of the two terms in the right-hand side of (5.2) becomes, therefore,

$$\epsilon_0 \frac{|\partial \mathbf{E}/\partial t|}{J} \sim \frac{\epsilon_0}{\sigma \tau} \tag{6.3}$$

For most of the fluids normally considered in MHD problems, σ is very large, typically greater than 1 mho/m, whereas ϵ_0 is of the order of 10^{-11} Farad/m. Consequently,

$$\epsilon_0 \frac{|\partial \mathbf{E}/\partial t|}{J} \sim \frac{10^{-11}}{\tau} \tag{6.4}$$

with τ in seconds, which shows that this approximation is not valid only when we are considering *extremelly small* characteristic times.

It is also usually assumed that macroscopic electric neutrality is maintained with a high degree of accuracy and therefore the electric density ρ is set equal to zero.

A questionable approximation in the set of MHD equations is the generalized Ohm's law, in the form given in (5.36). In this form, the terms containing the time derivatives and pressure gradient (or divergence of the pressure dyad) are omitted, even though these terms are retained in other equations of the set. This approximation is not therefore justifiable in a direct manner. It is common to simply assume that all time derivatives are negligibly small and that the plasma is *almost* a cold plasma, so that the generalized Ohm's law reduces to the form given in (5.36).

For convenience, we collect here the following set of simplified magnetohydrodynamic èquations

$$\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \mathbf{u}) = 0 \tag{6.5}$$

$$\rho_m \frac{D\mathbf{u}}{Dt} = \mathbf{J} \times \mathbf{B} - \nabla p \tag{6.6}$$

$$\nabla p = V_s^2 \nabla \rho_m \tag{6.7}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{6.8}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{6.9}$$

$$\mathbf{J} = \sigma_0(\mathbf{E} + \mathbf{u} \times \mathbf{B}) - \frac{\sigma_0}{ne}\mathbf{J} \times \mathbf{B} \tag{6.10}$$

In this set of equations, viscosity and thermal conductivity are neglected, so that the pressure dyad reduces to a scalar pressure. Note that (6.9) implies in

$$\nabla \cdot \mathbf{J} = 0 \tag{6.11}$$

which is the equation of conservation of electric charge in the absence of changes in the total macroscopic charge density ρ. It is for this reason that the equation of conservation of electric charge is not explicity considered in this set of MHD equations. Except in some special circunstances, it is also common to neglect the Hall effect term $(\sigma_0/en)\mathbf{J} \times \mathbf{B}$ in (6.10).

In some situations where the fluid electrical conductivity is extremelly large, it is convenient to consider the idealized approximation of a *perfectly*

conducting fluid, in which the conductivity approaches infinity. In this case Ohm's law reduces to

$$\mathbf{E} = -\mathbf{u} \times \mathbf{B} \tag{6.12}$$

and the corresponding set of equations is commonly referred to as the *ideal MHD equations.*

PROBLEMS

9.1 - Show that the total kinetic energy density of all species in a fluid can be written as the sum of the thermal energy density of the whole fluid plus the kinetic energy of the mass motion, that is

$$\sum_\alpha \tfrac{1}{2}\rho_{m\alpha} < v^2 >_\alpha = \frac{3p}{2} + \sum_\alpha \tfrac{1}{2}\rho_{m\alpha} u_\alpha^2$$

where

$$\frac{3p}{2} = \sum_\alpha \tfrac{1}{2}\rho_{m\alpha} < c_{\alpha 0}^2 > = \sum_\alpha \tfrac{1}{2}\rho_{m\alpha} < c_\alpha^2 > + \sum_\alpha \tfrac{1}{2}\rho_{m\alpha} w_\alpha^2$$

9.2 - Show that when there is no heat flow ($\mathbf{q} = 0$), no Joule heating ($\mathbf{J}' \cdot \mathbf{E}' = 0$) and when the pressure tensor is isotropic given by $\mathcal{P} = p\mathbf{1}$, the energy equation (4.14) reduces to the following adiabatic equation

$$p\rho_m^{-5/3} = constant$$

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9.3 - From the momentum conservation equation with the MHD approximation [see (6.6)], and the generalized Ohm's law in the simplified form (6.10), but without considering the Hall effect term, derive the following equation:

$$\rho_m \frac{D\mathbf{u}}{Dt} = \sigma_0(\mathbf{E} \times \mathbf{B}) + \sigma_0(\mathbf{u} \times \mathbf{B}) \times \mathbf{B} - \nabla p$$

Solve this equation, considering that $\mathbf{E} = 0$ and $p = constant$, to show that the fluid velocity perpendicular to $\mathbf{B}$ is given by

$$u_\perp(t) = u_\perp(0) \exp\,(-t/\tau)$$

where τ is a characteristic time for diffusion of the fluid across the magnetic field lines, given by

$$\tau = \frac{\rho_m}{\sigma_0 B^2}$$

9.4 - In equations (1.5) and (1.6), explain the reason why the mass flux $\mathbf{J}_m$ is given by $\rho_m \mathbf{u}$, whereas the electric charge flux $\mathbf{J}$ is *not* given by $\rho \mathbf{u}$.

9.5 - Obtain an expression for the heat flux triad $\mathcal{Q}$ for the plasma as a whole, defined as

$$\mathcal{Q} = \sum_\alpha \rho_{m\alpha} < \mathbf{c}_{\alpha 0}\ \mathbf{c}_{\alpha 0}\ \mathbf{c}_{\alpha 0} >$$

where $\mathbf{c}_{\alpha 0} = \mathbf{c}_\alpha + \mathbf{w}_\alpha$, in terms of a summation over the heat flux triad for each species $\mathcal{Q}_\alpha$ and of terms involving the diffusion velocity $\mathbf{w}_\alpha$. Then, simplify this expression for the isotropic case.

9.6 - Derive an energy equation, of higher order than (4.14), involving the total time rate of change of the total pressure dyad, that is, $D\mathcal{P}/Dt$.

9.7 - For a perfectly conducting fluid characterized by a scalar pressure, under steady state conditions, use the equation of motion (6.6) and the generalized Ohm's law (6.10) to derive the following equation for the fluid velocity component perpendicular to $\mathbf{B}$,

$$\mathbf{u}_\perp = \frac{1}{B^2}\left(\mathbf{E} - \frac{1}{ne}\nabla p\right) \times \mathbf{B}$$

10

PLASMA CONDUCTIVITY AND DIFFUSION

1. INTRODUCTION

In the previous chapters we have introduced the fundamental elements of kinetic theory and the macroscopic transport equations necessary for the study of a variety of important phenomena in plasmas. Many plasma phenomena can be analyzed using the macroscopic transport equations, either considering the plasma as a multi-constituent fluid or by treating the whole plasma as a single conducting fluid. In some cases, however, a satisfactory description can only be obtained through the use of kinetic theory.

In this and in the following chapters we investigate some basic plasma phenomena which illustrate the use of the cold and warm plasma models, and of the phase space distribution function. Phenomena that can be analyzed treating the whole plasma as a single conducting fluid are usually studied under the general title of magnetohydrodynamics (MHD), and will be considered in Chapters 12, 13 and 15.

2. THE LANGEVIN EQUATION

Before we consider the phenomena of plasma conductivity and diffusion, it is convenient to introduce a very simple form of the equation of motion for a weakly ionized cold plasma, known as the *Langevin equation.* In a weakly ionized plasma the number density of the charged particles is much smaller than that of the neutral particles. In this case the charge-neutral interactions are dominant. The *macroscopic* equation of motion

for the electrons under the action of the Lorentz force and the collisional forces, can be written as

$$m_e \frac{D\mathbf{u}_e}{Dt} = -e(\mathbf{E} + \mathbf{u}_e \times \mathbf{B}) + (\mathbf{F}_{coll})_e \tag{2.1}$$

where $\mathbf{u}_e(\mathbf{r}, t)$ is the average electron velocity and $(\mathbf{F}_{coll})_e$ denotes symbolically the rate of change of the average electron momentum due to collisions with neutral particles.

The macroscopic collision term $(\mathbf{F}_{coll})_e$ can be expressed in a phenomenological way as the product of the average electron momentum with an *effective* constant collision frequency ν_c for momentum transfer between the electrons and the heavy (neutral) particles,

$$(\mathbf{F}_{coll})_e = -\nu_c m_e \mathbf{u}_e \tag{2.2}$$

In this expression we are neglecting the average motion of the neutral particles, as they are much more massive than the electrons. Note that this does not mean that the velocities of the individual neutral particles are zero, but only that they are completely *random* so that their *average* velocity is zero. Using this expression for the collision term, we obtain the following equation, known as the *Langevin equation*,

$$m_e \frac{D\mathbf{u}_e}{Dt} = -e(\mathbf{E} + \mathbf{u}_e \times \mathbf{B}) - \nu_c m_e \mathbf{u}_e \tag{2.3}$$

The physical meaning of this collision term can be seen as follows. In the absence of electric and magnetic fields, (2.3) reduces to

$$\frac{D\mathbf{u}_e}{Dt} = -\nu_c \mathbf{u}_e \tag{2.4}$$

whose solution is

$$\mathbf{u}_e(t) = \mathbf{u}_e(0) \exp(-\nu_c t) \tag{2.5}$$

Thus, the electron-neutral collisions decrease the average electron velocity exponentially, at a rate governed by the collision frequency.

An equation analogous to (2.3) can be written for the ions,

$$m_i \frac{D\mathbf{u}_i}{Dt} = Ze(\mathbf{E} + \mathbf{u}_i \times \mathbf{B}) + (\mathbf{F}_{coll})_i \tag{2.6}$$

where $\mathbf{u}_i$ denotes the average ion velocity and Ze the ion charge. In many cases of interest, as in high-frequency phenomena, we can neglect the ion

motion and assume $\mathbf{u}_i = 0$, since the ion mass is typically about 10^3 or 10^4 times greater than the electron mass. The type of plasma, in which only the electron motion is important, is usually called a *Lorentz gas*. When dealing with very low frequencies, however, the motion of the ions must be considered.

Despite the approximations implicit in the Langevin equation, it has been successfully used to describe a variety of phenomena in plasmas, including the propagation of electromagnetic waves in cold magnetoplasmas. Particularly, the analysis of the characteristics of electromagnetic wave propagation in the Earth's ionosphere has been quite successful. A great advantage of this equation is its simplicity.

3. LINEARIZATION OF THE LANGEVIN EQUATION

In the form presented in (2.3) the Langevin equation contains nonlinear terms which involve the product of two variables. In many situations of interest the difficulty inherent in the nonlinear terms can be eliminated through a linearization approximation, which is valid for small-amplitude variations.

The total time derivative contains the nonlinear term $(\mathbf{u}_e \cdot \nabla)\mathbf{u}_e$, which is called the inertial term in fluid dynamics. The omission of this inertial term is justified when the average velocity and its space derivatives are small, or when $\mathbf{u}_e$ is normal to its gradient (such as in the case of transverse waves).

For the nonlinear term $\mathbf{u}_e \times \mathbf{B}$, we can separate the magnetic flux density $\mathbf{B}(\mathbf{r}, t)$ into two parts

$$\mathbf{B}(\mathbf{r}, t) = \mathbf{B}_0 + \mathbf{B}'(\mathbf{r}, t) \tag{3.1}$$

where $\mathbf{B}_0$ is constant and $\mathbf{B}'(\mathbf{r}, t)$ is the variable component, so that

$$q(\mathbf{E} + \mathbf{u}_e \times \mathbf{B}) = q(\mathbf{E} + \mathbf{u}_e \times \mathbf{B}_0 + \mathbf{u}_e \times \mathbf{B}') \tag{3.2}$$

For situations in which

$$|\mathbf{u}_e \times \mathbf{B}'| \ll |\mathbf{E}| \tag{3.3}$$

the nonlinear term $\mathbf{u}_e \times \mathbf{B}'$ in (3.2) can be neglected. With these linearization approximations the Langevin equation becomes

$$m_e \frac{\partial \mathbf{u}_e}{\partial t} = -e(\mathbf{E} + \mathbf{u}_e \times \mathbf{B}_0) - m_e \nu_c \mathbf{u}_e \tag{3.4}$$

A case of great practical interest is the one in which the variables $\mathbf{E}$, $\mathbf{B}'$ and $\mathbf{u}_e$ vary harmonically in space and time. The treatment in terms of plane waves has the advantage of great mathematical simplicity, besides the fact that any complex and physically realizable wave motion can be synthesized in terms of a superposition of plane waves. Let us consider therefore plane wave solutions for $\mathbf{E}$, $\mathbf{B}'$ and $\mathbf{u}_e$ in the form

$$\mathbf{E},\ \mathbf{B}',\ \mathbf{u}_e \propto \exp\left[i(\mathbf{k}\cdot\mathbf{r} - \omega t)\right] \tag{3.5}$$

where ω denotes the wave angular frequency, $\mathbf{k}$ is the wave propagation vector (normal to the wave front) and $\mathbf{r}$ is a position vector drawn from the origin of a coordinate system to the point considered on the wave front (refer to Fig. 1).

For the space and time dependence given in (3.5), the differential operators ∇ and $\partial/\partial t$ are transformed into simple algebraic operators, according to $\nabla \to i\mathbf{k}$ and $\partial/\partial t \to -i\omega$. Substituting (3.1) into Maxwell equation $\nabla \times \mathbf{E} = -\partial\mathbf{B}/\partial t$, we obtain

$$i\mathbf{k} \times \mathbf{E} = i\omega\mathbf{B}' \tag{3.6}$$

where $\partial\mathbf{B}_0/\partial t = 0$, since $\mathbf{B}_0$ is constant. Therefore,

$$\mathbf{B}' = \frac{\mathbf{k} \times \mathbf{E}}{\omega} \tag{3.7}$$

and plugging this result back into (3.3) yields the condition

$$|\mathbf{u}_e \times (\mathbf{k} \times \mathbf{E})/\omega| \ll |\mathbf{E}| \tag{3.8}$$

The magnitude of the nonlinear term $\mathbf{u}_e \times \mathbf{B}'$ may be equal to or smaller than $|(u_e k E)/\omega|$. Hence, the nonlinear term can be neglected if

$$|u_e(k/\omega)| \ll 1 \tag{3.9}$$

or, equivalently, if

$$|u_e| \ll |\omega/k| \tag{3.10}$$

The term (ω/k) represents the phase velocity of the plane wave. Since this term is usually of the order of the speed of light c, whereas the magnitude of the mean velocity of the electrons u_e is much less than c, the nonlinear term can generally be neglected. However, in cases of resonance ω/k is very

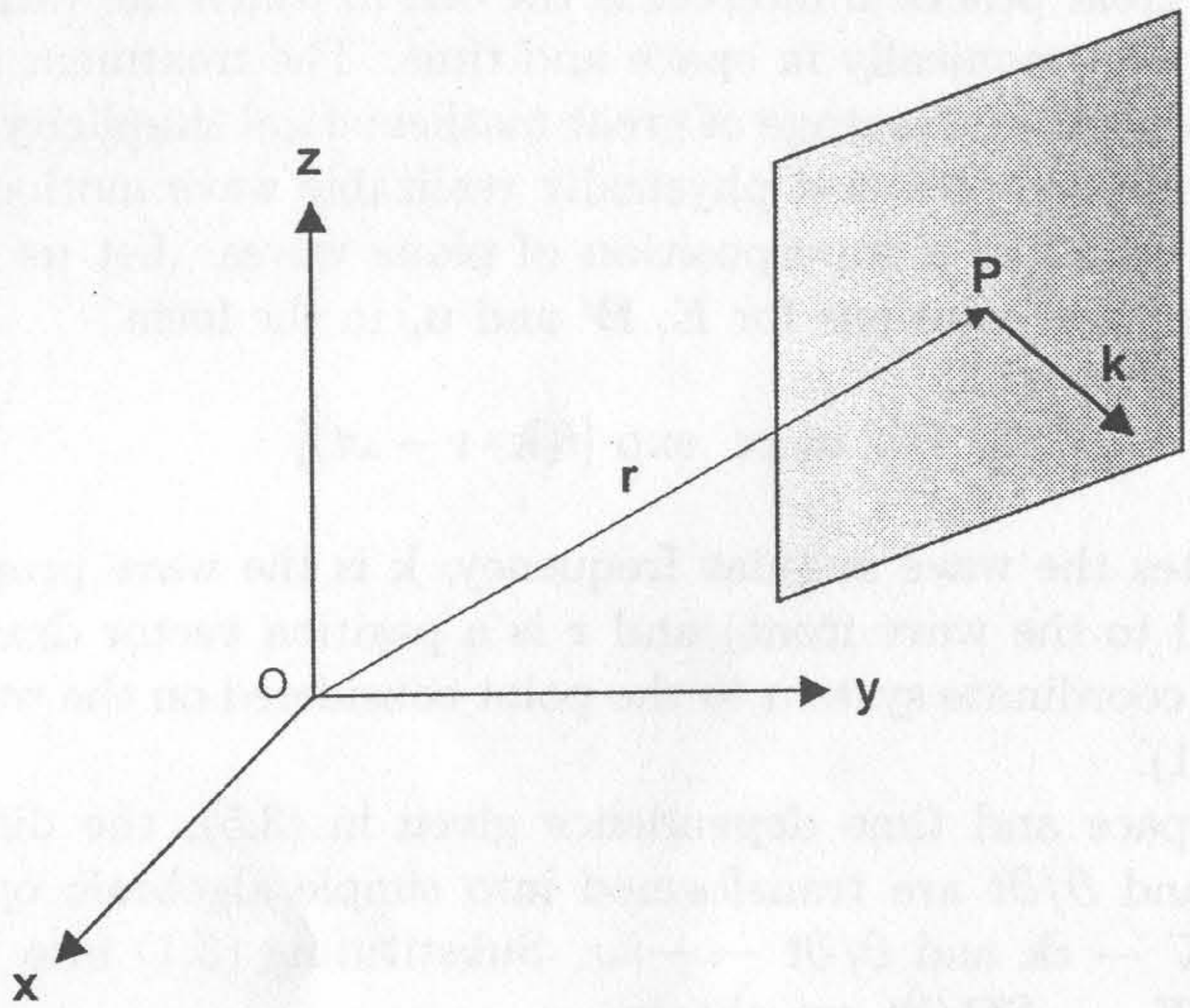

Fig. 1 *Position vector* **r** *drawn from the origin of a coordinate system* (x, y, z) *to a point* P *on the wave front, whose normal is given by the wave propagation vector* **k**.

small, whereas u_e becomes large. Under these conditions the nonlinear terms are important and a nonlinear analysis must be used.

4. DC CONDUCTIVITY AND ELECTRON MOBILITY

In this section we apply the steady state Langevin equation to derive an expression for the DC (direct current) conductivity of a weakly ionized homogeneous plasma, for which the Lorentz model (electron gas) is applicable. The applied electric field is assumed to be constant and uniform.

4.1 Isotropic Plasma

In the absence of a magnetic field the steady state Langevin equation for the electrons becomes

$$-e\mathbf{E} - m_e \nu_c \mathbf{u}_e = 0 \tag{4.1}$$

In this case the action of the applied electric field is balanced dynamically by the electron-neutral collisions. The electric current density associated

with the electron motion is

$$\mathbf{J} = -en_e\mathbf{u}_e \tag{4.2}$$

Combining (4.1) and (4.2), gives

$$\mathbf{J} = \frac{n_e e^2}{m_e \nu_c}\mathbf{E} \tag{4.3}$$

From Ohm's law, $\mathbf{J} = \sigma_0\mathbf{E}$, we identify the following expression for the *DC conductivity* of an isotropic electron gas

$$\sigma_0 = \frac{n_e e^2}{m_e \nu_c} \tag{4.4}$$

The *electron mobility* $\mathcal{M}_e$ is defined as the ratio of the mean velocity of the electrons to the applied electric field,

$$\mathcal{M}_e = \frac{u_e}{E} \tag{4.5}$$

Therefore, from (4.1) we obtain

$$\mathcal{M}_e = -\frac{e}{m_e \nu_c} = -\frac{\sigma_0}{n_e e} \tag{4.6}$$

4.2 Anisotropic Magnetoplasma

In the presence of a magnetic field the plasma becomes spatially anisotropic. The steady state Langevin equation can be written as

$$-e(\mathbf{E} + \mathbf{u}_e \times \mathbf{B}_0) - m_e \nu_c \mathbf{u}_e = 0 \tag{4.7}$$

where $\mathbf{B}_0$ is a constant and uniform magnetic field. Using (4.2),

$$\frac{m_e \nu_c}{n_e e}\mathbf{J} = e(\mathbf{E} + \mathbf{u}_e \times \mathbf{B}_0) \tag{4.8}$$

which may be written in the form

$$\mathbf{J} = \sigma_0(\mathbf{E} + \mathbf{u}_e \times \mathbf{B}_0) \tag{4.9}$$

where σ_0 is given in (4.4). This last equation is a simplified form of the generalized Ohm's law (see Chapter 9).

At this point it is worth to consider a useful result which arises when the collisional effects are negligible. When $\nu_c \to 0$ the DC conductivity becomes very large ($\sigma_0 \to \infty$) so that we must have, from (4.9),

$$\mathbf{E} + \mathbf{u}_e \times \mathbf{B}_0 = 0 \tag{4.10}$$

This expression represents therefore the simplified form of the generalized Ohm's law for a plasma with a very large conductivity. In this case, taking the cross product of (4.10) with $\mathbf{B}_0$ and noting that

$$(\mathbf{u}_e \times \mathbf{B}_0) \times \mathbf{B}_0 = -\mathbf{u}_{e\perp} B_0^2 \tag{4.11}$$

we obtain

$$\mathbf{u}_{e\perp} = \frac{\mathbf{E} \times \mathbf{B}_0}{B_0^2} \tag{4.12}$$

This result shows that, in the absence of collisions, the electrons have a *drift velocity* $\mathbf{u}_{e\perp}$ perpendicular to both the electric and the magnetic fields. Since this result is independent of the particle mass and charge, the same result will be obtained for the ions if their motion is taken into account. This can be easily shown considering the Langevin equation for the ions. Thus, in the absence of collisions, both electrons and ions move together with the drift velocity (4.12), and there is no electric current ($\mathbf{J} = 0$) associated with their motion. When the collisional effects are not negligible, the motion of the ions suffers a larger retardation than that of the electrons as a result of collisions. In this case, there is an electric current given by (assuming $n_e = n_i$)

$$\mathbf{J}_\perp = e n_e (\mathbf{u}_{i\perp} - \mathbf{u}_{e\perp}) \tag{4.13}$$

which is perpendicular to both $\mathbf{E}$ and $\mathbf{B}_0$, known as the *Hall current.* Note that, since $\mathbf{u}_{e\perp} > \mathbf{u}_{i\perp}$, this current is in the direction of $-(\mathbf{E} \times \mathbf{B}_0)$, that is, opposite to the drift velocity of both types of particles.

Returning now to the generalized Ohm's law in the simplified form (4.9), let us rewrite it in a way which relates the current density directly to the applied electric field. We define therefore a *DC conductivity dyad* (or *tensor*) $\mathcal{S}$ by the equation

$$\mathbf{J} = \mathcal{S} \cdot \mathbf{E} \tag{4.14}$$

In order to obtain an expression for $\mathcal{S}$, consider a Cartesian coordinate system with the z axis parallel to the magnetic field, $\mathbf{B}_0 = B_0\widehat{\mathbf{z}}$. Replacing $\mathbf{u}_e$ in (4.9) by $-\mathbf{J}/(en_e)$, we get

$$\mathbf{J} = \sigma_0\mathbf{E} - \frac{\sigma_0 B_0}{en_e}(\mathbf{J}\times\widehat{\mathbf{z}}) \tag{4.15}$$

Noting that

$$\mathbf{J}\times\widehat{\mathbf{z}} = J_y\widehat{\mathbf{x}} - J_x\widehat{\mathbf{y}} \tag{4.16}$$

we obtain the following set of equations for the x, y and z components of (4.15)

$$\widehat{\mathbf{x}}: \qquad J_x = \sigma_0 E_x - \frac{\Omega_{ce}}{\nu_c}J_y \tag{4.17}$$

$$\widehat{\mathbf{y}}: \qquad J_y = \sigma_0 E_y + \frac{\Omega_{ce}}{\nu_c}J_x \tag{4.18}$$

$$\widehat{\mathbf{z}}: \qquad J_z = \sigma_0 E_z \tag{4.19}$$

where Ω_{ce} denotes the electron cyclotron frequency. We can combine (4.17) and (4.18) to eliminate J_y from the first one and J_x from the second one, obtaining

$$J_x = \frac{\nu_c^2}{(\nu_c^2+\Omega_{ce}^2)}\sigma_0 E_x - \frac{\nu_c\Omega_{ce}}{(\nu_c^2+\Omega_{ce}^2)}\sigma_0 E_y \tag{4.20}$$

$$J_y = \frac{\nu_c\Omega_{ce}}{(\nu_c^2+\Omega_{ce}^2)}\sigma_0 E_x + \frac{\nu_c^2}{(\nu_c^2+\Omega_{ce}^2)}\sigma_0 E_y \tag{4.21}$$

In matrix form we can write, therefore,

$$\begin{pmatrix} J_x \\ J_y \\ J_z \end{pmatrix} = \sigma_0 \begin{pmatrix} \dfrac{\nu_c^2}{(\nu_c^2+\Omega_{ce}^2)} & -\dfrac{\nu_c\Omega_{ce}}{(\nu_c^2+\Omega_{ce}^2)} & 0 \\ \dfrac{\nu_c\Omega_{ce}}{(\nu_c^2+\Omega_{ce}^2)} & \dfrac{\nu_c^2}{(\nu_c^2+\Omega_{ce}^2)} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} \tag{4.22}$$

which is now in the form given in (4.14). The DC conductivity dyad is therefore given by

$$\mathcal{S} = \begin{pmatrix} \sigma_\perp & -\sigma_H & 0 \\ \sigma_H & \sigma_\perp & 0 \\ 0 & 0 & \sigma_\parallel \end{pmatrix} \tag{4.23}$$

where we have used the notation

$$\sigma_\perp = \frac{\nu_c^2}{(\nu_c^2 + \Omega_{ce}^2)}\sigma_0 \tag{4.24}$$

$$\sigma_H = \frac{\nu_c \Omega_{ce}}{(\nu_c^2 + \Omega_{ce}^2)}\sigma_0 \tag{4.25}$$

$$\sigma_\parallel \equiv \sigma_0 = \frac{n_e e^2}{m_e \nu_c} \tag{4.26}$$

To illustrate the physical meaning of the components of $\mathcal{S}$ it is convenient to separate the applied electric field in a component parallel to $\mathbf{B}_0$, $\mathbf{E}_\parallel$, and a component in the plane normal to $\mathbf{B}_0$, $\mathbf{E}_\perp$, as shown in Fig. 2. The element $\sigma_\perp$ is called the *perpendicular* or *transverse conductivity* (also known as *Pedersen conductivity*), since it governs the flow of electric current in the direction of the electric field component *normal* to the magnetic field ($\parallel \mathbf{E}_\perp$, $\perp \mathbf{B}_0$), while σ_H (known as the *Hall conductivity*) governs the flow of electric current in the direction *perpendicular* to *both* the electric and magnetic fields ($\perp \mathbf{E}$, $\perp \mathbf{B}_0$). The element σ_0 is the *longitudinal conductivity*, since it governs the electric current flow in the direction of the electric field component *along* the magnetic field ($\parallel \mathbf{E}_\parallel$, $\parallel \mathbf{B}_0$). Note that the electric current along $\mathbf{B}_0$ is governed by the same conductivity (σ_0) as in the isotropic plasma.

The dependence of $\sigma_\perp$ and σ_H on the ratio of the cyclotron frequency to the collision frequency is shown in Fig. 3. As the ratio (Ω_{ce}/ν_c) increases, $\sigma_\perp$ and σ_H decrease rapidly, the effect being more pronounced for $\sigma_\perp$. Thus, when (Ω_{ce}/ν_c) is relatively large, very little current is produced across the magnetic field lines, as compared to the current produced along the field lines, for the same applied electric field. Note that σ_0 increases as ν_c decreases and it is independent of the magnitude of $\mathbf{B}$ and therefore of Ω_{ce}. Thus, in a rarefied plasma immersed in a relatively strong magnetic field, the electric current flows essentially along the magnetic field lines.

Note that in the absence of a magnetic field ($\Omega_{ce} = 0$), (4.24), (4.25) and (4.26) give $\sigma_\perp = \sigma_0$ and $\sigma_H = 0$, so that the plasma becomes isotropic.

We deduce next an expression for the electron mobility. Due to the anisotropy introduced by the magnetic field we have in this case a *mobility dyad* $\mathcal{M}_e$. We define the electron mobility dyad by the equation

$$\mathbf{u}_e = \mathcal{M}_e \cdot \mathbf{E} \tag{4.27}$$

Since $\mathbf{J} = -en_e\mathbf{u}_e = \mathcal{S} \cdot \mathbf{E}$, we find that

$$\mathcal{M}_e = -\frac{1}{n_e e}\mathcal{S} \tag{4.28}$$

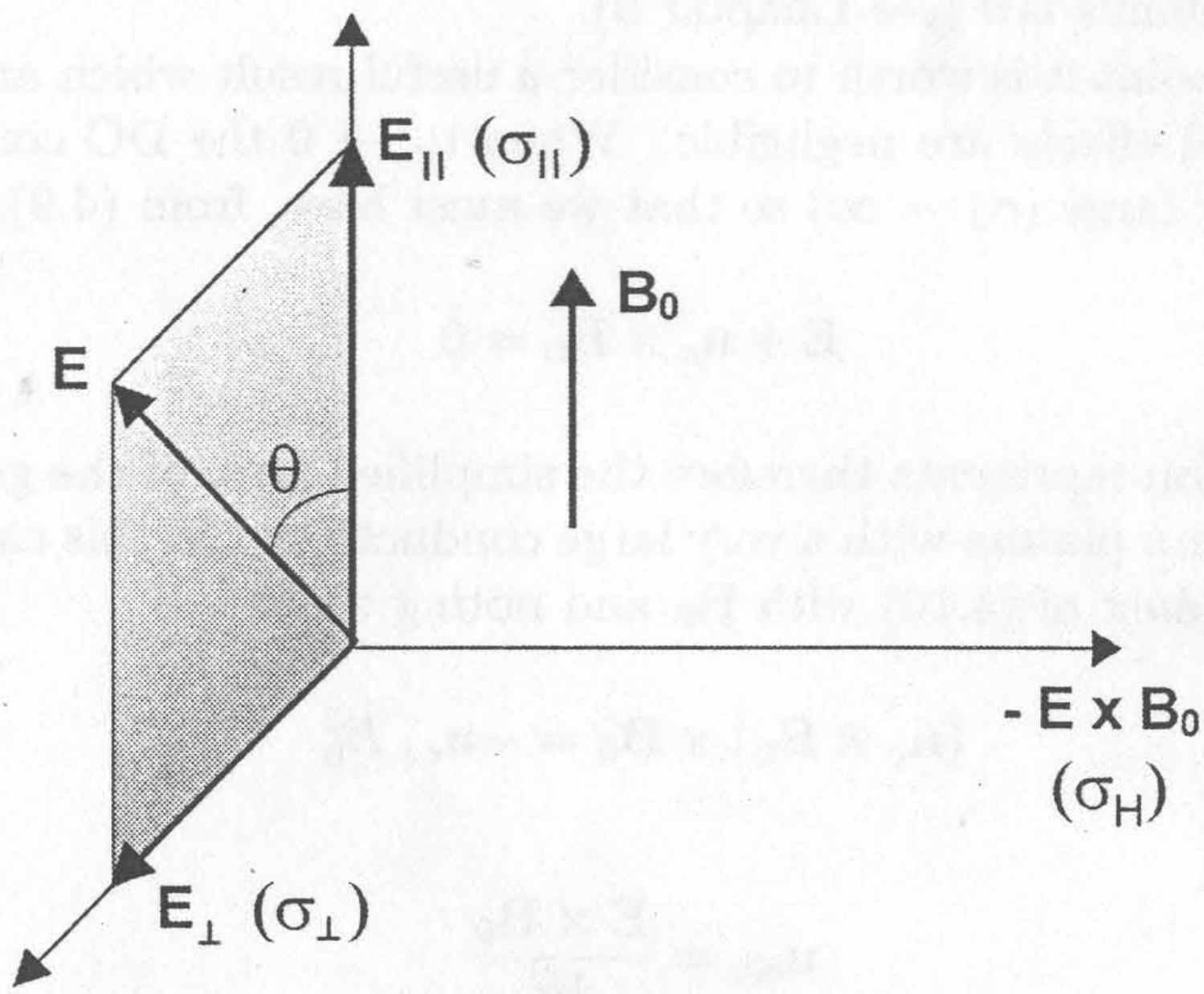

Fig. 2 *Relative orientation of the vector fields* $\mathbf{E}_{\parallel}$, $\mathbf{E}_{\perp}$ *and of* $-\mathbf{E} \times \mathbf{B}_0$. *The conductivities* $\sigma_{\parallel}$, $\sigma_{\perp}$ *and* σ_H *govern the magnitude of the electric currents flowing along these directions, respectively.*

Explicit expressions for the components of $\mathcal{M}_e$ can be easily written down considering (4.23), (4.24), (4.25) and (4.26).

5. AC CONDUCTIVITY AND ELECTRON MOBILITY

Consider now the case when the electric field $\mathbf{E}(\mathbf{r}, t)$ and the mean electron velocity $\mathbf{u}_e(\mathbf{r}, t)$ vary harmonically in time, as $exp\ (-i\omega t)$. We have seen that for time harmonic disturbances $\partial/\partial t$ is replaced by $-i\omega$. Therefore, the linearized Langevin equation (3.4) becomes

$$-i\omega m_e \mathbf{u}_e = -e(\mathbf{E} + \mathbf{u}_e \times \mathbf{B}_0) - m_e \nu_c \mathbf{u}_e \tag{5.1}$$

which can be written as

$$-e(\mathbf{E} + \mathbf{u}_e \times \mathbf{B}_0) - m_e(\nu_c - i\omega)\mathbf{u}_e = 0 \tag{5.2}$$

This equation is identical to (4.7), except for the change in the collision frequency ν_c to $(\nu_c - i\omega)$. We obtain, therefore, solutions similar to the ones

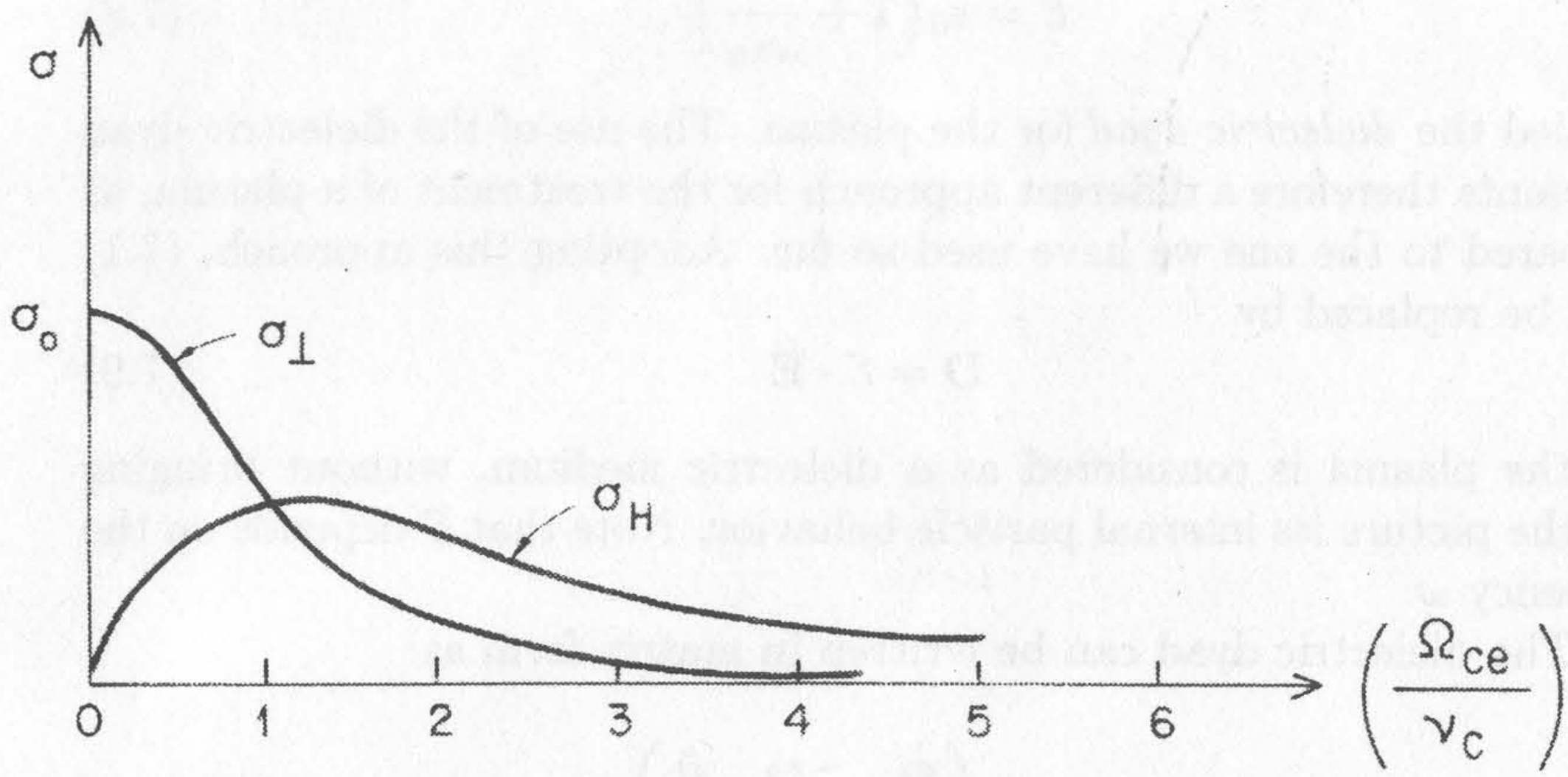

Fig. 3 *Dependence of the Hall conductivity σ_H and of the perpendicular conductivity $\sigma_\perp$ on the ratio of the cyclotron frequency Ω_{ce} to the collision frequency ν_c.*

obtained for the DC conductivity dyad in the previous section, except that now we must replace ν_c by $(\nu_c - i\omega)$ in each element of the dyad. Therefore, the expressions for the *frequency-dependent* perpendicular conductivity, Hall conductivity and longitudinal conductivity are, respectively,

$$\sigma_\perp = \frac{(\nu_c - i\omega)^2}{(\nu_c - i\omega)^2 + \Omega_{ce}^2}\sigma_0 \tag{5.3}$$

$$\sigma_H = \frac{(\nu_c - i\omega)\Omega_{ce}}{(\nu_c - i\omega)^2 + \Omega_{ce}^2}\sigma_0 \tag{5.4}$$

$$\sigma_0 = \frac{n_e e^2}{m_e(\nu_c - i\omega)} = \frac{n_e e^2(\nu_c + i\omega)}{m_e(\nu_c^2 + \omega^2)} \tag{5.5}$$

When the electron-neutral collisions can be neglected ($\nu_c = 0$), the expressions for the components of the AC (alternating current) conductivity dyad become

$$\sigma_\perp = \frac{\omega^2}{(\omega^2 - \Omega_{ce}^2)}\sigma_0 \tag{5.6}$$

$$\sigma_H = \frac{i\omega\Omega_{ce}}{(\omega^2 - \Omega_{ce}^2)}\sigma_0 \tag{5.7}$$

$$\sigma_0 = i\frac{n_e e^2}{m_e \omega} \tag{5.8}$$

A complex conductivity means that there is a phase difference between the current density and the applied electric field.

The electron mobility, in any of the cases considered in this section, can be easily written down considering the relation (4.28).

6. CONDUCTIVITY WITH ION MOTION

The evaluation of the conductivity dyad, when the contribution due to the motion of the ions is included, can be performed in a straightforward way. For this purpose, consider the linearized Langevin equation for the type α species,

$$m_\alpha \frac{\partial \mathbf{u}_\alpha}{\partial t} = q_\alpha(\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}_0) - m_\alpha \nu_{c\alpha} \mathbf{u}_\alpha \tag{6.1}$$

where $\nu_{c\alpha}$ is an effective collision frequency or damping term for the type α species resulting from collisions with *neutral* particles. Note that the Langevin equations, for each charged particle species, are uncoupled. Therefore, the net current density is given by

$$\mathbf{J} = \sum_\alpha n_\alpha q_\alpha \mathbf{u}_\alpha = \sum_\alpha \mathbf{J}_\alpha = \Big(\sum_\alpha \mathcal{S}_\alpha\Big) \cdot \mathbf{E} \tag{6.2}$$

and the total conductivity dyad is simply

$$\mathcal{S} = \sum_\alpha \mathcal{S}_\alpha \tag{6.3}$$

For a plasma with electrons and several types of ions (index j) we obtain, using (5.3), (5.4) and (5.5), in terms of the plasma frequency $\omega_{p\alpha}$ and ϵ_0,

$$\sigma_\perp = \epsilon_0 \Big[\frac{\omega_{pe}^2(\nu_{ce} - i\omega)}{(\nu_{ce} - i\omega)^2 + \Omega_{ce}^2} + \sum_j \frac{\omega_{pj}^2(\nu_{cj} - i\omega)}{(\nu_{cj} - i\omega)^2 + \Omega_{cj}^2}\Big] \tag{6.4}$$

$$\sigma_H = \epsilon_0 \Big[\frac{\omega_{pe}^2 \Omega_{ce}}{(\nu_{ce} - i\omega)^2 + \Omega_{ce}^2} - \sum_j \frac{\omega_{pj}^2 \Omega_{cj}}{(\nu_{cj} - i\omega)^2 + \Omega_{cj}^2}\Big] \tag{6.5}$$

$$\sigma_{\parallel} = \epsilon_0 \Big[\frac{\omega_{pe}^2}{(\nu_{ce} - i\omega)} + \sum_j \frac{\omega_{pj}^2}{(\nu_{cj} - i\omega)} \Big] \tag{6.6}$$

7. PLASMA AS A DIELECTRIC MEDIUM

The plasma can also be treated as a dielectric medium characterized by a *dielectric dyad*, in which the internal particle behavior is not considered. So far, we have treated the plasma as a collection of charged and neutral particles moving about in their own internal fields. Thus, as far as the *constitutive relations* are concerned, we have taken

$$\mathbf{D} = \epsilon_0 \mathbf{E} \tag{7.1}$$

$$\mathbf{B} = \mu_0 \mathbf{H} \tag{7.2}$$

which are applicable for free space, and the plasma effects show up through the motion and interaction of the charged particles inside the plasma.

A different approach is provided by the use of a dielectric dyad, in which we are concerned only with the gross macroscopic properties of the plasma and not with the elementary particle motions. Thus, instead of the Langevin equation, let us consider the following Maxwell equation

$$\nabla \times \mathbf{B} = \mu_0 \Big(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \Big) \tag{7.3}$$

and incorporate the effects of the plasma in the conductivity dyad $\mathcal{S}$, defined by the equation

$$\mathbf{J} = \mathcal{S} \cdot \mathbf{E} \tag{7.4}$$

Substituting (7.4) into (7.3), and assuming time-harmonic variations of the form $exp\ (-i\omega t)$, we obtain

$$\nabla \times \mathbf{B} = \mu_0 \mathcal{S} \cdot \mathbf{E} - i\omega\mu_0\epsilon_0 \mathbf{E} \tag{7.5}$$

If we let $\mathbf{1}$ denote the unit dyad, we can write

$$\nabla \times \mathbf{B} = -i\omega\mu_0\epsilon_0 \Big(\mathbf{1} + \frac{i\mathcal{S}}{\omega\epsilon_0} \Big) \cdot \mathbf{E} \tag{7.6}$$

or, equivalently, as

$$\nabla \times \mathbf{B} = -i\omega\mu_0 \mathcal{E} \cdot \mathbf{E} \tag{7.7}$$

where

$$\mathcal{E} = \epsilon_0\left(\mathbf{1} + \frac{i\mathcal{S}}{\omega\epsilon_0}\right) \tag{7.8}$$

is called the *dielectric dyad* for the plasma. The use of the dielectric dyad represents therefore a different approach for the treatment of a plasma, as compared to the one we have used so far. Adopting this approach, (7.1) must be replaced by

$$\mathbf{D} = \mathcal{E} \cdot \mathbf{E} \tag{7.9}$$

and the plasma is considered as a dielectric medium, without bringing into the picture its internal particle behavior. Note that $\mathcal{E}$ depends on the frequency ω.

The dielectric dyad can be written in matrix form as

$$\mathcal{E} = \epsilon_0 \begin{pmatrix} \epsilon_1 & -\epsilon_2 & 0 \\ \epsilon_2 & \epsilon_1 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix} \tag{7.10}$$

where the following notation was introduced

$$\epsilon_1 = 1 + \frac{i}{\omega\epsilon_0}\sigma_\perp \tag{7.11}$$

$$\epsilon_2 = \frac{i}{\omega\epsilon_0}\sigma_H \tag{7.12}$$

$$\epsilon_3 = 1 + \frac{i}{\omega\epsilon_0}\sigma_0 \tag{7.13}$$

For the case of a multi-species plasma the total conductivity must be used in (7.8), so that the expressions to be substituted for $\sigma_\perp$, σ_H and σ_0 are those given in (6.4), (6.5) and (6.6).

8. FREE ELECTRON DIFFUSION

The presence of a pressure gradient term in the momentum transport equation provides a force which tends to smooth out any inhomogeneities in the plasma density. The diffusion of particles in a plasma results from this pressure gradient force. To deduce the expression for the electron diffusion coefficient for a warm weakly ionized plasma we will use the momentum transport equation for the electrons, with a constant electron-neutral collision frequency. We assume that the deviations from the equilibrium

state caused by the inhomogeneities in the density are very small, so that they may be considered as first order quantities. This means that the mean velocity of the electrons $\mathbf{u}_e$ is also a first order quantity, and considering that the velocity distribution is approximately isotropic, we can replace the pressure dyad $\mathcal{P}_e$ by a scalar pressure p_e.

Consider the case in which $\mathbf{E}$ and $\mathbf{B}$ are zero and the electron temperature T_e is constant. For a *slightly nonuniform* electron number density, we can write

$$n_e(\mathbf{r},t) = n_0 + n'_e(\mathbf{r},t) \tag{8.1}$$

$$p_e(\mathbf{r},t) = n_e(\mathbf{r},t)kT_e = (n_0 + n'_e)kT_e \tag{8.2}$$

where $|n'_e| \ll n_0$ is a first order quantity and n_0 is constant. Since $\mathbf{u}_e$ is also a first order quantity, the continuity equation for the electron gas becomes

$$\frac{\partial n'_e}{\partial t} + n_0 \nabla \cdot \mathbf{u}_e = 0 \tag{8.3}$$

where the second order term $n'_e \mathbf{u}_e$ has been neglected. Similarly, for the momentum transport equation,

$$n_e m_e \left[\frac{\partial \mathbf{u}_e}{\partial t} + (\mathbf{u}_e \cdot \nabla)\mathbf{u}_e \right] = -\nabla p_e - n_e m_e \nu_c \mathbf{u}_e \tag{8.4}$$

we obtain, after linearization,

$$n_0 \frac{\partial \mathbf{u}_e}{\partial t} = -\frac{kT_e}{m_e} \nabla n'_e - n_0 \nu_c \mathbf{u}_e \tag{8.5}$$

Taking the divergence of this equation, we obtain

$$n_0 \frac{\partial (\nabla \cdot \mathbf{u}_e)}{\partial t} = -\frac{kT_e}{m_e} \nabla^2 n'_e - n_0 \nu_c \nabla \cdot \mathbf{u}_e \tag{8.6}$$

Using (8.3) to substitute for $n_0 \nabla \cdot \mathbf{u}_e$, yields

$$\frac{\partial^2 n'_e}{\partial t^2} = \frac{kT_e}{m_e} \nabla^2 n'_e - \nu_c \frac{\partial n'_e}{\partial t}. \tag{8.7}$$

This equation may also be written in the form

$$\frac{\partial n'_e}{\partial t} = D_e \nabla^2 n'_e - \frac{1}{\nu_c} \frac{\partial^2 n'_e}{\partial t^2} \tag{8.8}$$

where we have defined

$$D_e = \frac{kT_e}{m_e \nu_c} \tag{8.9}$$

which is called the *electron free-diffusion coefficient.*

To obtain a rough estimate of the order of magnitude of the various terms in (8.8), let τ and L represent, respectively, a characteristic time and a characteristic length over which n'_e varies significantly. Thus, any spatial derivative is of the order of L^{-1} and any time derivative is of the order τ^{-1}, so that the order of magnitude of the terms in (8.8) are

$$\frac{\partial n'_e}{\partial t} \sim \frac{n'_e}{\tau} \tag{8.10}$$

$$D_e \nabla^2 n'_e \sim D_e \frac{n'_e}{L^2} \tag{8.11}$$

$$\frac{1}{\nu_c}\frac{\partial^2 n'_e}{\partial t^2} \sim \frac{n'_e}{\nu_c \tau^2} \tag{8.12}$$

Comparing (8.10) and (8.12) we see that if $\nu_c \tau \gg 1$, that is, if the average number of electron-neutral collisions is large during the time interval τ, then the last term in (8.8) can be neglected and it reduces to the following *diffusion equation*

$$\frac{\partial n'_e}{\partial t} = D_e \nabla^2 n'_e \tag{8.13}$$

Therefore, when the rate of change in the number density is slow compared to the collision frequency, the number density of the electrons is governed by a diffusion equation with a free-diffusion coefficient as given by (8.9).

The condition $\nu_c \tau \gg 1$ implies in the omission of the acceleration term in the momentum transport equation, that is, $\partial \mathbf{u}_e / \partial t$ is neglected. From the linearized equation (8.5), when there are no time variations in $\mathbf{u}_e$, we obtain,

$$n_0 \nu_c \mathbf{u}_e = -\frac{kT_e}{m_e} \nabla n'_e \tag{8.14}$$

which can be written as

$$\mathbf{\Gamma}_e = -D_e \nabla n'_e \tag{8.15}$$

where $\mathbf{\Gamma}_e = n_0 \mathbf{u}_e$ denotes the linearized electron flux.

Expression (8.15) is analogous to the simple Ohm's law $\mathbf{J} = \sigma_0 \mathbf{E}$, replacing $\mathbf{J}$ by $\mathbf{\Gamma}_e$, σ_0 by D_e, and $\mathbf{E}$ by $-\nabla n'_e$. Thus, we see that the electron flux $\mathbf{\Gamma}_e$ is caused by a density gradient, in a way analogous to the

electric current caused by an electric field, under steady state conditions for $\mathbf{u}_e$.

9. ELECTRON DIFFUSION IN A MAGNETIC FIELD

Consider now the problem of electron diffusion in the presence of a constant and uniform magnetic field B_0. We shall make the same assumptions as in the previous section and neglect the acceleration term $\partial \mathbf{u}_e/\partial t$ in the equation of motion. In the linearized momentum transport equation (8.5), with the time derivative set equal to zero, we include now a magnetic force term, which results in

$$\mathbf{\Gamma}_e = -D_e \nabla n'_e - \frac{e}{m_e \nu_c}(\mathbf{\Gamma}_e \times \mathbf{B}_0) \tag{9.1}$$

Choosing a Cartesian coordinate system with the z axis pointing in the direction of the constant $\mathbf{B}_0$ field, that is, $\mathbf{B}_0 = B_0 \widehat{\mathbf{z}}$, we have

$$\mathbf{\Gamma}_e = -D_e \nabla n'_e - \frac{\Omega_{ce}}{\nu_c}(\mathbf{\Gamma}_e \times \widehat{\mathbf{z}}) \tag{9.2}$$

This equation is analogous to (4.15), with $\mathbf{\Gamma}_e$ replacing $\mathbf{J}$, D_e replacing σ_0, and $-\nabla n'_e$, replacing $\mathbf{E}$. Note that $\Omega_{ce}/\nu_c = \sigma_0 B_0/(en_e)$. Therefore, in analogy with the expression $\mathbf{J} = \mathcal{S} \cdot \mathbf{E}$, we can write

$$\mathbf{\Gamma}_e = -\mathcal{D} \cdot \nabla n'_e \tag{9.3}$$

where $\mathcal{D}$ is the *dyad coefficient* for *free-diffusion* given in matrix form by

$$\mathcal{D} = \begin{pmatrix} D_\perp & D_H & 0 \\ -D_H & D_\perp & 0 \\ 0 & 0 & D_\parallel \end{pmatrix} \tag{9.4}$$

where the following notation is used

$$D_\perp = \frac{\nu_c^2}{\nu_c^2 + \Omega_{ce}^2} D_e \tag{9.5}$$

$$D_H = \frac{\nu_c \Omega_{ce}}{\nu_c^2 + \Omega_{ce}^2} D_e \tag{9.6}$$

$$D_\parallel \equiv D_e = \frac{kT_e}{m_e \nu_c} \tag{9.7}$$

A diffusion equation for n'_e, when there is a constant and uniform magnetic field present, can also be derived in the same way as in the previous section. First, we write the continuity equation (8.3) in the form

$$\frac{\partial n'_e}{\partial t} + \nabla \cdot \mathbf{\Gamma}_e = 0 \tag{9.8}$$

Substituting (9.3) for $\mathbf{\Gamma}_e$, yields

$$\frac{\partial n'_e}{\partial t} = \nabla \cdot (\mathcal{D} \cdot \nabla n'_e) \tag{9.9}$$

Using (9.4) we find, by direct calculation in Cartesian coordinates,

$$\mathcal{D} \cdot \nabla n'_e = \widehat{\mathbf{x}}\Big(D_\perp \frac{\partial n'_e}{\partial x} + D_H \frac{\partial n'_e}{\partial y}\Big) +$$

$$+ \widehat{\mathbf{y}}\Big(-D_H \frac{\partial n'_e}{\partial x} + D_\perp \frac{\partial n'_e}{\partial y}\Big) + \widehat{\mathbf{z}} D_e \frac{\partial n'_e}{\partial z} \tag{9.10}$$

Substituting this result into (9.9), yields

$$\frac{\partial n'_e}{\partial t} = D_\perp \Big(\frac{\partial^2 n'_e}{\partial x^2} + \frac{\partial^2 n'_e}{\partial y^2}\Big) + D_e \frac{\partial^2 n'_e}{\partial z^2}. \tag{9.11}$$

Since $D_\perp < D_e$ and since $D_\perp$ decreases with increasing values of Ω_{ce}/ν_c (similarly to $\sigma_\perp$, as shown in Fig. 3) the diffusion of particles in a direction perpendicular to $\mathbf{B}$ is always less than that in the direction parallel to $\mathbf{B}$. For values of Ω_{ce} much larger than ν_c, the diffusion of particles across the magnetic field lines is greatly reduced. Note that, from (9.5) and (9.6), it can be seen that for $\Omega_{ce} \gg \nu_c$ we have, approximately, $D_\perp \propto 1/B^2$ and $D_H \propto 1/B$.

As a final point in this section we note that the momentum transport equation for an electron gas, neglecting the acceleration term but including the electromagnetic force, and when the temperature is constant, can be written in the general form

$$\mathbf{\Gamma}_e = \mathcal{M}_e(n_e \mathbf{E} + \mathbf{\Gamma}_e \times \mathbf{B}) - D_e \nabla n_e \tag{9.12}$$

From this equation we can see that the electron flux is produced by either, or both, electromagnetic fields and density gradients. The ratio of the scalar mobility $\mathcal{M}_e$ to the diffusion coefficient is known as the *Einstein relation* and is given by

$$\frac{\mathcal{M}_e}{D_e} = -\frac{e}{kT_e} \tag{9.13}$$

10. AMBIPOLAR DIFFUSION

We have seen in section 8 that the steady state momentum equation, in the absence of electromagnetic forces and when the temperature is constant, gives the following diffusion equation for the electrons

$$\mathbf{\Gamma}_e = -D_e \nabla n_e' \tag{10.1}$$

where the *electron free-diffusion coefficient* is given by

$$D_e = \frac{kT_e}{m_e \nu_{ce}} \tag{10.2}$$

The subscript e has been added here to ν_c to indicate that the effective collision frequency ν_{ce} refers to electron-neutral collisions.

If we consider similar equations for the ions in a weakly ionized plasma, under the same assumptions, we obtain the following ion diffusion equation

$$\mathbf{\Gamma}_i = -D_i \nabla n_i' \tag{10.3}$$

where

$$D_i = \frac{kT_i}{m_i \nu_{ci}} \tag{10.4}$$

denotes the *ion free-diffusion coefficient* and ν_{ci} is the effective ion-neutral collision frequency.

In deriving the results given by (10.1) and (10.3), the interaction between the electrons and the ions were not taken into account. Since the diffusion coefficient is inversely proportional to the particle mass, the electrons tend to diffuse faster than the ions leaving an excess of positive charge behind them. This gives rise to a space charge electric field, which points in the same direction as the particle diffusion, and which accelerates the diffusion of the ions and slow down that of the electrons. The diffusion in which the effect of this space charge electric field is *not* included is known as *free-diffusion.*

For most problems of plasma diffusion, however, the space charge electric field *cannot* be neglected. According to Maxwell equation

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} = \frac{e(n_i - n_e)}{\epsilon_0} \tag{10.5}$$

it is clear that an electric field is present whenever the electron density differs from the ion density. To estimate the importance of the space

charge electric field in diffusion problems let us use dimensional analysis and let L represent a characteristic length over which the number density changes significantly. Thus, from (10.5) we may write

$$E \sim \frac{enL}{\epsilon_0} \tag{10.6}$$

so that the electric force per unit mass f_E is of the order

$$f_E = \frac{eE}{m} \sim \frac{e^2 nL}{m\epsilon_0} \tag{10.7}$$

The diffusion force per unit mass f_D, obtained from (10.1), is of the order

$$f_D = \frac{kT}{mn_0}|\nabla n| \sim \frac{kTn}{mn_0 L} \tag{10.8}$$

Therefore, the space charge electric field can be neglected only if $f_E \ll f_D$, or equivalently, if

$$L^2 \ll \frac{\epsilon_0 kT}{n_0 e^2} = \lambda_D^2 \tag{10.9}$$

where λ_D is the *Debye length.* Since the Debye length is generally very small (see Fig. 2, Chapter 1), the condition $L \ll \lambda_D$ is rarely satisfied and for most plasma diffusion problems we cannot neglect the space charge electric field. In what follows we will reexamine therefore the problem of plasma diffusion taking into account the motion of both ions and electrons and including the space charge electric field **E**. The combined diffusion of the electrons and the ions, forced by the space charge **E** field, is known as *ambipolar diffusion.* Since the electric field is such that the electrons are deccelerated and the ions are accelerated, the two kinds of charged particles tend to reach a diffusion rate which is intermediate in value to their *free* diffusion rates.

To investigate the characteristics of *ambipolar diffusion* we assume that the variations for both electrons and ions are small first order quantities, so that (for $\alpha = e,\ i$)

$$n_\alpha(\mathbf{r}, t) = n_0 + n'_\alpha(\mathbf{r}, t) \tag{10.10}$$

with $|n'_\alpha| \ll n_0$, and that the mean velocities $\mathbf{u}_\alpha$ are of very small amplitude. We obtain, under these assumptions, the following linearized mass conservation equations (for $\alpha = e,\ i$),

$$\frac{\partial n'_\alpha}{\partial t} + n_0 \nabla \cdot \mathbf{u}_\alpha = 0 \tag{10.11}$$

The linearized momentum equations, assuming that the temperatures are constant, and without a magnetic field, become (with $\alpha = e,\ i$)

$$\frac{\partial \mathbf{u}_\alpha}{\partial t} = \frac{q_\alpha}{m_\alpha}\mathbf{E} - \frac{kT_\alpha}{m_\alpha n_0}\nabla n'_\alpha - \nu_{c\alpha}\mathbf{u}_\alpha \tag{10.12}$$

where the space charge $\mathbf{E}$ field satisfies Maxwell equation (10.5). We are assuming that the neutral mean velocity is zero and we are neglecting electron-ion collisions, since the plasma is weakly ionized. Taking the divergence of (10.12) and using (10.11), we obtain

$$\frac{\partial^2 n'_\alpha}{\partial t^2} = -\frac{q_\alpha n_0}{m_\alpha}(\nabla \cdot \mathbf{E}) + \frac{kT_\alpha}{m_\alpha}\nabla^2 n'_\alpha - \nu_{c\alpha}\frac{\partial n'_\alpha}{\partial t} \tag{10.13}$$

If we replace $\nabla \cdot \mathbf{E}$ from (10.5), we obtain the following set of coupled equations for the two variables n'_e and n'_i,

$$\frac{\partial^2 n'_e}{\partial t^2} = \omega_{pe}^2(n'_i - n'_e) + \frac{kT_e}{m_e}\nabla^2 n'_e - \nu_{ce}\frac{\partial n'_e}{\partial t} \tag{10.14}$$

$$\frac{\partial^2 n'_i}{\partial t^2} = -\omega_{pi}^2(n'_i - n'_e) + \frac{kT_i}{m_i}\nabla^2 n'_i - \nu_{ci}\frac{\partial n'_i}{\partial t} \tag{10.15}$$

These equations, however, are still too complicated for a detailed analytical treatment and to go further we will make some additional simplifying assumptions.

Recall that, if $\nu_c\tau \gg 1$, that is, if the average electron or ion has many collisions with neutral particles during the characteristic time for diffusion τ, the term $\partial^2 n'_\alpha/\partial t^2$ (originated from the acceleration term in the momentum equation) can be neglected. With this assumption we neglect the term in the left-hand side of (10.14) and (10.15). Combining these equations, we obtain

$$kT_e\nabla^2 n'_e + kT_i\nabla^2 n'_i - m_e\nu_{ce}\frac{\partial n'_e}{\partial t} - m_i\nu_{ci}\frac{\partial n'_i}{\partial t} = 0 \tag{10.16}$$

As a second approximation we will set $n'_e = n'_i = n'$ in (10.16) to obtain the following diffusion equation

$$k(T_e + T_i)\nabla^2 n' - (m_e\nu_{ce} + m_i\nu_{ci})\frac{\partial n'}{\partial t} = 0 \tag{10.17}$$

which can be written in the form

$$\frac{\partial n'}{\partial t} = D_a \nabla^2 n' \tag{10.18}$$

where

$$D_a = \frac{k(T_e + T_i)}{m_e \nu_{ce} + m_i \nu_{ci}} \tag{10.19}$$

is the *ambipolar diffusion coefficient.* Note that the coupling of the two equations (10.14) and (10.15) is a consequence of the electric field term, and that the simplifying approximation $n'_i = n'_e$ was introduced only after the two equations were combined into (10.16). This approximation implies that the space charge electric field becomes a negligible perturbation with the result that both ions and electrons diffuse together. This situation is known as *perfect ambipolar diffusion*, since the coupling between the two types of charged particles is complete.

Instead of taking $n'_i = n'_e$, a less restrictive simplifying approximation would be to assume

$$n'_i = C n'_e \tag{10.20}$$

where C is a constant. Using this approximation in (10.16) we obtain

$$k(T_e + CT_i)\nabla^2 n'_e = (m_e \nu_{ce} + C m_i \nu_{ci}) \frac{\partial n'_e}{\partial t} = 0 \tag{10.21}$$

or

$$\frac{\partial n'_e}{\partial t} = D_a \nabla^2 n'_e \tag{10.22}$$

where the ambipolar diffusion coefficient is now given by

$$D_a = \frac{k(T_e + CT_i)}{m_e \nu_{ce} + C m_i \nu_{ci}} \tag{10.23}$$

The space charge density is now

$$\rho = e(n'_i - n'_e) = e n'_e (C - 1) \tag{10.24}$$

and the electric field can be obtained from Maxwell equation

$$\nabla \cdot \mathbf{E} = \frac{e n'_e (C - 1)}{\epsilon_0} \tag{10.25}$$

The effect of this electric field is to accelerate the diffusion of the ions and to retard the diffusion of the electrons, as compared to their individual

free-diffusion rates, so that to a good approximation both species diffuse together.

Whenever there is a significant deviation from charge neutrality, the electric field force becomes very strong as can be seen from the following dimensional argument. A comparison of the magnitude of the electric force per unit mass, $\mathbf{f}_E = q_\alpha \mathbf{E}/m_\alpha$, and the diffusion force per unit mass, $\mathbf{f}_D = -(kT_\alpha/m_\alpha n_0)\nabla n'_\alpha$, which are of the order

$$f_E \sim \frac{Le^2 n'_e(C-1)}{m\epsilon_0} \tag{10.26}$$

$$f_D \sim \frac{kTn'_e}{mn_0 L} \tag{10.27}$$

shows that

$$\frac{f_E}{f_D} \sim \frac{L^2(C-1)}{\lambda_D^2} \tag{10.28}$$

Since in most cases L^2 is much larger than λ_D^2, we see that if n'_i differs significantly from n'_e then the electric field force (which tends to equalize n'_i and n'_e) becomes very strong.

11. DIFFUSION IN A FULLY IONIZED PLASMA

Consider now the problem of diffusion in a fully ionized plasma. For simplicity, we shall describe the plasma as a single conducting fluid for which the equation of motion, under steady state conditions, in the presence of magnetic and pressure gradient forces, is

$$\mathbf{J} \times \mathbf{B} = \nabla p \tag{11.1}$$

where $\mathbf{J}$ denotes the total electric current density, $\mathbf{B}$ is the magnetic induction, and p represents the total scalar pressure of the conducting fluid. Note that the electric force is zero, since the plasma, as a whole, is macroscopically neutral ($\rho = 0$). This equation is complemented by the generalized Ohm's law in the following simplified form,

$$\mathbf{J} = \sigma_0(\mathbf{E} + \mathbf{u} \times \mathbf{B}) \tag{11.2}$$

where σ_0 is the longitudinal electric conductivity and $\mathbf{u}$ is the total macroscopic fluid velocity.

Taking the cross-product of (11.2) with $\mathbf{B}$, yields

$$\mathbf{J} \times \mathbf{B} = \sigma_0(\mathbf{E} \times \mathbf{B} - B^2\mathbf{u}_\perp) \tag{11.3}$$

where $\mathbf{u}_\perp$ is the component of $\mathbf{u}$ in a direction normal to the external field $\mathbf{B}$. Using (11.1) and rearranging, (11.3) gives

$$\mathbf{u}_\perp = \frac{\mathbf{E} \times \mathbf{B}}{B^2} - \frac{\nabla p}{\sigma_0 B^2} \tag{11.4}$$

This result shows that the total fluid velocity across the magnetic field is given by the $\mathbf{E} \times \mathbf{B}$ drift of the whole plasma plus a diffusion velocity in the direction of $-\nabla p$.

The flux associated with the diffusion velocity only, is given by

$$\mathbf{\Gamma}_\perp = n\mathbf{u}_\perp = -\frac{n\nabla p}{\sigma_0 B^2} \tag{11.5}$$

where n denotes the electron (or total ion) number density. Considering a two-fluid plasma (electrons and one type of ions), we have

$$p = p_e + p_i = nk(T_e + T_i) \tag{11.6}$$

so that (11.5) becomes, assuming the temperatures to be constant,

$$\mathbf{\Gamma}_\perp = -\frac{nk(T_e + T_i)}{\sigma_0 B^2}\nabla n = -D_\perp \nabla n \tag{11.7}$$

The quantity

$$D_\perp = \frac{nk(T_e + T_i)}{\sigma_0 B^2} \tag{11.8}$$

is known as the *classical diffusion coefficient* for a fully ionized plasma.

This diffusion coefficient is proportional to $1/B^2$, just as in the case of a weakly ionized plasma. Nevertheless, there are some fundamental differences between $D_\perp$, as given by (11.8), and the corresponding coefficient for a partially ionized plasma. Initially note that in a fully ionized plasma $D_\perp$ is not constant, but depends on the number density n. Further, since it can be shown that σ_0 is proportional to $T^{3/2}$ for a Maxwellian distribution of velocities, $D_\perp$ decreases with increasing temperature in a fully ionized plasma, while the opposite situation holds for a weakly ionized plasma. Finally, the diffusion coefficient $D_\perp$ in (11.8) was derived for the whole plasma as a conducting fluid and, since both ions and electrons diffuse together, there is no ambipolar electric field.

In some experiments it has been observed a dependence of $D_\perp$ on the magnetic field as B^{-1}, rather than B^{-2}, and the decay of the plasma was

found to be exponential, rather than reciprocal with time. Furthermore, the absolute value of $D_\perp$ was found to be much larger than that given in (11.8). This anomalously poor magnetic confinement was first noted in laboratory by Bohm, in 1946, who arrived at the following semiempirical formula

$$D_\perp \equiv D_B = \frac{kT_e}{16\ eB} \tag{11.9}$$

Since this diffusion coefficient does not depend on the density, the decay of the plasma density is exponential with time. This type of diffusion in plasmas is known as *Bohm diffusion.*

PROBLEMS

10.1 Consider a solid state plasma with the same number of electrons (e) and holes (h). Using the linearized Langevin equation (with $\alpha = e,\ h$)

$$m_\alpha \frac{\partial \mathbf{u}_\alpha}{\partial t} = q_\alpha(\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}_0) - \nu_{c\alpha} m_\alpha \mathbf{u}_\alpha$$

taking $m_e = m_h$, $\nu_{ce} = \nu_{ch}$, assuming a time dependence for both $\mathbf{E}$ and $\mathbf{u}_\alpha$ of the form exp $(-i\omega t)$, and choosing a Cartesian coordinate system with the z axis pointing along the constant and uniform magnetic field $\mathbf{B}_0$, show that the conductivity dyad is given by

$$\mathcal{S} = 2\begin{pmatrix} \sigma_\perp & 0 & 0 \\ 0 & \sigma_\perp & 0 \\ 0 & 0 & \sigma_0 \end{pmatrix}$$

with $\sigma_\perp$ and σ_0 given by (5.3) and (5.5), respectively. Explain, in physical terms, why we have $\sigma_H = 0$ in this case.

10.2 Assume that the average velocities of the electrons and ions in a completely ionized plasma, in the presence of constant and uniform electric ($\mathbf{E}$) and magnetic ($\mathbf{B}_0$) fields, satisfy, respectively, the following equations of motion

$$m_e \frac{\partial \mathbf{u}_e}{\partial t} = -e(\mathbf{E} + \mathbf{u}_e \times \mathbf{B}_0) - m_e \nu(\mathbf{u}_e - \mathbf{u}_i)$$

$$m_i \frac{\partial \mathbf{u}_i}{\partial t} = e(\mathbf{E} + \mathbf{u}_i \times \mathbf{B}_0) - m_e \nu(\mathbf{u}_i - \mathbf{u}_e)$$

(a) Determine expressions for the steady state DC conductivities σ_H, $\sigma_\perp$ and σ_0.
(b) For $\mathbf{u}_e$, $\mathbf{u}_i$ and $\mathbf{E}$ all proportional to exp $(-i\omega t)$ and $\mathbf{B}_0$ constant, calculate the AC conductivity dyad for the plasma.

10.3 Consider the equation $\mathbf{J} = \mathcal{S} \cdot \mathbf{E}$, with $\mathcal{S}$ as given in (4.23). If we choose a Cartesian coordinate system such that $E_x = E_\perp$, $E_y = 0$, $E_z = E_\parallel$ and $\mathbf{B}_0 = B_0\widehat{\mathbf{z}}$ (refer to Fig. 2), verify that in this coordinate system we have

$$J_x = \sigma_\perp E_\perp$$

$$J_y = \sigma_H E_\perp$$

$$J_z = \sigma_\parallel E_\parallel$$

Interpret physically this result with reference to Fig. 2.

10.4 What is the physical meaning of a complex conductivity, as given in (5.7) and (5.8)? Consider, for example, that $\mathbf{E}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r})$ exp $(-i\omega t)$, and calculate the real parts of $\mathbf{E}(\mathbf{r}, t)$ and of $\mathbf{J}(\mathbf{r}, t) = \mathcal{S} \cdot \mathbf{E}(\mathbf{r}, t)$. Interpret physically the results considering the phase differences between $\mathbf{J}$ and $\mathbf{E}$.

10.5 Write expressions for the components of the dielectric dyad $\mathcal{E}$ of a multi-constituent magnetized plasma.

10.6 Consider the electrons in a plasma acted upon by a small, constant and uniform external electric field $\mathbf{E}$. Under steady state conditions with no spatial gradients, obtain an expression for the nonequilibrium distribution function f for the electrons, by applying a perturbation technique to the Boltzmann equation (take $f = f_0 + f_1$ with $|f_1| \ll f_0$ and neglect all second order terms), using the relaxation model for the collision term

$$\left(\frac{\delta f}{\delta t}\right)_{coll} = -\nu(f - f_0)$$

where ν is a relaxation collision frequency and f_0 is the equilibrium Maxwellian distribution function. Assuming that ν is independent of velocity, obtain an expression for the electric conductivity σ_0 of the plasma, by taking $\mathbf{J} = \sigma_0 \mathbf{E}$.

10.7 Same as Problem 10.6, but including also a constant and uniform magnetic field $\mathbf{B}_0$.

10.8 Imagine a horizontally stratified ionosphere in the absence of a magnetic field, constituted only of electrons (density n, temperature T, charge $-e$, mass m_e) and one type of ions (density n, temperature T, charge $+e$, mass m_i), subjected to the gravitational field ($\mathbf{g}$), vertical pressure gradient (∇p) and the internal electric field ($\mathbf{E}$) due to the charge separation associated with ambipolar diffusion. Neglect the gravitational force for the electrons and consider the system in equilibrium. Using the collisionless equations of motion for the electrons and the ions, show that the internal electric field acts downward on the electrons with a force $m_i g/2$, and upward on the ions with the same force. Consequently, the net effect is the same as if both ions and electrons had mass $m_i/2$.

10.9 (a) In order to solve the *diffusion equation*

$$\frac{\partial n(\mathbf{r},t)}{\partial t} = D\nabla^2 n(\mathbf{r},t)$$

by the method of separation of variables, let

$$n(\mathbf{r},t) = S(\mathbf{r})\ T(t)$$

and show that

$$T_k(t) = T_0 \exp\left(-Dk^2 t\right)$$

$$(\nabla^2 + k^2)S(\mathbf{r}) = 0$$

where k^2 is the separation constant and T_0 is a constant.
(b) Assuming that S depends only on the x coordinate show that

$$S(x) = c(k) \exp\left(ikx\right)$$

where k can be either positive or negative, and that

$$n(x,t) = \int_{-\infty}^{+\infty} c(k) \exp\left(ikx - Dk^2 t\right)\, dk$$

$$n_0(x) = \int_{-\infty}^{+\infty} c(k) \exp\left(ikx\right)\, dk$$

where $n_0(x) = n(x,0)$ is the known *initial* density distribution.
(c) Using Fourier transform theory, show that

$$c(k) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} n_0(x) \exp\left(-ikx\right)\, dx$$

and, consequently, that

$$n(x,t) = \frac{1}{2(\pi D t)^{1/2}} \int_{-\infty}^{+\infty} n_0(x') \exp\left[-\frac{(x-x')^2}{4Dt}\right] dx'$$

(d) Taking as initial condition

$$n_0(x) = \exp\left(-x^2/x_0^2\right)$$

show that

$$n(x,t) = \left(\frac{\tau_D}{\tau_D + 4t}\right)^{1/2} \exp\left[-\frac{x^2}{x_0^2}\left(\frac{\tau_D}{\tau_D + 4t}\right)\right]$$

where $\tau_D = x_0^2/D$ is a characteristic time for diffusion to smooth out the density n.
(e) Generalize the problem for the three-dimensional case in Cartesian coordinates, when $\mathrm{S} = \mathrm{S}(\mathbf{r})$.

10.10 Consider the solution of the diffusion equation by separation of variables in the linear geometry of the plasma slab indicated in Fig. 4. Show that the solutions of the equation

$$\frac{d^2 S(x)}{dx^2} + k^2\, S(x) = 0$$

which satisfy the boundary condition $S = 0$ at $x = \pm L$, are

$$S(x) = \sum_m a_m \cos\left[(m + 1/2)\pi x/L\right]$$

and

$$S(x) = \sum_m b_m \sin\left(m\pi x/L\right)$$

Explain why the solution as a sine series is not a physically acceptable solution for this case. Consequently, from $n(x,t) = S(x)\, T(t)$, show that the number density can be expressed as

$$n(x,t) = \sum_m a_m \exp\left[-D\pi^2(m+1/2)^2 t/L^2\right] \cos\left[\pi(m+1/2)x/L\right]$$

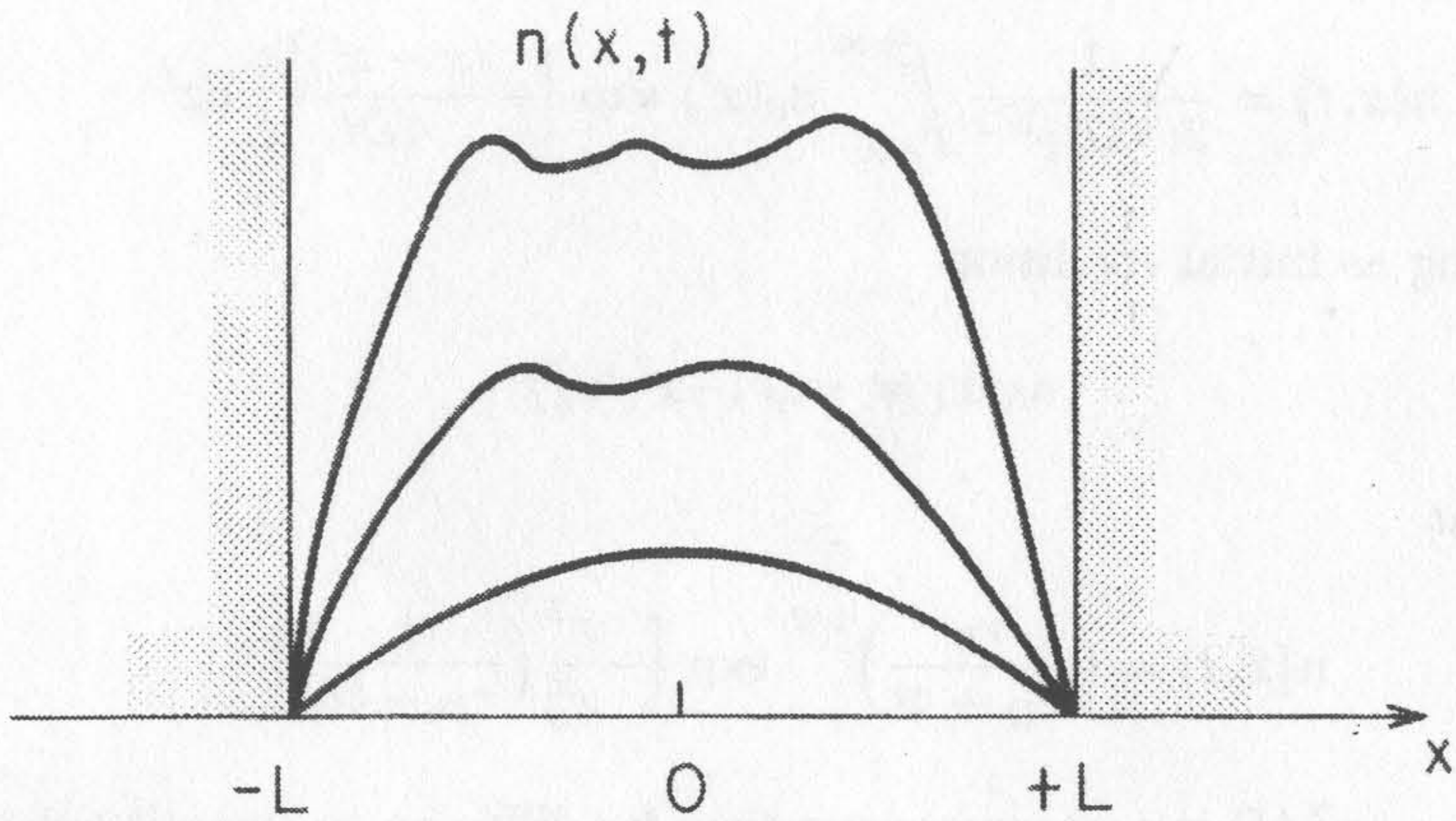

Fig. 4 *Geometry of the plasma slab for the solution of the diffusion equation considered in Problem 10.10.*

Therefore, the *decay time constant* for the mth mode is

$$\tau_m = \left[\frac{L}{\pi(m+1/2)}\right]^2 \frac{1}{D}$$

This result shows that the higher modes decay faster than the lower ones. How are the coefficients a_m determined in terms of $n_0(x)$?

10.11 Show that the solution of the diffusion equation in the case of cylindrical geometry (see Fig. 5),

$$\frac{d^2S(r)}{dr^2} + \frac{1}{r}\frac{dS(r)}{dr} + k^2S(r) = 0$$

can be written in terms of Bessel's functions $J_m(kr)$. Explain how k must be determined so that $n(\mathbf{r},t)$ satisfies the boundary condition $n = 0$ at $r = R_0$.

10.12 Verify that plane wave solutions to the diffusion equation

$$\frac{\partial n(\mathbf{r},t)}{\partial t} = D\nabla^2 n(\mathbf{r},t)$$

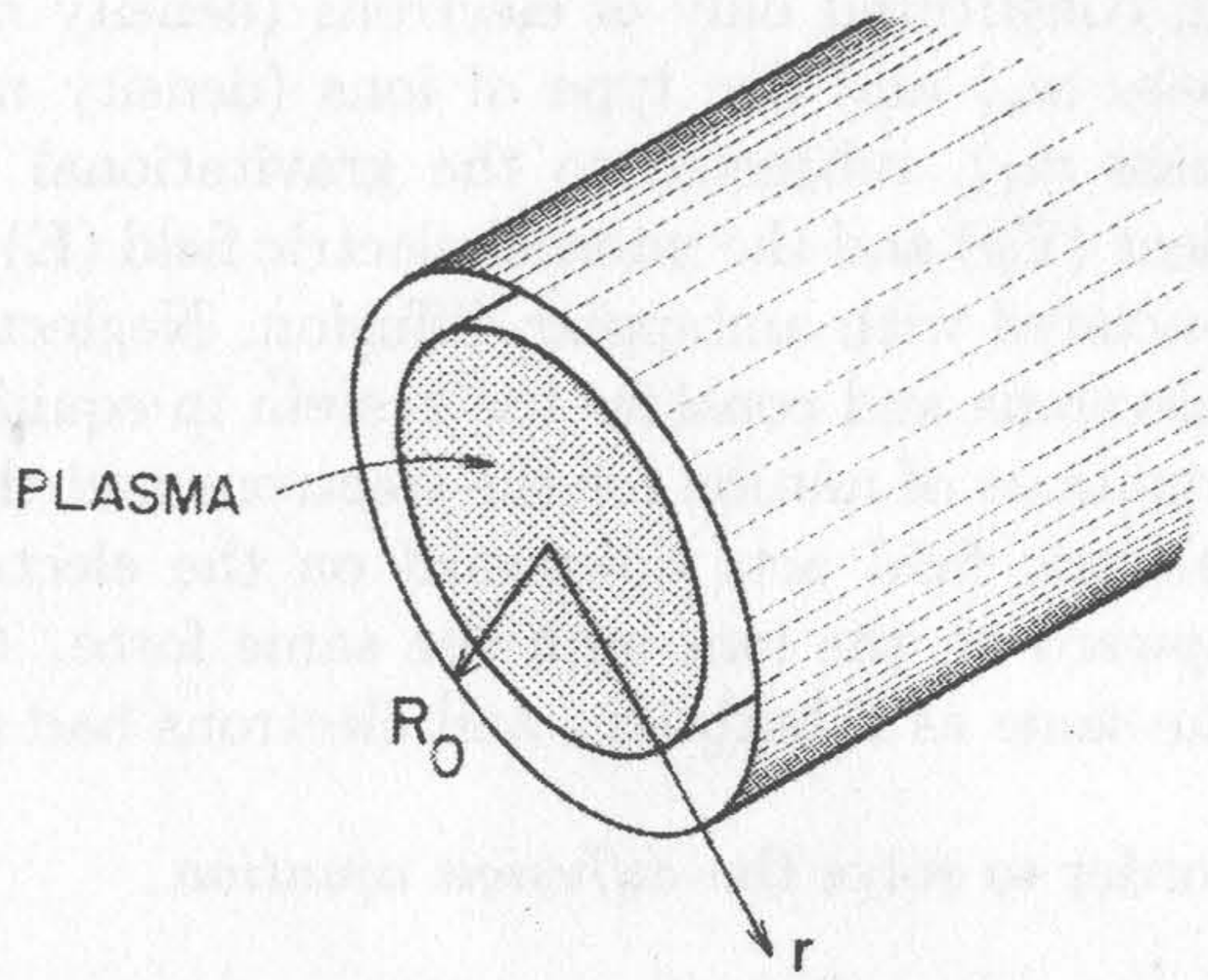

Fig. 5 *Cylindrical geometry of the plasma column for the solution of the diffusion equation considered in Problem 10.11.*

yields the following *dispersion relation* between k and ω,

$$k^2 D = i\omega$$

Then, show that for *free electron diffusion* we obtain

$$k^2 V_{se}^2 = i\omega\nu_{ce}$$

where $V_{se} = (k_B T_e/m_e)^{1/2} = (p_e/\rho_{me})^{1/2}$ is the isothermal speed of sound in the electron gas and k_B is Boltzmann's constant. Next, show that for *ambipolar diffusion* we obtain

$$k^2 V_{sp}^2 = i\omega\nu_{ci}$$

where

$$V_{sp} = \left[\frac{k_B(T_e + T_i)}{m_i}\right]^{1/2} = \left(\frac{p_e + p_i}{\rho_{mi}}\right)^{1/2}$$

is the isothermal plasma sound speed. Calculate the phase velocity and the damping factor for these waves and verify if they are longitudinal or transverse waves.

10.13 Consider a weakly ionized plasma immersed in a uniform magnetostatic field $\mathbf{B}_0$ oriented along the z axis of a Cartesian coordinate system.

(a) Show that the diffusion equation for the electrons (with $D\mathbf{u}_e/Dt = 0$) in the presence of the space charge electric field is given by

$$\mathbf{\Gamma}_e = -\nabla \cdot (\mathcal{D}_e n_e) + n_e \mathcal{M}_e \cdot \mathbf{E}$$

where

$$\mathcal{D}_e = \begin{pmatrix} D_{e\perp} & D_{eH} & 0 \\ -D_{eH} & D_{e\perp} & 0 \\ 0 & 0 & D_{e\parallel} \end{pmatrix}$$

with $D_{e\perp}$, D_{eH} and $D_{e\parallel}$ given by (9.5), (9.6) and (9.7), respectively, and where

$$\mathcal{M}_e = -\frac{e}{kT_e}\mathcal{D}_e$$

(b) Deduce the corresponding equation for the ions in the presence of the space charge electric field $\mathbf{E}$. Combine the equations for the electrons and for the ions in order to eliminate the space charge electric field. Then, assuming that the electron and ion fluxes are equal, $\mathbf{\Gamma}_e = \mathbf{\Gamma}_i$, and that their number densities are also equal, $n_e = n_i$, determine the ambipolar diffusion coefficient. Verify that it is not affected by the presence of the magnetostatic field.

10.14 Consider the following heat flow equation, derived in Problem 8.11, for a stationary electron gas immersed in a magnetic field,

$$\mathbf{q}_e + \frac{\Omega_{ce}}{\nu}(\mathbf{q}_e \times \mathbf{B}) = -K_0 \nabla T_e$$

Show that this equation can be written in the form

$$\mathbf{q}_e = -\mathcal{K} \cdot \nabla T_e$$

where $\mathcal{K}$ denotes the dyadic thermal conductivity coefficient, given by

$$\mathcal{K} = \begin{pmatrix} K_\perp & -K_H & 0 \\ k_H & K_\perp & 0 \\ 0 & 0 & K_0 \end{pmatrix}$$

where

$$K_\perp = \frac{\nu^2}{(\nu^2 + \Omega_{ce}^2)} K_0$$

$$K_H = \frac{\nu \Omega_{ce}}{(\nu^2 + \Omega_{ce}^2)} K_0$$

$$K_0 = \frac{5kp_e}{2m_e\nu}$$

11

SOME BASIC PLASMA PHENOMENA

1. ELECTRON PLASMA OSCILLATIONS

One of the fundamental properties of a plasma is its tendency to maintain electric charge neutrality on a macroscopic scale under equilibrium conditions. When this macroscopic charge neutrality is disturbed, such as to temporarily produce a significant imbalance of charge, large Coulomb forces come into play, which tend to restore the macroscopic charge neutrality. Since these Coulomb forces cannot be naturally sustained in the plasma, it breaks into high frequency electron plasma oscillations, which enable the plasma to maintain on the average its electrical neutrality.

As a simple example, consider a small spherical region inside a plasma and suppose that a perturbation in the form of an excess of negative charge is introduced in this small region. Because of spherical symmetry, the corresponding electric field is radial and points towards the center (see Fig. 1), forcing the electrons to move radially outward. After a small time interval, since the electrons gain kinetic energy in the course of their motion, more electrons leave the spherical plasma region (due to their inertia) than is necessary to resume the state of electrical neutrality. An excess of positive charge results therefore inside this region and the reversed (outward, now) electric field causes the electrons to move inward. This sequence of outward and inward electron movement in the spherical plasma region continues periodically, resulting in electron plasma oscillations. In this way the plasma maintains its macroscopic neutrality on the average, since the total charge inside the spherical region, averaged over one period of these oscillations, is zero. The frequency of these oscillation is usually very

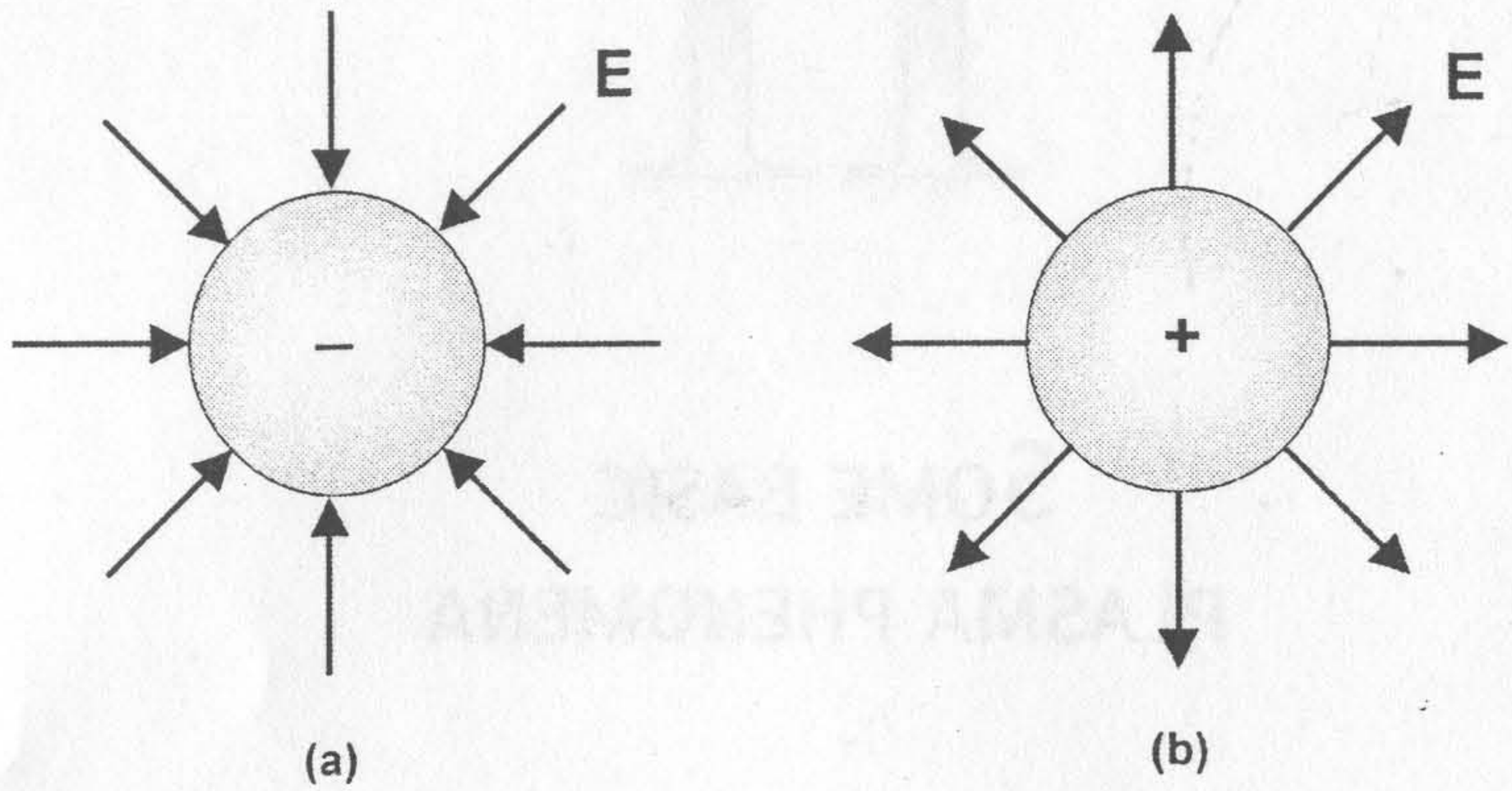

Fig. 1 *The radial electric field* **E** *produced by a spherical distribution of negative charge (a) forces the electrons to move radially outwards, whereas the field produced by a spherical distribution of positive charge (b) forces the electrons to move radially inwards.*

high and since the ions (in view of their much higher mass) are unable to follow the rapidity of the electron oscillations, their motion is often neglected.

To study the characteristics of the electron plasma oscillations we can use the cold plasma model, in which the particle thermal motion and the pressure gradient force are not taken into account. We shall neglect ion motion and assume a very small electron density perturbation such that

$$n_e(\mathbf{r}, t) = n_0 + n'_e(\mathbf{r}, t) \tag{1.1}$$

where n_0 is a constant number density and $|n'_e| \ll n_0$. Similarly, we assume that the electric field produced, $\mathbf{E}(\mathbf{r}, t)$, and the average electron velocity, $\mathbf{u}_e(\mathbf{r}, t)$, are first order perturbations, so that the linearized equations can be used. The linearized continuity and momentum equations become, respectively,

$$\frac{\partial n'_e(\mathbf{r}, t)}{\partial t} + n_0 \nabla \cdot \mathbf{u}_e(\mathbf{r}, t) = 0 \tag{1.2}$$

$$\frac{\partial \mathbf{u}_e(\mathbf{r}, t)}{\partial t} = -\frac{e}{m_e}\mathbf{E}(\mathbf{r}, t) \tag{1.3}$$

In the momentum equation we have assumed that the rate of momentum loss from the electron gas due to collisions is negligible. Considering singly charged ions, the charge density is given by

$$\rho(\mathbf{r},t) = -e[n_0 + n'_e(\mathbf{r},t)] + en_0 = -en'_e(\mathbf{r},t) \tag{1.4}$$

where the ion density was considered to be constant and uniform, and equal to n_0 (neglecting ion motion). Therefore,

$$\nabla \cdot \mathbf{E}(\mathbf{r},t) = \frac{\rho(\mathbf{r},t)}{\epsilon_0} = -\frac{e}{\epsilon_0} n'_e(\mathbf{r},t) \tag{1.5}$$

Eqs. (1.2), (1.3) and (1.5) constitute a complete set of equations to be solved for the variables $n'_e(\mathbf{r},t)$, $\mathbf{u}_e(\mathbf{r},t)$ and $\mathbf{E}(\mathbf{r},t)$. Taking the divergence of (1.3) and using (1.2) to substitute for $\nabla \cdot \mathbf{u}_e$, we obtain

$$\frac{\partial^2 n'_e(\mathbf{r},t)}{\partial t^2} - \frac{en_0}{m_e} \nabla \cdot \mathbf{E}(\mathbf{r},t) = 0 \tag{1.6}$$

Combining (1.5) and (1.6) to eliminate $\nabla \cdot \mathbf{E}$, yields

$$\frac{\partial^2 n'_e(\mathbf{r},t)}{\partial t^2} + \omega_{pe}^2 n'_e(\mathbf{r},t) = 0 \tag{1.7}$$

where

$$\omega_{pe} = \left(\frac{(n_0 e^2}{m_e \epsilon_0}\right)^{1/2} \tag{1.8}$$

is called the *electron plasma frequency.* Equation (1.7) shows that $n'_e(\mathbf{r},t)$ varies harmonically in time at the electron plasma frequency,

$$n'_e(\mathbf{r},t) = n'_e(\mathbf{r}) \exp\,(-i\omega_{pe} t) \tag{1.9}$$

In fact, all first-order perturbations have a harmonic time variation at the plasma frequency ω_{pe}. To justify this statement it is convenient to start with the assumption that all first order quantities vary harmonically in time, as $\exp\,(-i\omega t)$. Eqs. (1.2) and (1.3) become, in this case,

$$n'_e = -\frac{i}{\omega} n_0 \nabla \cdot \mathbf{u}_e \tag{1.10}$$

$$\mathbf{u}_e = -\frac{ie}{\omega m_e} \mathbf{E} \tag{1.11}$$

which can be combined into

$$n_e' = -\frac{n_0 e}{\omega^2 m_e} \nabla \cdot \mathbf{E} \tag{1.12}$$

Substituting this expression for n_e' into (1.5), yields

$$\left(1 - \frac{\omega_{pe}^2}{\omega^2}\right) \nabla \cdot \mathbf{E} = 0 \tag{1.13}$$

which shows that a nontrivial solution requires $\omega = \omega_{pe}$. Therefore, all the perturbations vary harmonically in time at the electron plasma frequency. Further, for all variables there is no change in phase from point to point, implying in the absence of wave propagation. The oscillations are therefore *stationary*. Also, (1.11) shows that the electron velocity is in the same direction as the electric field, so that these oscillations are *longitudinal*.

The electron plasma oscillations are also *electrostatic* in character. In order to show this aspect of the oscillations consider Maxwell curl equations with a harmonic time variation,

$$\nabla \times \mathbf{E} = i\omega \mathbf{B} \tag{1.14}$$

$$\nabla \times \mathbf{B} = \mu_0 (\mathbf{J} - i\omega \epsilon_0 \mathbf{E}) \tag{1.15}$$

The electric current density is given by

$$\mathbf{J} = -e n_0 \mathbf{u}_e = \frac{i n_0 e^2}{\omega m_e} \mathbf{E} \tag{1.16}$$

where we have used (1.11) for $\mathbf{u}_e$. Therefore,

$$\nabla \times \mathbf{B} = -i\omega \mu_0 \epsilon_0 \epsilon_r \mathbf{E} \tag{1.17}$$

where we have defined a relative permittivity by

$$\epsilon_r = 1 - \frac{\omega_{pe}^2}{\omega^2} \tag{1.18}$$

For the electron plasma oscillations we have $\omega = \omega_{pe}$, so that $\epsilon_r = 0$, and (1.17) reduces to

$$\nabla \times \mathbf{B} = 0 \tag{1.19}$$

Since the curl of the gradient of any scalar function vanishes identically,

we may write

$$\mathbf{B} = \nabla\psi \tag{1.20}$$

where ψ is a magnetic scalar potential. Substituting (1.20) into (1.14) and taking the divergence of both sides we obtain the Laplace equation

$$\nabla \cdot (\nabla\psi) = \nabla^2\psi = 0 \tag{1.21}$$

since the divergence of the curl of any vector function vanishes identically. The only solution of this equation, which is not singular and finite at infinity, is $\psi = constant$, so that $\mathbf{B} = 0$. Hence, there is no magnetic field associated with these space charge oscillations.

In summary, the electron plasma oscillations are *stationary*, *longitudinal* and *electrostatic*. They are also referred to as *Langmuir oscillations*. When the effect of the pressure gradient force is included in the equation of motion (1.3), complemented by an adiabatic energy equation, these oscillations become propagating disturbances, commonly known as *space charge waves* or *Langmuir waves*. Characteristic values of the electron plasma frequency for various laboratory and cosmic plasmas are given in Fig. 2 of Chapter 1.

2. THE DEBYE SHIELDING PROBLEM

To examine the mechanism by which the plasma strives to shield its interior from a disturbing electric field, consider a plasma whose equilibrium state is perturbed by an electric field due to an external charged particle. For that matter, this electric field may also be considered to be due to one of the charged particles inside the plasma, isolated for observation. For definiteness, we assume this *test particle* to have a positive charge $+Q$, and choose a spherical coordinate system whose origin coincides with the position of the test particle. We are interested in determining the electrostatic potential $\phi(\mathbf{r})$ that is established near the test charge Q, due to the *combined* effects of the test charge and the distribution of charged particles surrounding it. Since the positive test charge Q attracts the negatively charged particles and repels the positively charged ones, the number densities of the electrons $n_e(\mathbf{r})$ and of the ions $n_i(\mathbf{r})$ will be slightly different near the origin (test particle), whereas at large distances from the origin the electrostatic potential vanishes, so that $n_e(\infty) = n_i(\infty) = n_0$. Since this is a steady state problem under the action of a conservative electric field, we have

$$\mathbf{E}(\mathbf{r}) = -\nabla\phi(\mathbf{r}) \tag{2.1}$$

and from (7.5.16) it follows that

$$n_e(\mathbf{r}) = n_0 \exp\left[\frac{e\phi(\mathbf{r})}{kT}\right] \tag{2.2}$$

$$n_i(\mathbf{r}) = n_0 \exp\left[-\frac{e\phi(\mathbf{r})}{kT}\right] \tag{2.3}$$

where we have assumed that the electrons and ions (of charge e) have the same temperature T.

The total electric charge density $\rho(\mathbf{r})$, including the test charge Q, can be expressed as

$$\rho(\mathbf{r}) = -e[n_e(\mathbf{r}) - n_i(\mathbf{r})] + Q\,\delta(\mathbf{r}) \tag{2.4}$$

where $\delta(\mathbf{r})$ denotes the *Dirac delta function.* Using (2.2) and (2.3),

$$\rho(\mathbf{r}) = -en_0\left\{\exp\left[\frac{e\phi(\mathbf{r})}{kT}\right] - \exp\left[-\frac{e\phi(\mathbf{r})}{kT}\right]\right\} + Q\,\delta(\mathbf{r}) \tag{2.5}$$

Substituting (2.1) and (2.5) into the following Maxwell equation

$$\nabla \cdot \mathbf{E}(\mathbf{r}) = \frac{\rho(\mathbf{r})}{\epsilon_0} \tag{2.6}$$

gives the differential equation

$$\nabla^2\phi(\mathbf{r}) - \frac{en_0}{\epsilon_0}\left\{\exp\left[\frac{e\phi(\mathbf{r})}{kT}\right] - \exp\left[-\frac{e\phi(\mathbf{r})}{kT}\right]\right\} = -\frac{Q}{\epsilon_0}\,\delta(\mathbf{r}) \tag{2.7}$$

which allows the evaluation of the electrostatic potential $\phi(\mathbf{r})$.

In order to proceed analytically, we assume now that the perturbing electrostatic potential is weak so that the electrostatic potential energy is much less than the mean thermal energy, that is,

$$e\phi(\mathbf{r}) \ll kT \tag{2.8}$$

Under this condition we can use the approximation (making a series expansion)

$$\exp\left[\pm\frac{e\phi(\mathbf{r})}{kT}\right] \simeq 1 \pm \frac{e\phi(\mathbf{r})}{kT} \tag{2.9}$$

Therefore, (2.7) simplifies to

$$\nabla^2\phi(\mathbf{r}) - \frac{2}{\lambda_D^2}\phi(\mathbf{r}) = -\frac{Q}{\epsilon_0}\,\delta(\mathbf{r}) \tag{2.10}$$

where λ_D denotes the Debye length

$$\lambda_D = \left(\frac{\epsilon_0 kT}{n_0 e^2}\right)^{1/2} = \frac{1}{\omega_{pe}}\left(\frac{kT_e}{m_e}\right)^{1/2} \tag{2.11}$$

Since the problem has spherical symmetry, the electrostatic potential depends only on the radial distance r measured from the position of the test particle, being independent of the spatial orientation of $\mathbf{r}$. Thus, using spherical coordinates, (2.10) can be written (for $r \neq 0$) as

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d}{dr}\phi(\mathbf{r})\right] - \frac{2}{\lambda_D^2}\phi(\mathbf{r}) = 0 \qquad (r \neq 0) \tag{2.12}$$

In order to solve this equation we note initially that for an isolated particle of charge $+Q$, in free space, the electric field is directed radially outward and is given by

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\widehat{\mathbf{r}} \tag{2.13}$$

so that the electrostatic Coulomb potential $\phi_c(r)$ due to this isolated charged particle in free space is

$$\phi_c(r) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r} \tag{2.14}$$

In the very close proximity of the test particle the electrostatic potential should be the same as that for an isolated particle in free space. Hence, it is appropriate to seek the solution of (2.12) in the form

$$\phi(r) = \phi_c(r)F(r) = \frac{Q}{4\pi\epsilon_0}\frac{F(r)}{r} \tag{2.15}$$

where the function $F(r)$ must be such that $F(r) \to 1$ when $r \to 0$. Furthermore, the electrostatic potential $\phi(r)$ is required to vanish at infinity, that is, $\phi \to 0$ when $r \to \infty$. Substituting (2.15) into (2.12) yields the following differential equation for $F(r)$

$$\frac{d^2F(r)}{dr^2} = \frac{2}{\lambda_D^2}F(r) \tag{2.16}$$

This simple differential equation for $F(r)$ has the solution

$$F(r) = A\exp\left(\frac{\sqrt{2}\,r}{\lambda_D}\right) + B\exp\left(-\frac{\sqrt{2}\,r}{\lambda_D}\right) \tag{2.17}$$

The condition that $\phi(r)$ vanishes for large values of r requires $A = 0$. Also, the condition that $F(r)$ tends to one when r tends to zero requires $B = 1$. Therefore, the solution of (2.12) is

$$\phi(r) = \phi_c(r) \exp\left(-\frac{\sqrt{2}\, r}{\lambda_D}\right) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r} \exp\left(-\frac{\sqrt{2}\, r}{\lambda_D}\right) \tag{2.18}$$

This result is commonly known as the *Debye potential*, since this nonrigorous derivation was first presented by Debye and Huckel in their theory of electrolytes. It shows that $\phi(r)$ becomes much less than the ordinary Coulomb potential once r exceeds the distance λ_D, called the *Debye length* (see Fig. 2). Hence, we can say in a crude way that a charged particle in a plasma interacts effectively only with particles situated at distances less than one Debye length away, and it has a negligible influence on particles lying at distances greater than one Debye length.

The charge Q of the test particle is neutralized by the charge distribution surrounding the test particle. From (2.5) and (2.9) we obtain for the charge density

$$\rho(\mathbf{r}) = -2\frac{n_0 e^2 \phi(\mathbf{r})}{kT} + Q\, \delta(\mathbf{r}) \tag{2.19}$$

Substituting $\phi(\mathbf{r})$ by the Debye potential (2.18), we obtain

$$\rho(\mathbf{r}) = -\frac{Q}{2\pi r \lambda_D^2} \exp\left(-\frac{\sqrt{2}\, r}{\lambda_D}\right) + Q\, \delta(\mathbf{r}) \tag{2.20}$$

To obtain the total charge q_t we integrate (2.20) over all space,

$$q_t = \int\int\int \rho(\mathbf{r})\, d^3r = -\frac{Q}{2\pi\lambda_D^2}\int_0^\infty \frac{1}{r} \exp\left(-\frac{\sqrt{2}\, r}{\lambda_D}\right) 4\pi r^2\, dr + \\ + Q\int\int\int \delta(\mathbf{r})\, d^3r \tag{2.21}$$

Since the first integral gives $-Q$, whereas the second one is equal to $+Q$, we find $q_t = 0$. The principal contribution to the first integral in (2.21) comes from the plasma particles lying in the very close neighborhood of the test particle, since the integrand falls off exponentially with increasing values of r. Thus, the neutralization of the test particle takes place effectively on account of the charged particles inside the Debye sphere. From (2.2) and (2.3) we see that in the neighborhood of the test particle the electron number density is larger than the ion number density, on account of the fact that the positive test particle attracts the electrons and repels the ions. Therefore, in the close proximity of the test particle there is an imbalance of charge and, consequently, an electric field. We have seen

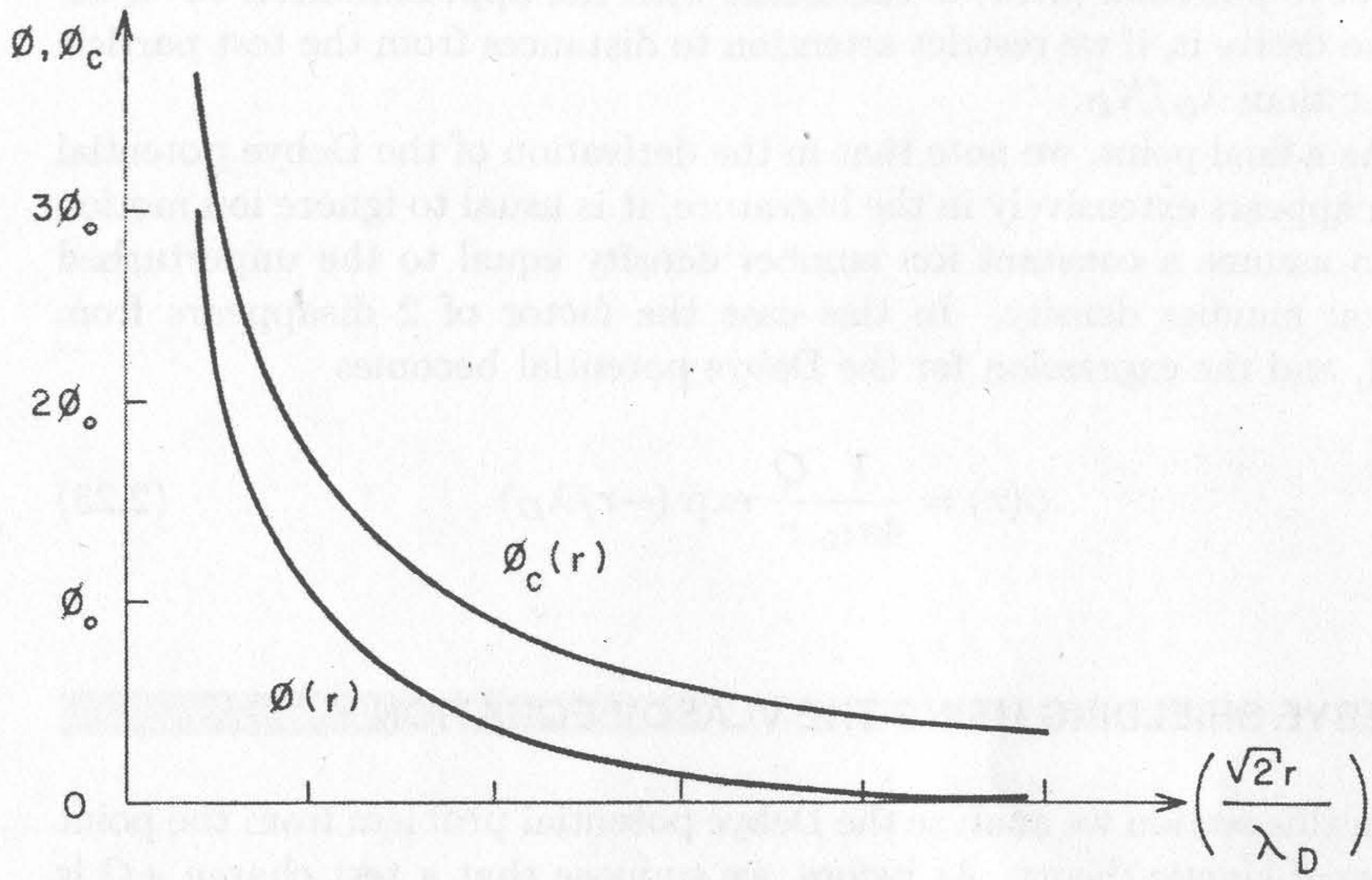

Fig. 2 *Electrostatic Coulomb potential $\phi_c(r)$ and Debye potential $\phi(r)$ as a function of distance r from the test charge Q. Here we have $\phi_0 = \sqrt{2}\, Q/(4\pi\epsilon_0\lambda_D)$.*

that the shielding of this electric field is effectively completed over a distance of the order of λ_D. Thus, for macroscopic neutrality, it is necessary that the plasma typical characteristic dimension L be much greater than λ_D. This quantitative *criterion* for the definition of a plasma has been previously discussed in Chapter 1.

An important point to be noted in the result (2.18) is that, for $r \to 0$, the Debye potential becomes very large and the assumption $e\phi(r) \ll kT$ is unlikely to be fulfilled. To verify the validity of this approximation, and consequently of (2.18), note that using (2.18) with $Q = e$ we have

$$\frac{e\phi}{kT} = \frac{e^2 \exp\left(-\sqrt{2}\, r/\lambda_D\right)}{4\pi\epsilon_0 rkT} = \frac{\lambda_D}{3N_D}\frac{\exp\left(-\sqrt{2}\, r/\lambda_D\right)}{r} \tag{2.22}$$

where N_D is the number of electrons inside a Debye sphere. Since N_D is very large for virtually all plasmas, it is evident that the ratio given in (2.22) is much less than one, except when r is less than λ_D/N_D. Therefore,

the Debye potential (2.18) is consistent with the approximation $e\phi \ll kT$ used to derive it, if we restrict attention to distances from the test particle greater than λ_D/N_D.

As a final point, we note that in the derivation of the Debye potential which appears extensively in the literature, it is usual to ignore ion motion and to assume a constant ion number density equal to the unperturbed electron number density. In this case the factor of 2 disappears from (2.10), and the expression for the Debye potential becomes

$$\phi(r) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}\exp\left(-r/\lambda_D\right) \tag{2.23}$$

3. DEBYE SHIELDING USING THE VLASOV EQUATION

In this section we analyze the Debye potential problem from the point of view of kinetic theory. As before, we suppose that a test charge $+Q$ is placed at the origin of a spherical coordinate system inside the plasma. In order to determine the steady state electron and ion distribution functions, f_e and f_i, and the electrostatic potential $\phi(\mathbf{r})$ near the test charge, let us consider the steady state Vlasov equations for the electrons and the ions, with only the electric field $\mathbf{E}(\mathbf{r}) = -\nabla\phi(\mathbf{r})$ in the Lorentz force term,

$$\mathbf{v}\cdot\nabla f_e + \frac{e}{m_e}(\nabla\phi)\cdot\nabla_v f_e = 0 \tag{3.1}$$

$$\mathbf{v}\cdot\nabla f_i - \frac{e}{m_i}(\nabla\phi)\cdot\nabla_v f_i = 0 \tag{3.2}$$

Since

$$n_\alpha(\mathbf{r}) = \int_v f_\alpha(\mathbf{r}, v)\, d^3v \tag{3.3}$$

the total charge density (including the test particle) can be expressed as

$$\rho(\mathbf{r}) = -e\int_v (f_e - f_i)\, d^3v + Q\,\delta(\mathbf{r}) \tag{3.4}$$

and Poisson equation for this case becomes

$$\nabla^2\phi - \frac{e}{\epsilon_0}\int_v (f_e - f_i)\, d^3v = -\frac{Q}{\epsilon_0}\,\delta(\mathbf{r}) \tag{3.5}$$

Equations (3.1), (3.2) and (3.5) constitute three equations to be solved simultaneously to determine f_e, f_i and ϕ.

The solution of the Vlasov equations (3.1) and (3.2) can be expressed in terms of the Maxwellian distribution function and the Boltzmann factor (see section 5, Chapter 7), as

$$f_\alpha(\mathbf{r}, v) = f_{0\alpha}(v) \exp\left[-\frac{q_\alpha \phi(\mathbf{r})}{kT}\right] \tag{3.6}$$

When the electrostatic potential vanishes, this distribution function goes into the Maxwellian distribution $f_{0\alpha}(v)$, with zero drift velocity. Substituting (3.6), with $\alpha = e,\ i$, into (3.5), yields

$$\nabla^2\phi - \frac{e}{\epsilon_0}\Big[\exp\left(\frac{e\phi}{kT}\right)\int_v f_{0e}\, d^3v - \\ - \exp\left(-\frac{e\phi}{kT}\right)\int_v f_{0i}\, d^3v\Big] = -\frac{Q}{\epsilon_0}\,\delta(\mathbf{r}) \tag{3.7}$$

Denoting the electron and ion number densities under equilibrium conditions (when ϕ vanishes) by n_0,

$$n_0 = \int_v f_{0\alpha}(v)\, d^3v \qquad \alpha = e, i \tag{3.8}$$

equation (3.7) becomes

$$\nabla^2\phi - \frac{en_0}{\epsilon_0}\Big[\exp\left(\frac{e\phi}{kT}\right) - \exp\left(-\frac{e\phi}{kT}\right)\Big] = -\frac{Q}{\epsilon_0}\,\delta(\mathbf{r}) \tag{3.9}$$

This equation is identical to (2.7), yielding the same result as before for the Debye potential.

4. PLASMA SHEATH

When a material body is immersed in a plasma, the body acquires a net negative charge and therefore a negative potential with respect to the plasma potential. In the region near the wall of the body there is a boundary layer, known as the *plasma sheath*, in which the electron and the ion number densities are different. Inside the plasma sheath the potential increases monotonically from a negative value on the wall to the value corresponding to the unperturbed plasma. The thickness of the plasma

sheath, where departures from macroscopic electric neutrality occur, is found to be of the order of a Debye length.

A satisfactory mathematical treatment of this phenomenon is quite involved and cannot be presented here. However, the basic physical factors responsible for the formation of the plasma sheath are fairly simple to understand. In this section we will set up the problem mathematically and obtain some approximate results. The problem is strongly dependent on the particular geometry under consideration. For simplicity, we shall assume that the wall bounding the plasma is an infinite plane surface at $x = 0$, with the plasma in the region $x > 0$, and that there are no variations with respect to the coordinates y and z.

4.1 Physical Mechanism

We begin with a descriptive account of the physical mechanism responsible for the formation of the plasma sheath. The charged particles in the plasma that strike the wall in virtue of their random thermal motions are for the most part lost to the plasma. The ions generally recombine at the wall and return to the plasma as neutral particles, whereas the electrons may either recombine there or they may enter the conduction band if the surface is a metal. We have seen in section 4, of Chapter 7, that the *random* particle flux, that is, the number of particles that hit the surface per unit area and per unit time, from one side only, for the case of an isotropic velocity distribution function, is given by [see Eqs. (7.4.35) and (7.4.18)]

$$\Gamma_\alpha = n_\alpha < v >_\alpha /4 \tag{4.1}$$

where $< v >_\alpha$ is the average particle speed for the α species. For the Maxwell-Boltzmann velocity distribution function we have found that [see Eq. (7.4.20)]

$$< v >_\alpha = \left(\frac{8}{\pi}\right)^{1/2} \left(\frac{kT_\alpha}{m_\alpha}\right)^{1/2} \tag{4.2}$$

so that a typical value for the random particle flux in this case is

$$\Gamma_\alpha = n_\alpha \left(\frac{kT_\alpha}{2\pi m_\alpha}\right)^{1/2} \tag{4.3}$$

It is evident from this result that if initially the electron and the ion number densities are equal, then the random particle flux for the electrons (Γ_e) greatly exceeds that for the ions (Γ_i), since in general $(T_e/m_e)^{1/2}$ is much larger than $(T_i/m_i)^{1/2}$. For the least heavy ion, hydrogen, for

example, $m_i/m_e = 1836$. Therefore, the wall in contact with the plasma rapidly accumulates a negative charge, since initially more electrons reach the wall than positive ions. This negative potential repels the electrons and attracts the ions so that the electron flux diminishes and the ion flux increases. Eventually, the negative potential at the wall becomes large enough in magnitude to equalize the rate at which electrons and ions hit the surface. At this *floating negative potential* the wall and the plasma reach a dynamical equilibrium such that the net current at the wall is zero.

4.2 Electric Potential on the Wall

To estimate the value of the potential on the wall after the plasma sheath has been established, consider a steady state situation and let the electric potential $\phi(x)$ at the wall ($x = 0$) be given by

$$\phi(0) = \phi_w \tag{4.4}$$

Let us choose the reference potential inside the plasma, at a very large distance from the wall, equal to zero,

$$\phi(\infty) = 0 \tag{4.5}$$

The electrons and the ions are assumed to be in thermodynamic equilibrium at the same temperature T, under the action of the conservative electric field associated with the negative potential on the wall. At $x \to \infty$ the plasma is unperturbed and the electron and ion number densities are each equal to n_0. According to the results of section 5, Chapter 7, the electron and ion number densities can be expressed as

$$n_e(\mathbf{r}) = n_0 \exp\left[\frac{e\phi(\mathbf{r})}{kT}\right] \tag{4.6}$$

$$n_i(\mathbf{r}) = n_0 \exp\left[-\frac{e\phi(\mathbf{r})}{kT}\right] \tag{4.7}$$

It is important to note at this point that (4.6) and (4.7) do not take into account the particle drift velocity towards the wall. Since the electrons and the ions impinging on the wall surface are for the most part lost to the plasma, there must be a steady flux of both species towards the wall to replenish this charged particle loss. Despite this inadequacy, (4.6)

and (4.7) will still be used to obtain an approximate expression for the potential on the wall. Afterwards, in order to study the inner structure of the plasma sheath, we will take into account the particle drift in an approximate manner using the hydrodynamic equations.

One of the boundary conditions of the problem is that under equilibrium conditions there must be no charge build up at the wall ($x = 0$), so that

$$J_e(0) = J_i(0) \tag{4.8}$$

Using (4.3), (4.6) and (4.7), considering singly charged ions,

$$\left(\frac{1}{m_e}\right)^{1/2} \exp\left(\frac{e\phi_w}{kT}\right) = \left(\frac{1}{m_i}\right)^{1/2} \exp\left(-\frac{e\phi_w}{kT}\right) \tag{4.9}$$

which may be written as

$$\exp\left(-\frac{2e\phi_w}{kT}\right) = \left(\frac{m_i}{m_e}\right)^{1/2} \tag{4.10}$$

Taking the natural logarithm of both sides, and solving for the wall potential, we obtain

$$\phi_w = -\left(\frac{kT}{4e}\right) \ln\left(\frac{m_i}{m_e}\right) \tag{4.11}$$

Other more accurate methods of calculating the wall potential yield results which, for $T_e = T_i$, agree qualitatively with the one given in (4.11), despite the inadequacy of (4.6) and (4.7) which neglect the particle drift velocity towards the wall.

Note from (4.11) that the magnitude of the potential energy near the wall $|e\phi_w|$ is of the same order as the average thermal energy kT of the particles in the plasma, since

$$\frac{|e\phi_w|}{kT} = \frac{1}{4} \ln\left(\frac{m_i}{m_e}\right) \tag{4.12}$$

For a hydrogen ion, for example, $|e\phi_w|/(kT)$ is approximately equal to 2, whereas for heavier ions it may be close to 3.

4.3 Inner Structure of the Plasma Sheath

To investigate the inner structure of the plasma sheath consider the equations of conservation of particles and momentum for the electrons

and ions, under steady state conditions, with spatial dependence only on the x direction. The equation of conservation of particles becomes (with $\alpha = e, i$)

$$\frac{d(n_\alpha u_\alpha)}{dx} = n_\alpha \frac{du_\alpha}{dx} + u_\alpha \frac{dn_\alpha}{dx} = 0 \tag{4.13}$$

In the momentum conservation equation we neglect viscosity effects and approximate the kinetic pressure dyad by a scalar pressure. The ideal gas equation of state, $p_\alpha = n_\alpha k T_\alpha$, can be used to introduce the temperature, which is assumed to be constant. Collisions are neglected, since the thickness of the plasma sheath is much less than the mean free path for the plasma particles. With these assumptions and in the absence of a magnetic field, the equation of motion becomes [taking $\mathbf{E}(\mathbf{r}) = -\nabla\phi(\mathbf{r})$, and $D/Dt = \partial/\partial t + \mathbf{u}_\alpha \cdot \nabla = u_\alpha d/dx$]

$$m_\alpha u_\alpha \frac{du_\alpha}{dx} = -\frac{kT_\alpha}{n_\alpha}\frac{dn_\alpha}{dx} - q_\alpha \frac{d\phi}{dx} \tag{4.14}$$

In order to simplify the analysis we shall make two approximations. From (4.13), we can write

$$\frac{dn_\alpha}{dx} = -\frac{n_\alpha}{u_\alpha}\frac{du_\alpha}{dx} \tag{4.15}$$

and the ratio of the magnitude of the term in the left-hand side of (4.14) to the first term in the right-hand side can be expressed as

$$\frac{|m_\alpha u_\alpha (du_\alpha/dx)|}{|(kT_\alpha/n_\alpha)(dn_\alpha/dx)|} = \frac{m_\alpha u_\alpha^2}{kT_\alpha} \tag{4.16}$$

The two approximations consist in neglecting the left-hand side term of (4.14) for the electrons, whereas for the ions we neglect the first term in the right-hand side of (4.14). Explicitly, we take for the electrons (neglecting electron inertia)

$$\frac{kT_e}{n_e}\frac{dn_e}{dx} - e\frac{d\phi}{dx} = 0 \tag{4.17}$$

and for the ions (assuming cold ions)

$$m_i u_i \frac{du_i}{dx} + e\frac{d\phi}{dx} = 0 \tag{4.18}$$

These two approximations are justified only if the thermal energy of the *electrons* is much *larger* than their kinetic energy, and if the thermal energy

of the *ions* is much *smaller* than their kinetic energy. Thus, we require from (4.15) that

$$m_e u_e^2 \ll kT \ll m_i u_i^2 \tag{4.19}$$

in order to justify the approximations in (4.17) and (4.18). We shall assume that the condition (4.19) is satisfied in the plasma sheath but it remains to be justified later in this section.

If we integrate (4.17), we obtain

$$e\phi(x) = kT \ln n_e(x) + (\text{constant}) \tag{4.20}$$

and using the condition that $n_e = n_0$ when $\phi = 0$, we find

$$n_e(x) = n_0 \exp\left[\frac{e\phi(x)}{kT}\right] \tag{4.21}$$

This result is identical to (4.6), which is not surprising, since the condition $m_e u_e^2 \ll kT$ implies in neglecting the electron inertia ($m_e = 0$) and consequently their kinetic energy.

For the ions, we first integrate (4.13) to find

$$n_i(x)\ u_i(x) = C_1 \tag{4.22}$$

and then integrate (4.18) to obtain

$$e\phi(x) + \tfrac{1}{2} m_i u_i^2(x) = C_2 \tag{4.23}$$

where C_1 and C_2 are constants. The boundary conditions require that at $x \to \infty$ we must have $\phi(\infty) = 0$, $n_i(\infty) = n_0$ and $u_i(\infty) = u_{0i}$. Thus,

$$C_i = n_0\ u_{0i}; \qquad C_2 = \tfrac{1}{2} m_i u_{0i}^2 \tag{4.24}$$

and using these results in (4.22) and (4.23), we find

$$n_i(x)\ u_i(x) = n_0\ u_{0i} \tag{4.25}$$

$$e\phi(x) + \tfrac{1}{2} m_i u_i^2(x) = \tfrac{1}{2} m_i u_{0i}^2 \tag{4.26}$$

These two equations can be combined to eliminate $u_i(x)$ and solved for $n_i(x)$, giving

$$n_i(x) = n_0 \left[1 - \frac{2e\phi(x)}{m_i u_{0i}^2}\right]^{-1/2} \tag{4.27}$$

This expression for $n_i(x)$ is substantially different from the one given in (4.7) and this difference is due to the importance of the ion drift velocity. We now find that, since $\phi(x) < 0$ in the sheath, $n_i(x)$ decreases slowly towards the wall rather than increasing as predicted by (4.7). Physically, this behavior is due to the fact that the negative potential on the wall causes $u_i(x)$ to increase as the ions approach the wall and since the ion flux $n_i(x)\ u_i(x)$ must stay constant, in virtue of (4.25), it turns out that $n_i(x)$ must decrease according to (4.27). This behavior is illustrated schematically in Fig. 3.

To obtain the differential equation satisfied by the electrostatic potential $\phi(x)$, we substitute (4.21) and (4.27) into Poisson equation

$$\nabla^2\phi = \frac{e}{\epsilon_0}(n_e - n_i) \tag{4.28}$$

to obtain

$$\frac{d^2\phi}{dx^2} = \frac{n_0 e}{\epsilon_0}\Big[\ exp\left(\frac{e\phi}{kT}\right) - \left(1 - \frac{2e\phi}{m_i u_{0i}^2}\right)^{-1/2}\Big] \tag{4.29}$$

In this equation, the drift velocity u_{0i}, far away from the wall, still needs to be determined. This equation is nonlinear and in order to facilitate its analytical solution we need to make one more approximation. Since we have seen that $|e\phi|$ ranges from zero in the plasma to a value of the order of kT on the wall and since we have also assumed that $m_i u_{0i}^2$ is larger than kT, we will restrict our attention to the region near the plasma edge of the sheath and assume further that $|e\phi|$ is small compared to both kT and $m_i u_{0i}^2$. Thus, in the region near the edge of the sheath adjacent to the plasma, we can expand the terms in the right-hand side of (4.29), for $e\phi/(kT) \ll 1$ and $e\phi/(m_i u_{0i}^2) \ll 1$, as

$$\exp\left(\frac{e\phi}{kT}\right) \simeq 1 + \frac{e\phi}{kT} \tag{4.30}$$

$$\left(1 - \frac{2e\phi}{m_i u_{0i}^2}\right)^{-1/2} \simeq 1 + \frac{e\phi}{m_i u_{0i}^2} \tag{4.31}$$

with the result that (4.29) reduces to

$$\frac{d^2\phi}{dx^2} = \frac{\phi}{X^2} \tag{4.32}$$

where

$$X^2 = \lambda_D^2\left(1 - \frac{kT}{m_i u_{0i}^2}\right)^{-1} \tag{4.33}$$

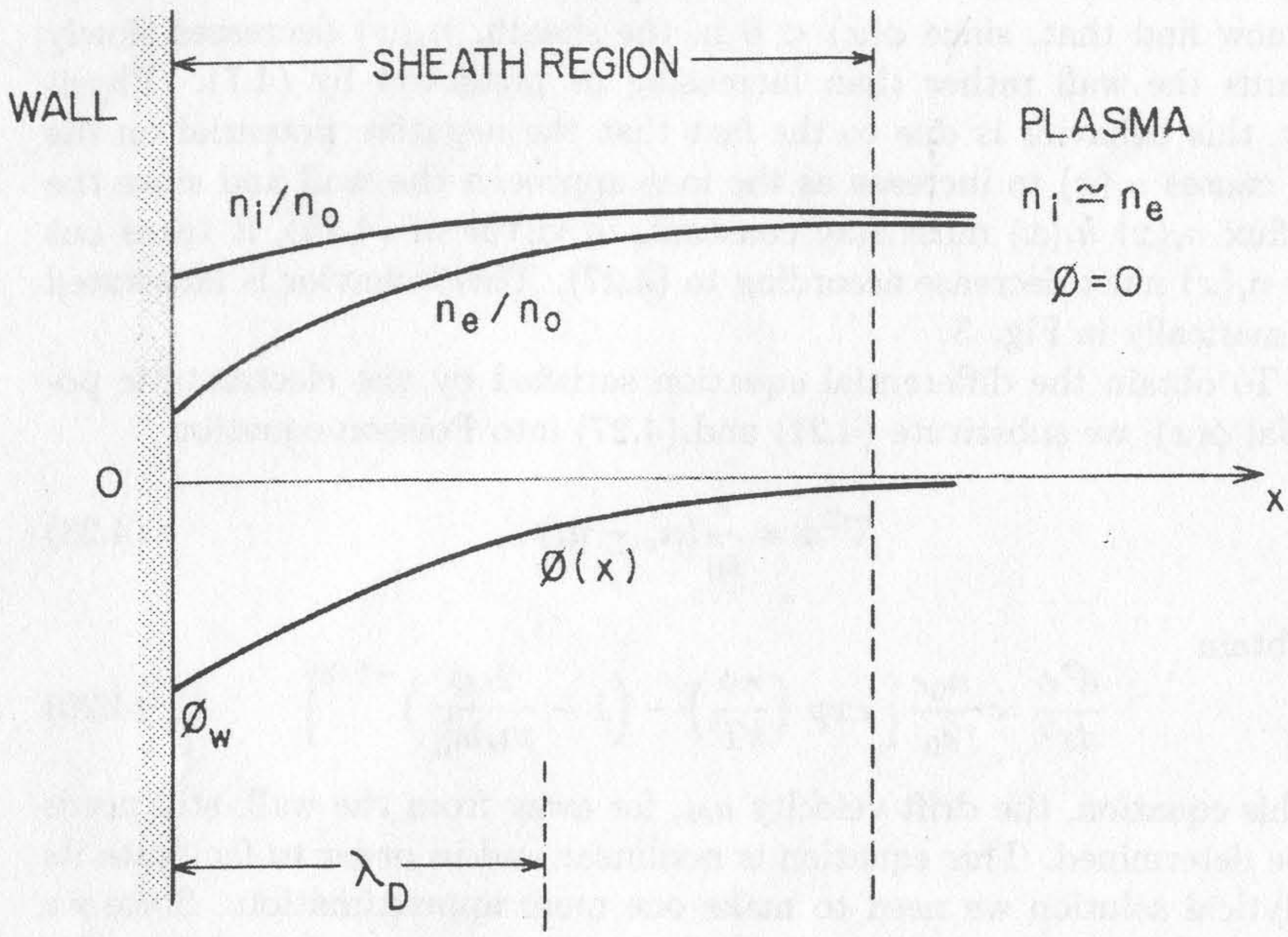

Fig. 3 *Diagram showing the variation of the electrostatic potential $\phi(x)$ and the number densities $n_e(x)$ and $n_i(x)$ inside the plasma sheath near an infinite plane wall.*

The solution of (4.32), with the boundary condition $\phi(\infty) = 0$, is

$$\phi(x) = A \exp(-x/X) \tag{4.34}$$

where A is a constant. Since we have assumed that $kT \ll m_i u_{0i}^2$, it follows from (4.33) that X is real and approximately equal to λ_D.

Therefore, we find that the *absolute* value of $\phi(x)$ decreases exponentially (note that A must be negative, so that $\phi(x)$ actually increases) as we move from inside the sheath into the plasma and goes asymptotically to zero at very large distances from the wall. Since $X \simeq \lambda_D$, this variation effectively takes place within a distance of the order of a Debye length. This solution for $\phi(x)$ is strictly valid only near the plasma edge of the sheath, but if it is continued to apply throughout the plasma sheath, we can impose the boundary condition on the wall, that is, $\phi(0) = \phi_w$, which requires that $A = \phi_w$.

If kT were greater than $m_i u_{0i}^2$, than X would be imaginary and the electric potential would be an oscillating function of distance near the wall. The condition

$$kT < m_i u_{0i}^2 \tag{4.35}$$

must therefore be satisfied for the formation of a plasma sheath. It is known as the *Bohm criterion.*

It is not a trivial matter to determine the potential on the wall using the hydrodynamic equations. All the approximate methods that have been suggested give results which, for $T_e = T_i$, agree reasonably well with the approximate value for ϕ_w given in (4.11). Furthermore, there is no consistent way to determine the ion drift velocity u_{0i} at $x = \infty$, but an approximate estimate can be obtained as follows. Since the ion flux must be constant, from (4.25) we can equate the ion flux $n_0 u_{0i}$ at $x = \infty$ to its value $\Gamma_i(0)$ at the wall. Using (4.3) with n_i as given by (4.7) evaluated at the wall, we find

$$u_{0i} = \left(\frac{kT}{2\pi m_i}\right)^{1/2} \exp\left(-\frac{e\phi_w}{kT}\right) \tag{4.36}$$

Similarly, for the electrons we can equate the electron flux $n_0 u_{0e}$ at $x = \infty$ to its value $\Gamma_e(0)$ at the wall. From (4.3), with n_e given by (4.6) evaluated at the wall, we find

$$u_{0e} = \left(\frac{kT}{2\pi m_e}\right)^{1/2} \exp\left(\frac{e\phi_w}{kT}\right) \tag{4.37}$$

and using (4.9) we see that

$$u_{0e} = u_{0i} \tag{4.38}$$

To verify the validity of the approximations indicated in (4.19), we note first that from the mass conservation equation the particle flux, given by $n_\alpha(x)\, u_\alpha(x)$, must be constant for all x and equal to the value $n_0\, u_{0\alpha}$. From (4.21) we see that the *minimum* value of $n_e(x)$ is $n_0\, exp\, (e\phi_w/(kT)$ since ϕ_w is negative. Therefore,

$$u_e = \left(\frac{n_0}{n_e}\right) u_{0e} \;\leq\; u_{0e} \exp\left(-\frac{e\phi_w}{kT}\right) \tag{4.39}$$

and using (4.37), we get

$$u_e \;\leq\; \left(\frac{kT}{2\pi m_e}\right)^{1/2} \tag{4.40}$$

or

$$\frac{kT}{m_e u_e^2} \geq 2\pi \tag{4.41}$$

in agreement with the assumption (4.19). Similarly, from (4.27) the *maximum* value of $n_i(x)$ is n_0, so that

$$u_i = \left(\frac{n_0}{n_i}\right) u_{0i} \geq u_{0i} \tag{4.42}$$

and using (4.36), we obtain

$$u_i \geq \left(\frac{kT}{2\pi m_i}\right)^{1/2} \exp\left(-\frac{e\phi_w}{kT}\right) \tag{4.43}$$

or

$$\frac{kT}{m_i u_i^2} \leq 2\pi \exp\left(\frac{2e\phi_w}{kT}\right) \simeq 0.1 \tag{4.44}$$

where the result on the right has been obtained substituting ϕ_w by the value given in (4.11). Therefore, in view of (4.41) and (4.44), the thermal energy of the electrons is seen to be greater than their kinetic energy, whereas for the ions the opposite situation is verified. From (4.44) we can also see that the Bohm criterion, for the formation of the plasma sheath, is satisfied.

Although the quantitative aspects of the discussion presented here are very approximate, it provides, however, a satisfactory qualitative picture of the plasma sheath.

5. THE PLASMA PROBE

The plasma probe is a device that has been widely used to measure the temperature and density of a plasma, both in the laboratory and in space. The electrostatic probe was originally developed by Langmuir and Mott-Smith, and the physical mechanism of its operation can be well explained using the theory of plasma sheaths presented in the previous section.

A conducting probe, or electrode, is immersed in a plasma and the current that flows through it is measured for various potentials applied to the probe. The temperature and number density of the electrons can be obtained from the characteristics of the resulting current-potential curve. When the surface of the probe is plane, the current-potential curve has a shape like that illustrated in Fig. 4.

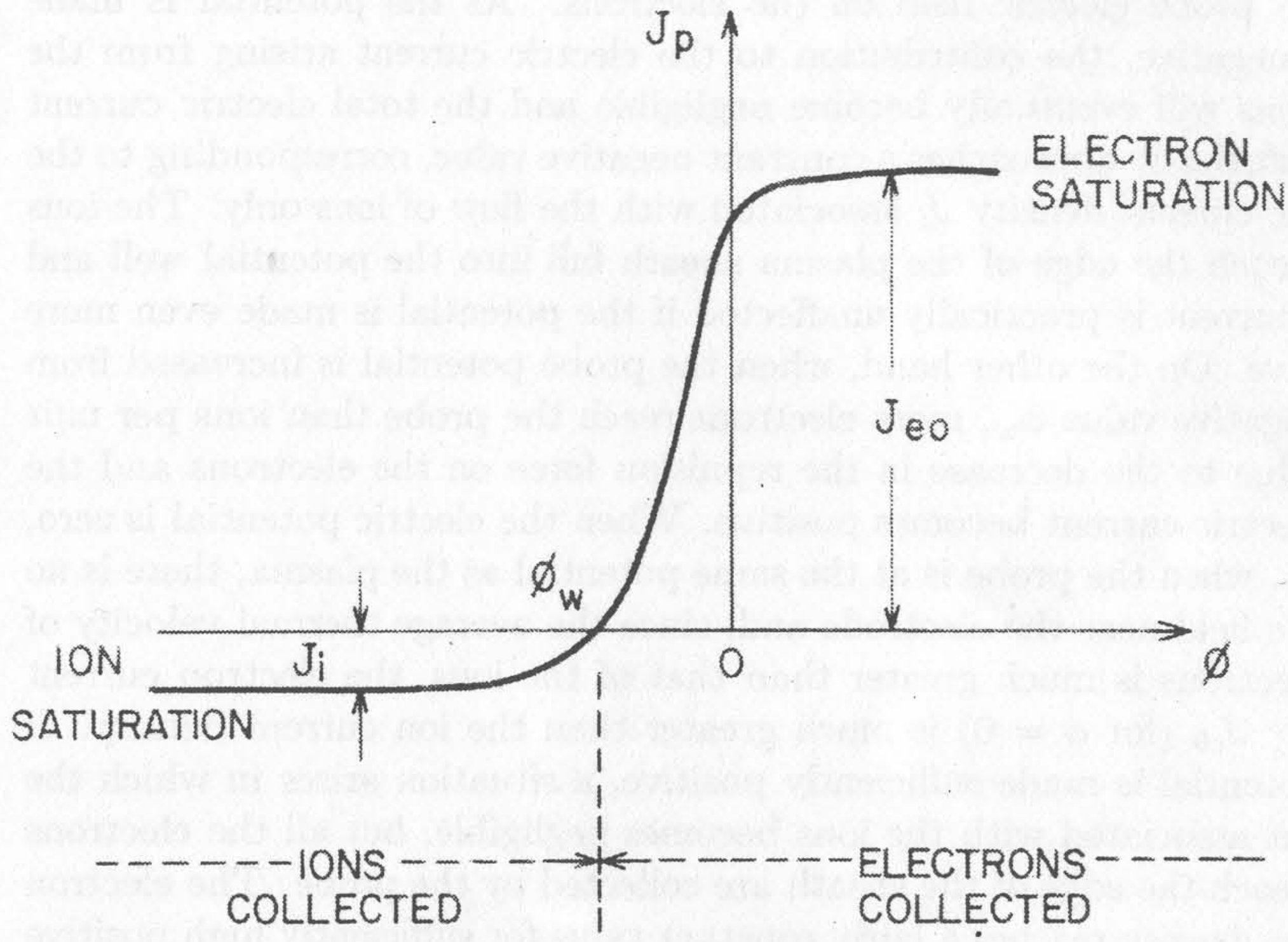

Fig. 4 *Characteristics of the current-potential curve of an electrostatic plane probe immersed in a plasma. The probe floating potential, with reference to the plasma potential, is denoted by* ϕ_w.

The probe, when inserted in the plasma, is surrounded by a plasma sheath which shields the major portion of the plasma from the disturbing probe field. The thickness of the sheath is of the order of a Debye length. When no current flows through the electrode, it stays at the negative *floating potential* ϕ_w, which is the wall potential discussed in the previous section. Under these equilibrium conditions, the number of electrons reaching the probe per unit time is equal to the number of positive ions reaching the probe per unit time. We assume the current to be positive when it flows in the direction away from the probe. The current associated with the electron flow is directed away from the probe and therefore it is considered positive. Consequently, the electric current associated with the flow of ions is negative. Under equilibrium conditions there is no net current flowing through the probe and its potential is the floating potential ϕ_w. When the probe potential is made more negative than ϕ_w, the

electron current is reduced due to the increased repulsive force imposed by the probe electric field on the electrons. As the potential is made more negative, the contribution to the electric current arising from the electrons will eventually become negligible and the total electric current asymptotically approaches a constant negative value, corresponding to the electric current density J_i associated with the flow of ions only. The ions that reach the edge of the plasma sheath fall into the potential well and their current is practically unaffected if the potential is made even more negative. On the other hand, when the probe potential is increased from the negative value ϕ_w, more electrons reach the probe than ions per unit time due to the decrease in the repulsion force on the electrons and the net electric current becomes positive. When the electric potential is zero, that is, when the probe is at the same potential as the plasma, there is no electric field near the electrode and, since the average thermal velocity of the electrons is much greater than that of the ions, the electron current density J_{e0} (for $\phi = 0$) is much greater than the ion current density. If the potential is made sufficiently positive, a situation arises in which the current associated with the ions becomes negligible, but all the electrons that reach the edge of the sheath are collected by the probe. The electron current density reaches a fairly constant value for sufficiently high positive values of ϕ. This plateau region in the probe current-potential curve is called the region of *saturation* of the electron current. For higher positive values of ϕ there are complications in the current-potential characteristic of the electrode due to the occurrence of another phenomenon.

An approximate expression for the magnitude of the electron current density, away from the region of saturation, can be obtained from (4.6) as

$$J_e = J_{e0} \exp\left(\frac{e\phi}{kT_e}\right) \tag{5.1}$$

where J_{e0} is the electron current density when the electric potential is zero. Since for $\phi = 0$ we have $\Gamma_e = n_e < v >_e /4$, and using (4.2) for the average electron speed, we obtain,

$$J_{e0} = en_e\left(\frac{kT_e}{2\pi m_e}\right)^{1/2} \tag{5.2}$$

where n_e is the electron number density in the unperturbed plasma region. Note that when ϕ is negative the ions reaching the edge of the sheath continue to fall into the negative potential of the probe and hence the ion current density is a constant (J_i) in the negative potential region. Thus,

we can express the probe current density, in the region where $\phi < 0$, as

$$J_p = J_{e0} \exp\left(\frac{e\phi}{kT_e}\right) - J_i \qquad (for\ \phi < 0) \tag{5.3}$$

From this equation we can deduce the result

$$T_e = \frac{e}{k}\left\{\frac{d}{d\phi}[\ \ln\ (J_p + J_i)]\right\}^{-1} \tag{5.4}$$

This expression can be used to determine the electron temperature as follows. First, the electrode is made sufficiently negative with reference to the plasma potential, so that the current that flows through the probe is due to the ions only. The measurement of this current gives directly the value J_i. Then, the current-potential characteristic curve of the probe is measured and a plot of $\ln\ (J_p + J_i)$ as a function of ϕ is made. This curve has a straight-line section corresponding to the probe potential less than the plasma potential, and the slope of this straight line gives the value of $(d/d\phi)[\ \ln\ (J_p + J_i)]$ which, when substituted in (5.4), gives the electron temperature in the plasma.

After the electron temperature T_e has been determined, we can evaluate the electron number density from (5.2), which can be written as,

$$n_e = \frac{J_{e0}}{e}\left(\frac{2\pi m_e}{kT_e}\right)^{1/2} \tag{5.5}$$

The value of J_{e0} is determined by measuring the probe current corresponding to the plateau (electron saturation) region of the current-potential characteristic of the probe.

PROBLEMS

11.1 Consider a stationary plasma (electrons and one type of ions) under steady state conditions at a uniform temperature T_0, when perturbed by a point charge $+Q$ placed at the origin of a coordinate system. Using the collisionless hydrodynamic equation for the electrons and ions ($\alpha = e, i$)

$$\rho_{m\alpha}\frac{D\mathbf{u}_\alpha}{Dt} = n_\alpha q_\alpha \mathbf{E} - \nabla p_\alpha$$

with the ideal gas law $p_\alpha = n_\alpha k T_0$, and Poisson equation

$$\nabla^2 \phi(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\epsilon_0}$$

obtain the following differential equation

$$\nabla^2 \phi(\mathbf{r}) - \frac{2}{\lambda_D^2}\phi(\mathbf{r}) = -\frac{Q}{\epsilon_0}\,\delta(\mathbf{r})$$

for the Debye potential $\phi(\mathbf{r})$. Assume that the number density of each species can be expressed as $n_\alpha = n_0 + n'_\alpha$, where n_0 is constant and $|n'_\alpha| \ll n_0$. What are the approximations necessary to obtain this result?

11.2 Analyze the Debye potential problem considering only the motion of the electrons (ions stay immobile) and show that in this case the differential equation for the electric potential $\phi(\mathbf{r})$ is

$$\nabla^2 \phi(\mathbf{r}) - \frac{1}{\lambda_D^2}\phi(\mathbf{r}) = -\frac{Q}{\epsilon_0}\,\delta(\mathbf{r})$$

11.3 When the macroscopic neutrality of a plasma is instantaneously perturbed by external means, the electrons react in a such a way as to give rise to oscillations at the electron plasma frequency $\omega_{pe} = [n_0 e^2/(m_e \epsilon_0)]^{1/2}$. Consider these oscillations, but including the motion of the ions. Show that in this case the natural frequency of oscillation of the net charge density is given by

$$\omega = (\omega_{pe}^2 + \omega_{pi}^2)^{1/2}$$

where $\omega_{pi} = [n_0 e^2/(m_i \epsilon_0)]^{1/2}$. Use the linearized equations of continuity and of momentum for each species, and Poisson equation, considering only the electric force due to the internal charge separation.

11.4 Evaluate the negative electrostatic potential ϕ_w which appears on an infinite plane wall immersed in a plasma consisting of electrons of charge $-e$ and ions of charge Ze, under steady state conditions. Denote the electron and ion temperatures by T_e and T_i, respectively.

11.5 Deduce an expression for the Debye potential for a test particle of charge $+Q$ immersed in a plasma consisting of electrons (charge $-e$) and ions of charge Ze, the temperature of the electrons and the ions being

T_e and T_i, respectively. Show that, if $T_e \gg T_i$, then the Debye length is governed by the ion temperature T_i.

11.6 Using the following expressions for the electron and ion number densities

$$n_e(\mathbf{r}) = n_0 \exp\left[\frac{e\phi(\mathbf{r})}{kT}\right]$$

$$n_i(\mathbf{r}) = n_0 \exp\left[-\frac{e\phi(\mathbf{r})}{kT}\right]$$

in the plasma sheath region formed between an infinite plane and a semi-infinite plasma, deduce the differential equation satisfied by the electric potential $\phi(x)$ in the plasma sheath. Show that this differential equation can be written in the form

$$\frac{d^2F}{d\xi^2} = \sinh(F)$$

where $F = e\phi/kT$ and $\xi = \sqrt{2}\, x/\lambda_D$. Assuming that at $x = \infty$ we have $n_e = n_i = n_0$, $F = 0$ and $dF/d\xi = 0$, show that

$$F(\xi) = 4 \tanh^{-1} \{ exp\, [-(\xi - \xi_0)]\}$$

where ξ_0 is a constant. Denoting the potential at the wall by ϕ_w and assuming that $e\phi/kT \ll 1$, show that

$$\phi(x) = \phi_w \exp\left(\frac{\sqrt{2}\, x}{\lambda_D}\right)$$

with

$$\phi_w = \frac{4kT}{e} \exp(\xi_0)$$

11.7 For the plasma sheath region formed in the vicinity of a plane wall immersed in a plasma, assume that the ions at the plasma edge of the sheath can be described by a shifted Maxwellian distribution function

$$f_i(v) = n_0\left(\frac{m_i}{2\pi kT}\right)^{3/2} \exp\left[-\frac{m_i(\mathbf{v} - \mathbf{u}_0)^2}{2kT}\right]$$

with drift velocity $\mathbf{u}_0 = u_0\hat{\mathbf{x}}$. Prove that the ion flux Γ_{ix}, at the edge of the sheath, is given by

$$\Gamma_{ix} = n_0\left(\frac{kT}{2\pi m_i}\right)^{1/2} \{ \exp(-y^2) + y\sqrt{\pi}\, [1 + \mathrm{erf}\,(y)]\}$$

where $y = u_0[m_i/(2kT)]^{1/2}$ and erf (y) is the *error function*, defined by

$$\text{erf}\ (y) = \frac{2}{\sqrt{\pi}} \int_0^y \exp\ (-s^2)\ ds$$

Calculate $d\Gamma_{ix}/dy$. Note that the error function vanishes for $y = 0$, increases monotonically as y increases and tends asymptotically to unity as $y \to \infty$. Also note that

$$\frac{d}{dy}[\ \text{erf}\ (y)] = \frac{2}{\sqrt{\pi}} \exp\ (-y^2)$$

11.8 From an experimental current-potential curve of a Langmuir probe of area A immersed in a plasma, such as shown in Fig. 5, where the electric potentials are measured with respect to a fixed reference potential, explain how you can determine J_i, J_{eo}, the space potential ϕ_s, the floating potential ϕ_f with respect to ϕ_s (note that $\phi_f - \phi_s = \phi_w$), T_e and n_e.

11.9 The Langmuir plasma probe has been widely used in satellites to measure space plasma properties. In one valuable technique, circuits are arranged that measure directly $dI_p/d\phi$ and $d^2I_p/d\phi^2$, where $I_p = J_pA$ and A is the probe's area. Use (5.3) to show that

$$\frac{(dI_p/d\phi)}{(d^2I_p/d\phi^2)} = \frac{kT_e}{e}$$

which gives directly the electron temperature T_e. Next show that J_{e0} can be calculated from T_e and $dI_p/d\phi$ at a known value of ϕ, according to

$$J_{e0} = \frac{kT_e}{Ae} \exp\left(-\frac{e\phi}{kT_e}\right)\frac{dI_p}{d\phi}$$

The electron density n_e can then be calculated from (5.5).

11.10 An electron gas (Lorentz gas) in a background of stationary ions, is acted upon by a weak, externally applied electric field $\mathbf{E}$, under steady state conditions. Using the Boltzmann equation for the electrons, with the relaxation model for the collision term (considering a constant collision frequency ν),

$$\left(\frac{\delta f}{\delta t}\right)_{coll} = -\nu(f - f_0)$$

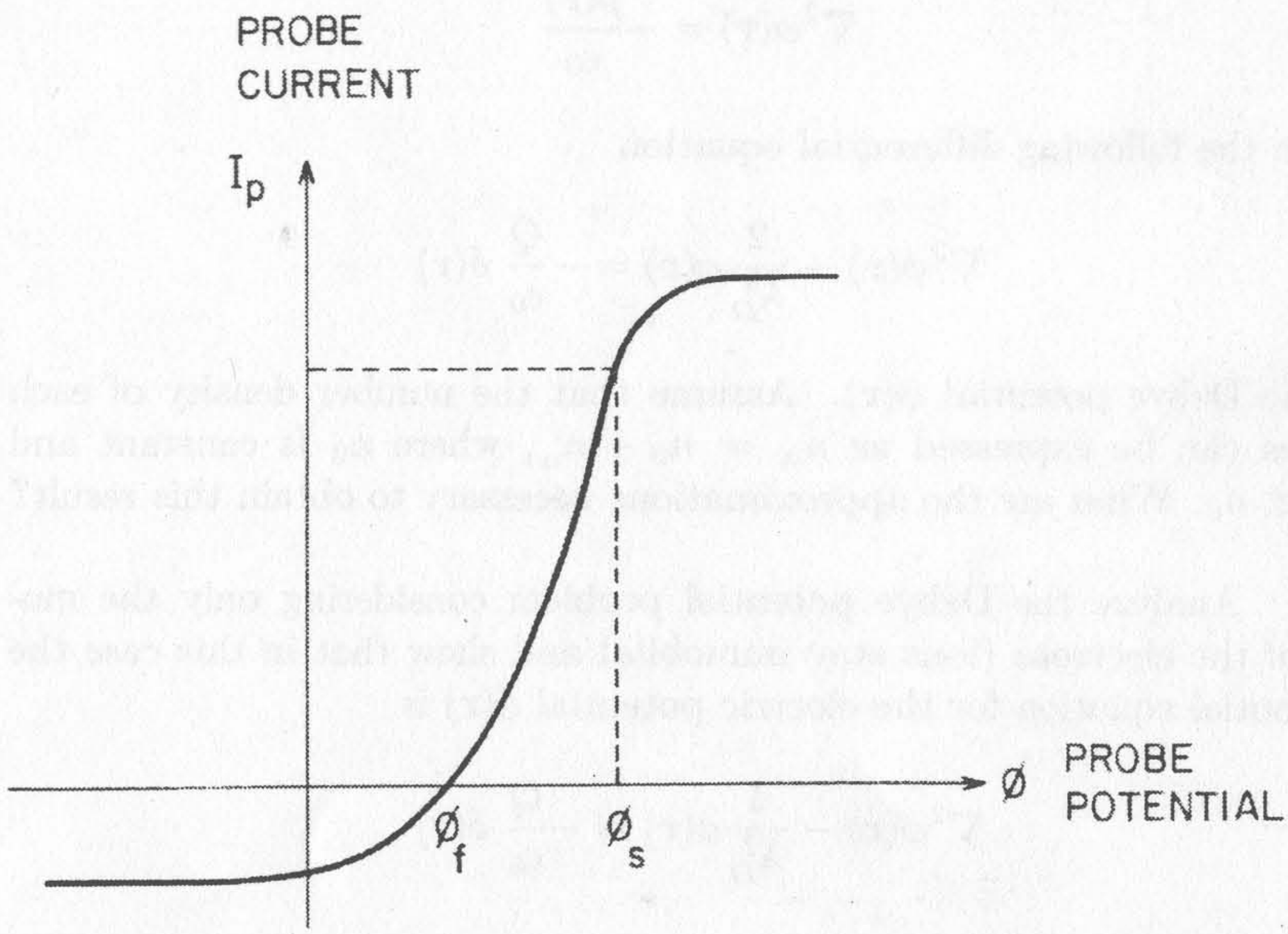

Fig. 5 *Typical current-potential curve of a plane Langmuir probe immersed in a plasma.*

and considering the adiabatic case for which

$$n(\mathbf{r})[T(\mathbf{r})]^{-3/2} = \text{constant}$$

show that the electron distribution function is given by

$$f = f_0\left\{1 - \frac{1}{\nu}\left[\frac{mv^2}{2kT}\left(\mathbf{v}\cdot\frac{\nabla T}{T}\right) + \frac{e}{kT}(\mathbf{v}\cdot\mathbf{E})\right]\right\}$$

Assume that $f = f_0 + f_1$, where $|f_1| \ll f_0$ and where f_0 is the following modified Maxwellian distribution

$$f_0(\mathbf{r}, v) = n(\mathbf{r})\left[\frac{m}{2\pi k\, T(\mathbf{r})}\right]^{3/2} \exp\left[-\frac{mv^2}{2k\, T(\mathbf{r})}\right]$$

and neglect all second-order terms in the Boltzmann equation. Consider the term involving ∇f_1 as a second-order quantity.

11.11 Using the distribution function of the previous problem, evaluate the electric current density $\mathbf{J}$ to show that the presence of a temperature gradient gives rise to an electric current associated with thermoelectric effects.

11.12 Consider problem 11.10, but taking $\mathbf{E} = 0$ and, instead of the adiabatic case, consider a constant kinetic pressure

$$p = n(\mathbf{r})\ k\ T(\mathbf{r}) = \text{constant}$$

(a) Show that the electron distribution function is given by

$$f = f_0\Big[1 - \frac{1}{\nu}\Big(-\frac{5}{2} + \frac{mv^2}{2kT}\Big)\mathbf{v}\cdot\frac{\nabla T}{T}\Big]$$

(b) Evaluate the heat flux vector $\mathbf{q}$ and show that it can be written as

$$\mathbf{q} = -K\nabla T(\mathbf{r})$$

where the thermal conductivity K is equal to $5kp/(2m\nu)$.
(c) What is the value of the electric current density $\mathbf{J}$ in this case?

11.13 In the previous problem, consider that $n =$ constant and that f_0 is the following modified Maxwell-Boltzmann distribution function

$$f_0(\mathbf{r}, v) = n\Big[\frac{m}{2\pi k\ T(\mathbf{r})}\Big]^{3/2} \exp\Big[-\frac{mv^2}{2k\ T(\mathbf{r})}\Big]$$

Calculate the electron distribution function $f(\mathbf{r}, \mathbf{v})$ and show that the heat flux vector is given by

$$\mathbf{q} = -K\nabla T(\mathbf{r})$$

Determine the expression for the thermal conductivity K.

11.14 In Problem 11.12, include the presence of an external magnetostatic field $\mathbf{B}$ in the z direction and deduce the following expression for the nonequilibrium distribution function

$$f = f_0\Big\{1 - \frac{1}{\nu}\Big(-\frac{5}{2} + \frac{mv^2}{2kT}\Big)\Big[\frac{\nu\Omega_{ce}}{(\nu^2 + \Omega_{ce}^2)}v_x\Big(\frac{\nu}{\Omega_{ce}}\hat{\mathbf{x}} - \hat{\mathbf{y}}\Big) +$$

$$+ \frac{\nu\Omega_{ce}}{(\nu^2 + \Omega_{ce}^2)}v_y\Big(\hat{\mathbf{x}} + \frac{\nu}{\Omega_{ce}}\hat{\mathbf{y}}\Big) + v_z\hat{\mathbf{z}}\Big]\cdot\frac{\nabla T}{T}\Big\}$$

Show that the heat flux vector $\mathbf{q}$ can be expressed as

$$\mathbf{q} = -\mathcal{K} \cdot \nabla T(\mathbf{r})$$

where $\mathcal{K}$ is the dyadic thermal conductivity, which in matrix form can be written as

$$\mathcal{K} = \begin{pmatrix} K_\perp & -K_H & 0 \\ K_H & K_\perp & 0 \\ 0 & 0 & K_\parallel \end{pmatrix}$$

with

$$K_\perp = \frac{\nu^2}{(\nu^2 + \Omega_{ce}^2)} K_0$$

$$K_H = \frac{\nu \Omega_{ce}}{(\nu^2 + \Omega_{ce}^2)} K_0$$

$$K_\parallel = \frac{5kp}{2m\nu} \equiv K_0$$

Note: The solution of the differential equation

$$\frac{df_1}{d\phi} + \frac{\nu}{\Omega_{ce}} f_1 = -\frac{1}{\Omega_{ce}} \mathbf{v} \cdot \nabla f_0$$

is given by

$$f_1 = -\exp\left(-\frac{\nu}{\Omega_{ce}}\phi\right) \frac{1}{\Omega_{ce}} \int_{-\infty}^{\phi} d\phi' \exp\left(\frac{\nu}{\Omega_{ce}}\phi'\right) \mathbf{v} \cdot \nabla f_0$$

11.15 The *coefficient of viscosity* η is defined as the shear stress produced by unit velocity gradient. For the p_{xz} component of the kinetic pressure dyad, for example, we have

$$p_{xz} = -\eta \frac{d}{dz} u_x(z)$$

Assume the following form for the equilibrium velocity distribution function of the electrons

$$f_0(\mathbf{r}, \mathbf{v}) = n\left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left[-\frac{m}{2kT}\{[v_x - u_x(z)]^2 + v_y^2 + v_z^2\}\right]$$

which indicates the presence of an average velocity $u_x(z)$ in the x direction having a gradient in the z direction. In the absence of external forces and

using the relaxation model for the collision term with a constant collision frequency ν, let $f = f_0 + f_1$ with $|f_1| \ll f_0$ in the Boltzmann equation, under steady state conditions, to show that

$$f_1(\mathbf{r}, \mathbf{v}) = -\frac{m}{kT\nu}\, f_0(\mathbf{r}, \mathbf{v})\, v_z[v_x - u_x(z)]\, \frac{\partial}{\partial z} u_x(z)$$

Next, calculate p_{xz} and show that the coefficient of viscosity is given by

$$\eta = \frac{nkT}{\nu}$$

12

SIMPLE APPLICATIONS OF MAGNETOHYDRODYNAMICS

1. FUNDAMENTAL EQUATIONS OF MAGNETOHYDRODYNAMICS

The basic equations governing the behavior of a conducting fluid have been presented and discussed in Chapter 9. For convenience, we reproduce here the simplified form of the magnetohydrodynamic equations. They include the equation of continuity for the whole conducting fluid

$$\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \mathbf{u}) = 0 \tag{1.1}$$

the equation of motion in the form

$$\rho_m \frac{D\mathbf{u}}{Dt} = \mathbf{J} \times \mathbf{B} - \nabla p \tag{1.2}$$

and the adiabatic equation of conservation of energy

$$\nabla p = V_s^2 \nabla \rho_m \tag{1.3}$$

where ρ_m denotes the total mass density, $\mathbf{u}$ is the average fluid velocity, $\mathbf{J}$ is the electric current density, $\mathbf{B}$ is the magnetic flux density, p is the total scalar pressure, and V_s is the adiabatic sound speed, which is equal to $(\gamma p/\rho_m)^{1/2}$, where γ is the ratio of the specific heats at constant pressure and at constant volume. To these equations we must add Maxwell curl equations, in the following reduced form,

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{1.4}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1.5}$$

and the generalized Ohm's law, in the simplified form,

$$\mathbf{J} = \sigma_0(\mathbf{E} + \mathbf{u} \times \mathbf{B}) \tag{1.6}$$

where σ_0 denotes the electric conductivity of the fluid, and $\mathbf{E}$ is the electric field.

In this closed set of simplified MHD equations it has been assumed that macroscopic electrical neutrality is maintained to a high degree of approximation, so that the electric charge density ρ vanishes. As discussed in section 6 of Chapter 9, the term $\partial \mathbf{E}/\partial t$, in Maxwell equation (1.4), can be neglected for the case of *very low frequency* phenomena and *highly conducting fluids.* As far as the generalized Ohm's law (1.6) is concerned, it is assumed that the time derivatives and pressure gradients are negligible, even though these terms are retained in some of the other MHD equations. Also, viscosity and thermal conductivity are neglected and the pressure dyad reduces to a scalar pressure.

The advantage of this approximate set of equations is that they reduce substantially the mathematical complexity of the more general equations for a conducting fluid and therefore facilitate the understanding of the physical processes that take place in highly conducting fluids at very low frequencies.

1.1 Parker Modified Momentum Equation

In the presence of a strong magnetic field the pressure tensor of an inviscid conducting fluid is *anisotropic.* When the cyclotron frequency is much larger than the collision frequency, a charged particle gyrates many times around a magnetic force line during the time between collisions, so that there is equipartition between the particle kinetic energies in the two independent directions normal to $\mathbf{B}$ but not, in general, in the direction along $\mathbf{B}$. If we denote by $p_\perp$ and $p_\parallel$ the scalar pressures in the plane normal to $\mathbf{B}$ and along $\mathbf{B}$, respectively, and consider a *local* coordinate system in which the third axis is in the direction of $\mathbf{B}$, we can write the pressure tensor of an inviscid fluid as

$$\mathcal{P} = \begin{pmatrix} p_\perp & 0 & 0 \\ 0 & p_\perp & 0 \\ 0 & 0 & p_\parallel \end{pmatrix} \tag{1.7}$$

Note that the parallel and perpendicular indexes used here do not refer to vector components but only indicates the part of the scalar pressures associated with the kinetic energy densities of the particle motions along $\mathbf{B}$ and in the plane perpendicular to $\mathbf{B}$, respectively.

When the magnetic field is not constant, the orientation of the axes of the local coordinate system changes from point to point and this change in direction must be taken into account when evaluating the divergence of the pressure tensor. We can express $\mathcal{P}$, in (1.7), as the sum of a hydrostatic scalar pressure $p_\perp$ and another tensor referred to the local coordinate system, as

$$\mathcal{P} = p_\perp \mathbf{1} + (p_\parallel - p_\perp)\widehat{\mathbf{B}}\widehat{\mathbf{B}} \tag{1.8}$$

where $\mathbf{1}$ is the unit dyad

$$\mathbf{1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.9}$$

and $\widehat{\mathbf{B}}\widehat{\mathbf{B}} = \mathbf{B}\mathbf{B}/B^2$ is the dyad formed from the unit vector $\widehat{\mathbf{B}}$,

$$\widehat{\mathbf{B}}\widehat{\mathbf{B}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.10}$$

The momentum equation, in the form (1.2), must be modified to include the anisotropy of the pressure dyad. Thus, we write

$$\rho_m \frac{D\mathbf{u}}{Dt} = \mathbf{J} \times \mathbf{B} - \nabla \cdot \mathcal{P} \tag{1.11}$$

To evaluate $\nabla \cdot \mathcal{P}$, with $\mathcal{P}$ as given by (1.8), we note that

$$\nabla \cdot (p_\perp \mathbf{1}) = \nabla p_\perp \tag{1.12}$$

and using the following identity

$$\nabla \cdot [(p_\parallel - p_\perp)\widehat{\mathbf{B}}\widehat{\mathbf{B}}] = (\mathbf{B} \cdot \nabla)\left[(p_\parallel - p_\perp)\frac{\mathbf{B}}{B^2}\right] + \\ + \left[(p_\parallel - p_\perp)\frac{\mathbf{B}}{B^2}\right](\nabla \cdot \mathbf{B}) \tag{1.13}$$

where the second term in the right-hand side vanishes in virtue of $\nabla \cdot \mathbf{B} = 0$, we obtain

$$\nabla \cdot \mathcal{P} = \nabla p_\perp + (\mathbf{B} \cdot \nabla)\Big[(p_\parallel - p_\perp)\frac{\mathbf{B}}{B^2}\Big] \tag{1.14}$$

Furthermore, using Maxwell equation (1.4) we can write the *magnetic force per unit volume* as

$$\mathbf{J} \times \mathbf{B} = \frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \mathbf{B} \tag{1.15}$$

The term in the right-hand side can be expanded, using a vector identity, with the result

$$\mathbf{J} \times \mathbf{B} = \frac{1}{\mu_0}[(\mathbf{B} \cdot \nabla)\mathbf{B} - \nabla(\tfrac{1}{2}B^2)] \tag{1.16}$$

Substituting expressions (1.14) and (1.16) into the momentum equation (1.11), we obtain, finally,

$$\rho_m \frac{D\mathbf{u}}{Dt} = -\nabla\Big(p_\perp + \frac{B^2}{2\mu_0}\Big) + (\mathbf{B} \cdot \nabla)\Big[\frac{\mathbf{B}}{\mu_0} - \frac{(p_\parallel - p_\perp)\mathbf{B}}{B^2}\Big] \tag{1.17}$$

This equation differs from the usual momentum equation (1.2), for a highly conducting inviscid fluid, only through the term $(p_\parallel - p_\perp)/B^2$. It was derived, although in a quite different way, by E.N. Parker in 1957 and, for this reason, it is usually referred to as the *Parker modified momentum equation.*

1.2 The Double Adiabatic Equations of Chew, Goldberger and Low (CGL)

To complement the momentum equation (1.17), we need equations for the time rate of change of $p_\parallel$ and $p_\perp$. These equations will take the place of the familiar adiabatic energy equation (1.3) which applies for the isotropic case. From the general energy equation (9.4.14) for a conducting fluid, if we do not take into account heat conduction and Joule heating, we have

$$\frac{D}{Dt}\Big(\frac{3p}{2}\Big) + \Big(\frac{3p}{2}\Big)(\nabla \cdot \mathbf{u}) + (\mathcal{P} \cdot \nabla) \cdot \mathbf{u} = 0 \tag{1.18}$$

with the pressure dyad $\mathcal{P}$ as given by (1.8) and where the scalar pressure p is one third the trace of $\mathcal{P}$, that is,

$$p = \frac{1}{3}(2p_\perp + p_\parallel) \tag{1.19}$$

Note that $3p/2$ represents the *total thermal energy density.* By direct expansion, using (1.8) for $\mathcal{P}$, we find that

$$(\mathcal{P} \cdot \nabla) \cdot \mathbf{u} = [p_\perp \nabla + (p_\parallel - p_\perp)(\widehat{\mathbf{B}}\widehat{\mathbf{B}} \cdot \nabla)] \cdot \mathbf{u} \tag{1.20}$$

and taking this expression, together with (1.19), into (1.18), we obtain

$$\frac{D}{Dt}(2p_\perp + p_\parallel) + (p_\parallel + 4p_\perp)\nabla \cdot \mathbf{u} + 2(p_\parallel - p_\perp)(\widehat{\mathbf{B}}\widehat{\mathbf{B}} \cdot \nabla) \cdot \mathbf{u} = 0 \tag{1.21}$$

A strong magnetic field constrains the charged particle motion only in the direction transverse to $\mathbf{B}$, but they are still free to move large distances along $\mathbf{B}$. Thus, it is reasonable to suppose that the contribution to the total thermal energy, arising from the particle motion parallel to $\mathbf{B}$, also satisfy an energy conservation equation similar to (1.18). This leads to the following equation for the part of the total thermal energy due to the random particle motions along $\mathbf{B}$,

$$\frac{Dp_\parallel}{Dt} + p_\parallel \nabla \cdot \mathbf{u} + 2p_\parallel(\widehat{\mathbf{B}}\widehat{\mathbf{B}} \cdot \nabla) \cdot \mathbf{u} = 0 \tag{1.22}$$

Therefore, decoupling the parallel and perpendicular motions, the equation for $p_\perp$ becomes

$$\frac{Dp_\perp}{Dt} + 2p_\perp \nabla \cdot \mathbf{u} - p_\perp(\widehat{\mathbf{B}}\widehat{\mathbf{B}} \cdot \nabla) \cdot \mathbf{u} = 0 \tag{1.23}$$

Equations (1.21) and (1.22) can also be obtained from an energy equation of higher order than (9.4.14), involving the total time rate of change of the pressure dyad $\mathcal{P}$. When this equation, involving $D\mathcal{P}/Dt$, is contracted with the unit dyad $\mathbf{1}$ we obtain (1.21), and when contracted with the dyad $\widehat{\mathbf{B}}\widehat{\mathbf{B}}$ results in (1.22).

Equations (1.22) and (1.23) enable $p_\parallel$ and $p_\perp$ to be calculated. They can be written in a more succint form, as follows. First we note that using Maxwell curl equation

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1.24}$$

and considering a perfectly conducting fluid for which

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = 0 \tag{1.25}$$

we have

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) \tag{1.26}$$

Expanding the right-hand side using the vector identity $\nabla \times (\mathbf{u} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{u} - \mathbf{B}(\nabla \cdot \mathbf{u}) - (\mathbf{u} \cdot \nabla)\mathbf{B} + \mathbf{u}(\nabla \cdot \mathbf{B})$, and noting that $\nabla \cdot \mathbf{B} = 0$, we obtain

$$\frac{D\mathbf{B}}{Dt} = (\mathbf{B} \cdot \nabla)\mathbf{u} - \mathbf{B}(\nabla \cdot \mathbf{u}) \tag{1.27}$$

If we now take the scalar product of (1.27) with $\mathbf{B}/B^2$, we obtain

$$\frac{1}{2B^2}\frac{D(B^2)}{Dt} = \widehat{\mathbf{B}} \cdot (\widehat{\mathbf{B}} \cdot \nabla)\mathbf{u} - \nabla \cdot \mathbf{u} \tag{1.28}$$

which may be written as

$$\frac{1}{B}\frac{DB}{Dt} = (\widehat{\mathbf{B}}\widehat{\mathbf{B}} \cdot \nabla) \cdot \mathbf{u} - \nabla \cdot \mathbf{u} \tag{1.29}$$

Furthermore, from the equation of continuity (1.1), we get

$$\nabla \cdot \mathbf{u} = -\frac{1}{\rho_m}\frac{D\rho_m}{Dt} \tag{1.30}$$

and using (1.29) and (1.30), to eliminate $(\widehat{\mathbf{B}}\widehat{\mathbf{B}} \cdot \nabla) \cdot \mathbf{u}$ and $\nabla \cdot \mathbf{u}$ in (1.22) and (1.23), we obtain

$$\frac{1}{p_\parallel}\frac{Dp_\parallel}{Dt} - \frac{3}{\rho_m}\frac{D\rho_m}{Dt} + \frac{2}{B}\frac{DB}{Dt} = 0 \tag{1.31}$$

$$\frac{1}{p_\perp}\frac{Dp_\perp}{Dt} - \frac{1}{\rho_m}\frac{D\rho_m}{Dt} - \frac{1}{B}\frac{DB}{Dt} = 0 \tag{1.32}$$

These two equations can be written in compact form as

$$\frac{D}{Dt}\left(\frac{p_\parallel B^2}{\rho_m^3}\right) = 0 \tag{1.33}$$

$$\frac{D}{Dt}\left(\frac{p_\perp}{\rho_m B}\right) = 0 \tag{1.34}$$

They are known as the *double adiabatic equations* for a conducting fluid in a strong magnetic field and are due to G.F. Chew, M.L. Goldberger and F.E. Low (1956). They are also known as the CGL equations. They take the place of the adiabatic energy equation for isotropic plasmas

$$\frac{D}{Dt}(p\rho_m^{-\gamma}) = 0 \tag{1.35}$$

1.3 Special Cases of the Double Adiabatic Equations

As a simple application of the double adiabatic equations, consider initially the case in which the only variations are *parallel* to the magnetic field as, for example, in sound waves travelling along the field lines. This situation is usually referred to as *linear compression* parallel to the magnetic field or one-dimensional compression. The magnetic field is assumed to be straight and uniform, and directed along the z axis. Thus, $B_x = B_y = 0$ and $\mathbf{B} = B_z\widehat{\mathbf{z}}$, as well as $\partial/\partial x = \partial/\partial y = 0$. In this case, we find

$$(\widehat{\mathbf{B}}\widehat{\mathbf{B}} \cdot \nabla) \cdot \mathbf{u} = \frac{\partial u_z}{\partial z} = \nabla \cdot \mathbf{u} \tag{1.36}$$

and from (1.29), we see that B stays constant. Equations (1.31) and (1.32), with $DB/Dt = 0$, then yields

$$\frac{D}{Dt}\left(\frac{p_\parallel}{\rho_m^3}\right) = 0 \tag{1.37}$$

$$\frac{D}{Dt}\left(\frac{p_\perp}{\rho_m}\right) = 0 \tag{1.38}$$

If we compare these results with (1.35), we find that γ may be assigned the value 3 along the field lines (one-dimensional compression), and the value 1 across the field lines.

It is useful to introduce a parallel and a perpendicular temperature through the relations

$$p_\parallel = nkT_\parallel \tag{1.39}$$

$$p_\perp = nkT_\perp \tag{1.40}$$

Therefore, for the case of *one-dimensional compression parallel* to $\mathbf{B}$, we have

$$T_\parallel \propto n^2 \tag{1.41}$$

$$T_\perp = \text{constant} \tag{1.42}$$

which shows that this type of compression is isothermal with respect to the perpendicular temperature $T_\perp$. The changes in $p_\perp$ are therefore entirely due to the changes in the number density n, whereas those of $p_\parallel$ are due to changes in both n and $T_\parallel$.

Another special case of interest is the *two-dimensional compression perpendicular* to the magnetic field, in which all motion is transverse to the field lines. This situation can be pictured as the motion of magnetic flux

tubes, identified by the particles contained in them. Assuming straight field lines along the z axis ($B_x = B_y = 0$, $\mathbf{B} = B_z\widehat{\mathbf{z}}$) and variations only in the transverse direction ($\partial/\partial z = 0$), we find

$$(\widehat{\mathbf{B}}\widehat{\mathbf{B}} \cdot \nabla) \cdot \mathbf{u} = \left(\widehat{\mathbf{z}}\frac{\partial}{\partial z}\right) \cdot \mathbf{u} = 0 \tag{1.43}$$

and (1.22) and (1.23) yield

$$\frac{Dp_\parallel}{Dt} - \frac{p_\parallel}{\rho_m}\frac{D\rho_m}{Dt} = 0 \tag{1.44}$$

$$\frac{Dp_\perp}{Dt} - \frac{2p_\perp}{\rho_m}\frac{D\rho_m}{Dt} = 0 \tag{1.45}$$

Therefore, in the case of cylindrical compression perpendicular to $\mathbf{B}$ the adiabatic equations reduce to

$$\frac{D}{Dt}\left(\frac{p_\parallel}{\rho_m}\right) = 0 \tag{1.46}$$

$$\frac{D}{Dt}\left(\frac{p_\perp}{\rho_m^2}\right) = 0 \tag{1.47}$$

Comparing with (1.35), we see that γ takes the value 1 parallel to the magnetic field and 2 transverse to it. From (1.39) and (1.40) we find that for a two-dimensional (cylindrically symmetric) compression perpendicular to $\mathbf{B}$,

$$T_\parallel = \text{constant} \tag{1.48}$$

$$T_\perp \propto n \tag{1.49}$$

so that this type of compression is isothermal with respect to the parallel temperature. The changes in $p_\parallel$ are due entirely to variations in the number density n, whereas those of $p_\perp$ result from variations in n as well as in $T_\perp$.

In the case of *three-dimensional spherically symmetric* compression, we have

$$p_\perp = p_\parallel = p \tag{1.50}$$

and (1.21) reduces to

$$3\frac{Dp}{Dt} - \frac{5p}{\rho_m}\frac{D\rho_m}{Dt} = 0 \tag{1.51}$$

Thus, we obtain

$$\frac{D}{Dt}\left(\frac{p}{\rho_m^{5/3}}\right) = 0 \tag{1.52}$$

which is the familiar adiabatic equation (1.35) of gas dynamics, with $\gamma = 5/3$. In any of the cases of adiabatic compression, the fluid has to be subjected to a certain system of forces in order to achieve the desired type of adiabatic compression. The required system of forces has to be determined from the momentum equation in conjunction with the conditions appropriate to the particular problem under analysis.

1.4 Energy Integral

As a final consideration in this section, we will show that the system of hydromagnetic equations (1.1) to (1.6) possesses an energy integral. Using Maxwell equation (1.4), to substitute $\mathbf{J}$ in the equation of motion (1.2), yields

$$\rho_m \frac{D\mathbf{u}}{Dt} = \frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \mathbf{B} - \nabla p \tag{1.53}$$

Now, we take the dot product of this equation with $\mathbf{u}$,

$$\rho_m \mathbf{u} \cdot \frac{D\mathbf{u}}{Dt} = \frac{1}{\mu_0}\mathbf{u} \cdot (\nabla \times \mathbf{B}) \times \mathbf{B} - \mathbf{u} \cdot \nabla p \tag{1.54}$$

The term on the left-hand side can be expanded as

$$\rho_m \mathbf{u} \cdot \frac{D\mathbf{u}}{Dt} = \rho_m \mathbf{u} \cdot \left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}\right] = \tfrac{1}{2}\rho_m \left[\frac{\partial u^2}{\partial t} + (\mathbf{u} \cdot \nabla)u^2\right] =$$

$$= \frac{\partial}{\partial t}(\tfrac{1}{2}\rho_m u^2) - \tfrac{1}{2}u^2 \frac{\partial \rho_m}{\partial t} + \tfrac{1}{2}\rho_m (\mathbf{u} \cdot \nabla)u^2 \tag{1.55}$$

Using the continuity equation (1.1) to eliminate $\partial \rho_m / \partial t$, yields

$$\rho_m \mathbf{u} \cdot \frac{D\mathbf{u}}{Dt} = \frac{\partial}{\partial t}(\tfrac{1}{2}\rho_m u^2) + \tfrac{1}{2}u^2 \nabla \cdot (\rho_m \mathbf{u}) + \tfrac{1}{2}\rho_m (\mathbf{u} \cdot \nabla)u^2 =$$

$$= \frac{\partial}{\partial t}(\tfrac{1}{2}\rho_m u^2) + \nabla \cdot (\tfrac{1}{2}\rho_m u^2 \mathbf{u}) \tag{1.56}$$

In order to transform the term $(\mathbf{u} \cdot \nabla)p$, let us write the adiabatic energy equation (1.35) as

$$\rho_m^{-\gamma} \frac{Dp}{Dt} - \gamma p \, \rho_m^{-(\gamma+1)} \frac{D\rho_m}{Dt} = 0 \tag{1.57}$$

and use the continuity equation in the form

$$\frac{D\rho_m}{Dt} = -\rho_m(\nabla \cdot \mathbf{u}) \tag{1.58}$$

Combining these two equations, we obtain

$$\frac{\partial p}{\partial t} + (\mathbf{u} \cdot \nabla)p + \gamma p(\nabla \cdot \mathbf{u}) = 0 \tag{1.59}$$

which may also be written as

$$\frac{\partial p}{\partial t} + (1-\gamma)(\mathbf{u} \cdot \nabla)p + \gamma \nabla \cdot (p\mathbf{u}) = 0 \tag{1.60}$$

from which we get

$$(\mathbf{u} \cdot \nabla)p = \frac{1}{(\gamma-1)}\frac{\partial p}{\partial t} + \frac{\gamma}{(\gamma-1)}\nabla \cdot (p\mathbf{u}) \tag{1.61}$$

Finally, for the $\mathbf{u} \cdot (\nabla \times \mathbf{B}) \times \mathbf{B}$ term in (1.54), considering a perfectly conducting fluid for which $\mathbf{E} = -\mathbf{u} \times \mathbf{B}$, and using a vector identity, we can write

$$\begin{aligned} \mathbf{u} \cdot (\nabla \times \mathbf{B}) \times \mathbf{B} &= -(\mathbf{u} \times \mathbf{B}) \cdot (\nabla \times \mathbf{B}) = \mathbf{E} \cdot (\nabla \times \mathbf{B}) = \\ &= \mathbf{B} \cdot (\nabla \times \mathbf{E}) - \nabla \cdot (\mathbf{E} \times \mathbf{B}) \end{aligned} \tag{1.62}$$

Using Maxwell equation (1.5) we arrive at

$$\frac{1}{\mu_0}\mathbf{u} \cdot (\nabla \times \mathbf{B}) \times \mathbf{B} = -\frac{\partial}{\partial t}\left(\frac{B^2}{2\mu_0}\right) - \frac{1}{\mu_0}\nabla \cdot (\mathbf{E} \times \mathbf{B}) \tag{1.63}$$

Substituting (1.56), (1.61) and (1.63), into (1.54), yields the following *energy conservation equation*

$$\begin{aligned} &\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho_m u^2 + \frac{p}{\gamma-1} + \frac{B^2}{2\mu_0}\right) + \\ &+ \nabla \cdot \left(\tfrac{1}{2}\rho_m u^2 \mathbf{u} + \frac{\gamma}{\gamma-1}p\mathbf{u} + \mathbf{E} \times \mathbf{H}\right) = 0 \end{aligned} \tag{1.64}$$

The first three terms in this equation represent the kinetic energy density associated with the macroscopic motion of the fluid, the thermal

energy density and the energy density stored in the magnetic field, respectively, whereas the last three terms denote the flux of macroscopic kinetic energy, the flux of thermal energy transported at the macroscopic mean velocity $\mathbf{u}$ and the flux of electromagnetic energy (given by the Poynting vector $\mathbf{E} \times \mathbf{H}$), respectively.

If we integrate (1.64) over the entire fluid-plus-vacuum volume and use Gauss' divergence theorem to transform the divergence term into a surface integral, we find that the first two terms in the surface integral vanish, since ρ_m, p and $\mathbf{u}$ are zero outside the fluid. The remaining surface term is the surface integral of the Poynting vector which, for an isolated system, also vanishes. Therefore, we obtain the following *energy conservation integral*

$$\int_V \left(\tfrac{1}{2}\rho_m u^2 + \frac{p}{\gamma - 1} + \frac{B^2}{2\mu_0} \right) d^3r = \text{constant} \tag{1.65}$$

The first integral represents the macroscopic kinetic energy of the fluid, the second term is the thermal free energy and the last one represents the total energy of the magnetic field. It is usually useful to separate (1.65) into a kinetic energy part

$$K = \int_V \tfrac{1}{2}\rho_m u^2 \, d^3r \tag{1.66}$$

and a potential energy part

$$U = \int_V \left(\frac{p}{\gamma - 1} + \frac{B^2}{2\mu_0} \right) d^3r \tag{1.67}$$

with the energy conservation law $K + U = \text{constant}$. In these equations the integration extends over the entire fluid-plus-vacuum volume.

2. MAGNETIC VISCOSITY AND REYNOLDS NUMBER

The behavior of the magnetic field is of great importance in many MHD problems. To obtain a simple equation for the variations of $\mathbf{B}$, let us start by taking the curl of the generalized Ohm's law (1.6),

$$\nabla \times \mathbf{J} = \sigma_0 [\nabla \times \mathbf{E} + \nabla \times (\mathbf{u} \times \mathbf{B})] \tag{2.1}$$

Replacing $\mathbf{J}$ and $\nabla \times \mathbf{E}$, using Maxwell curl equations (1.4) and (1.5),

$$\nabla \times (\nabla \times \mathbf{B}) = \mu_0 \sigma_0 \left[-\frac{\partial \mathbf{B}}{\partial t} + \nabla \times (\mathbf{u} \times \mathbf{B}) \right] \tag{2.2}$$

Making use of the following identity (with $\nabla \cdot \mathbf{B} = 0$)

$$\nabla \times (\nabla \times \mathbf{B}) = -\nabla^2 \mathbf{B} \tag{2.3}$$

equation (2.2) reduces to

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \eta_m \nabla^2 \mathbf{B} \tag{2.4}$$

where η_m is called the *magnetic viscosity*,

$$\eta_m = \frac{1}{\mu_0 \sigma_0} \tag{2.5}$$

The first term in the right-hand side of (2.4) is called the *flow term*, while the second term is called the *diffusion term*. To compare the relative magnitude of these two terms, we can use dimensional analysis and take, approximately,

$$|\nabla \times (\mathbf{u} \times \mathbf{B})| \simeq \frac{uB}{L} \tag{2.6}$$

$$\eta_m |\nabla^2 \mathbf{B}| \simeq \eta_m \frac{B}{L^2} \tag{2.7}$$

where L denotes some characteristic length for variation of the parameters. The ratio of the flow term to the diffusion term is called the *magnetic Reynolds number* and is given by

$$R_m = \frac{uL}{\eta_m} \tag{2.8}$$

In most MHD problems one or the other of these two terms is of predominant importance and R_m is either very large or very small compared to unity.

It is instructive to compare the magnetic viscosity (η_m) and the magnetic Reynolds number (R_m) with the ordinary hydrodynamic viscosity (η_k) and hydrodynamic Reynolds number (R). For this purpose, consider the *Navier-Stokes equation* of hydrodynamics

$$\frac{D\mathbf{u}}{Dt} = \mathbf{f} - \frac{1}{\rho_m} \nabla p + \eta_k [\nabla^2 \mathbf{u} + \frac{1}{3} \nabla (\nabla \cdot \mathbf{u})] \tag{2.9}$$

where $\mathbf{f}$ denotes the average force per unit mass of the fluid, and η_k is the kinematic viscosity (viscosity divided by density). Comparing this

equation with (2.4) we see that the role played by η_m, in the rate of change of $\mathbf{B}$, is completely analogous to the role played by η_k, in the rate of change of the mean fluid velocity $\mathbf{u}$. The ordinary Reynolds number is defined as the ratio of the inertia term $(\mathbf{u}\cdot\nabla)\mathbf{u}$ to the viscosity term $\eta_k\nabla^2\mathbf{u}$. Using dimensional analysis, we have

$$|(\mathbf{u}\cdot\nabla)\mathbf{u}| \simeq \frac{u^2}{L} \tag{2.10}$$

$$\eta_k|\nabla^2\mathbf{u}| \simeq \eta_k\frac{u}{L^2} \tag{2.11}$$

which gives the following expression (completely analogous to R_m) for the ordinary Reynolds number

$$R = \frac{uL}{\eta_k} \tag{2.12}$$

3. DIFFUSION OF MAGNETIC FIELD LINES

When $R_m \ll 1$, that is when the diffusion term dominates, (2.4) becomes approximately,

$$\frac{\partial\mathbf{B}}{\partial t} = \eta_m\nabla^2\mathbf{B} \qquad (R_m \ll 1) \tag{3.1}$$

This is the equation of diffusion of a magnetic field in a stationary conductor, resulting in the decay of the magnetic field. It is analogous to the particle diffusion equation studied in Chapter 10. The characteristic decay time of the magnetic field can be obtained by dimensional analysis, taking

$$|\partial\mathbf{B}/\partial t| \simeq \frac{B}{\tau_D} \tag{3.2}$$

$$|\eta_m\nabla^2\mathbf{B}| \simeq \eta_m\frac{B}{L^2} \tag{3.3}$$

where τ_D represents a characteristic time for variation of the plasma parameters. Thus, according to (3.1), the magnetic field diffuses away with a characteristic decay time of the order of

$$\tau_D = \frac{L^2}{\eta_m} = L^2\mu_0\sigma_0 \tag{3.4}$$

For ordinary conductors the time of decay is very small. If we consider, for example, a copper sphere of radius 1 meter, we find that τ_D is less than 10 seconds. For a celestial body, however, because of the large dimensions, the time of decay can be very large. For the Earth's core, considering it to be molten iron, the time of free decay is approximately

10^4 years, while for the general magnetic field of the Sun it is found to be of the order of 10^{10} years.

4. FREEZING OF MAGNETIC FIELD LINES TO THE PLASMA

A completely different type of behavior appears when $R_m \gg 1$. In this case, the flow term dominates over the diffusion term and (2.4) reduces to

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) \qquad (R_m \gg 1) \tag{4.1}$$

This equation implies that in a highly conducting fluid the magnetic field lines move along exactly with the fluid, rather than simply diffusing out. Alfvén has expressed this type of behavior by saying that the magnetic field lines are "frozen" in the conducting fluid. In effect, the fluid can flow freely along the magnetic field lines, but any motion of the conducting fluid, perpendicular to the field lines, carries them with the fluid.

In order to show this implication of (4.1), it is convenient to consider initially the concept of magnetic tubes of force which are used to visually describe the direction and magnitude of $\mathbf{B}$ at various points in space. One can think of the space pervaded by a magnetic field as divided into a large number of elementary magnetic tubes of force, all of them enclosing the same magnetic flux $\Delta\Phi_B$. If ΔS is the local cross sectional area of an elementary magnetic tube of force (see Fig. 1), then the magnitude of $\mathbf{B}$, at the local point P, is equal to $\Delta\Phi_B/\Delta S$. According to this definition, the magnitude of $\mathbf{B}$ is everywhere inversely proportional to the cross sectional area of the elementary tube of force.

Let us now consider a closed line whose points move with velocity $\mathbf{u}$ in a space pervaded by a magnetic field. Assume, for the moment, that $\mathbf{u}$ is an arbitrary function of position and time (not necessarily equal to the fluid velocity), with the result that the closed curve may change in shape, as well as undergo translational and rotational motion. Let C_1 denote this closed line at time t, bounding the open surface $\mathbf{S}(t) = \mathbf{S}_1$. At a time Δt later, let C_2 and $\mathbf{S}(t + \Delta t) = \mathbf{S}_2$ denote the corresponding closed line and open surface (refer to Fig. 2). The flux of the magnetic field through an open surface $\mathbf{S}$, at time t, is given by

$$\Phi_B(t) = \int_S \mathbf{B}(\mathbf{r}, t) \cdot d\mathbf{S} \tag{4.2}$$

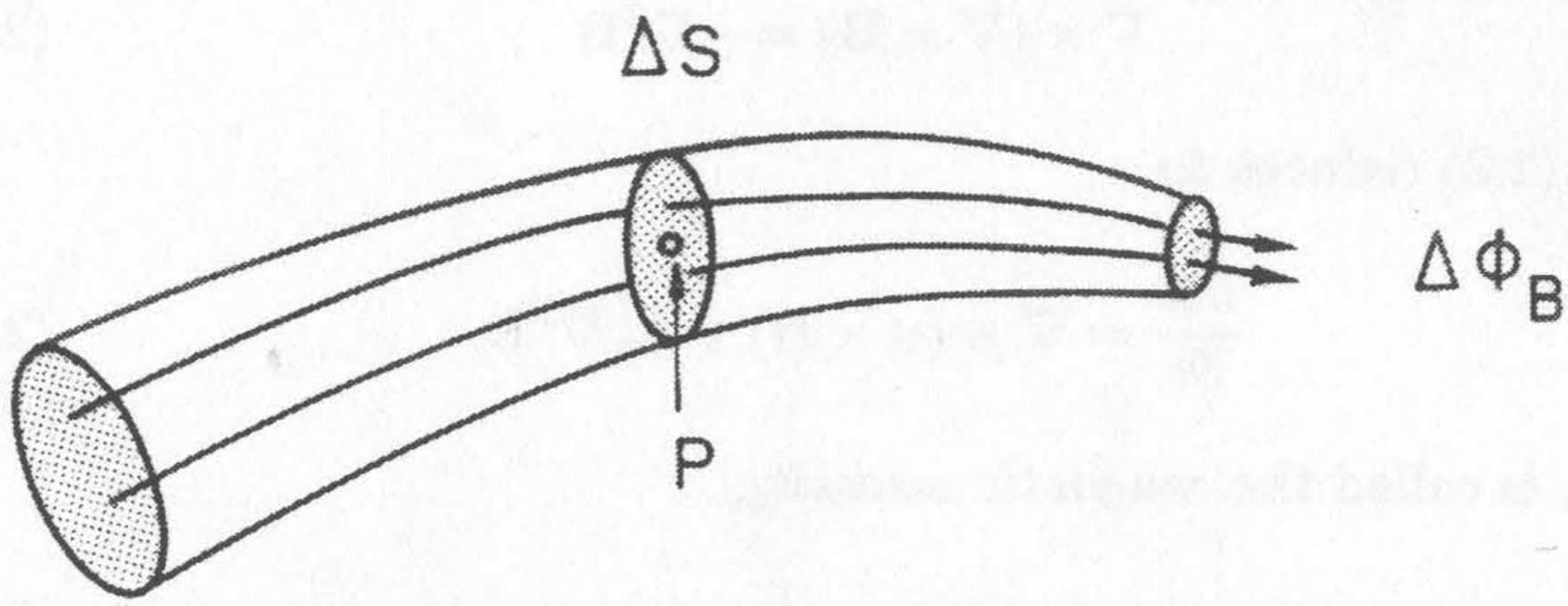

Fig. 1 *Elementary magnetic tube of force. The magnitude of* **B** *at the point P is equal to* $\Delta\Phi_B/\Delta S$.

The rate of change of the magnetic flux through an open surface **S** can be written as

$$\frac{d}{dt}\Big[\int_S \mathbf{B}(\mathbf{r},t)\cdot d\mathbf{S}\Big] =$$

$$= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\Big[\int_{S_2}\mathbf{B}(\mathbf{r},t+\Delta t)\cdot d\mathbf{S} - \int_{S_1}\mathbf{B}(\mathbf{r},t)\cdot d\mathbf{S}\Big] \tag{4.3}$$

If we expand $\mathbf{B}(\mathbf{r},t+\Delta t)$ in a Taylor series about $\mathbf{B}(\mathbf{r},t)$, we obtain

$$\mathbf{B}(\mathbf{r},t+\Delta t) = \mathbf{B}(\mathbf{r},t) + \frac{\partial\mathbf{B}(\mathbf{r},t)}{\partial t}\Delta t + \cdots \tag{4.4}$$

so that, in the limit as $\Delta t \to 0$, (4.3) reduces to

$$\frac{d}{dt}\Big[\int_S \mathbf{B}(\mathbf{r},t)\cdot d\mathbf{S}\Big] = \lim_{\Delta t\to 0}\Big\{\int_{S_2}\frac{\partial\mathbf{B}(\mathbf{r},t)}{\partial t}\cdot d\mathbf{S} +$$

$$+\frac{1}{\Delta t}\Big[\int_{S_2}\mathbf{B}(\mathbf{r},t)\cdot d\mathbf{S} - \int_{S_1}\mathbf{B}(\mathbf{r},t)\cdot d\mathbf{S}\Big]\Big\} \tag{4.5}$$

To evaluate the term within brackets in the right-hand side of this equation, we can use the fact that for any *closed surface* at time t we have, from Gauss' divergence theorem,

$$\oint_S \mathbf{B}\cdot d\mathbf{S} = \int_V \nabla\cdot\mathbf{B}\; d^3r = 0 \tag{4.6}$$

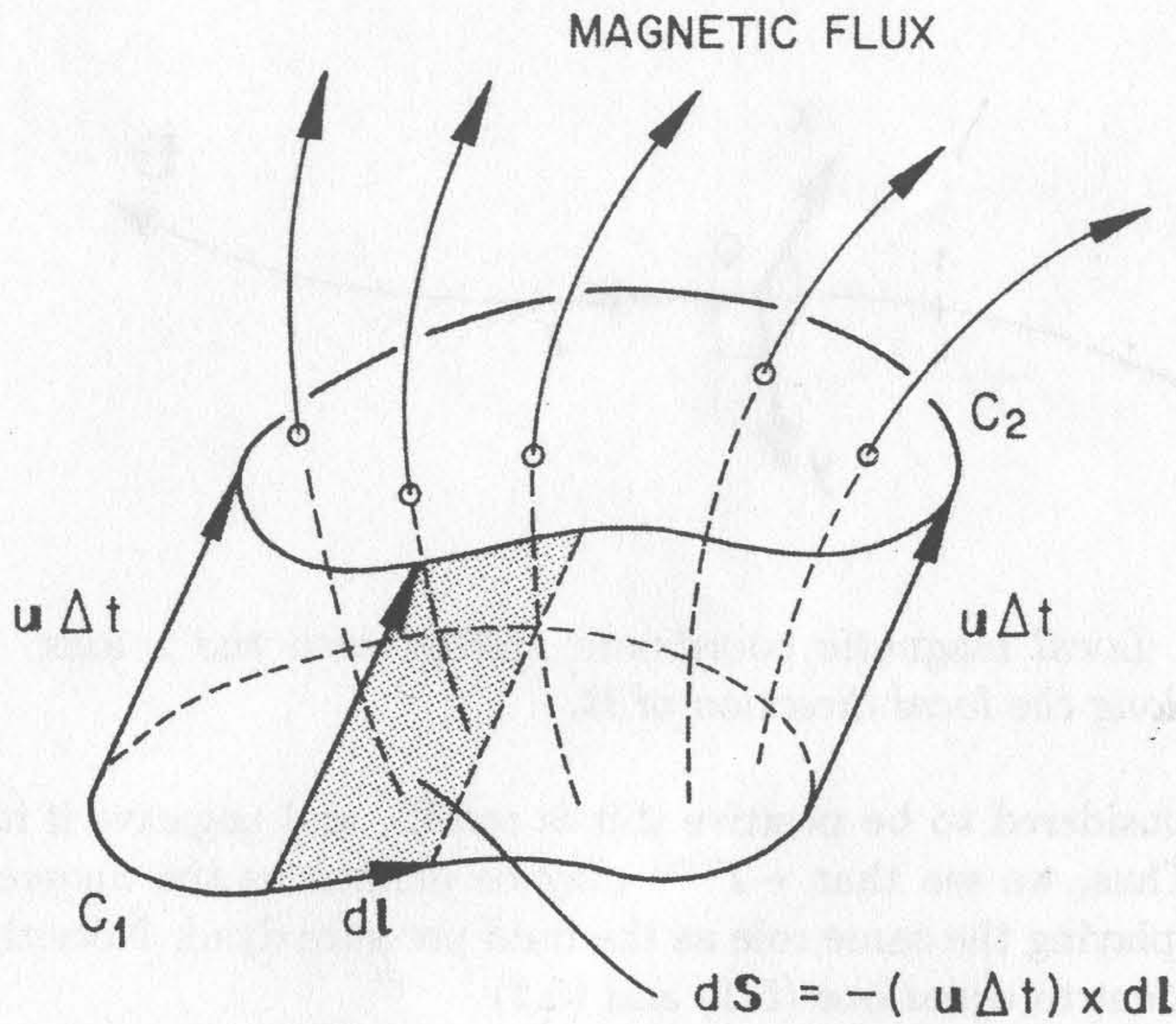

Fig. 2 *A closed line bounding an open surface moving in a magnetic field with velocity* $\mathbf{u}(\mathbf{r},t)$*, viewed at the instants of time* t *and* $t+\Delta t$*. The shaded area is the part of the cylindrical surface described by an element* $d\ell$ *of the contour curve.*

since $\nabla \cdot \mathbf{B} = 0$. Thus, if we apply this result to the closed surface consisting of $\mathbf{S}_1$, $\mathbf{S}_2$ and the sides of the cylindrical surface of length $\mathbf{u}\Delta t$ shown in Fig. 2, we obtain

$$-\int_{S_1} \mathbf{B}(\mathbf{r},t)\cdot d\mathbf{S} + \int_{S_2} \mathbf{B}(\mathbf{r},t)\cdot d\mathbf{S} - \oint_{C_1} \mathbf{B}(\mathbf{r},t)\cdot[(\mathbf{u}\Delta t)\times d\mathbf{l}] = 0 \quad (4.7)$$

where the minus sign in the first term on the left-hand side is due to the fact that the *outwardly drawn* unit normal to the surface S_1 is in a direction opposite to that of the surface S_2, and $-(\mathbf{u}\Delta t)\times d\mathbf{l}$ is the element of area (pointing outwards) covered by the vector element $d\mathbf{l}$ of the closed line in the time interval Δt. If (4.7) is substituted into (4.5) and the limit $\Delta t \to 0$ is evaluated, noting that in this limit $S_2 = S_1 = S(t)$, we obtain

$$\frac{d}{dt}\left[\int_S \mathbf{B}(\mathbf{r},t)\cdot d\mathbf{S}\right] = \int_S \frac{\partial \mathbf{B}(\mathbf{r},t)}{\partial t}\cdot d\mathbf{S} + \oint_C \mathbf{B}(\mathbf{r},t)\cdot(\mathbf{u}\times d\mathbf{l}) \quad (4.8)$$

The last term in the right-hand side of this equation can be transformed using the vector identity

$$\mathbf{B}(\mathbf{r},t)\cdot(\mathbf{u}\times d\mathbf{l}) = -[\mathbf{u}\times\mathbf{B}(\mathbf{r},t)]\cdot d\mathbf{l} \tag{4.9}$$

and from Stokes' theorem we can write

$$\oint_C [\mathbf{u}\times\mathbf{B}(\mathbf{r},t)]\cdot d\mathbf{l} = \int_S \nabla\times[\mathbf{u}\times\mathbf{B}(\mathbf{r},t)]\cdot d\mathbf{S} \tag{4.10}$$

Thus, using this expression in (4.8), we obtain the general result

$$\frac{d}{dt}\Big[\int_S \mathbf{B}(\mathbf{r},t)\cdot d\mathbf{S}\Big] = \int_S \Big\{\frac{\partial\mathbf{B}(\mathbf{r},t)}{\partial t} - \nabla\times[\mathbf{u}\times\mathbf{B}(\mathbf{r},t)]\Big\}\cdot d\mathbf{S} \tag{4.11}$$

Suppose now that the space is filled with a highly conducting fluid so that (4.1), which is valid for $R_m \gg 1$, applies. If the velocity $\mathbf{u}$ in (4.11) is taken as the fluid velocity, we conclude, from (4.1) and (4.11), that

$$\frac{d}{dt}\Big[\int_S \mathbf{B}(\mathbf{r},t)\cdot d\mathbf{S}\Big] = 0 \tag{4.12}$$

which is a mathematical statement of the fact that the magnetic flux linked by a closed line (bounding the open surface S) moving with the fluid velocity $\mathbf{u}$ is constant. Note that this conclusion requires that only the velocity component of the *closed line* perpendicular to $\mathbf{B}$ be the same as the *fluid* velocity component perpendicular to $\mathbf{B}$, since the velocity component parallel to $\mathbf{B}$ gives no contribution to the term $\mathbf{u}\times\mathbf{B}$. Thus, (4.1) implies that the lines of magnetic flux are frozen into the highly conducting fluid and are carried by any motion of the fluid *perpendicular* to the field lines. There is no restriction, however, on the motion along the field lines so that the conducting fluid can flow freely in the direction *parallel* to $\mathbf{B}$.

This result is expected on physical grounds since, as the conducting fluid moves across the magnetic field, it induces an electric field which is proportional to the fluid velocity component perpendicular to $\mathbf{B}$. However, if the fluid conductivity is infinite, this perpendicular velocity component must be infinitesimally small if the flow of electric current is to remain finite.

In a fluid of *finite* conductivity the result (4.12) is no longer true. Using (2.4) in the general result (4.11), yields

$$\frac{d\Phi_B}{dt} = \frac{1}{\mu_0\sigma_0}\int_S \nabla^2\mathbf{B}\cdot d\mathbf{S} \tag{4.13}$$

where the right-hand side of this equation gives rise to a slipping of magnetic flux through a closed material line.

5. MAGNETIC PRESSURE

5.1 Concept of Magnetic Pressure

The concept of *magnetic pressure* is very useful in the study of high temperature plasma confinement. Under steady state conditions the MHD equations reduce to the following closed set of *magnetohydrostatic* equations

$$\nabla p = \mathbf{J} \times \mathbf{B} \tag{5.1}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{5.2}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{5.3}$$

If we eliminate $\mathbf{J}$ from these equations, we obtain the following equivalent set of magnetohydrostatic equations involving only p and $\mathbf{B}$,

$$\nabla p = \frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \mathbf{B} \tag{5.4}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{5.5}$$

The term in the right-hand side of (5.4) can be written as the divergence of the magnetic part of the *electromagnetic stress dyad.* Using the vector identity

$$(\nabla \times \mathbf{B}) \times \mathbf{B} = (\mathbf{B} \cdot \nabla)\mathbf{B} - \tfrac{1}{2}\nabla(B^2) = \nabla \cdot (\mathbf{BB}) - \nabla \cdot (\tfrac{1}{2}\mathbf{1}B^2) \tag{5.6}$$

where $\mathbf{1}$ is the unit dyad, and using the following definition of the *magnetic stress dyad*

$$\mathcal{T}^{(m)} = \frac{1}{\mu_0}(\mathbf{BB} - \tfrac{1}{2}\mathbf{1}B^2) \tag{5.7}$$

which written out in matrix form (in a Cartesian coordinate system) is

$$\mathcal{T}^{(m)} = \frac{1}{\mu_0}\begin{pmatrix} (B_x^2 - B^2/2) & B_xB_y & B_xB_z \\ B_yB_x & (B_y^2 - B^2/2) & B_yB_z \\ B_zB_x & B_zB_y & (B_z^2 - B^2/2) \end{pmatrix} \tag{5.8}$$

we can write (5.4) as

$$\nabla p = \nabla \cdot \mathcal{T}^{(m)} \tag{5.9}$$

or, equivalently,

$$\nabla \cdot (\mathbf{1}p - \mathcal{T}^{(m)}) = 0 \tag{5.10}$$

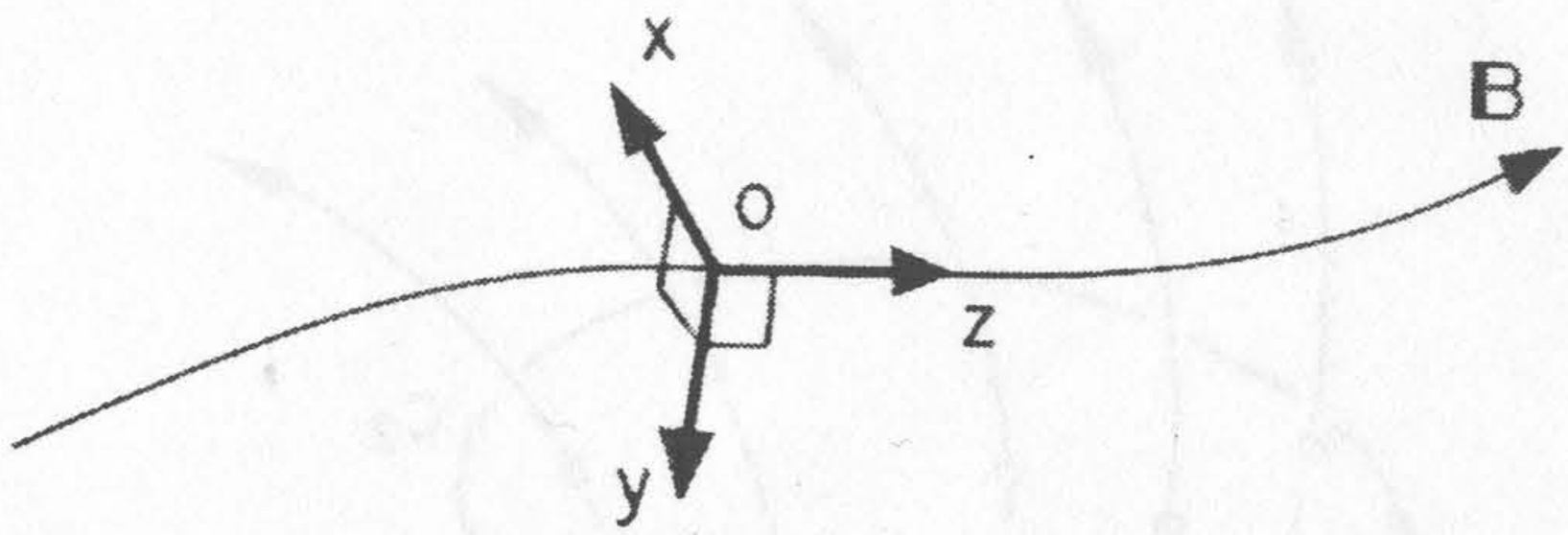

Fig. 3 *Local magnetic coordinate system with the z axis pointing along the local direction of* $\mathbf{B}$.

The stress is considered to be positive if it is tensile, and negative if it is compressive. Thus, we see that $-\mathcal{T}^{(m)}$ may be defined as the *magnetic pressure* dyad, playing the same role as the fluid pressure dyad. Note that (5.10) is equivalent to equations (5.1) and (5.2).

It is instructive to consider a *local* magnetic coordinate system in which the third axis points along the local direction of $\mathbf{B}$, as shown in Fig. 3. For this local coordinate system, the off-diagonal elements of the magnetic stress dyad vanish, since $\mathbf{B} = B\widehat{\mathbf{z}}$, so that

$$\mathcal{T}^{(m)} = \begin{pmatrix} -B^2/2\mu_0 & 0 & 0 \\ 0 & -B^2/2\mu_0 & 0 \\ 0 & 0 & B^2/2\mu_0 \end{pmatrix} \tag{5.11}$$

Therefore, the principal stresses are equivalent to a *tension* $B^2/2\mu_0$ along the magnetic field lines, and a *pressure* $B^2/2\mu_0$ perpendicular to the magnetic field lines, similar to a mutual repulsion of the field lines. Alternatively, we can express (5.11) in the form

$$\mathcal{T}^{(m)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & B^2/\mu_0 \end{pmatrix} + \begin{pmatrix} -B^2/2\mu_0 & 0 & 0 \\ 0 & -B^2/2\mu_0 & 0 \\ 0 & 0 & -B^2/2\mu_0 \end{pmatrix} \tag{5.12}$$

so that the stress caused by the magnetic flux can also be thought of as an *isotropic magnetic pressure* $B^2/2\mu_0$ and a *tension* B^2/μ_0 along the magnetic flux lines as if they were elastic cords (see Fig. 4). This latter representation is very useful, since the isotropic pressure $B^2/2\mu_0$ can

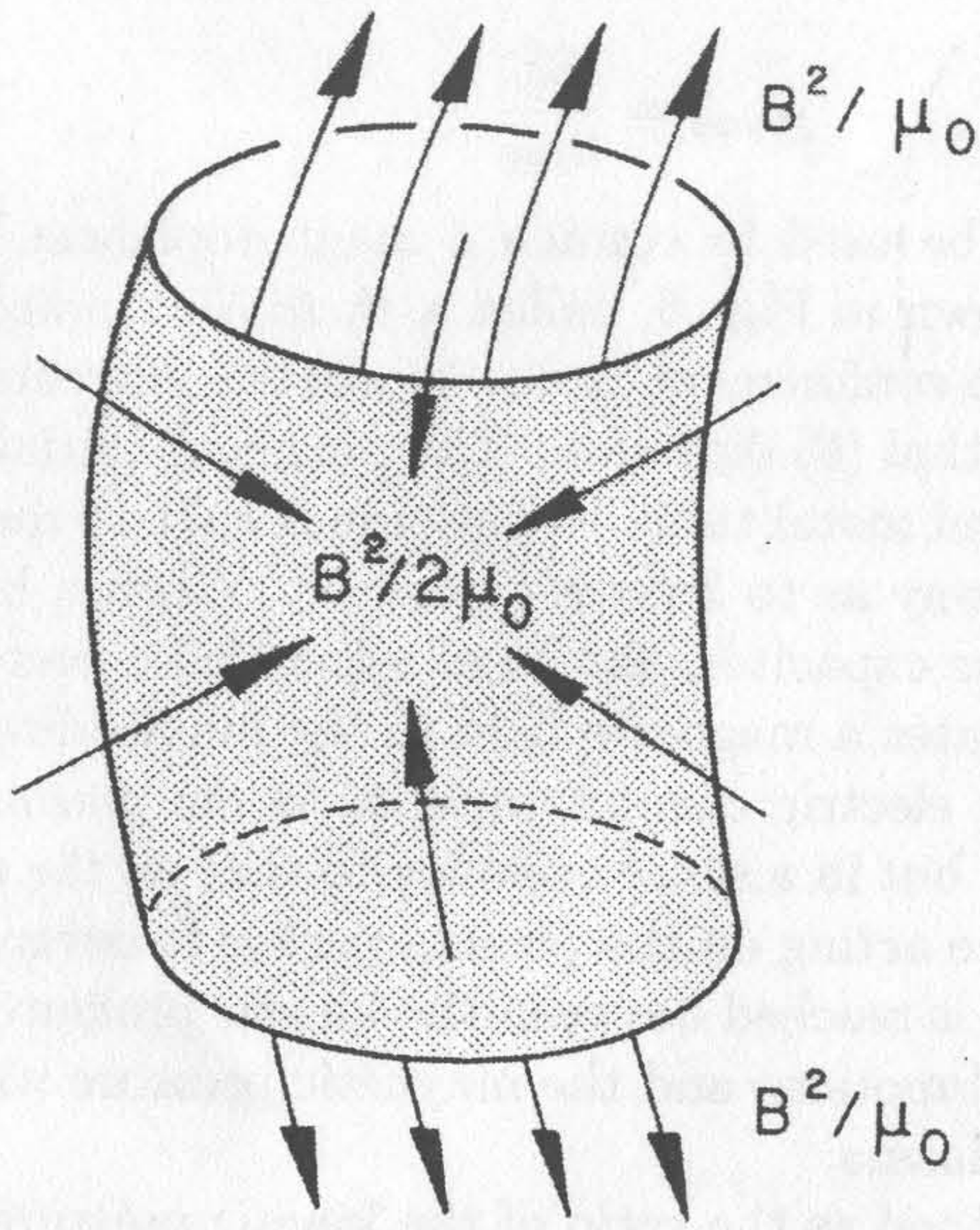

Fig. 4 *The stress caused by the magnetic flux can be decomposed into an isotropic magnetic pressure $B^2/2\mu_0$ and a magnetic tension B^2/μ_0 along the field lines.*

always be superposed on the fluid pressure, resulting in a decrease in the pressure exerted by the fluid.

5.2 Isobaric Surfaces

It is convenient to consider in the plasma hypothetical surfaces over which the kinetic pressure is constant, called *isobaric surfaces.* At any point, the vector ∇p is normal to the isobaric surface passing through the point considered. From (5.1) we see that ∇p is normal to the plane containing $\mathbf{J}$ and $\mathbf{B}$, that is

$$\mathbf{J} \cdot \nabla p = 0 \tag{5.13}$$

$$\mathbf{B} \cdot \nabla p = 0 \tag{5.14}$$

Therefore, both **J** and **B** lie on isobaric surfaces. To illustrate this point, consider the particular case in which the isobaric surfaces are closed concentric cylindrical surfaces, with the kinetic pressure increasing in the direction towards the central axis of the concentric cylindrical surfaces. Thus, ∇p is along a radial line directed towards the axis. From (5.13) and (5.14) we see that neither **B**, nor **J**, pass through the isobaric surfaces and therefore it follows that the cylindrical isobaric surfaces are formed by a network of magnetic field lines and electric currents. Further, in view of (5.1), the magnetic field lines and electric currents, lying on the isobaric surfaces, must cross each other in such a way that $\mathbf{J} \times \mathbf{B}$ is equal to ∇p. This situation is shown in Fig. 5. The maximum kinetic pressure occurs along the central axis, which also coincides with a magnetic field line. For this reason, this axis is usually called the *magnetic axis* of the magnetoplasma configuration.

6. PLASMA CONFINEMENT IN A MAGNETIC FIELD

The subject of plasma confinement by magnetic fields is of considerable interest in the theory of controlled thermonuclear fusion. Consider, for simplicity, the special case in which the magnetic field is along the z axis, that is $\mathbf{B} = B\widehat{\mathbf{z}}$, so that (5.10) simplifies to

$$\nabla \cdot \begin{pmatrix} (p + B^2/2\mu_0) & 0 & 0 \\ 0 & (p + B^2/2\mu_0) & 0 \\ 0 & 0 & (p - B^2/2\mu_0) \end{pmatrix} = 0 \qquad (6.1)$$

from which we obtain

$$\frac{\partial}{\partial x}\left(p + \frac{B^2}{2\mu_0}\right) = 0 \qquad (6.2)$$

$$\frac{\partial}{\partial y}\left(p + \frac{B^2}{2\mu_0}\right) = 0 \qquad (6.3)$$

$$\frac{\partial}{\partial z}\left(p - \frac{B^2}{2\mu_0}\right) = 0 \qquad (6.4)$$

Also, from $\nabla \cdot \mathbf{B} = 0$, we have

$$\frac{\partial B}{\partial z} = 0 \qquad (6.5)$$

since, in the local coordinate system, **B** is along the z axis. This last equation, together with (6.4), implies that both p and **B** do not vary in

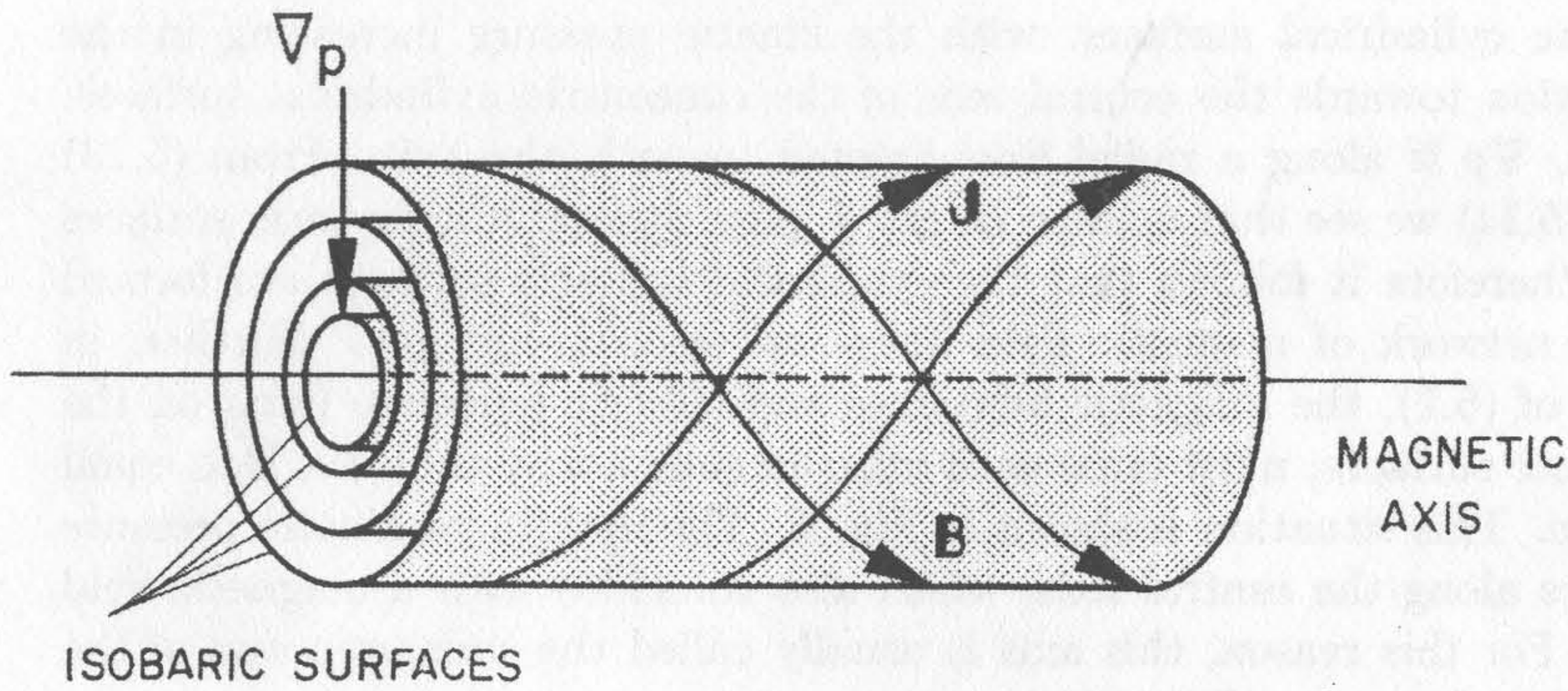

Fig. 5 *Isobaric concentric cylindrical surfaces, with ∇p along a radial directed towards the magnetic axis. The lines of* **B** *and* **J** *lie on the isobaric surfaces and cross each other in a such a manner that* $\mathbf{J} \times \mathbf{B}$ *is equal to* ∇p.

the z direction. The solutions of (6.2) and (6.3), combined with this result, give

$$\left(p + \frac{B^2}{2\mu_0}\right) = constant \tag{6.6}$$

Therefore, in the presence of an externally applied magnetic field, if the plasma is bounded, the plasma kinetic pressure decreases from the axis radially outwards, whereas the magnetic pressure increases in the same direction in such a manner that their sum remains constant at each point, according to (6.6). The plasma kinetic pressure can be forced to vanish on an outer surface if the applied magnetic field is sufficiently strong, with the result that the plasma is confined within this outer surface by the magnetic field.

Let $\mathbf{B}_0$ be the value of the magnetic induction at the plasma boundary. Since the kinetic pressure at the plasma boundary is zero (ideally), we can evaluate the constant in (6.6) from the pressure equilibrium condition at the plasma boundary. Therefore,

$$p + \frac{B^2}{2\mu_0} = \frac{B_0^2}{2\mu_0} \tag{6.7}$$

The maximum fluid pressure that can be confined for a given applied field $\mathbf{B}_0$ is, consequently,

$$p_{max} = \frac{B_0^2}{2\mu_0} \tag{6.8}$$

A device that can be used to confine a magnetoplasma by straight parallel field lines is shown in Fig. 6, called a *theta* (θ) - *pinch*, since the effect responsible for the confinement is due to electric currents flowing in the plasma in the azimuthal (θ) direction. The plasma is initially confined inside a hollow cylindrical metal tube, whose side is split in the longitudinal direction in such a way as to form a capacitor. When a high voltage is discharged through the capacitor, the large azimuthal current produced in the metal tube generates a magnetic field in the longitudinal direction inside the plasma. The electric current *induced in the plasma* is also in the azimuthal direction, but in a sense opposite to that on the metal tube. The resulting $\mathbf{J} \times \mathbf{B}$ force acting on the plasma pushes it inwards, towards the axis, until a balance is reached between the kinetic pressure due to the random particle thermal motions and the magnetic pressure which acts to constrict or *pinch* the plasma.

A parameter β, defined as the ratio of the kinetic pressure at a point inside the plasma, to the confining magnetic pressure at the plasma boundary, is usually introduced as a measure of the relative magnitudes of the kinetic and magnetic pressures. It is given by

$$\beta = \frac{p}{B_0^2/2\mu_0} \tag{6.9}$$

Note that β ranges between 0 and 1, since the field inside the plasma is less than B_0. From (6.7) we can also express the parameter β as

$$\beta = 1 - \left(\frac{B}{B_0}\right)^2 \tag{6.10}$$

Two special cases of plasma confinement schemes are the so called *low* β and *high* β devices. In the low β devices, the kinetic pressure is small in comparison to the magnetic pressure at the plasma boundary, whereas in the high β devices they are of an equal order of magnitude ($\beta \simeq 1$).

An important property of a plasma is its *diamagnetic* character. Equation (6.7) implies that the magnetic field inside the plasma is less than its value at the plasma boundary. As the kinetic pressure increases inside the plasma, the magnetic field decreases. Under the action of the externally aplied $\mathbf{B}$ field, the particle motions give rise to internal electric

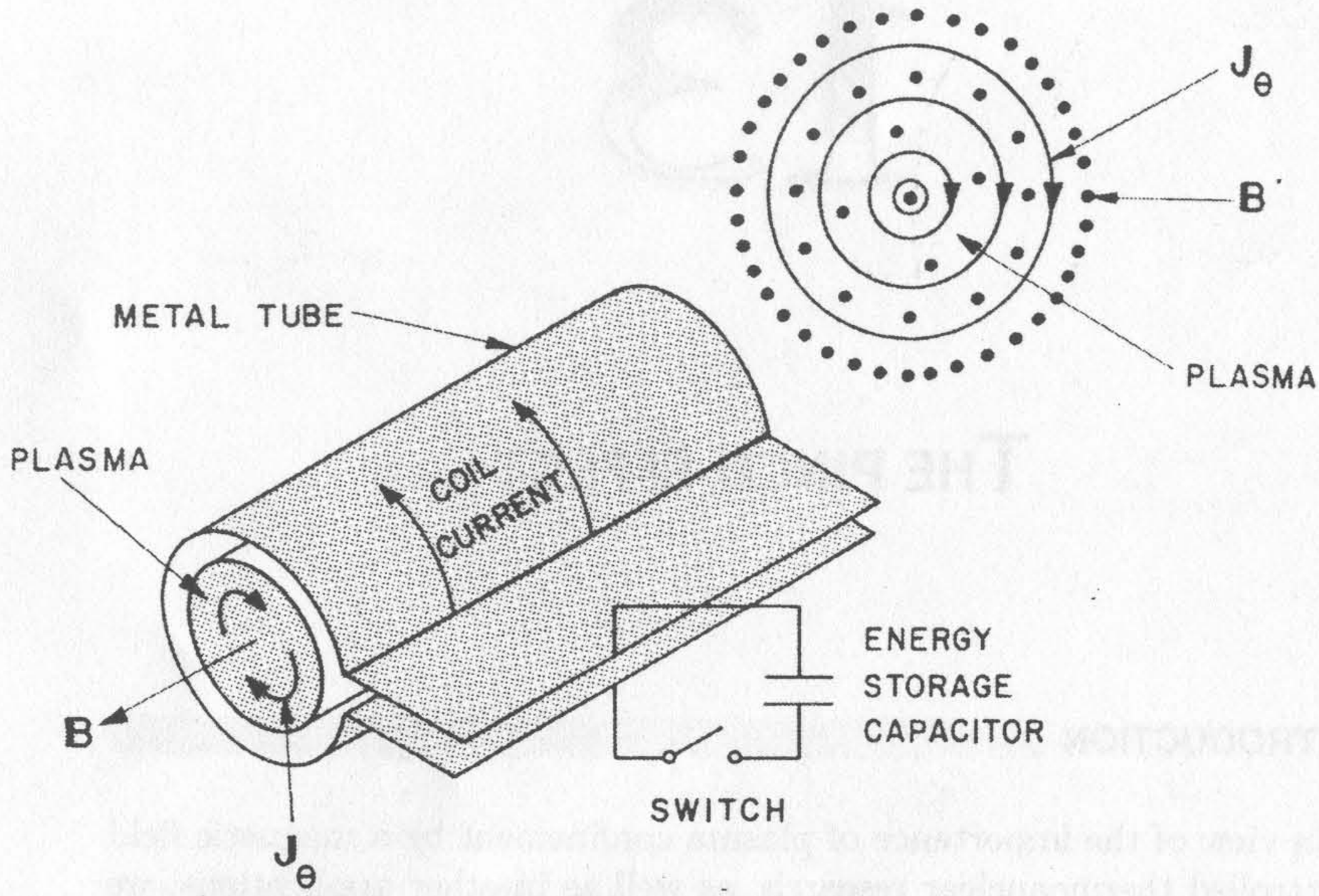

Fig. 6 *Magnetoplasma confined by straight parallel field lines in a theta-pinch device.*

currents which induce a magnetic field opposite to the externally applied field. Consequently, the *resultant* magnetic field inside the plasma is reduced to a value less than that at the plasma boundary. The electric current, induced in the plasma, depends on the number density of the charged particles and on their velocity. Therefore, as the plasma kinetic pressure increases, the induced electric current and the induced magnetic field also increase, thus enhancing the diamagnetic effect.

PROBLEMS

12.1 Consider the energy equation involving the time rate of change of the total pressure dyad $\mathcal{P}$, derived in Problem 9.6. Show that when this equation is contracted with the unit dyad **1** results in (1.21), whereas when contracted with the dyad **BB** yields (1.22).

12.2 Derive an energy conservation equation, similar to (1.64), but considering the Parker modified momentum equation and the CGL energy equations, instead of (1.2) and (1.3).

12.3 Calculate the *minimum* intensity of the magnetic induction ($\mathbf{B}_0$) necessary to confine a plasma at:
(a) an internal pressure of 100 atm;
(b) a temperature of 10 keV and density of $8 \times 10^{21}\mathrm{m}^{-3}$.

12.4 A plasma is confined by a unidirectional magnetic induction $\mathbf{B}$ of magnitude 5 weber/m^2. Considering that the plasma temperature is 10 keV and $\beta = 0.4$, calculate the particle number density. If the temperature increases to 50 keV, what is the value of the $\mathbf{B}$ field necessary to confine the plasma, assuming that β stays the same?

12.5 Calculate the diffusion time (τ_D) and the magnetic Reynolds number (R_m) for a typical MHD generator, with $L = 0.1$ m, $u = 10^3$ m/s and $\sigma_0 = 100$ mho/m. Verify that in this case τ_D is very short, so that inhomogeneities in the magnetic field are smoothed out rapidly.

12.6 Consider a plasma in the form of a straight circular cylinder with a helical magnetic field given by

$$\mathbf{B} = B_\theta(r)\widehat{\boldsymbol{\Theta}} + B_z(r)\widehat{\mathbf{z}}$$

Show that the force per unit volume, associated with the inward magnetic pressure for this configuration, is

$$-\nabla_\perp\left(\frac{B^2}{2\mu_0}\right) = -\widehat{\mathbf{r}}\frac{\partial}{\partial r}\left[\frac{B^2(r)}{2\mu_0}\right]$$

and the force per unit volume, associated with the magnetic tension due to the curvature of the magnetic field lines, is

$$\left(\frac{B^2}{\mu_0}\right)(\mathbf{B}\cdot\nabla)\mathbf{B} = -\widehat{\mathbf{r}}\frac{B_\theta^2(r)}{\mu_0 r}$$

12.7 Use (4.1), for a perfectly conducting fluid, and the nonlinear equation of continuity (1.1), to show that the change of $\mathbf{B}$ with time in a fluid element is related to changes of density according to

$$\frac{D}{Dt}\left(\frac{\mathbf{B}}{\rho_m}\right) = \frac{1}{\rho_m}(\mathbf{B}\cdot\nabla)\mathbf{u}$$

Use this relation to establish that, in a perfectly conducting fluid, the fluid elements which lie initially on a magnetic flux line continue to lie on a flux line.

12.8 The boundary of the Earth's magnetosphere, in the direction of the Earth-Sun line, occurs at a distance where the kinetic pressure of the solar wind particles is equal to the (modified) Earth's magnetic field pressure. Show that the distance of the magnetopause from the center of the Earth, along the Earth-Sun line, is given approximately by

$$R_M = \left(\frac{2B_0^2}{\mu_0 \rho_m u_s^2}\right)^{1/6} R_E$$

where R_E is the Earth's radius, ρ_m is the mass density of the solar wind, u_s is its undisturbed speed and B_0 is the surface value of the undisturbed Earth's magnetic field.

12.9 Consider a cylindrically symmetric plasma column ($\partial/\partial z = 0$, $\partial/\partial\theta = 0$) under equilibrium conditions, confined by a magnetic field. Verify that in cylindrical coordinates the radial component of (5.1) becomes

$$\frac{dp(r)}{dr} = J_\theta(r)\ B_z(r) - J_z(r)\ B_\theta(r)$$

Using Maxwell equation (5.2), show that

$$J_\theta = -\frac{1}{\mu_0}\frac{dB_z}{dr}$$

$$J_z = \frac{1}{\mu_0 r}\frac{d(rB_\theta)}{dr}$$

From these results, obtain the following basic equation for the equilibrium of a plasma column with cylindrical symmetry

$$\frac{d}{dr}\left(p + \frac{B_z^2}{2\mu_0} + \frac{B_\theta^2}{2\mu_0}\right) = -\frac{1}{\mu_0}\frac{B_\theta^2}{r}$$

Give a physical interpretion for the various terms in this equation.

13

THE PINCH EFFECT

1. INTRODUCTION

In view of the importance of plasma confinement by a magnetic field in controlled thermonuclear research, as well as in other applications, we present in this chapter a detailed treatment of plasma confinement for the special case in which the confinement is produced by an azimuthal (θ) self-magnetic field, due to an axial current in the plasma generated by an appropriately applied electric field.

Consider an infinite cylindrical column of conducting fluid with an axial current density $\mathbf{J} = J_z(r)\ \widehat{\mathbf{z}}$ and a resulting azimuthal magnetic induction $\mathbf{B} = B_\theta(r)\ \widehat{\boldsymbol{\Theta}}$, as depicted in Fig. 1. The $\mathbf{J} \times \mathbf{B}$ force, acting on the plasma, forces the column to contract radially. This radial constriction of the plasma column is known as the *pinch effect.* In this case the isobaric surfaces, for which $p = constant$, are concentric cylinders.

As the plasma is compressed radially, the plasma number density and the temperature increase. The plasma kinetic pressure counteracts to hinder the constriction of the plasma column, whereas the magnetic force acts to confine the plasma. When these counteracting forces are balanced, a steady state condition results in which the plasma is mainly confined within a certain radius R, which remains constant in time. This situation is commonly referred to as the *equilibrium pinch.* When the self-magnetic pressure exceeds the plasma kinetic pressure, the column radius changes with time resulting in a situation known as the *dynamic pinch.* In what follows we investigate first the equilibrium pinch and afterwards the dynamic pinch.

20

PARTICLE INTERACTIONS IN PLASMAS

1. INTRODUCTION

The fundamental properties of a plasma depend upon the interactions between the plasma particles and the existing force fields. These fields may be externally applied or they can be internal fields associated with the nature and motion of the particles themselves. In this text the words *collision* and *interaction* are used synonymously. The notion of a collision as a physical contact between bodies loses its utility in the microscopic world. In the atomic level a *collision* between particles must be regarded as an *interaction* between the force fields associated with each of the interacting particles.

Collisional phenomena can be broadly divided into two categories: *elastic* and *inelastic*. In *elastic collisions* there is conservation of mass, of momentum and of energy in such a way that there are no changes in the internal states of the particles involved and there is neither creation nor annihilation of particles. In *inelastic collisions* the internal states of some or all of the particles involved are changed and particles may be created as well as destroyed. In inelastic collisions a charged particle may *recombine* with another to form a neutral particle or it can *attach* itself with a neutral particle to form a heavier charged particle. Also, the energy state of an electron in an atom may be raised and electrons can be removed from their atoms resulting in *ionization*.

In plasmas there is an important distinction to be made between charge-charge and charge-neutral interactions. Electrically charged particles interact with one another according to Coulomb's law. This Coulomb

interaction, in view of its $1/r^2$ dependence, is a *long-range interaction*, so that the field of one particle interacts simultaneously with a large number of other particles. Therefore, it involves *multiple* interactions. In contrast, the fields associated with neutral particles are significantly strong only within the electronic shells of the particles. Thus, they are *short-range fields* and a neutral particle only occasionally interacts with another particle and very rarely it interacts simultaneously with more than one particle. Therefore, these short-range fields result primarily in *binary* interactions.

The multiple-particle Coulomb interaction, however, can be thought of as a number of simultaneous binary interactions. In fact, one way of dealing with multiple interactions is to consider that a series of consecutive small-angle binary interactions describe the situation. The multiple interactions, which result from the Coulomb force, are of essential importance in understanding the behavior of plasmas and underlines the validity of describing a plasma as the fourth state of matter. Nevertheless, binary collisions adequately describe plasma phenomena in the case of *weakly ionized plasmas.* In fact, we use the term weakly ionized plasma to mean a plasma in which multiple particle interactions can be ignored. In these plasmas the electrons tend to dominate the situation, since they respond quickly to the influence of electric and magnetic fields, in view of their low inertia.

In this chapter we deal with collisional processes that are of importance in plasmas, from the point of view of classical dynamics. The results are valid to a good approximation, even though the internal structure of the particles is ignored. More important, however, the procedures to be developed are useful whether the mechanics is classical or quantum.

2. BINARY COLLISIONS

Consider an elastic collision between two particles of mass m and m_1, having velocities $\mathbf{v}$ and $\mathbf{v}_1$ before collision, and $\mathbf{v}'$ and $\mathbf{v}_1'$, after collision. This binary interaction is illustrated in Fig. 1, as seen from the laboratory system. In what follows, the variables indicated with a *prime* are *after-collision* variables.

It is convenient to adopt a coordinate system in which the particle having mass m is at rest and the particle having mass m_1 approaches with the *relative velocity*

$$\mathbf{g} = \mathbf{v}_1 - \mathbf{v} \tag{2.1}$$

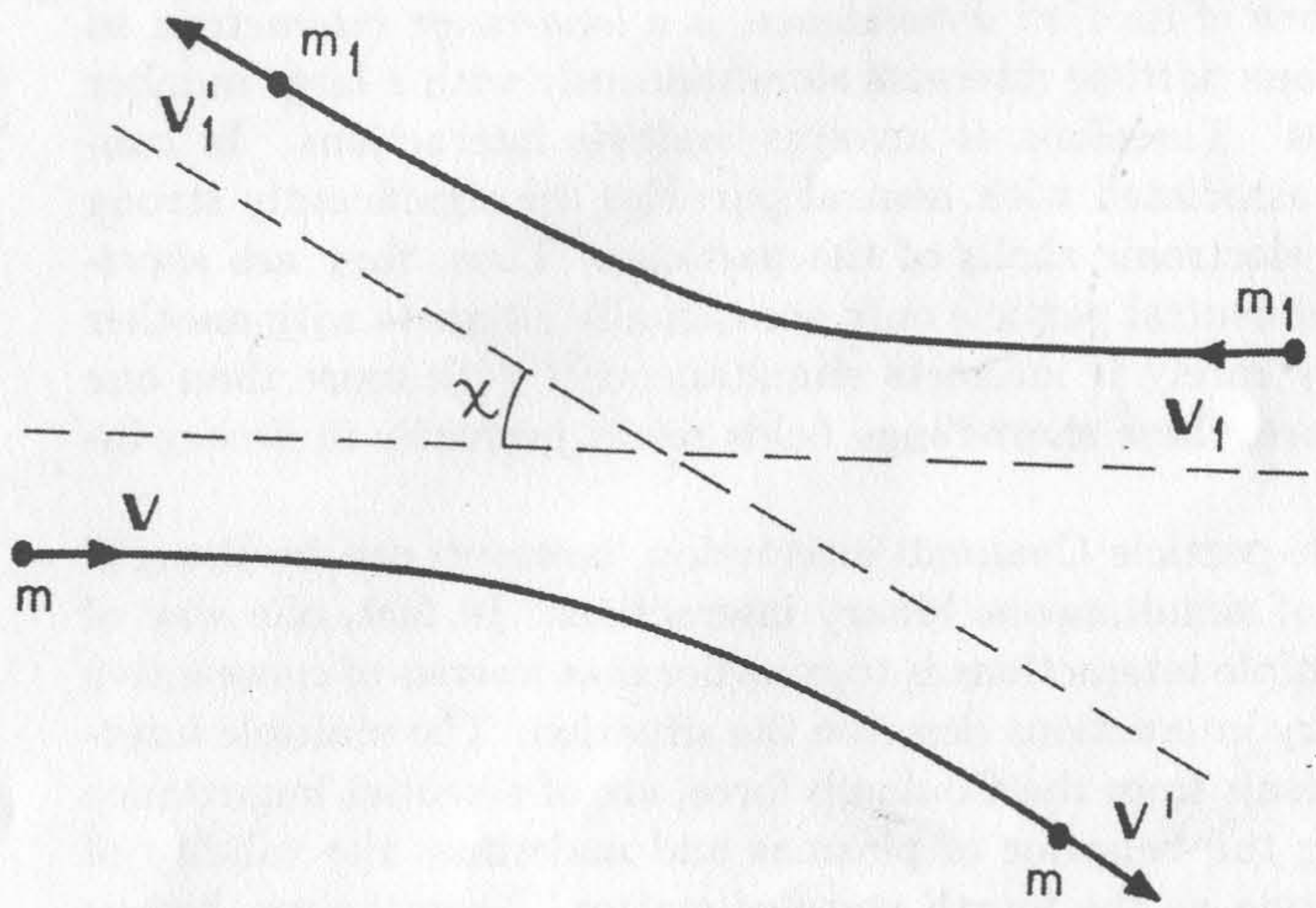

Fig. 1 *Binary interaction between two particles having mass m and m_1, with velocities $\mathbf{v}$ and $\mathbf{v}_1$, respectively, as seen from the laboratory system.*

After collision, the relative velocity is

$$\mathbf{g}' = \mathbf{v}_1' - \mathbf{v}' \tag{2.2}$$

The geometry of this interaction is shown in Fig. 2. The *impact parameter*, defined as the minimum distance of approach if there were no interaction, is denoted by b, the *scattering angle* by χ and the orientation of the *orbital plane* (or *collision plane*), with respect to some given direction in a plane normal to the orbital plane, is denoted by ϵ.

The velocity of the *center of mass* of the colliding particles, before collision, is defined by

$$\mathbf{c}_0 = \frac{m\mathbf{v} + m_1\mathbf{v}_1}{m + m_1} \tag{2.3}$$

and, after collision, by

$$\mathbf{c}_0' = \frac{m\mathbf{v}' + m_1\mathbf{v}_1'}{m + m_1} \tag{2.4}$$

We can express the initial velocities in terms of $\mathbf{c}_0$ and $\mathbf{g}$. From (2.3) and (2.1), which define $\mathbf{c}_0$ and $\mathbf{g}$, respectively, we find

$$\mathbf{v} = \mathbf{c}_0 - \frac{\mu}{m}\,\mathbf{g} \tag{2.5}$$

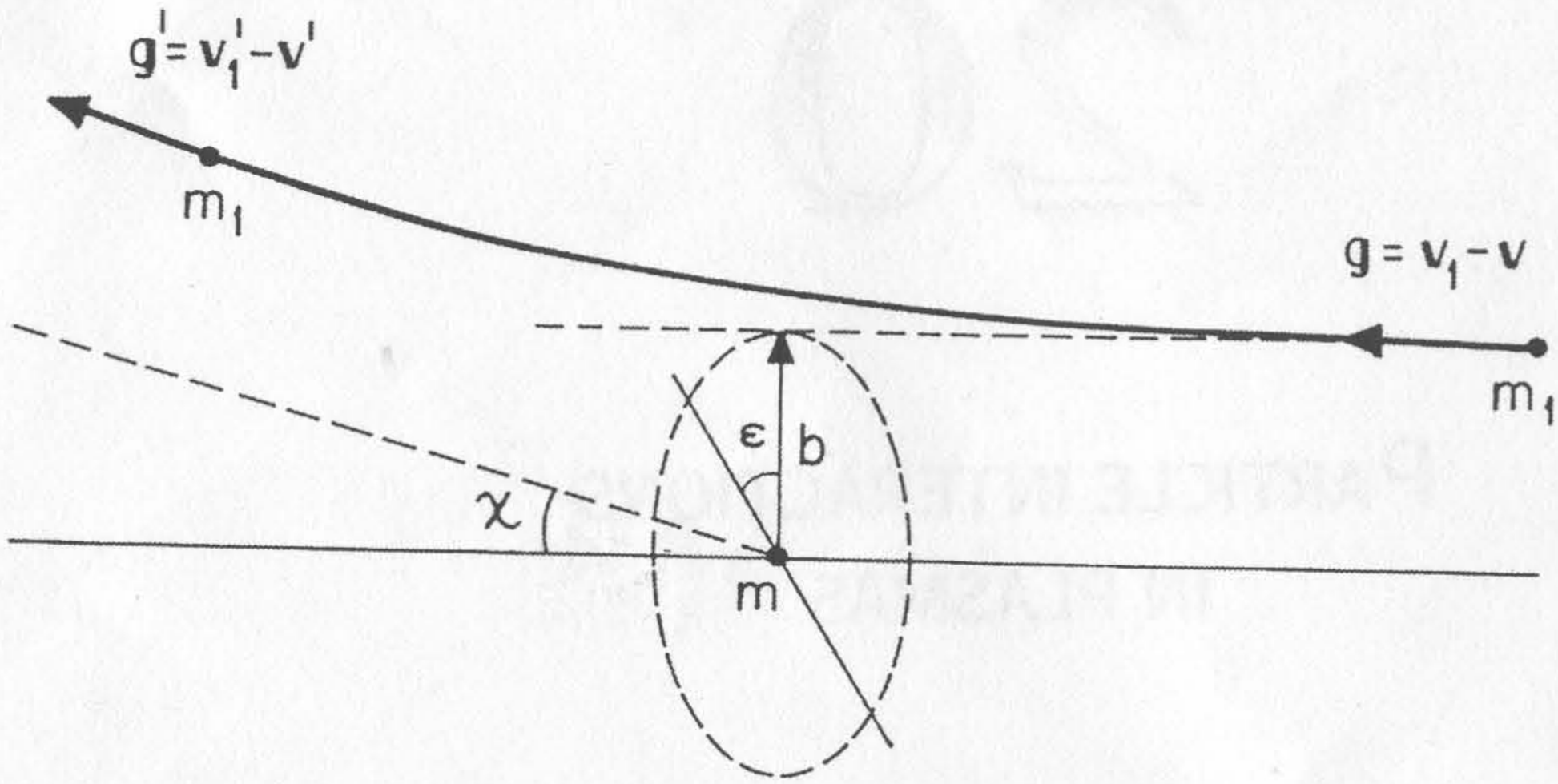

Fig. 2 *Geometry of a collision between a particle of mass m and velocity $\mathbf{v}$, and a particle of mass m_1 and velocity $\mathbf{v}_1$, viewed from a coordinate system in which the first particle is at rest.*

$$\mathbf{v}_1 = \mathbf{c}_0 + \frac{\mu}{m_1}\,\mathbf{g} \tag{2.6}$$

where μ denotes the *reduced mass*, defined by

$$\mu = \frac{mm_1}{m + m_1} \tag{2.7}$$

Similarly, from (2.4) and (2.2) we obtain, for the after-collision velocities,

$$\mathbf{v}' = \mathbf{c}_0' - \frac{\mu}{m}\,\mathbf{g}' \tag{2.8}$$

$$\mathbf{v}_1' = \mathbf{c}_0' + \frac{\mu}{m_1}\,\mathbf{g}' \tag{2.9}$$

From the law of *conservation of momentum* for the collision event, we have

$$m\mathbf{v} + m_1\mathbf{v}_1 = m\mathbf{v}' + m_1\mathbf{v}_1' \tag{2.10}$$

or, using (2.3) and (2.4),

$$(m + m_1)\,\mathbf{c}_0 = (m + m_1)\,\mathbf{c}_0' \tag{2.11}$$

Thus, we conclude that

$$\mathbf{c}_0 = \mathbf{c}_0' \tag{2.12}$$

that is, the velocity of the center of mass is the same before and after the collision event.

From the law of *conservation of energy* for elastic collisions, we have

$$\tfrac{1}{2}\left(mv^2 + m_1 v_1^2\right) = \tfrac{1}{2}\left[m(v')^2 + m_1(v_1')^2\right] \tag{2.13}$$

and using (2.5), (2.6), (2.8) and (2.9) we find, by direct calculation,

$$\tfrac{1}{2}\left(mv^2 + m_1 v_1^2\right) = \tfrac{1}{2}\left(m + m_1\right) c_0^2 + \tfrac{1}{2}\mu g^2 \tag{2.14}$$

$$\tfrac{1}{2}\left[m(v')^2 + m_1(v_1')^2\right] = \tfrac{1}{2}\left(m + m_1\right)(c_0')^2 + \tfrac{1}{2}\mu\,(g')^2 \tag{2.15}$$

Now, since $\mathbf{c}_0 = \mathbf{c}_0'$, we conclude that

$$g = g' \tag{2.16}$$

Thus, the *magnitude*, but not the direction of the relative velocity is conserved in a binary elastic collision. Equations (2.14) and (2.15) show that the total instantaneous kinetic energy of the two-particle system is equivalent to that associated with the motion of the center of mass plus the motion of one particle relative to the other, but using the reduced mass.

The angle between $\mathbf{g}$ and $\mathbf{g}'$ is the *scattering angle*, or *deflection angle*, which is denoted by χ. To relate the relative velocity vectors $\mathbf{g}$ and $\mathbf{g}'$, we can choose, for instance, a Cartesian coordinate system with the z axis along $\mathbf{g}$, as shown in Fig. 3. Thus, we have

$$g_x = g_y = 0 \tag{2.17}$$

$$g_z = g = g' \tag{2.18}$$

$$g_x' = g \sin\chi \cos\epsilon \tag{2.19}$$

$$g_y' = g \sin\chi \sin\epsilon \tag{2.20}$$

$$g_z' = g \cos\chi \tag{2.21}$$

where ϵ defines the relative orientation of the *collision plane*. Therefore, knowing the initial velocities and the scattering angle χ we can determine the after-collision velocities. The opposite is also true, that is, if we know the final velocities and the scattering angle, we can find the initial velocities.

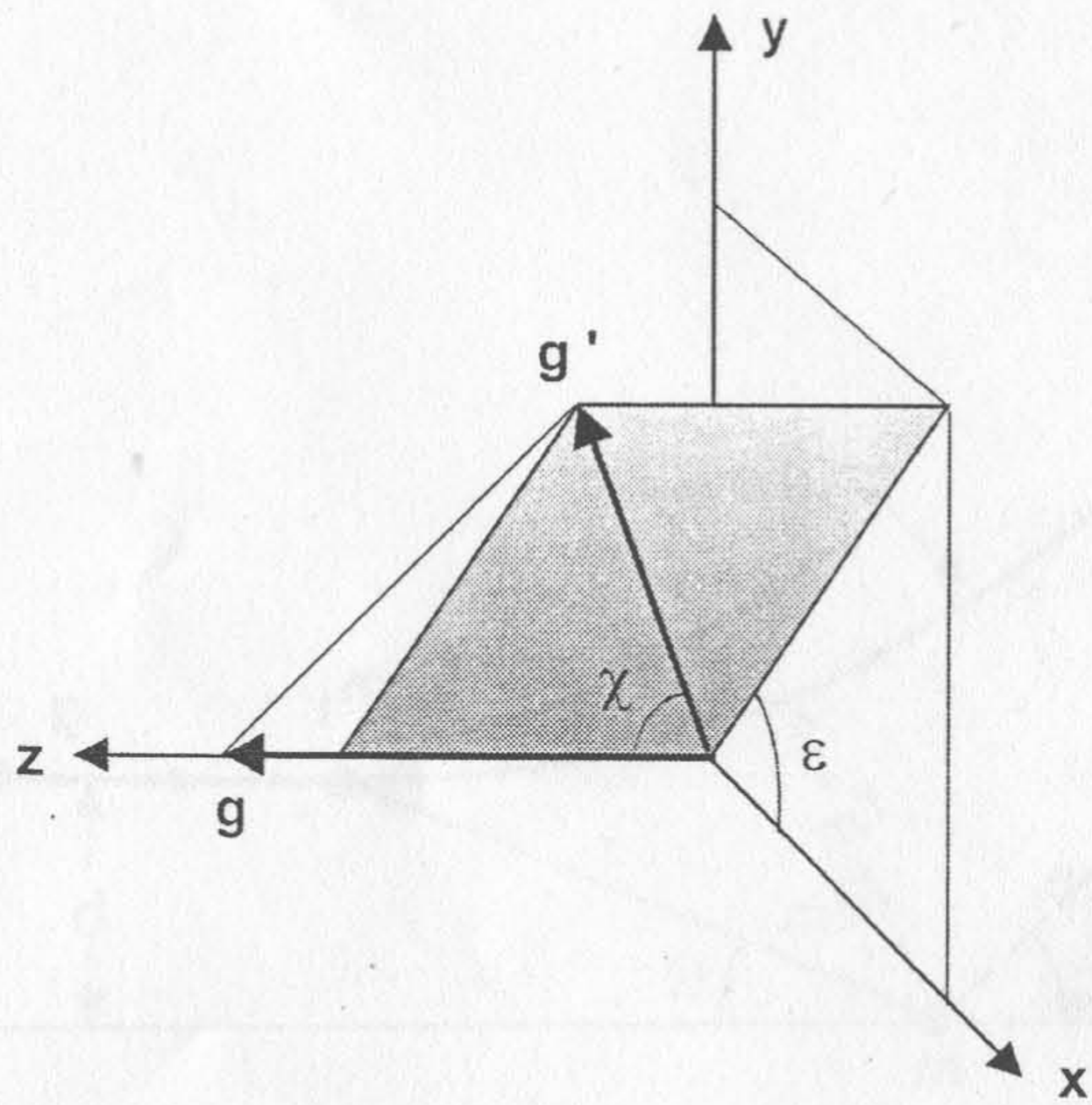

Fig. 3 *Relative orientations of the velocities* $\mathbf{g}$ *and* $\mathbf{g}'$ *in a Cartesian coordinate system chosen such that* $\mathbf{g} = g\widehat{\mathbf{z}}$. *The angle* ϵ *defines the relative orientation of the plane containing the trajectory of the particle.*

It is of interest to consider the *inverse collision* (refer to Fig. 1 of Chapter 7), in which a particle with initial velocity $\mathbf{v}'$ collides with another particle having initial velocity $\mathbf{v}_1'$, the velocities *after* collision being $\mathbf{v}$ and $\mathbf{v}_1$, respectively. For the inverse collision, the scattering angle χ is the same as that for the direct collision, since the impact parameter b, the interparticle force law and the relative speed $\mathbf{g}$ are all the same.

The scattering angle is the only quantity appearing in the analysis presented in this section that depends on the details of the collision process. For interparticle force laws which depend only on the distance between the interacting particles, χ depends on the following quantities:

(a) Interparticle force law;
(b) Magnitude of the relative velocity g;
(c) The value of the impact parameter b.

Therefore, in order to determine χ we must analyze the classical dynamics of binary collisions.

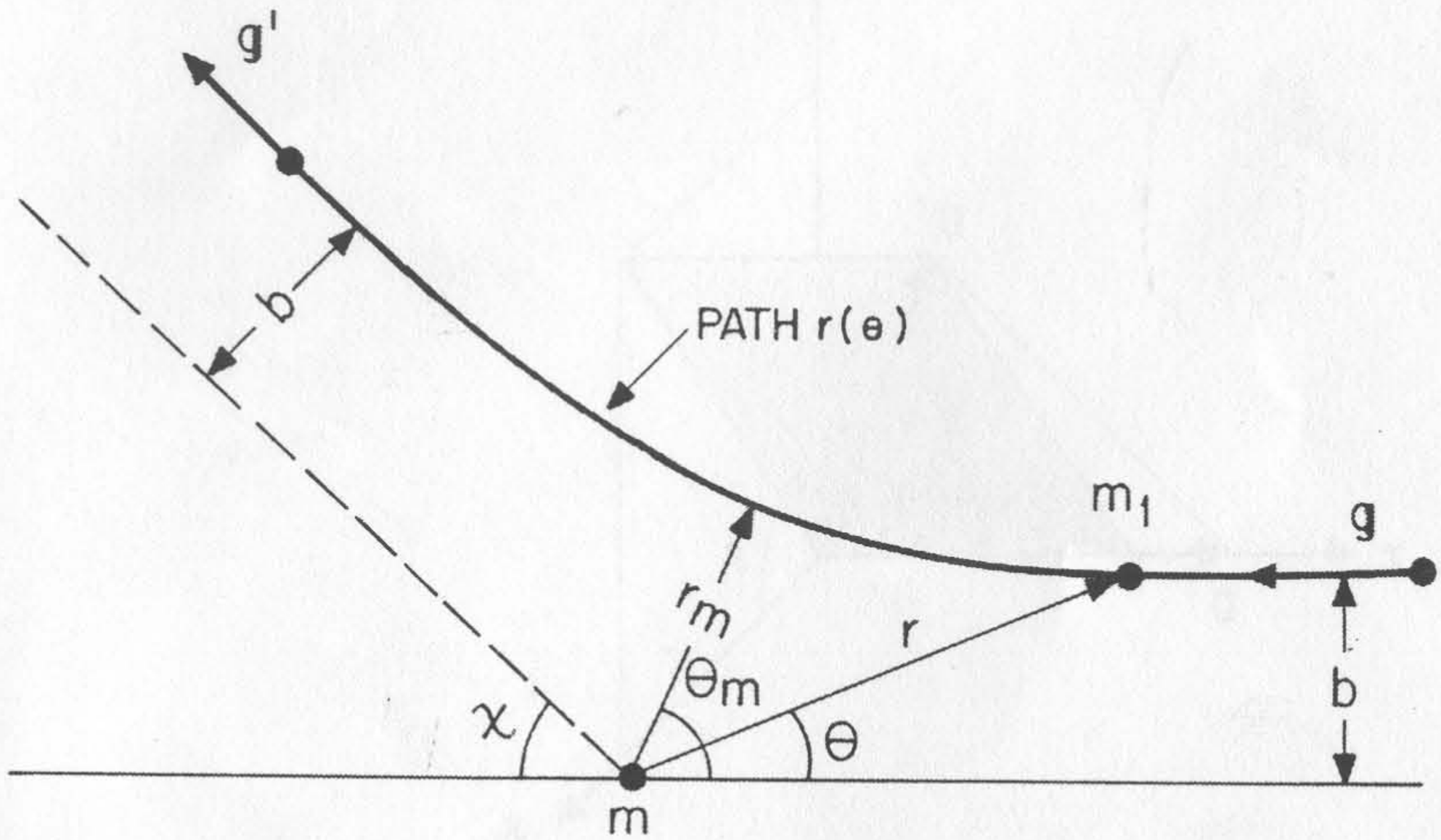

Fig. 4 *Path $r(\theta)$ of the particle of mass m_1 relative to the particle of mass m.*

3. DYNAMICS OF BINARY COLLISIONS

The dynamics of a binary collision is governed by the interparticle force law. For each impact parameter b there will be associated a given scattering angle χ, the relation being dependent on the interparticle force law. This information is contained in the *differential cross section*, which we shall define in section 5.

Consider the collision between two particles having mass m and m_1, viewed from a reference system in which the first particle is at rest. Let $\mathbf{r}$ be the position vector of the particle of mass m_1 with respect to that of mass m (see Fig. 4). The force of interaction between the two particles is assumed to be a central force, which acts along the straight line joining the two particles, that is,

$$\mathbf{F}(\mathbf{r}) = F(r)\,\widehat{\mathbf{r}} \tag{3.1}$$

This force is related to the potential energy $U(\mathbf{r})$ of the interaction by the condition

$$\mathbf{F}(\mathbf{r}) = -\nabla U(\mathbf{r}) = -\frac{\partial U(r)}{\partial r}\,\widehat{\mathbf{r}} \tag{3.2}$$

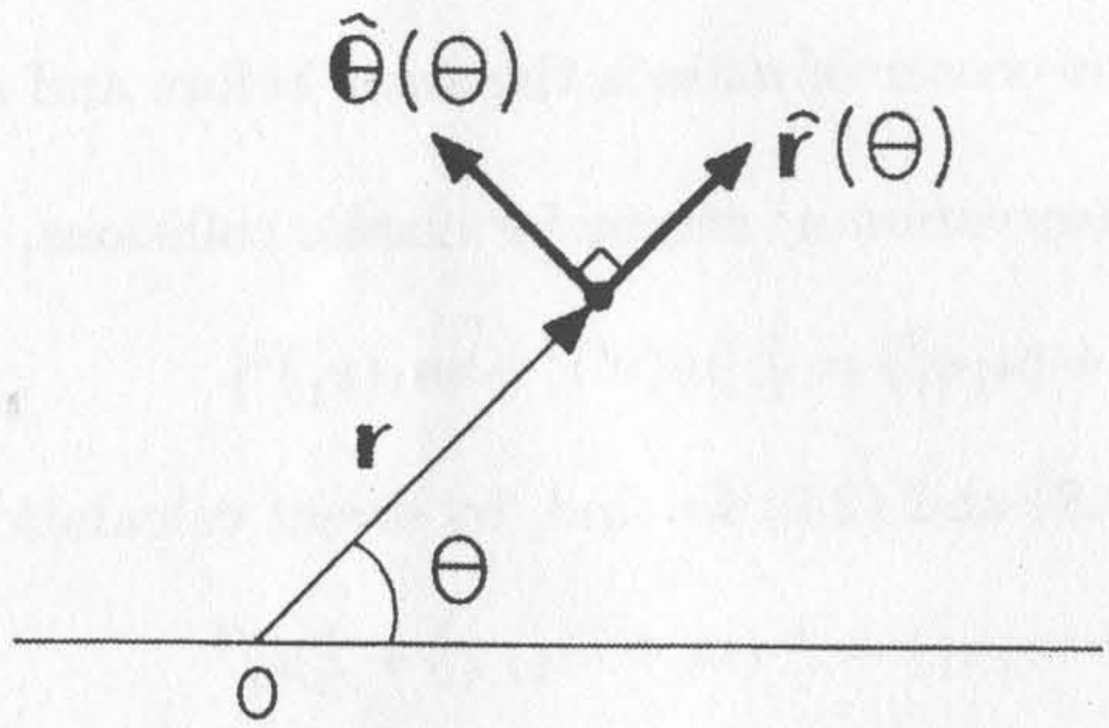

Fig. 5 *Polar coordinates (r, θ), showing that the directions of the unit vectors $\widehat{\mathbf{r}}$ and $\widehat{\boldsymbol{\Theta}}$ depend on θ.*

For a central force the torque $\mathbf{N} = \mathbf{r} \times \mathbf{F}(\mathbf{r})$ vanishes, because $\mathbf{F}(\mathbf{r})$ is parallel to $\mathbf{r}$. Since the torque is the time rate of change of the angular momentum $\mathbf{L}$,

$$\mathbf{N} = \frac{d\mathbf{L}}{dt} \tag{3.3}$$

we conclude that the angular momentum is a constant of the motion. Furthermore, since $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, we see that $\mathbf{r}$ is always normal to the constant direction of $\mathbf{L}$ in space so that the motion lies in a plane.

Using polar coordinates (r, θ) and noting that the unit vectors $\widehat{\mathbf{r}}$ and $\widehat{\boldsymbol{\Theta}}$ depend on θ, as illustrated in Fig.5, we have for the instantaneous relative velocity,

$$\frac{d\mathbf{r}}{dt} = \frac{dr}{dt}\,\widehat{\mathbf{r}} + r\,\frac{d\widehat{\mathbf{r}}}{dt} = \frac{dr}{dt}\,\widehat{\mathbf{r}} + r\,\frac{d\widehat{\mathbf{r}}}{d\theta}\,\frac{d\theta}{dt} \tag{3.4}$$

Since it can be shown that $d\widehat{\mathbf{r}}/d\theta = \widehat{\boldsymbol{\Theta}}$, we obtain

$$\frac{d\mathbf{r}}{dt} = \frac{dr}{dt}\,\widehat{\mathbf{r}} + r\,\frac{d\theta}{dt}\,\widehat{\boldsymbol{\Theta}} \tag{3.5}$$

or, using a dot over the variable to denote the time derivative,

$$\dot{\mathbf{r}} = \dot{r}\,\widehat{\mathbf{r}} + r\,\dot{\theta}\,\widehat{\boldsymbol{\Theta}} \tag{3.6}$$

The trajectory of the particle in the equivalent one body problem can be easily found by using the laws of conservation of *energy* and of *angular momentum*. The kinetic energy of the relative motion is given by

$$K = \tfrac{1}{2}\mu\,\dot{\mathbf{r}}\cdot\dot{\mathbf{r}} = \tfrac{1}{2}\mu\,(\dot{r}^2 + r^2\dot{\theta}^2) \tag{3.7}$$

From the law of conservation of energy we can equate the kinetic plus potential energy, at any point, to the initial kinetic energy $\frac{1}{2}\mu g^2$, since the initial potential energy is zero. Thus, we have

$$\tfrac{1}{2}\mu\,(\dot{r}^2 + r^2\dot{\theta}^2) + U(r) = \tfrac{1}{2}\mu g^2 \tag{3.8}$$

The angular momentum relative to the origin is given by

$$\mathbf{L} = \mathbf{r} \times (\mu\dot{\mathbf{r}}) = \mu r^2\dot{\theta}\,(\widehat{\mathbf{r}} \times \widehat{\boldsymbol{\Theta}}) \tag{3.9}$$

Setting the angular momentum, at any point, equal to its initial value, $b\mu g\,(\widehat{\mathbf{r}} \times \widehat{\boldsymbol{\Theta}})$, we get

$$r^2\dot{\theta} = bg \tag{3.10}$$

From (3.8) and (3.1) we can easily obtain a differential equation for the orbit $r(\theta)$. First, we write

$$\frac{dr}{dt} = \frac{dr}{d\theta}\,\frac{d\theta}{dt} \tag{3.11}$$

and use (3.10) and (3.8) to eliminate $d\theta/dt$ and dr/dt. The resulting differential equation for the trajectory $r(\theta)$ is found to be

$$\left(\frac{dr}{d\theta}\right)^2 = \frac{r^4}{b^2}\left[1 - \frac{b^2}{r^2} - \frac{2U(r)}{\mu g^2}\right] \tag{3.12}$$

Rearranging (3.12) yields the following result

$$d\theta = \pm\frac{b}{r^2}\left[1 - \frac{b^2}{r^2} - \frac{2U(r)}{\mu g^2}\right]^{-1/2} dr \tag{3.13}$$

The choice of sign must be made on physical grounds. The coordinates of the position of the particle, when it is at the distance of closest approach, are denoted by r_m and θ_m (refer to Fig. 4). This position is called the *vertex* of the trajectory and the line connecting the origin to the vertex is called the *apse line*. Thus, θ_m specifies the orientation of the apse line. The plus sign in (3.13) must be used when θ is greater than θ_m, since for $\theta > \theta_m$ we see that r increases with θ. On the other hand, for $\theta < \theta_m$ we see that r decreases as θ increases, so that the minus sign in (3.13) must be used when θ is less than θ_m. This result also shows that the particle trajectory is *symmetrical* about the apse line.

The *distance of closest approach* r_m can be obtained from (3.12) by noting that $dr/d\theta = 0$ when $r = r_m$. Thus, we have

$$1 - \frac{b^2}{r_m^2} - \frac{2U(r_m)}{\mu g^2} = 0 \tag{3.14}$$

or

$$r_m = b\left[1 - \frac{2U(r_m)}{\mu g^2}\right]^{-1/2} \tag{3.15}$$

To determine the *scattering angle* χ, we first note, from Fig. 5, that

$$\chi = \pi - 2\theta_m \tag{3.16}$$

In order to determine θ_m we integrate (3.13) from θ_m to some other angle θ, obtaining

$$\theta - \theta_m = \pm \int_{r_m}^{r} \frac{b}{(r')^2} \left[1 - \frac{b^2}{(r')^2} - \frac{2U(r')}{\mu g^2}\right]^{-1/2} dr' \tag{3.17}$$

where the plus sign is to be used when $\theta > \theta_m$ and the minus sign when $\theta < \theta_m$. When $r \to \infty$ we have $\theta_{(-)} \to 0$, while $\theta_{(+)} \to 2\theta_m$, so that (3.17) gives for the orientation of the apse line,

$$\theta_m = \int_{r_m}^{\infty} \frac{b}{r^2} \left[1 - \frac{b^2}{r^2} - \frac{2U(r)}{\mu g^2}\right]^{-1/2} dr \tag{3.18}$$

The scattering angle is therefore given by

$$\chi(b, g) = \pi - 2\int_{r_m}^{\infty} \frac{b}{r^2} \left[1 - \frac{b^2}{r^2} - \frac{2U(r)}{\mu g^2}\right]^{-1/2} dr \tag{3.19}$$

To compute χ from this equation we must know the impact parameter b, the magnitude of the initial relative velocity g and the interparticle potential energy function $U(r)$.

4. EVALUATION OF THE SCATTERING ANGLE

In this section we shall present two examples of the use of (3.19) to determine the scattering angle χ in terms of the impact parameter b and of the initial relative speed g. First, we will consider the collision between

two perfectly elastic hard spheres and afterwards the case of the Coulomb potential interaction.

4.1 Two Perfectly Elastic Hard Spheres

Consider the collision between two perfectly elastic hard spheres of radii R_1 and R_2 (see Fig. 6). The potential energy of interaction is given by

$$\begin{aligned} U(r) &= 0 \quad for \quad r > R_1 + R_2 \\ &= \infty \quad for \quad r < R_1 + R_2 \end{aligned} \tag{4.1}$$

For $b > R_1 + R_2$ there is no interaction and we must have $r_m = b$, whereas for $b < R_1 + R_2$ the particles collide and we have $r_m = R_1 + R_2$. In either case, however, since the spheres are impenetrable, we have $r > R_1 + R_2$, so that (3.19) becomes

$$\chi = \pi - 2 \int_{r_m}^{\infty} \frac{b}{r^2} \left(1 - \frac{b^2}{r^2}\right)^{-1/2} dr \tag{4.2}$$

To solve this integral it is convenient to define a new variable by $y = b/r$ and write (4.2) in the form

$$\chi = \pi - 2 \int_0^{b/r_m} (1 - y^2)^{-1/2} \, dy \tag{4.3}$$

which gives

$$\chi = \pi - 2 \, sin^{-1}(b/r_m) \tag{4.4}$$

Therefore, we find that

$$\begin{aligned} \chi &= \pi - 2 \, sin^{-1}\left(\frac{b}{R_1 + R_2}\right) \quad for \quad b \leq R_1 + R_2 \\ &= 0 \quad for \quad b \geq R_1 + R_2 \end{aligned} \tag{4.5}$$

4.2 Coulomb Interaction Potential

Let us consider now the important case of the Coulomb potential field, whose interaction potential energy is given by

$$U(r) = \frac{1}{4\pi\epsilon_0} \frac{qq_1}{r} \tag{4.6}$$

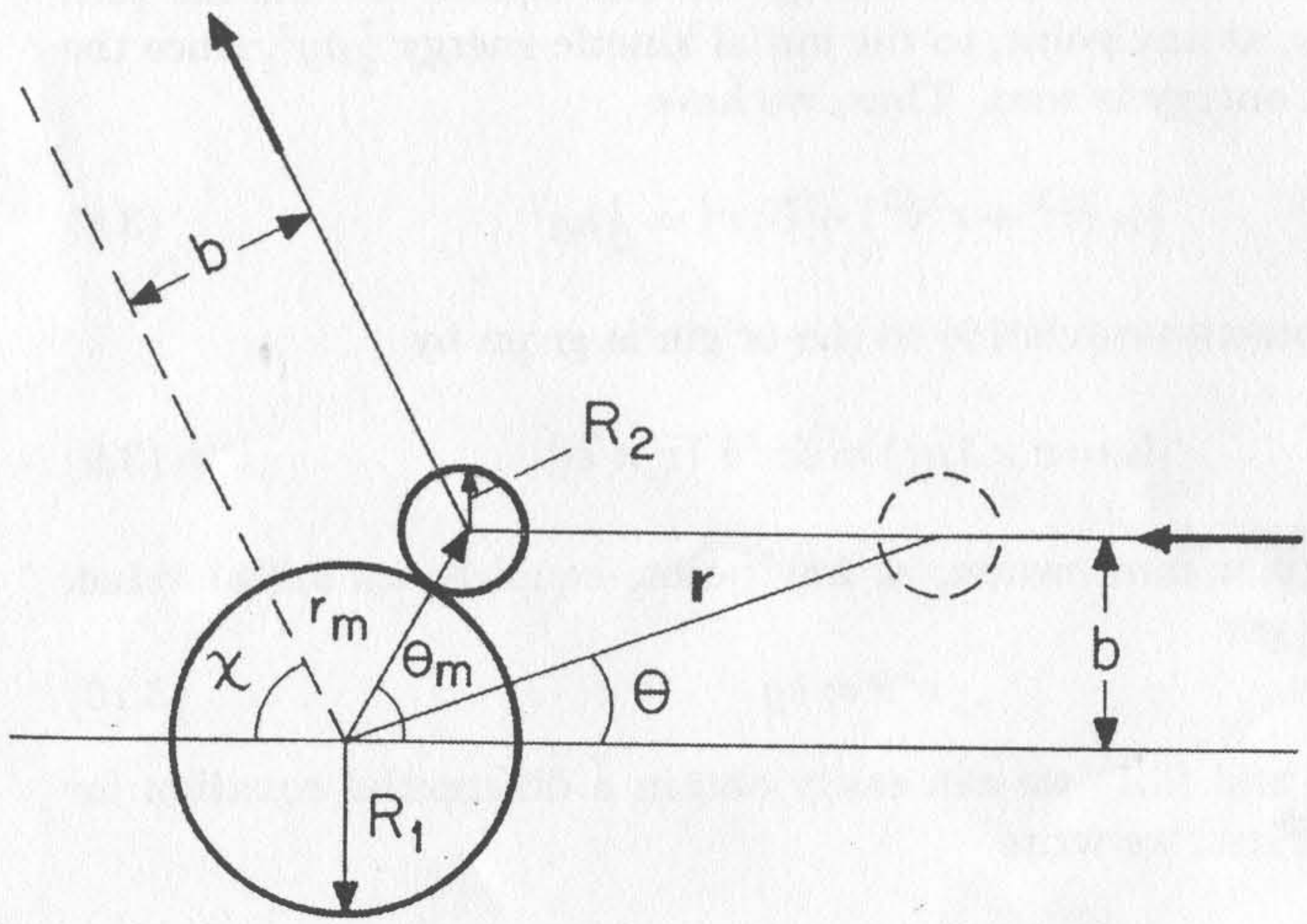

Fig. 6 *Collision between two perfectly elastic, impenetrable spheres.*

where q and q_1 denote the electric charge of the particles of mass m and m_1, respectively. Substituting (4.6) into (3.19), gives

$$\chi(b,g) = \pi - 2\int_{r_m}^{\infty} \frac{b}{r^2}\left(1 - \frac{b^2}{r^2} - \frac{qq_1}{2\pi\epsilon_0\mu g^2 r}\right)^{-1/2} dr \tag{4.7}$$

The distance of closest approach r_m in obtained from (3.15) and (4.6), and is found to be

$$r_m = \frac{b^2}{-b_0 + (b_0^2 + b^2)^{1/2}} \tag{4.8}$$

where, for convenience, we have introduced the notation

$$b_0 = \frac{qq_1}{4\pi\epsilon_0\mu g^2} \tag{4.9}$$

Thus, b_0 represents the distance at which the electrical potential energy of interaction is twice the relative kinetic energy at infinity. Making the change of variable $y = 1/r$ and inserting the value for b_0, given by (4.9), into (4.7), gives for the deflection angle

$$\chi(b,g) = \pi - 2b\int_0^{1/r_m} (1 - b^2y^2 - 2b_0y)^{-1/2}\, dy \tag{4.10}$$

The integral appearing here is of the standard form

$$\int (\alpha + \beta x + \gamma x^2)^{-1/2}\, dx = \frac{1}{\sqrt{-\gamma}} \sin^{-1}\left[\frac{-2\gamma x - \beta}{(\beta^2 - 4\alpha\gamma)^{1/2}}\right] \tag{4.11}$$

where, in our case, $\alpha = 1$, $\beta = -2b_0$, and $\gamma = -b^2$. Applying the limits of integration, with r_m as given by (4.8), yields for the deflection angle

$$\chi(b, g) = 2 \sin^{-1}\left[\frac{b_0}{(b_0^2 + b^2)^{1/2}}\right] \tag{4.12}$$

This equation for $\chi(b, g)$ can be written in the following alternative form

$$\tan\left(\tfrac{1}{2}\chi\right) = \frac{b_0}{b} \tag{4.13}$$

Note that for $\chi = \pi/2$ we have $b = b_0$, that is, b_0 is the value of the impact parameter for a 90° deflection angle. If the sign of the two charged particles are the same, then b_0 and χ will both be positive. On the other hand, if the sign of the two charged particles are opposite, then b_0 and χ will be negative. These two situations are illustrated in Fig. 7 for a deflection angle of 90°. Further, we note, from (4.13), that $\chi = \pi$ for $b = 0$. Also, χ decreases as b increases and $\chi = 0$ only in the limit of $b \to \infty$. Therefore, scattering occurs for all (finite) values of the impact parameter b, so that there is no cut-off value for b.

5. CROSS SECTIONS

So far we have considered specifically only the interaction between two particles. Cross sections are usually defined in terms of a beam of identical particles incident on a center of force (target particle). Therefore, let us imagine a steady beam of identical particles of mass m_1, uniformly spread out in space, incident with velocity $\mathbf{g} = \mathbf{v}_1 - \mathbf{v}$ upon the center of force provided by the particle of mass m, in its rest frame of reference. For simplicity, the incident beam of particles is assumed to be monoenergetic, so that all particles in the beam have the same initial relative velocity, $\mathbf{g}$, with respect to the scattering particle. Since the particles in the incident beam are supposed to be identical, the interaction potential is the same for all particles in the beam.

The particles incident with an impact parameter b will be scattered through some deflection angle χ, whereas the particles incident with an

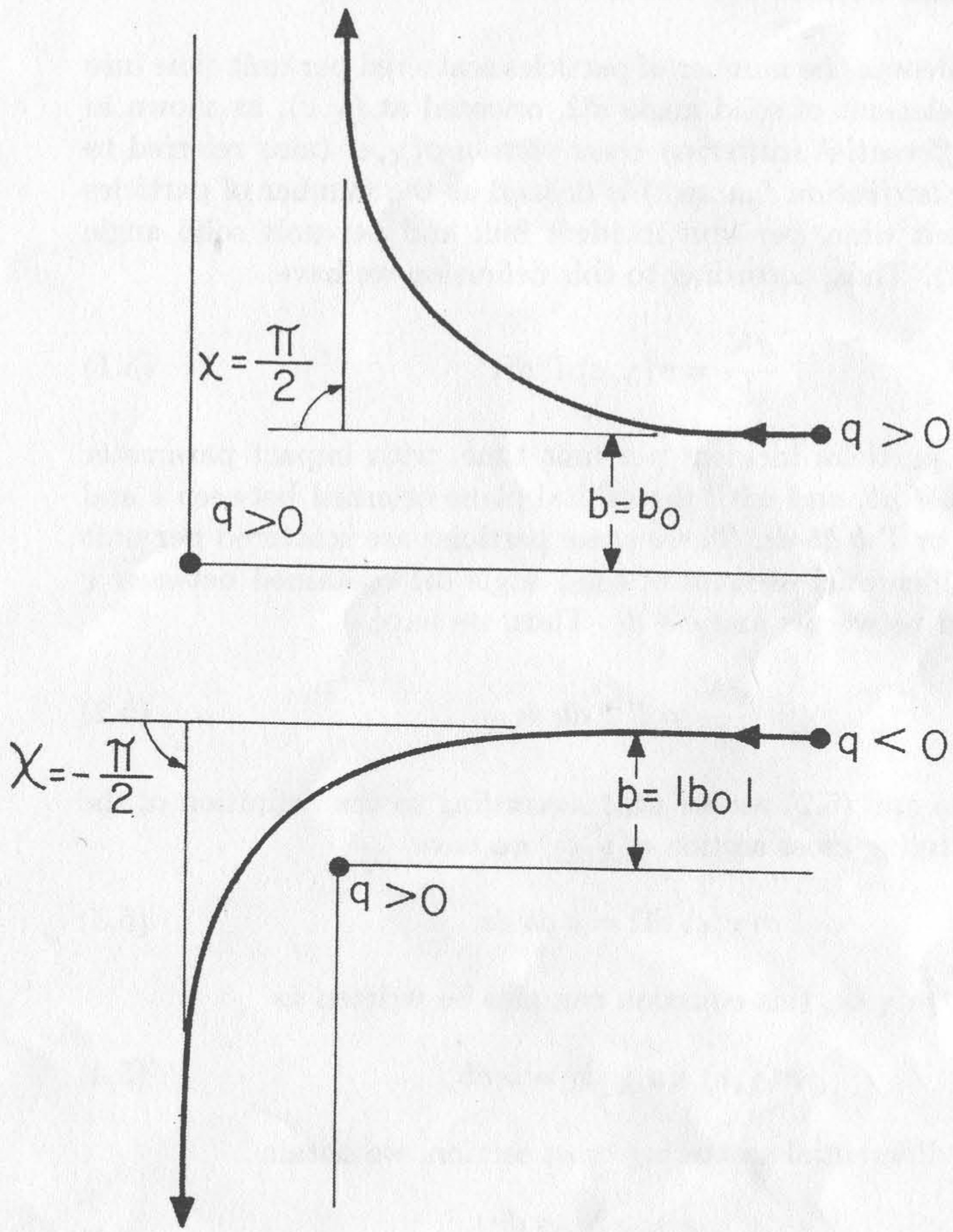

Fig. 7 *Scattering in a Coulomb potential field for a deflection angle χ of 90°.*

impact parameter $b + db$ will be scattered through the deflection angle $\chi + d\chi$, as illustrated in Fig. 8. The number of particles scattered per second, between the angles χ and $\chi + d\chi$, depends on the incident particle current density (particle flux) $\mathbf{\Gamma}$, that is, on the number of particles in the incident beam crossing a unit area normal to the beam per unit time.

5.1 Differential Scattering Cross Section

Let dN/dt denote the number of particles scattered per unit time into the differential element of solid angle $d\Omega$, oriented at (χ, ϵ), as shown in Fig. 8. The *differential scattering cross section* $\sigma(\chi, \epsilon)$ (also referred to as the *angular distribution function*) is defined as the number of particles scattered per unit time, per unit incident flux and per unit solid angle oriented at (χ, ϵ). Thus, according to this definition we have

$$\frac{dN}{dt} = \sigma(\chi, \epsilon)\ \Gamma\ d\Omega \tag{5.1}$$

The number of particles incident per unit time, with impact parameter between b and $b + db$, and with the orbital plane oriented between ϵ and $\epsilon + d\epsilon$, is given by $\Gamma\ b\ db\ d\epsilon$. These same particles are scattered per unit time into the differential element of solid angle $d\Omega$ contained between χ and $\chi + d\chi$, and between ϵ and $\epsilon + d\epsilon$. Thus, we have

$$\frac{dN}{dt} = \Gamma\ b\ db\ d\epsilon \tag{5.2}$$

Comparing (5.1) and (5.2) we see that, according to the definition of the differential scattering cross section $\sigma(\chi, \epsilon)$, we have

$$\sigma(\chi, \epsilon)\ d\Omega = b\ db\ d\epsilon \tag{5.3}$$

Since $d\Omega = \sin\chi\ d\chi\ d\epsilon$, this equation can also be written as

$$\sigma(\chi, \epsilon)\ \sin\chi\ d\chi = b\ db \tag{5.4}$$

Solving for the differential scattering cross section, we obtain

$$\sigma(\chi, \epsilon) = \frac{b}{\sin\chi}\left|\frac{db}{d\chi}\right| \tag{5.5}$$

The absolute value of $db/d\chi$ is used here, because χ normally decreases when b increases, whereas the differential scattering cross section $\sigma(\chi, \epsilon)$ is inherently a positive quantity, since it is associated with the number of particles being scattered. The quantity $db/d\chi$ can be obtained from (3.19), which gives $\chi(b, g)$ once the potential energy function $U(r)$ is known.

The differential cross section has dimensions of area and can be interpreted, in a geometrical way, such that the number of particles scattered

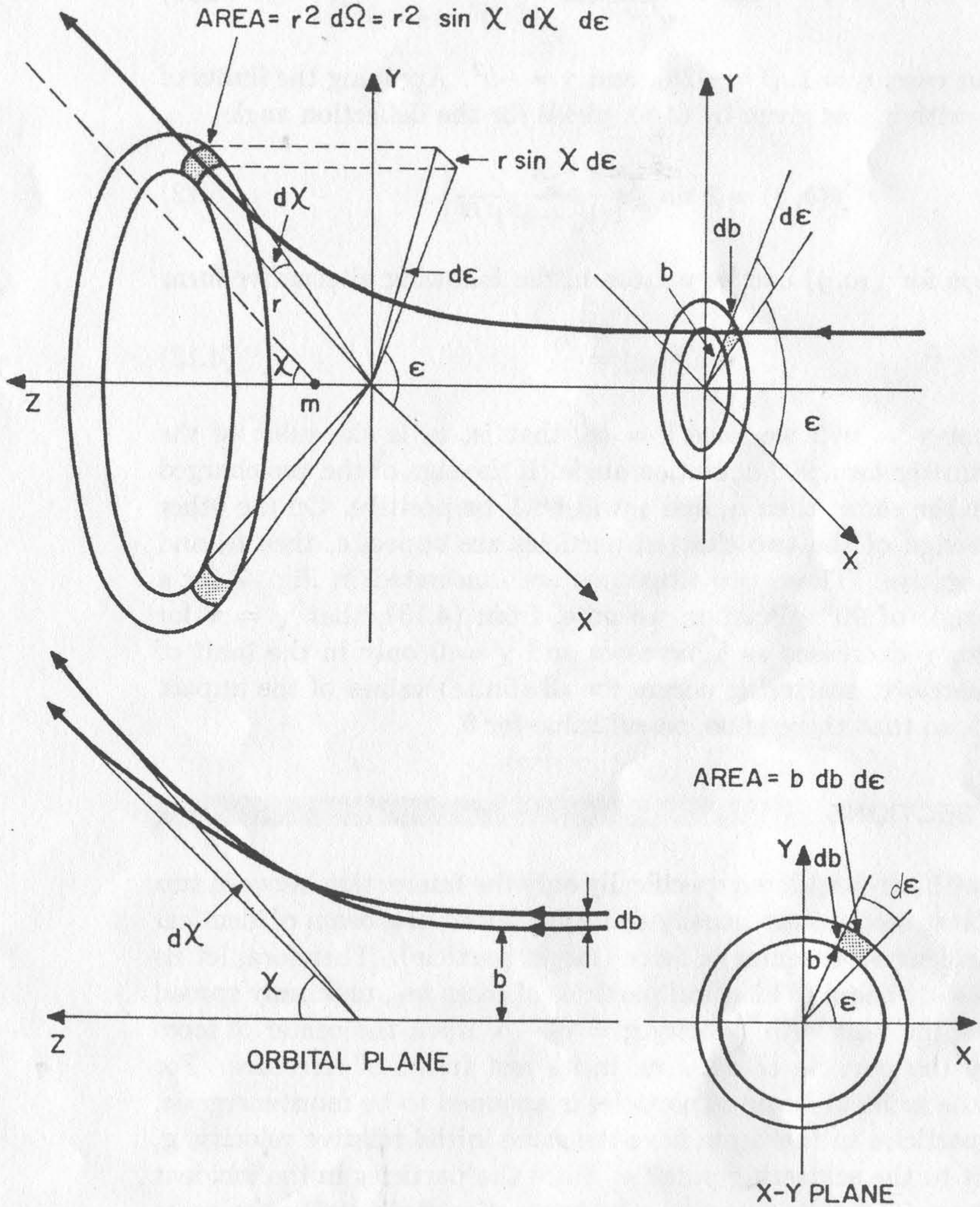

Fig. 8 *Particle scattering in a central field of force.*

into the solid angle element $d\Omega$ per unit time is equal to the number of particles crossing an area equal to $\sigma(\chi, \epsilon)d\Omega$ (or $b\ db\ d\epsilon$) in the incident beam, per unit time.

The form of $\sigma(\chi, \epsilon)$ depends on the interparticle force law and, if this force law is known, $\sigma(\chi, \epsilon)$ can be calculated. However, a quantum mechanical calculation is necessary for this purpose, since the quantum wave packets of the colliding particles necessarily overlap and the problem is no longer a classical one. For a collection of atoms or molecules to be regarded classically, with each particle having a rather well defined position and momentum, it is necessary that the particles be localized wave packets whose extensions are small compared to the average interparticle distance. For a classical treatment the average *de Broglie wavelength* of each particle must therefore be much smaller than the average interparticle separation.

The differential cross section is also a directly measurable quantity and can be obtained experimentally. For our purposes, we will consider the differential cross section $\sigma(\chi, \epsilon)$, which contains the nature of the collisional interaction, as a known quantity.

5.2 Total Scattering Cross Section

The *total scattering cross section* σ_t is defined as the number of particles scattered per unit time and per unit incident flux, in *all directions* from the scattering center. It is obtained by integrating $\sigma(\chi, \epsilon)\ d\Omega$ over the entire solid angle,

$$\sigma_t = \int_\Omega \sigma(\chi, \epsilon)\ d\Omega = \int_0^{2\pi} d\epsilon \int_0^\pi \sigma(\chi, \epsilon) \sin\chi\ d\chi \tag{5.6}$$

Both $\sigma(\chi, \epsilon)$ and σ_t depend on the magnitude of the relative particle velocity g.

In the special case when the interaction potential is *isotropic*, that is, when the differential scattering cross section is independent of ϵ, we can immediately perform the integral over ϵ in (5.6), to get

$$\sigma_t = 2\pi \int_0^\pi \sigma(\chi) \sin\chi\ d\chi \tag{5.7}$$

This is the case, for example, of the Coulomb interaction potential.

5.3 Momentum Transfer Cross Section

A cross section can be defined for various interaction processes. We shall see later that the transfer of momentum, during a collision, is the basic microscopic event in the transport phenomena of diffusion and mobility. Hence, it is appropriate to define a cross section for the rate of momentum transfer σ_m as the total momentum transferred per unit time to the scattering center, per unit incident momentum flux (momentum per normal unit area per unit time),

$$\sigma_m = \frac{(\text{momentum transferred per second})}{(\text{incident momentum flux})} \tag{5.8}$$

The momentum of a particle in the beam, before interaction, is μg, where μ is the reduced mass and g is the initial relative velocity. The incident momentum flux is therefore $\Gamma \mu g$. After interaction, the momentum of a particle in the beam, in the direction of incidence, and which is scattered at an angle χ, is $\mu g \cos \chi$. Therefore, the momentum transferred by this particle to the scattering center is $\mu g(1 - \cos \chi)$. The total momentum transferred per second to the scattering center, by all the particles scattered in all directions in space, is given by the expression

$$\Gamma \mu g \int_\Omega (1 - \cos \chi)\, \sigma(\chi, \epsilon)\, d\Omega \tag{5.9}$$

Recall that $\sigma(\chi, \epsilon)$ can be considered as an angular distribution function. Since the total incident momentum flux is $\Gamma \mu g$, we obtain for the *momentum transfer cross section*,

$$\sigma_m = \int_\Omega (1 - \cos \chi)\, \sigma(\chi, \epsilon)\, d\Omega \tag{5.10}$$

For the special case of an *isotropic* interaction potential, and noting that $d\Omega = \sin \chi\, d\chi\, d\epsilon$, we can perform the integral over ϵ, in (5.10), obtaining

$$\sigma_m = 2\pi \int_0^\pi (1 - \cos \chi)\, \sigma(\chi) \sin \chi\, d\chi \tag{5.11}$$

Since $\sigma(\chi)$ is an angular distribution function, it can be also be considered as a weight function to calculate the *mean value* of any function $F(\chi)$ of the scattering angle. The contribution to the total value of $F(\chi)$, resulting from the particles scattered into $d\Omega$, is $F(\chi)\, \sigma(\chi)\, d\Omega$. Since the

total number of particles scattered is $\int \sigma(\chi)\, d\Omega$, it follows that the *mean value* of $F(\chi)$, averaged over all values of χ, is given by

$$< F(\chi) > = \frac{\displaystyle\int_\Omega F(\chi)\, \sigma(\chi)\, d\Omega}{\displaystyle\int_\Omega \sigma(\chi)\, d\Omega} \tag{5.12}$$

which may be written as

$$< F(\chi) > = \frac{2\pi}{\sigma_t} \int_0^\pi F(\chi)\, \sigma(\chi) \sin \chi\, d\chi \tag{5.13}$$

According to this definition of mean values we see that (5.11) can be written in the form

$$\sigma_m = \sigma_t < 1 - \cos \chi > \tag{5.14}$$

Thus, the cross section for momentum transfer is a weighted cross section in which scattering angles of zero degrees do not count at all, scattering of 90° count as one and scattering of 180° count as two. This weighting is proportional to the amount of momentum transferred from the incident beam to the scattering center.

6. CROSS SECTIONS FOR THE HARD SPHERE MODEL

6.1 Differential Scattering Cross Section

To calculate $\sigma(\chi, \epsilon)$, as given in (5.5), we first obtain from (4.5), for $b \leq R_1 + R_2$,

$$b = (R_1 + R_2) \cos\left(\tfrac{1}{2}\chi\right) \tag{6.1}$$

and

$$\left|\frac{db}{d\chi}\right| = \tfrac{1}{2}(R_1 + R_2) \sin\left(\tfrac{1}{2}\chi\right) \tag{6.2}$$

Substituting these last two expressions into (5.5), yields for the differential scattering cross section

$$\sigma = \tfrac{1}{4}(R_1 + R_2)^2 \tag{6.3}$$

6.2 Total Scattering Cross Section

Integrating (6.3) over the whole solid angle, we obtain

$$\sigma_t = 2\pi \int_0^\pi \tfrac{1}{4}(R_1 + R_2)^2 \sin\chi \, d\chi = \pi(R_1 + R_2)^2 \tag{6.4}$$

Two special simple cases may be mentioned here. For the collision between an electron and a molecule of radius R, we have $\sigma = R^2/4$ and $\sigma_t = \pi R^2$. For molecules colliding with themselves, their diameter being D, we have $\sigma = D^2/4$ and $\sigma_t = \pi D^2$.

Note that in the case of the hard sphere model there is a cut-off value for the impact parameter b, beyond which collisions do not occur. It is the existence of this cut-off value for b that leads to a finite value for the total scattering cross section σ_t. This conclusion will be made clear in the next section.

6.3 Momentum Transfer Cross Section

From (6.3) and (5.11) we obtain

$$\sigma_m = 2\pi \int_0^\pi \tfrac{1}{4}(R_1 + R_2)^2 (1 - \cos\chi) \sin\chi \, d\chi \; =$$

$$= \tfrac{1}{2}\pi(R_1 + R_2)^2 \left(\int_0^\pi \sin\chi \, d\chi - \int_0^\pi \cos\chi \sin\chi \, d\chi \right) \tag{6.5}$$

Performing the integrals, yields

$$\sigma_m = \pi(R_1 + R_2)^2 \tag{6.6}$$

The average value of momentum loss per particle is found, from (5.13), to be given by

$$< \mu g(1 - \cos\chi) > = \frac{2\pi}{\sigma_t} \int_0^\pi \mu g(1 - \cos\chi) \, \sigma(\chi) \sin\chi \, d\chi \tag{6.7}$$

Using (5.11), we get

$$< \mu g(1 - \cos\chi) > = \mu g \, \frac{\sigma_m}{\sigma_t} \tag{6.8}$$

Thus, from (6.4), (6.6) and (6.8) we deduce that the average value of momentum loss, per particle, for the hard sphere model, is

$$< \mu g(1 - \cos \chi) > = \mu g \tag{6.9}$$

For collisions between electrons and neutral particles, for example, in a weakly ionized plasma, the mass of the electron can be neglected as compared to the mass of the neutral particle, so that the reduced mass becomes equal to the electron mass. From (6.9) it is seen, in a first approximation, that the entire momentum of an electron is lost in a collision with a neutral particle. Assuming that the motion of the heavy particles can be ignored and denoting the collision frequency by ν (that is, the number of collisions between electrons and neutral particles per second), then the rate of momentum loss of an average electron is $\nu m_e \mathbf{u}$, where $\mathbf{u}$ denotes the electron velocity. However, in general an electron does not lose its entire momentum on a collision with a neutral particle. Furthermore, the perfectly elastic hard sphere model is not a very good representation for the interaction between an electron and a neutral particle. Consequently, the rate of loss of momentum is written as $\nu_{en} m_e \mathbf{u}$, where ν_{en} represents an *effective collision frequency* for momentum transfer between electrons and neutral particles. This is the term used in the *Langevin equation*, introduced in Chapter 10, to represent the time rate of momentum transfer due to collisions.

7. CROSS SECTIONS FOR THE COULOMB POTENTIAL

7.1 Differential Scattering Cross Section

Differentiating (4.13) we find

$$\left|\frac{db}{d\chi}\right| = \frac{b^2}{2b_0 \cos^2(\chi/2)} \tag{7.1}$$

Thus, the differential scattering cross section, given in (5.5), becomes

$$\sigma(\chi) = \frac{b^3}{2b_0 \sin \chi \cos^2(\chi/2)} \tag{7.2}$$

Using (4.13) this equation can be rearranged as

$$\sigma(\chi) = \frac{b_0^2}{4 \sin^4(\chi/2)} \tag{7.3}$$

This equation is known as the *Rutherford scattering formula.* Since we have $2\sin^2(\chi/2) = (1-\cos\chi)$, it can also be written as

$$\sigma(\chi) = \frac{b_0^2}{(1-\cos\chi)^2} \tag{7.4}$$

The Rutherford scattering formula shows that the differential scattering cross section is equal to $b_0^2/4$ for the deflection angle $\chi = \pi$, increases monotonically as χ is decreased and tends to infinity as χ tends to zero.

7.2 Total Scattering Cross Section

Since the differential scattering cross section increases rapidly to infinity as χ goes to zero, it turns out that the total scattering cross section σ_t becomes infinite. From (5.7) and (7.4), we obtain

$$\sigma_t = 2\pi \int_{\chi_{min}}^{\pi} \sigma(\chi)\sin\chi\, d\chi = 2\pi b_0^2 \int_{\chi_{min}}^{\pi} \frac{\sin\chi}{(1-\cos\chi)^2}\, d\chi \tag{7.5}$$

where $\chi_{min} = 0$. The lower limit has been written implicity for reasons that will become apparent in what follows. Evaluating the integral in (7.5), yields

$$\sigma_t = \pi b_0^2 \left[\frac{1}{\sin^2(\chi_{min}/2)} - 1\right] \tag{7.6}$$

which clearly gives $\sigma_t = \infty$ for $\chi_{min} = 0$. The particles with very small deflection angles contribute to make σ_t infinite.

7.3 Momentum Transfer Cross Section

The substitution of (7.4) into (5.11) gives the following expression for the momentum transfer cross section

$$\sigma_m = 2\pi \int_{\chi_{min}}^{\pi} (1-\cos\chi)\,\sigma(\chi)\sin\chi\, d\chi = 2\pi b_0^2 \int_{\chi_{min}}^{\pi} \frac{\sin\chi}{(1-\cos\chi)}\, d\chi \tag{7.7}$$

where, again, $\chi_{min} = 0$. Evaluating the integral we find

$$\sigma_m = 4\pi b_0^2 \ln\left[\frac{1}{\sin(\chi_{min}/2)}\right] \tag{7.8}$$

Setting $\chi_{min} = 0$ we also find that $\sigma_m = \infty$. Thus, the Coulomb potential gives infinite values for both σ_t and σ_m, the particles with very small deflection angles being responsible for this infinite result.

8. SCREENING OF THE COULOMB POTENTIAL

The infinite results for σ_t and σ_m, obtained in the previous section, may be interpreted as due to the absence of a cut-off value for the impact parameter b. Note that small values of χ correspond to large values of b, and that for $\chi_{min} = 0$ we must have, from (4.13), $b_{max} = \infty$. In order to obtain finite and meaningful values for σ_t and σ_m, it is necessary to modify the basis of the treatment of interactions between individual charged particles and introduce, on some plausible grounds, a cut-off value $b = b_c$ for the impact parameter.

From (5.3) and (5.6) we have for the total scattering cross section (considering σ independent of ϵ)

$$\sigma_t = 2\pi \int_0^{b_c} b \, db \tag{8.1}$$

where a cut-off value $b = b_c$ has been introduced for the impact parameter. With this cut-off, σ_t for the Coulomb potential is found to be

$$\sigma_t = \pi \, b_c^2 \tag{8.2}$$

The introduction of a cut-off value for b corresponds to the assumption that, for charged particles incident with b greater than b_c there will be no interaction, whereas for charged particles incident with b less than b_c there will be a Coulomb-type interaction with the target particle.

The deflections that yield scattering angles between $\pi/2$ and π, and which are associated with values of b between 0 and b_0, are usually called *large-angle deflections*, or *close encounters*. If only the large-angle deflections are taken into account, we obtain

$$\sigma_{t,\ large} = \pi \, b_0^2 \quad ; \quad (\pi/2 < \chi < \pi) \tag{8.3}$$

with b_0 as given by (4.9).

For a charged particle situated inside a plasma we know that it will be surrounded by a shielding cloud of particles of opposite sign. The scale

length for an effective shielding of the charged particle under consideration is the *Debye length* defined by

$$\lambda_D = \left(\frac{\epsilon_0 kT}{n_0 e^2}\right)^{1/2} \tag{8.4}$$

The sphere of radius λ_D, surrounding the charged particle under consideration, is called the *Debye sphere.* We have seen in Chapter 11 that the charged particles lying within the Debye sphere shield the Coulomb potential due to the charged particle under consideration, reducing significantly its effect on the particles lying outside its Debye sphere. Taking this screening effect into account, we find that the interaction potential energy is given by

$$U(r) = \frac{1}{4\pi\epsilon_0}\frac{qq_1}{r}\exp\left(-\frac{r}{\lambda_D}\right) \tag{8.5}$$

Thus, when $r \ll \lambda_D$ the Debye potential, as given by (8.5), is very nearly equal to the Coulomb potential, whereas when $r \gg \lambda_D$ the Debye potential is nearly equal to zero.

The analysis required for calculating σ_t, using the Debye potential, is excessively complicated and it must be done numerically. However, an alternative simple approach can be used that leads to results in very good agreement with those obtained numerically using the Debye potential. It consists in assuming that the interaction potential is exactly equal to the Coulomb potential for $r < \lambda_D$ and is equal to zero for $r > \lambda_D$, as illustrated in Fig. 9. Therefore, it is convenient and more legitimate to introduce the cut-off in the impact parameter at $b_c = \lambda_D$ and not at $b_c = b_0$. In general, we have

$$\lambda_D \gg b_0 \tag{8.6}$$

It is usual to denominate the deflections corresponding to $b_0 < b < \lambda_D$, leading to $\chi < \pi/2$, as *small-angle deflections.* The contribution to the total scattering cross section from the small-angle deflections is found to be given by

$$\sigma_{t,\ small} = 2\pi\int_{b_0}^{\lambda_D} b\ db = \pi(\lambda_D^2 - b_0^2) \quad ; \quad (\chi < \pi/2) \tag{8.7}$$

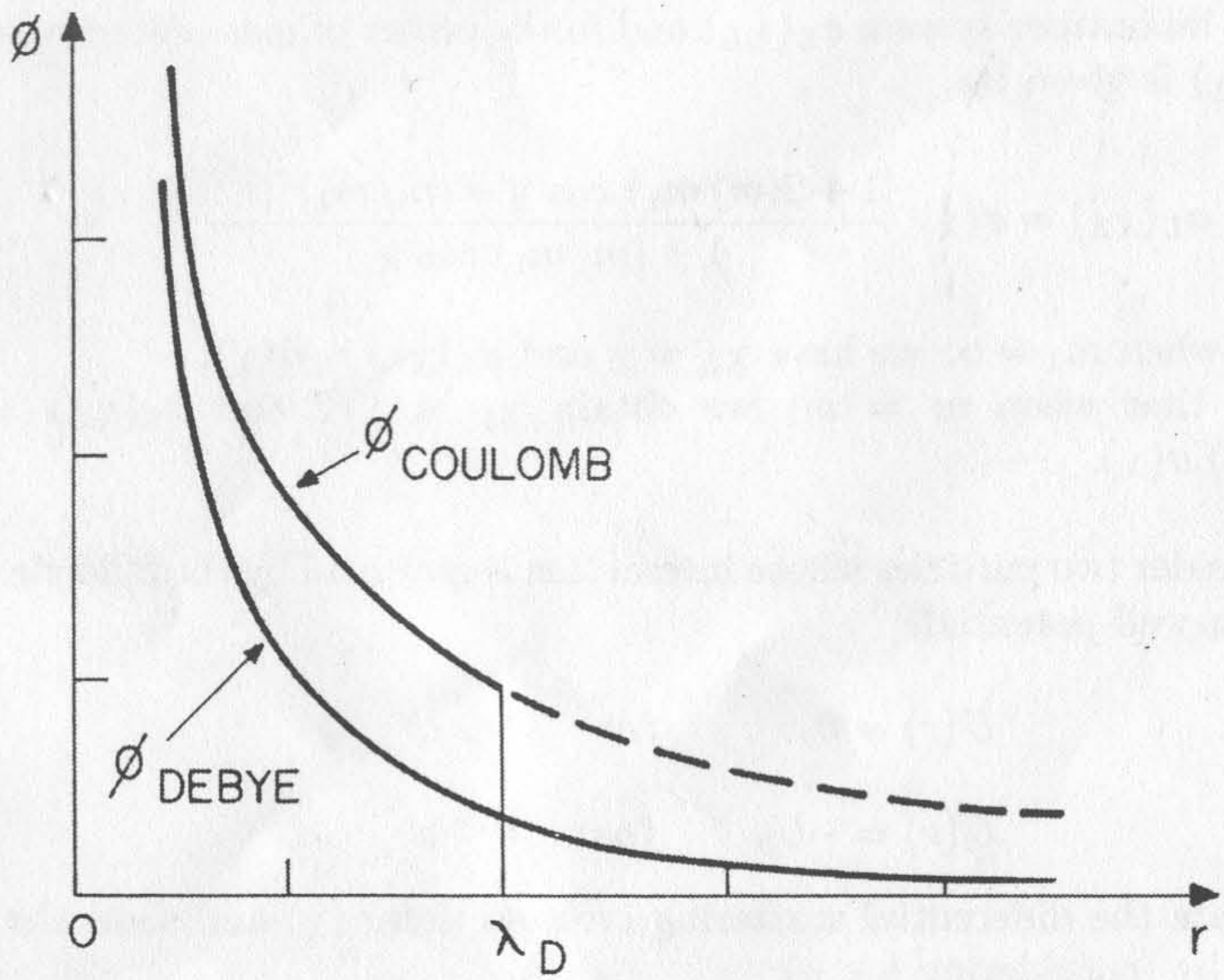

Fig. 9 *The approximation used to obtain $\sigma_t = \pi\lambda_D^2$ consists in assuming that for $r < \lambda_D$ the shielding effect is completely omitted so that the particles interact according to the Coulomb potential, whereas for $r > \lambda_D$ the shielding of the target particle is complete so that there is no interaction.*

Therefore, it follows from (8.3) and (8.7) that

$$\frac{\sigma_{t,\ small}}{\sigma_{t,\ large}} = \frac{\lambda_D^2}{b_0^2} - 1 \simeq \frac{\lambda_D^2}{b_0^2} \tag{8.8}$$

since $\lambda_D \gg b_0$. This result shows that the large number of particles interacting mildly with the target particle, and therefore producing only small-angle deflections, are much more important than the small number of particles interacting strongly with the target particle and producing large-angle deflections. Therefore, if the impact parameter is cut-off at $b_c = \lambda_D$ we obtain, from (8.1), the following result for the total scattering cross section

$$\sigma_t = \pi\ \lambda_D^2 \tag{8.9}$$

For the momentum transfer cross section, introducing the cut-off at $b_c = \lambda_D$, we obtain from (7.8),

$$\sigma_m = 2\pi b_0^2 \ln\left(1 + \frac{\lambda_D^2}{b_0^2}\right) \tag{8.10}$$

since from (4.12), letting $\chi = \chi_c$ for $b = b_c$, we have

$$\sin\left(\tfrac{1}{2}\chi_c\right) = \left(1 + \frac{b_c^2}{b_0^2}\right)^{-1/2} \tag{8.11}$$

Using the notation

$$\Lambda = \frac{\lambda_D}{b_0} \tag{8.12}$$

and noting that in general $\Lambda \gg 1$, (8.10) becomes

$$\sigma_m = 4\pi\, b_0^2 \ln \Lambda \tag{8.13}$$

It can be shown that for σ_m, as well as for σ_t, the large number of particle producing small-angle deflections are much more important than the small number of particles producing large-angle deflections.

The function $\ln \Lambda$ varies very slowly over a large range of variation of the parameters on which Λ depends. For most laboratory plasmas $\ell n\ \Lambda$ lies between 10 and 20. In order to calculate Λ it is usual to make some approximations. For this purpose consider the interaction between an electron gas (charge $q = -e$) and a singly charged ion gas ($q_1 = e$, $Z = 1$). Let n_0 denote the number density of both the electrons and the ions in the gas, which we assume to constitute a plasma, and let the temperature of both species be equal to T. If we further assume that the electron and ion velocities have a Maxwellian equilibrium distribution function with no drift velocity, then we find, by direct calculation,

$$< g^2 > = \frac{1}{n_0^2} \int_v \int_{v_1} f_e\, f_{i1} (\mathbf{v}_1 - \mathbf{v})^2\, d^3v\, d^3v_1 =$$

$$= \frac{1}{n_0} \int_v f_e \left(\frac{3kT}{m_i} + v^2\right) d^3v = \frac{3kT}{\mu} \tag{8.14}$$

where k is Boltzmann's constant and μ is the reduced mass. Replacing g^2, in (4.9), by its average value, we find (for $q_1 = -q = e$)

$$b_0 = \frac{e^2}{4\pi\epsilon_0 \mu < g^2 >} = \frac{e^2}{12\pi\epsilon_0 kT} \tag{8.15}$$

TABLE 1

VALUES OF ln Λ FOR $Z = 1$
AS A FUNCTION OF $T(K)$ AND n_e (cm^{-3})

$\mathbf{T} \setminus \mathbf{n}_e$	10^3	10^6	10^9	10^{12}	10^{15}	10^{18}	10^{21}
10^2	12.8	9.43	5.97				
10^3	16.3	12.8	9.43	5.97			
10^4	19.7	16.3	12.8	9.43	5.97		
10^5	23.2	19.7	16.3	12.8	9.43	5.97	
10^6	26.3	22.8	19.3	15.9	12.4	8.96	5.54
10^7	28.5	25.1	21.6	18.1	14.7	11.2	7.85
10^8	30.9	27.4	24.0	20.5	17.0	13.6	10.1

Substituting this result into the expression (8.12), with λ_D as given by (8.4), gives

$$\Lambda = \frac{12\pi\epsilon_0 kT}{e^2}\,\lambda_D = 12\pi n_0 \lambda_D^3 = 9N_D \tag{8.16}$$

where N_D denotes the number of electrons in a Debye sphere. Table 1 presents values for ln Λ, for various values of the electron number density n_e and the electron temperature T.

PROBLEMS

DÙ

20.1 For a differential scattering cross section with an angular dependence given by

$$\sigma(\chi) = \tfrac{1}{2}\sigma_0(3\cos^2\chi - 1)$$

where σ_0 is a constant, calculate the total cross section and the momentum transfer cross section.

20.2 Consider a collision between two particles of mass m and m_1, in which the particle of mass m_1 is initially at rest. Denote the scattering angle in the *center of mass* coordinate system by χ and in the *laboratory* coordinate system (as seen by an observer at rest) by χ_L.
(a) Show that

$$\tan\chi_L = \frac{\sin\chi}{\cos\chi + (m/m_1)}$$

(b) Show that the relationship between the differential scattering cross section in the laboratory system $\sigma_L(\chi_L)$ and in the center of mass coordinate system $\sigma(\chi)$ is given by

$$\sigma_L(\chi_L) = \sigma(\chi)\ \frac{[1 + 2(m/m_1)\cos\chi + (m/m_1)^2]^{3/2}}{1 + (m/m_1)\cos\chi}$$

Note that when $m_1 = \infty$ we have $\chi_L = \chi$ and $\sigma_L(\chi_L) = \sigma(\chi)$.
(c) Verify that when $m = m_1$ we obtain $\chi_L = \chi/2$ and $\sigma_L(\chi_L) = 4\cos(\chi/2)\,\sigma(\chi)$.

20.3 Consider two particles whose interaction is governed by the following rectangular-well potential,

$$U(r) = 0 \qquad for \qquad r > a$$

$$U(r) = -U_0 \qquad for \qquad r \leq a$$

(a) Calculate the differential scattering cross section $\sigma(\chi)$ and show that it is given by (considering $b < a$)

$$\sigma(\chi) = \frac{p^2 a^2 [p\cos(\chi/2) - 1]\,[p - \cos(\chi/2)]}{4\cos(\chi/2)\,[1 - 2p\cos(\chi/2) + p^2]^2}$$

where

$$p = \left(1 + \frac{2U_0}{\mu g^2}\right)^{1/2}.$$

(b) Show that the total scattering cross section is given by

$$\sigma_t = 2\pi \int_0^a b\ db = \pi a^2$$

20.4 Consider a general inverse-power interparticle force of the form

$$F(r) = \frac{K}{r^p}$$

where K is a constant and p is a positive integer number.
(a) Determine expressions for the scattering angle χ, for the differential scattering cross section $\sigma(\chi, \epsilon)$, for the total scattering cross section σ_t and for the momentum transfer cross section σ_m.

(b) Calculate χ, $\sigma(\chi, \epsilon)$, σ_t and σ_m for the case of *Maxwell molecules*, for which $p = 5$.

20.5 From the expression for σ_m, obtained in part (a) of Problem 20.4, verify that for $p = 2$ the momentum transfer cross section is given by

$$\sigma_m = 2\pi\ A_1(2) \left(\frac{K}{\mu}\right)^2 \frac{1}{g^4}$$

where $A_1(2)$ is given by (with $\ell = 1$ and $p = 2$)

$$A_\ell(p) = \int_0^\infty (1 - \cos^\ell \chi)\ v_0\ dv_0$$

$$v_0 = b \left(\frac{\mu g^2}{K}\right)^{1/(p-1)}$$

Consequently, the velocity-dependent collision frequency, defined by

$$\nu_r(g) = n\sigma_m g$$

varies as g^{-3}. This inverse dependence on g accounts for the so-called *electron runaway effect.* This implies that in the presence of a sufficiently large electric field $\mathbf{E}$, some electrons will gain enough kinetic energy between collisions so as to decrease their cross section and the collision frequency, which in turn allows them to pick up more energy from the field and decrease even further their cross section and collision frequency. If $\mathbf{E}$ is large enough, the collision frequency may fall so fast that these electrons will form an accelerated beam of runaway electrons.

20.6 Show that for the case of Coulomb interactions ($p = 2$) we have

$$A_1(2) = \left(\frac{\mu g^2}{K}\right)^2 b_0^2 \ln\,(1 + \Lambda^2)$$

$$A_2(2) = \left(\frac{\mu g^2}{k}\right)^2 2b_0^2 \left[\ln\,(1 + \Lambda^2) - \frac{\Lambda^2}{(1 + \Lambda^2)}\right]$$

where $b_0 = qq_1/(4\pi\epsilon_0\mu g^2)$, $A_\ell(p)$ is as defined in Problem 20.5, and $\Lambda = \lambda_D/b_0$. For $\Lambda \gg 1$ verify that

$$A_1(2) = \left(\frac{\mu g^2}{K}\right)^2 2b_0^2 \ln \Lambda = 2 \ln \Lambda$$

$$A_2(2) = \left(\frac{\mu g^2}{K}\right)^2 2b_0^2\,(2 \ln \Lambda - 1) = 2\,(2 \ln \Lambda - 1)$$

Note that, since $K = qq_1/(4\pi\epsilon_0)$ for the Coulomb interaction potential, we have $(\mu g^2/K)^2\ b_0^2 = 1$.

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THE BOLTZMANN AND THE FOKKER-PLANCK EQUATIONS

1. INTRODUCTION

When the Boltzmann equation was first introduced in Chapter 5, the collisional effects were incorporated in its right-hand side [see equation (5.5.27)] through a general collision term $(\delta f_\alpha/\delta t)_{coll}$ still to be specified. We present now a derivation of the Boltzmann collision term, which takes into account only *binary collisions.* This collision term involves integrals over the particle velocities, so that the Boltzmann equation turns out to be an *integro-differential equation.* The fact that the Boltzmann collision integral takes into account only binary collisions limits considerably its applicability for a plasma, where each charged particle interacts simultaneously with a large number of neighboring charged particles. Although these multiple Coulomb collisions are very important for a plasma, there are some cases (as in weakly ionized plasmas) where the bynary charged-neutral collisions play a dominant role.

The collision term originally proposed by Boltzmann applies to a low density gas, in which only binary elastic collisions are important. These binary collisions may involve neutral atoms or molecules in a dilute gas, or charged and neutral particles in a plasma. We have seen that in a plasma these are not the only particle interactions of importance. The multiple Coulomb interactions need to be taken into account and in most cases are much more important than the binary collisions. Nevertheless, the Boltzmann collision term can in some cases be used for a plasma, but the results obtained have to be interpreted cautiously.

Furthermore, the Fokker-Planck collision term, which applies to charged particle interactions, can be derived from the Boltzmann collision term by considering the charged particle encounters as a series of consecutive weak (small deflection angle) binary collisions.

2. THE BOLTZMANN EQUATION

2.1 Derivation of the Boltzmann Collision Integral

The collision term $(\delta f_\alpha/\delta t)_{coll}$ represents the rate of change of the distribution function $f_\alpha(\mathbf{r}, \mathbf{v}, t)$ as a result of collisions. Some of the particles of type α originally situated inside the volume element $d^3r\ d^3v$ at $(\mathbf{r}, \mathbf{v})$ in phase space may leave this volume element, whereas some particles of type α originally situated outside this volume element may end inside it, as a result of collisions during the time interval dt. Let ΔN_α denote this net gain or loss of particles of type α in $d^3r\ d^3v$ at $(\mathbf{r}, \mathbf{v})$ during dt, that is,

$$\Delta N_\alpha = \left(\frac{\delta f_\alpha}{\delta t}\right)_{coll} d^3r\ d^3v\ dt \tag{2.1}$$

It is convenient to separate ΔN_α into two parts

$$\Delta N_\alpha = \Delta N_\alpha^+ - \Delta N_\alpha^- \tag{2.2}$$

where ΔN_α^+ represents the gain term due to collisions in which a particle of type α situated inside d^3r about $\mathbf{r}$ has, *after* collision, a velocity lying within d^3v about $\mathbf{v}$ and ΔN_α^- represents the loss term due to collisions in which a particle of type α situated inside d^3r about $\mathbf{r}$ has, *before* collision, a velocity lying within d^3v about $\mathbf{v}$. We proceed now to determinte ΔN_α, defined in (2.1), by calculating initially ΔN_α^- and afterwards ΔN_α^+.

To calculate ΔN_α^- we consider the particles of type α situated within the volume element d^3r at $\mathbf{r}$, whose velocities lie within d^3v about $\mathbf{v}$, and which are scattered out of this velocity range, as a result of collisions with particles of some type (which may or may not be type α particles) lying in the *same* volume element d^3r at $\mathbf{r}$ and having some velocity within d^3v_1 about $\mathbf{v}_1$. Let us focus attention on a single particle of type α situated within the phase space volume element $d^3r\ d^3v$ at the coordinates $(\mathbf{r}, \mathbf{v})$. The particles of type β inside $d^3r\ d^3v_1$ at $(\mathbf{r}, \mathbf{v}_1)$ may be viewed as constituting a particle flux incident on this type α particle. Noting that $f_\beta(\mathbf{r}, \mathbf{v}_1, t)\ d^3v_1$ is the number of type β particles per unit volume,

with velocities within d^3v_1 about $\mathbf{v}_1$, the flux of this incident beam can be expressed as

$$\Gamma_\beta = f_\beta(\mathbf{r}, \mathbf{v}_1, t)\, d^3v_1\, |\mathbf{v}_1 - \mathbf{v}| = f_\beta(\mathbf{r}, \mathbf{v}_1, t)\, d^3v_1\, g \tag{2.3}$$

Consider the type β particles which approach with an impact parameter between b and $b + db$, in a collision plane lying between the angles ϵ and $\epsilon + d\epsilon$. The average number of interactions of these type β particles with the type α particle, occuring in the time interval dt, is equal to the number of particles crossing the element of area $b\, db\, d\epsilon$ during dt. This number can be obtained by multiplying the flux of type β particles, given in (2.3), by the element of area $b\, db\, d\epsilon$ and by the time interval dt,

$$\Gamma_\beta\, b\, db\, d\epsilon\, dt = f_\beta(\mathbf{r}, \mathbf{v}_1, t)\, d^3v_1\, g\, b\, db\, d\epsilon\, dt \tag{2.4}$$

This expression is just be number of particles of type β with velocities lying within d^3v_1 about $\mathbf{v}_1$ and situated inside the elementary cylindrical volume of length $g\, dt$ and cross sectional area $b\, db\, d\epsilon$, as shown in Fig. 1, and whose volume is $g\, b\, db\, d\epsilon\, dt$. It is assumed here that dt is large compared to the time of interaction of the colliding particles. To determine the number of collisions between the type β particles indicated in (2.4), with all the type α particles lying within $d^3r\, d^3v$ at $(\mathbf{r}, \mathbf{v})$, during dt, we multiply (2.4) by $f_\alpha(\mathbf{r}, \mathbf{v}, t)\, d^3r\, d^3v$, which is the number of type α particles lying within this phase space volume element, obtaining

$$f_\alpha(\mathbf{r}, \mathbf{v}, t)\, d^3r\, d^3v\, f_\beta(\mathbf{r}, \mathbf{v}_1, t)\, d^3v_1\, g\, b\, db\, d\epsilon\, dt \tag{2.5}$$

In deducing this expression it has been assumed that the number of collisions between the particles of the types α and β, with velocities in d^3v about $\mathbf{v}$, and in d^3v_1 about $\mathbf{v}_1$, respectively, lying in the same volume element d^3r about $\mathbf{r}$, is proportional to the product $f_\alpha(\mathbf{r}, \mathbf{v}, t)\, f_\beta(\mathbf{r}, \mathbf{v}_1, t)$. However, in a system of interacting particles the existence of a particle within a given volume element d^3r at $\mathbf{r}$, with a given velocity $\mathbf{v}$, affects the probability that another particle be found with a specified velocity $\mathbf{v}_1$ in the same volume element, at the same time. Thus, in expression (2.5) we are neglecting any possible correlation that may exist between the velocity of a particle and its position. This approximation, known as the *molecular chaos* assumption, is introduced as a mathematical convenience, but although it may represent a possible condition for a system of particles, it is not a general condition.

The total number of particles of type α in d^3r about $\mathbf{r}$ that are scattered *out* of the velocity space element d^3v about $\mathbf{v}$, during dt, is obtained

integrating expression (2.5) over all possible values of b, ϵ and $\mathbf{v}_1$, and summing over all species β,

$$\Delta N_\alpha^- = f_\alpha(\mathbf{r}, \mathbf{v}, t)\, d^3r\, d^3v\, dt \sum_\beta \int_{v_1} \int_b \int_\epsilon f_\beta(\mathbf{r}, \mathbf{v}_1, t)\, d^3v_1\, g\, b\, db\, d\epsilon \quad (2.6)$$

where the triple integral over $\mathbf{v}_1$ is represented again by a single integral sign.

To deternine the *gain* term ΔN_α^+, we proceed in a way similar to the determination of ΔN_α^-, by considering the *inverse collision*, in which a particle of type α with initial velocity in d^3v' about $\mathbf{v}'$, collides with a particle of type β having initial velocity in d^3v_1' about $\mathbf{v}_1'$, resulting in the particle of type α scattered into the velocity element d^3v about $\mathbf{v}$, the event occurring inside the volume element d^3r about $\mathbf{r}$. The average number of interactions between a single particle of type α, inside the volume element of phase space $d^3r\ d^3v'$ at $(\mathbf{r}, \mathbf{v}')$, with the particles of type β inside $d^3r\ d^3v_1'$ at $(\mathbf{r}, \mathbf{v}_1')$, which approach with an impact parameter between b and $b + db$, and with the collision plane oriented between the angles ϵ and $\epsilon + d\epsilon$, is given by

$$f_\beta(\mathbf{r}, \mathbf{v}_1', t)\, d^3v_1'\, g'\, b\, db\, d\epsilon\, dt \quad (2.7)$$

To take into account all collisions occurring within d^3r, at $\mathbf{r}$, during the time interval dt, between the particles of type α and β, which scatter the particles of type α into the volume element d^3v, about $\mathbf{v}$, we must multiply (2.7) by the number of particles of type α which lie initially inside $d^3r\ d^3v'$, that is, by $f_\alpha(\mathbf{r}, \mathbf{v}', t)\, d^3r\, d^3v'$, integrate the result over all possible values of b, ϵ and $\mathbf{v}_1'$, and sum over all species β,

$$\Delta N_\alpha^+ = f_\alpha(\mathbf{r}, \mathbf{v}', t)\, d^3r\, d^3v'\, dt \sum_\beta \int_{v_1'} \int_b \int_\epsilon f_\beta(\mathbf{r}, \mathbf{v}_1', t)\, d^3v_1'\, g'\, b\, db\, d\epsilon \quad (2.8)$$

We have seen that $g' = g = |\mathbf{v}_1 - \mathbf{v}|$ and from the theory of Jacobians

$$d^3v'\, d^3v_1' = |J|\, d^3v\, d^3v_1 \quad (2.9)$$

We will show in the following subsection that for this transformation of velocities we have $|J| = 1$, so that

$$d^3v'\, d^3v_1' = d^3v\, d^3v_1 \quad (2.10)$$

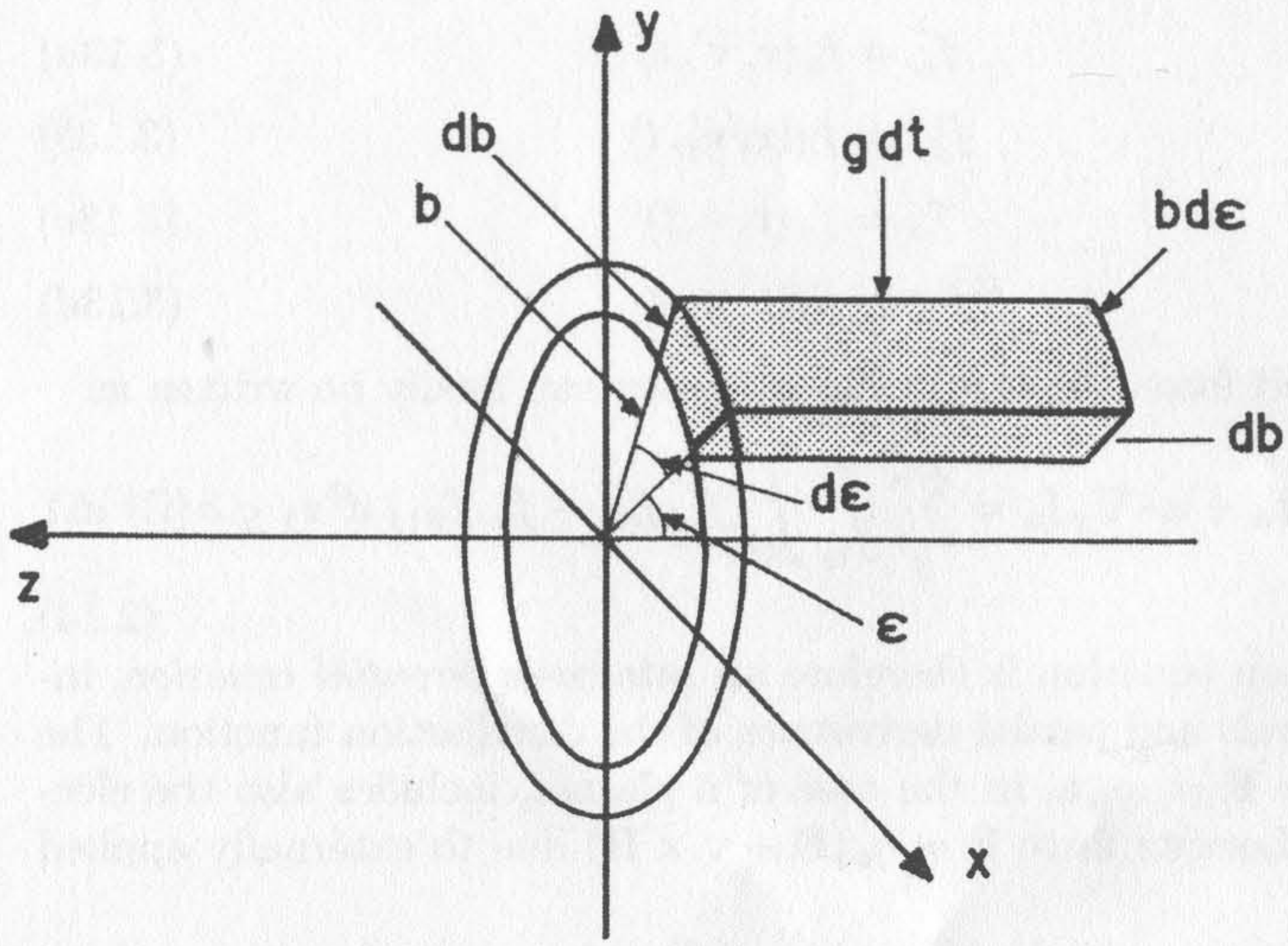

Fig. 1 *The volume element of height $g\ dt$ and cross sectional area $b\ db\ d\epsilon$, with sides lying between b and $b + db$, and between ϵ and $\epsilon + d\epsilon$.*

Consequently, (2.8) becomes

$$\Delta N_\alpha^+ = f_\alpha(\mathbf{r}, \mathbf{v}', t)\ d^3r\ d^3v\ dt \sum_\beta \int_{v_1} \int_b \int_\epsilon f_\beta(\mathbf{r}, \mathbf{v}_1', t)\ d^3v_1\ g\ b\ db\ d\epsilon \tag{2.11}$$

If we now combine the expressions for ΔN_α^- and ΔN_α^+, and substitute $b\ db\ d\epsilon$ by $\sigma(\Omega)\ d\Omega$, we obtain the following expression for the Boltzmann collision integral

$$\left(\frac{\delta f_\alpha}{\delta t}\right)_{coll} = \left(\frac{\Delta N_\alpha^+ - \Delta N_\alpha^-}{d^3r\ d^3v\ dt}\right) =$$

$$= \sum_\beta \int_{v_1} \int_\Omega (f_\alpha'\ f_{\beta 1}' - f_\alpha\ f_{\beta 1})\ d^3v_1\ g\ \sigma(\Omega)\ d\Omega \tag{2.12}$$

where we have introduced the notation

$$f'_\alpha = f_\alpha(\mathbf{r}, \mathbf{v}', t) \tag{3.13a}$$

$$f'_{\beta 1} = f_\beta(\mathbf{r}, \mathbf{v}'_1, t) \tag{3.13b}$$

$$f_\alpha = f_\alpha(\mathbf{r}, \mathbf{v}, t) \tag{3.13c}$$

$$f_{\beta 1} = f_\beta(\mathbf{r}, \mathbf{v}_1, t) \tag{3.13d}$$

In explicit form, the Boltzmann equation can finally be written as

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v} \cdot \nabla f_\alpha + \mathbf{a} \cdot \nabla_v f_\alpha = \sum_\beta \int_{v_1} \int_\Omega (f'_\alpha \, f'_{\beta 1} - f_\alpha \, f_{\beta 1}) \, d^3 v_1 \, g \, \sigma(\Omega) \, d\Omega \tag{2.14}$$

The Boltzmann equation is therefore an *integro-differential equation*, involving integrals and partial derivatives of the distribution function. The *external* force $\mathbf{F} = m_\alpha \mathbf{a}$, in the case of a plasma, includes also the electromagnetic Lorentz force $\mathbf{F} = q_\alpha(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ due to externally applied fields.

For a system consisting of various different species of particles, there is one equation for each species. For an ionized gas composed of electrons, one type of positive ions and one type of neutral particles, for example, we have a system of three Boltzmann equations coupled through the collision term. In the Boltzmann equation for the electrons, for example, the collision term contains the distribution function for the electrons f_e, the distribution function for the ions f_i and the distribution function for the neutral particles f_n. Since the collision term involves products of the distribution functions, the Boltzmann equation is also *nonlinear*. For a system consisting of only one type of particles, the summation over the type β particles disappears and the collision term involves only the product of distribution functions of the same particle species.

2.2 Jacobian of the Transformation

The relation between the velocity elements $d^3v \; d^3v_1$ and $d^3v' \; d^3v'_1$ is given by

$$d^3v' \; d^3v'_1 = |J| \; d^3v \; d^3v_1 \tag{2.15}$$

where J is the Jacobian of the transformation from the velocity variables $(\mathbf{v}, \mathbf{v}_1)$ to $(\mathbf{v}', \mathbf{v}'_1)$ and can be expressed as

$$J = \frac{\partial(\mathbf{v}', \mathbf{v}'_1)}{\partial(\mathbf{v}, \mathbf{v}_1)} = \frac{\partial(v'_x, v'_y, v'_z, v'_{1x}, v'_{1y}, v'_{1z})}{\partial(v_x, v_y, v_z, v_{1x}, v_{1y}, v_{1z})} \tag{2.16}$$

which corresponds to the determinant of the matrix

$$(J) = \begin{pmatrix} \frac{\partial v'_x}{\partial v_x} & \frac{\partial v'_y}{\partial v_x} & \cdots & \frac{\partial v'_{1z}}{\partial v_x} \\ \frac{\partial v'_x}{\partial v_y} & \frac{\partial v'_y}{\partial v_y} & \cdots & \frac{\partial v'_{1z}}{\partial v_y} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial v'_x}{\partial v_{1z}} & \frac{\partial v'_y}{\partial v_{1z}} & \cdots & \frac{\partial v'_{1z}}{\partial v_{1z}} \end{pmatrix} \tag{2.17}$$

Using (20.2.25) and (20.2.6) we can express d^3v and d^3v_1 in terms of d^3c_0 and d^3g,

$$d^3v\ d^3v_1 = |J_c|\ d^3c_0\ d^3g \tag{2.18}$$

where J_c denotes the Jacobian of the transformation indicated in equations (20.2.5) and (20.2.6). Let us consider initially only the x component of (2.18),

$$dv_x\ dv_{1x} = \left|\frac{\partial(v_x, v_{1x})}{\partial(c_{0x}, g_x)}\right|\ dc_{0x}\ dg_x \tag{2.19}$$

From (20.2.5) and (20.2.6) we can calculate the determinant of the 2×2 matrix indicated in (2.19), obtaining

$$dv_x\ dv_{1x} = \left(\frac{\mu}{m_1} + \frac{\mu}{m}\right)\ dc_{0x}\ dg_x = dc_{0x}\ dg_x \tag{2.20}$$

Taking the product of three such terms, corresponding to the components x, y and z, gives

$$d^3v\ d^3v_1 = d^3c_0\ d^3g \tag{2.21}$$

In a similar way, using (20.2.8) and (20.2.9) we find

$$d^3v'\ d^3v'_1 = d^3c'_0\ d^3g' \tag{2.22}$$

We have seen that $\mathbf{c}_0 = \mathbf{c}'_0$. Furthermore, g and g' differ only in direction, having the same magnitude, and since volume elements are not changed by a simple rotation of coordinates, we must have $d^3g = d^3g'$. Consequently, (2.21) and (2.22) yields

$$d^3v\ d^3v_1 = d^3v'\ d^3v'_1 \tag{2.23}$$

2.3 Assumptions in the Derivation of the Boltzmann Collision Integral

The derivation of the Boltzmann collision integral presented here involves four basic assumptions:

(a) The distribution function does not vary appreaciably over a distance of the order of the range of the interparticle force law, as well as over time scales of the order of the interaction time.
(b) Effects of the external force, on the magnitude of the collision cross section, are ignored.
(c) Only binary collisions are taken into account.
(d) The velocities of the interacting particles, before collision, are assumed to be uncorrelated.

The first assumption is quite reasonable and is incorporated in the calculation of $(\delta f_\alpha/\delta t)_{coll}$ when we evaluate all the distribution functions at the position $\mathbf{r}$ and at the instant t. The element of volume d^3r is considered to be large compared to the range of the interpaticle forces and the time interval dt is taken to be large compared to the time of interaction. On the other hand, as far as variations in the distribution functions are concerned, the elements d^3r and dt must be infinitesimally small quantities.

Next, the external force was assumed to have a negligible effect on the two-body collision problem. This is valid if the external force is negligibly small compared to the interaction force between the particles. When external forces of magnitude comparable to the short-range interparticle forces are present, the collision process is modified. The constancy of the relative speed $g = |\mathbf{v}_1 - \mathbf{v}|$ is strictly valid only in the absence of external forces.

The assumption of binary collisions is justified for a dilute gas, whose molecules interact through short-range forces. However, it is not valid for the Coulomb interactions in a plasma. Because of the long-range characteristic of the Coulomb force, a charged particle in a plasma interacts simultanelously with all the charged particles inside its Debye sphere. Since there is a large number of charged particles inside a Debye sphere, each charged particle in the plasma does not move freely, as does a neutral particle between collisions, but it is continuously interacting with a large number of charged particles. However, the Boltzmann collision integral may also be used as a starting point in the derivation of the Fokker-Plamck collision term for a plasma, under the assumption that each long-range individual interaction results only in a small deflection in the particle trajec-

tory and since each individual interaction is relatively weak, the collective effect of many simultaneous interactions can be considered as a cumulative succession of weak binary collisions. Thus, in general, the Boltzmann collision term is not strictly valid for a plasma and the results obtained for the case of charged-neutral particle interactions in weakly ionized plasmas must be interpreted cautiously.

Assumption (d) is known as the *molecular chaos* assumption. It is justified for a gas in which the density is sufficiently small, so that the mean free path is much larger than the characteristic range of the interparticle forces. This is certainly not a general situation for a plasma, in view of the long-range characteristic of the Coulomb force. Generally, the *joint* probability of having, at the position $\mathbf{r}$ and at the instant t, a particle of type α with velocity $\mathbf{v}$ and a particle of type β with velocity $\mathbf{v}_1$, is proportional to

$$f_\alpha(\mathbf{r}, \mathbf{v}, t)\ f_\beta(\mathbf{r}, \mathbf{v}_1, t)\ [1 + \psi_{\alpha\beta}(\mathbf{v}, \mathbf{v}_1, \mathbf{r}, t)] \tag{2.24}$$

where $\psi_{\alpha\beta}(\mathbf{v}, \mathbf{v}_1, \mathbf{r}, t)$ is known as the *correlation function.* In the derivation of the Boltzmann collision integral we have neglected the correlation effects and we have taken this joint probability as being proportional to the product $f_\alpha(\mathbf{r}, \mathbf{v}, t)\ f_\beta(\mathbf{r}, \mathbf{v}_1, t)$. The *irreversible* character of the Boltzmann equation, to be discussed in the next section, is a consequence of the molecular chaos assumption. In order to avoid this approximation the only alternative is to work with the *reversible* equations of the *BBGKY (Bogoliubov, Born, Green, Kirkwood and Yvon) hierarchy.* This kinetic treatment, however, is beyond the scope of this text.

For gaseous systems in which the characteristic interaction length is much less than the average interpaticle distance and where temporal and spatial gradients are not very large, the Boltzmann equation is nevertheless very well verified experimentally and, in this respect, constitutes one of the basic relations of the kinetic theory of gases.

2.4 Rate of Change of a Physical Quantity as a Result of Collisions

In section 2 of Chapter 8 we have denoted the time rate of change of a physical quantity $\chi(\mathbf{v})$ per unit volume, for the particles of type α, due to collisions with the other particles in the plasma, by

$$\left[\frac{\delta(n_\alpha < \chi >_\alpha)}{\delta t}\right]_{coll} = \int_v \chi \left(\frac{\delta f_\alpha}{\delta t}\right)_{coll} d^3v \tag{2.25}$$

Using the Boltzmann collision integral,

$$\left(\frac{\delta f_\alpha}{\delta t}\right)_{coll} = \sum_\beta \int_\Omega \int_{v_1} (f'_\alpha\, f'_{\beta 1} - f_\alpha\, f_{\beta 1})\, g\, \sigma(\Omega)\, d\Omega\, d^3v_1 \tag{2.26}$$

we obtain the following expression for (2.25),

$$\left[\frac{\delta(n_\alpha < \chi >_\alpha)}{\delta t}\right]_{coll} = \sum_\beta \int_\Omega \int_{v_1} \int_v (f'_\alpha\, f'_{\beta 1} - f_\alpha\, f_{\beta 1})\, \chi\, g\, \sigma(\Omega)\, d\Omega\, d^3v\, d^3v_1 \tag{2.27}$$

Recall that for each direct collision there is a corresponding inverse collision with the same cross section. Hence, the integrals over $\mathbf{v}$ and $\mathbf{v}_1$ can be replaced by integrals over $\mathbf{v}'$ and $\mathbf{v}'_1$, respectively, without altering the result. Therefore, the first group of integrals in (2.27) may be written as

$$\sum_\beta \int_\Omega \int_{v_1} \int_v f'_\alpha\, f'_{\beta 1}\, \chi\, g\, \sigma(\Omega)\, d\Omega\, d^3v\, d^3v_1 =$$

$$= \sum_\beta \int_\Omega \int_{v_1} \int_v f_\alpha\, f_{\beta 1}\, \chi'\, g\, \sigma(\Omega)\, d\Omega\, d^3v\, d^3v_1 \tag{2.28}$$

where we have replaced $d^3v'\, d^3v'_1$ by $d^3v\, d^3v_1$. Using this expression, we obtain the following alternative form for the collision term in (2.27),

$$\left[\frac{\delta(n_\alpha < \chi >_\alpha)}{\delta t}\right]_{coll} = \sum_\alpha \int_\Omega \int_{v_1} \int_v f_\alpha\, f_{\beta 1}\, (\chi' - \chi)\, g\, \sigma(\Omega)\, d\Omega\, d^3v\, d^3v_1 \tag{2.29}$$

Note that the property $\chi(\mathbf{v})$ is associated with the particles of type α and that χ' denotes $\chi(\mathbf{v}')$. Note also that only the quantity χ' on the right-hand side of (2.29) is a function of the *after* collision velocity $\mathbf{v}'$. Recall that the result just derived applies to the special case of binary elastic collisions in a dilute gas, when processes of particle creation and disappearance, as well as radiation losses are unimportant.

3. THE BOLTZMANN'S H FUNCTION

An important characteristic of the Boltzmann collision term is that it drives the distribution function towards the equilibrium state in an *irreversible* way. This irreversible characteristic of the Boltzmann collision

term, as mentioned before, is a consequence of the molecular chaos assumption, which neglects the correlation effects between the particles.

In order to investigate this aspect of the Boltzmann collision term, we introduce now the Boltzmann's function $H(t)$. For simplicity we will consider a gas consisting of only one particle species, with the particles uniformly distributed in space (having no density gradients) and isolated from the action of external forces. The distribution function is therefore independent of $\mathbf{r}$ and can be denoted by $f(\mathbf{v}, t)$. We define, according to Boltzmann, the function $H(t)$ by

$$H(t) = \int_v f(\mathbf{v}, t) \ln [f(\mathbf{v}, t)] \, d^3v \tag{3.1}$$

For problems involving spatial gradients, the function $H(t)$, defined in (3.1), corresponds to $H_{total}(t)$ per unit volume, where

$$H_{total}(t) = \int_r \int_v f(\mathbf{r}, \mathbf{v}, t) \ln [f(\mathbf{r}, \mathbf{v}, t)] \, d^3r \, d^3v \tag{3.2}$$

The function $H(t)$ is proportional to the *entropy* per unit volume of the system according to

$$\frac{S}{V} = - k \, H \tag{3.3}$$

where S denotes the total entropy, V is the volume of the system and k is Boltzmann's constant. More generally, for systems in which spatial gradients are present we have

$$S = - k \, H_{total} \tag{3.4}$$

3.1 Boltzmann's H Theorem

The Boltzmann's H theorem states that if $f(\mathbf{v}, t)$ is a solution of the Boltzmann equation, that is, if

$$\frac{\partial f(\mathbf{v}, t)}{\partial t} = \int_\Omega \int_{v_1} [f(\mathbf{v}', t) \, f(\mathbf{v}_1', t) - f(\mathbf{v}, t) \, f(\mathbf{v}_1, t)] \, g \, \sigma(\Omega) \, d\Omega \, d^3v_1 \tag{3.5}$$

then

$$\frac{\partial H(t)}{\partial t} \leq 0 \tag{3.6}$$

To prove this theorem let us take the derivative of (3.1) with respect to time, which gives

$$\frac{\partial H}{\partial t} = \int_v (1 + \ln f) \frac{\partial f}{\partial t} d^3v \tag{3.7}$$

Substituting (3.5) into (3.7) gives

$$\frac{\partial H}{\partial t} = \int_\Omega \int_v \int_{v_1} (1 + \ln f) (f' f_1' - f f_1) g \sigma(\Omega) d\Omega d^3v_1 d^3v \tag{3.8}$$

where the notation $f_1' = f(\mathbf{v}_1', t)$, and so on, has been used. The variables of integration $\mathbf{v}$ and $\mathbf{v}_1$ are dummy variables and can be interchanged in the integrand of (3.8) without changing the value of the integral, since $\sigma(\Omega)$ and $g = |\mathbf{v}_1 - \mathbf{v}|$ are also invariants. Thus, (3.8) can be written as

$$\frac{\partial H}{\partial t} = \int_\Omega \int_{v_1} \int_v (1 + \ln f_1) (f_1' f' - f_1 f) g \sigma(\Omega) d\Omega d^3v d^3v_1 \tag{3.9}$$

Adding equations (3.8) and (3.9), and dividing by 2, gives

$$\frac{\partial H}{\partial t} = \tfrac{1}{2} \int_\Omega \int_v \int_{v_1} [2 + \ln (f f_1)] (f' f_1' - f f_1) g \sigma(\Omega) d\Omega d^3v_1 d^3v \tag{3.10}$$

In this equation we can replace the velocities *before* collision, $\mathbf{v}$ and $\mathbf{v}_1$, by the velocities *after* collision, $\mathbf{v}'$ and $\mathbf{v}_1'$, respectively, without altering the value of the integral, since for each direct collision there exists an inverse collision with the same differential cross section $\sigma(\Omega)$. We have already seen that $d^3v' d^3v_1' = d^3v d^3v_1$ and $g = g'$. Consequently, (3.10) may be written as

$$\frac{\partial H}{\partial t} = \tfrac{1}{2} \int_\Omega \int_v \int_{v_1} [2 + \ln (f' f_1')] (f f_1 - f' f_1') g \sigma(\Omega) d\Omega d^3v_1 d^3v \tag{3.11}$$

We now combine (3.10) and (3.11) to obtain

$$\frac{\partial H}{\partial t} = \tfrac{1}{4} \int_\Omega \int_v \int_{v_1} \left[\ln \left(\frac{f f_1}{f' f_1'} \right) \right] (f' f_1' - f f_1) g \sigma(\Omega) d\Omega d^3v_1 d^3v \tag{3.12}$$

In this expression it is clear that if $f' f_1' > f f_1$ then $\ln (f f_1 / f' f_1') < 0$ and, consequently, $(\partial H / \partial t) < 0$, since all other factors appearing in the right-hand side of (3.12) are positive. On the other hand, if $f' f_1' < f f_1$

then $\ln(ff_1/f'f_1') > 0$ and, again, $(\partial H/\partial t) < 0$. When $f'f_1' = ff_1$, both factors are zero and $(\partial H/\partial t) = 0$, which corresponds to the equilibrium state.

This result proves the H theorem, showing that when f satisfies the Boltzmann equation, the functional $H(t)$ always decreases monotonically in time until it reaches a limiting value, which occurs when there is no further change with time in the system. This limiting value is reached *only* when

$$f'f_1' = ff_1 \tag{3.13}$$

so that this condition is *necessary* for $(\partial H/\partial t) = 0$ and, consequently, it is also a necessary condition for the equilibrium state. According to the Boltzmann equation (3.5), the *equilibrium distribution function* satisfies the following integral equation

$$\int_\Omega \int_{v_1} [f(\mathbf{v}')\, f(\mathbf{v}_1') - f(\mathbf{v})\, f(\mathbf{v}_1)]\, g\, \sigma(\Omega)\, d\Omega\, d^3v_1 = 0 \tag{3.14}$$

so that the condition (3.13) is also a *sufficient* condition for the equilibrium state.

It is instructive to note that (3.13) can be considered as an example of the *general principle of detailed balance* of statistical mechanics, as discussed in section 1 of Chapter 7, where it was used to derive the Maxwell-Boltzmann equilibrium distribution function. An important conclusion that can be drawn from (3.13) is that the equilibrium distribution function is independent of the differential collision cross section $\sigma(\Omega)$, considered to be nonzero. The Maxwell-Boltzmann distribution function is therefore the only distribution for the equilibrium state that can exist in a uniform gas in the absence of external forces.

3.2 Analysis of Boltzmann's H Theorem

According to (3.3), the H theorem states that the entropy of a given isolated system always increases with time until it reaches the equilibrium state. Although this *irreversible* behavior is compatible with the laws of thermodynamics, it is nevertheless in disagreement with the laws of mechanics, which are *reversible*. If, at a given instant of time, the velocities of all the particles in a system were reversed, the laws of mechanics predict that each particle would describe, in the opposite sense, its previous trajectory. However, we have seen that the Boltzmann collision term leads to a irreversible temporal evolution of the distribution function and of the

function $H(t)$. The existence of this paradox has its origin in the molecular chaos assumption that was used in the derivation of the Boltzmann collision term.

Recall that the molecular chaos assumption considers that, if $f(\mathbf{r}, \mathbf{v}, t)$ is proportional to the probability of finding in a given volume element d^3r, about $\mathbf{r}$, a particle with velocity $\mathbf{v}$, at the instant t, then the joint probability of simultaneously finding in the same volume element d^3r, about $\mathbf{r}$, a particle with velocity $\mathbf{v}$ and another particle with velocity $\mathbf{v}_1$, at the instant t, is proportional to the product $f(\mathbf{r}, \mathbf{v}, t)\ f(\mathbf{r}, \mathbf{v}_1, t)$. Thus, it neglects any possible correlation that may exist between the particle velocities. Generally, the state of the gas may or may not satisfy the molecular chaos assumption and consequently the distribution function describing the gas may or may not satisfy the Boltzmann equation. The distribution function, which characterizes the state of the gas, will obey the Boltzmann equation only at the instants of time when the molecular chaos assumption holds true for the gas. The H theorem, therefore, is also valid only when this condition is satisfied.

We shall show now that at the instants of time when the state of the gas satisfies the molecular chaos assumption, the function $H(t)$ is at a *local maximum*. For this purpose consider a gas not in equilibrium, which is in the state of molecular chaos at the instant $t = t_0$. The H theorem implies that at the instant $t_0 + dt$ we have $(\partial H/\partial t) \leq 0$. Consider a second gas which at the instant $t = t_0$ is exactly identical to the first one, except that the velocities of all the particles have directions opposite to the velocities of the first one, has the same function $H(t)$ as the first one and is in a state of molecular chaos at $t = t_0$. Consequently, at the instant $t_0 + dt$ we must have $(\partial H/\partial t) \leq 0$, according to the H theorem. On the other hand, due to the invariance of the equations of motion under time reversal, the time evolution of the second gas corresponds to the past of the first. This means that for the first gas we must have

$$\frac{\partial H}{\partial t} \leq 0 \qquad at \qquad t = t_0 + dt \tag{3.15a}$$

$$\frac{\partial H}{\partial t} \geq 0 \qquad at \qquad t = t_0 - dt \tag{3.15b}$$

which shows that, at the instant when the condition of molecular chaos is satisfied, the function $H(t)$ is at a local maximum. This situation is illustrated in Fig. 2 at the instant $t = t_0$ indicated by the number (2). At the instants when $H(t)$ does not present a local maximum, as for example at the instants indicated by the numbers (1) and (3) in Fig. 2, the gas is

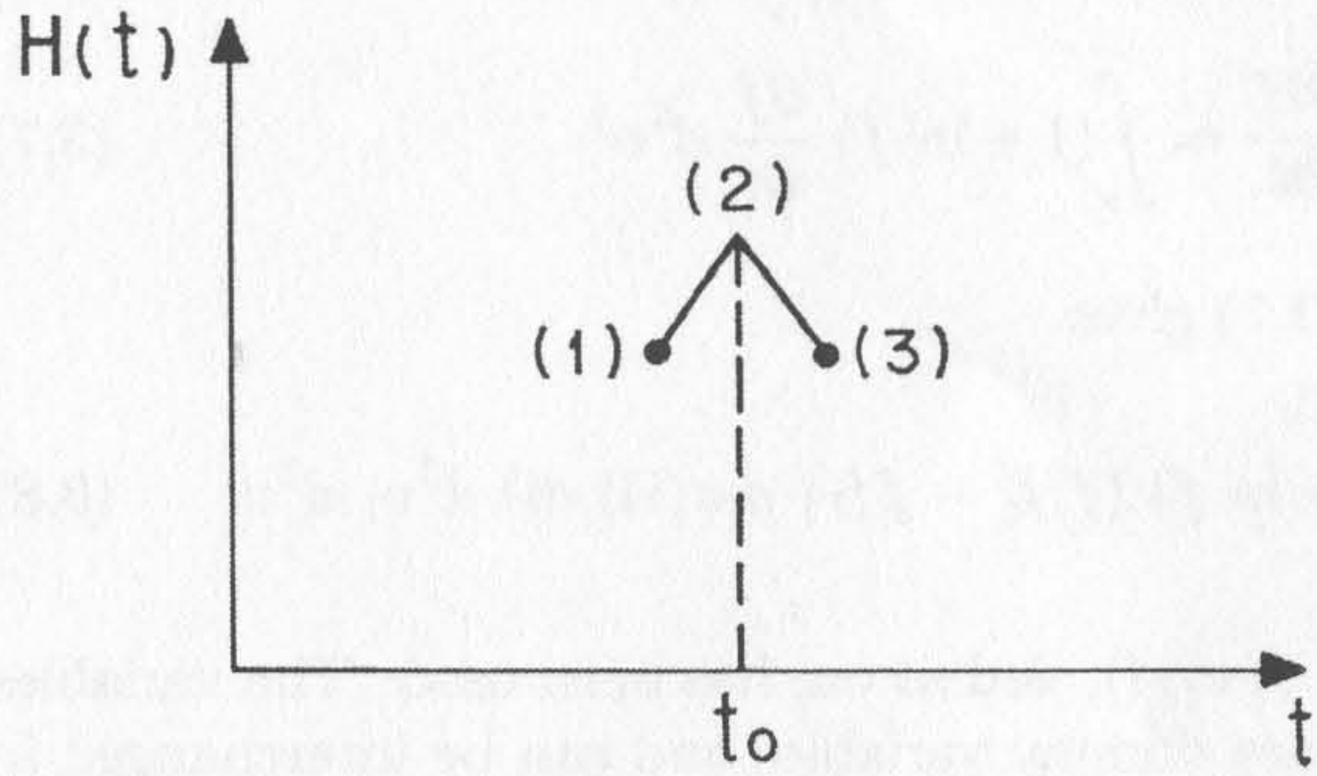

Fig. 2 *When the gas satisfies the molecular chaos condition, the function $H(t)$ is at a local maximum, indicated here by the point denoted (2).*

not in a state of molecular chaos. Note that $(\partial H/\partial t)$ need not necessarily be a continuous function of time and may change abruptly as a result of collisions.

The time evolution of $H(t)$ is governed by the collisional interactions between the particles, which occur at random and which can establish as well as destroy the state of molecular chaos as time passes. Fig. 3 illustrates how $H(t)$ may vary in time. Some of the instants when the condition of molecular chaos is satisfied are indicated by dots in the curve of $H(t)$. If the condition of molecular chaos prevails during most of the time, as in a dilute gas for example, $H(t)$ will be at a local maximum most of the time. Due to the random characteristic of the sequence of collisions, these instants of molecular chaos will probably be distributed in time in an almost uniform way. On the other hand, the time variation of $H(t)$, obtained using the distribution function that satisfies the Boltzmann equation, is represented by a smooth curve of negative slope which tries to fit, with a minimum deviation, all the points (instants) of the real curve of $H(t)$ in which the condition of molecular chaos is satisfied, as shown by the dashed line of Fig. 3. The state of molecular chaos can therefore be considered as a convenient mathematical model to describe a state not in equilibrium.

The Boltzmann equation, although strictly valid only at the instants

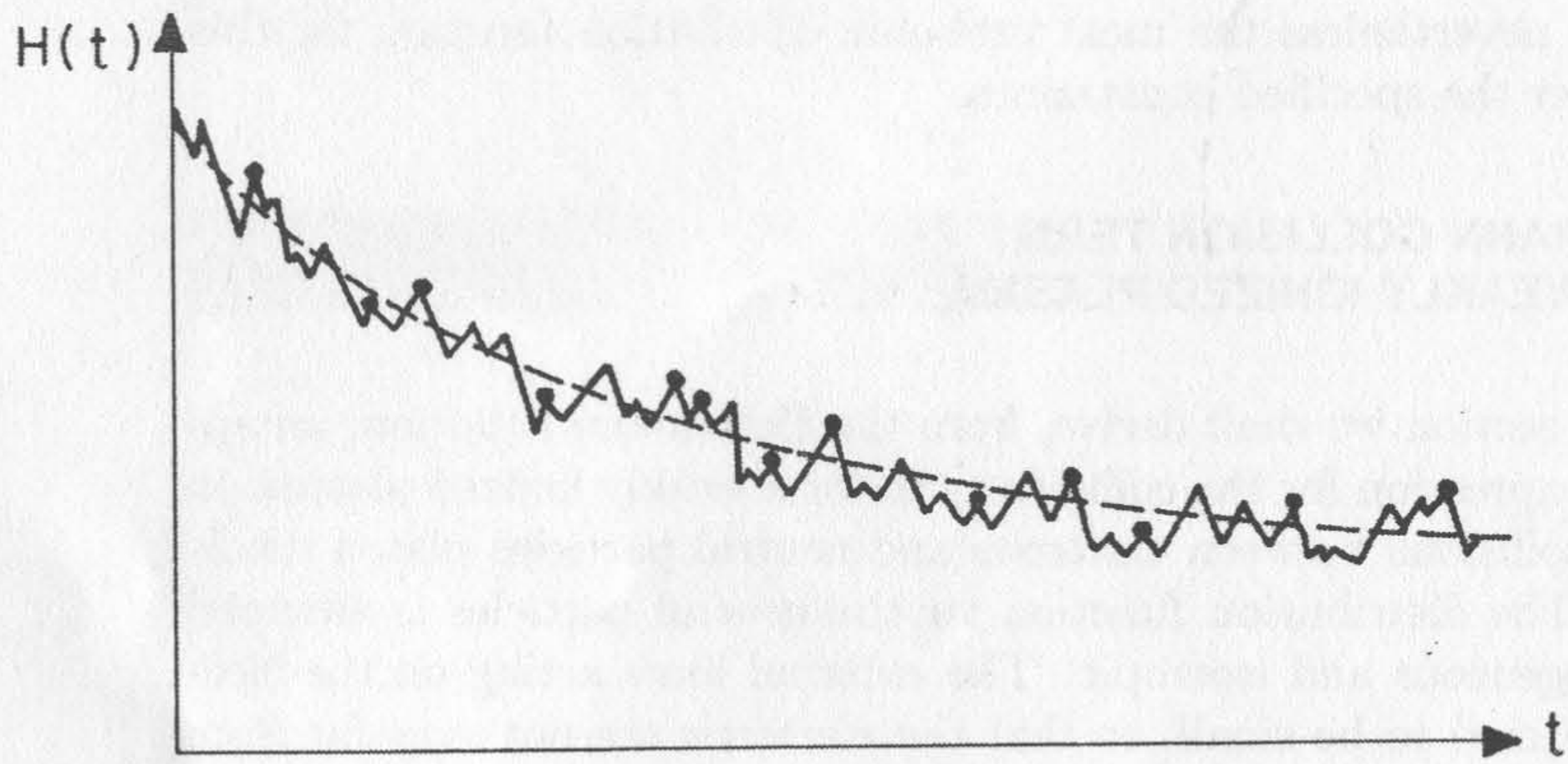

Fig. 3 *The time evolution of $H(t)$ for a gas, initially not in an equilibrium state, is indicated by the solid curve. The dashed curve represents the time variation of $H(t)$ predicted by the Boltzmann equation. The dots indicate some of the instants when the condition of molecular chaos is satisfied.*

when the gas is in the state of molecular chaos, can nevertheless be considered generally valid in a *statistical* sense at any instant and the same applies for the H theorem.

3.3 Maximum Entropy or Minimum H Approach for Deriving the Equilibrium Distribution Function

The Maxwell-Boltzmann equilibrium distribution function can also be derived by performing a *variational calculation* on the function $H(t)$. We have seen that, under equilibrium conditions, $H(t)$ is a minimum, so that for a one-component uniform gas we must have, at *equilibrium*,

$$\delta H = \delta \, [\int_v f \, \ln \, (f) \, d^3v] = 0 \tag{3.16}$$

where the symbol δ, before a given quantity, indicates a variation in this quantity as a result of a small change in the distribution function. Carrying

out the variation indicated in (3.16) in a formal way, we have

$$\delta H = \int_v (1 + \ln f)\ \delta f\ d^3v = 0 \tag{3.17}$$

There are, however, certain *macroscopic constraints* imposed on the system. When we vary f slightly, we cannot violate the basic laws of conservation of mass, of momentum and of energy for the system as a whole. Therefore, the variational integral (3.17) is subjected to the constraints that the total mass, momentum and energy densities of the uniform gas remain constant under the variation in f. The constancy of the mass density, under a small change δf in f, requires that

$$\delta(\rho_m) = m \int_v \delta f\ d^3v = 0 \tag{3.18}$$

Similarly, for the constancy of the momentum density,

$$\delta(\rho_m < \mathbf{v} >) = m \int_v \mathbf{v}\ \delta f\ d^3v = 0 \tag{3.19}$$

and for the energy density,

$$\delta(\tfrac{1}{2}\rho_m < v^2 >) = \tfrac{1}{2}m \int_v v^2\ \delta f\ d^3v = 0 \tag{3.20}$$

We can now solve the variational integral in (3.17), subjected to the constraints expressed by (3.18), (3.19) and (3.20), using the method of the *Langrange multipliers.* Multiplying (3.18) by the Lagrange multiplier a_1, the ith component of (3.19) by the Lagrange multiplier a_{2i} (for $i = x, y, z$), (3.20) by the Lagrange multiplier a_3, and adding the resulting equations together with (3.17), yields

$$m \int_v (1 + \ln f + a_1 + \mathbf{a}_2 \cdot \mathbf{v} + \tfrac{1}{2}a_3 v^2)\ \delta f\ d^3v = 0 \tag{3.21}$$

where we have used the notation $\mathbf{a}_2 \cdot \mathbf{v} = a_{2x}v_x + a_{2y}v_y + a_{2z}v_z$. The variation in f is now completely arbitrary, since all the constraints imposed on the system have been taken into account in (3.21). Thus, this integral can be equal to zero if and only if

$$\ln f = -(1 + a_1 + \mathbf{a}_2 \cdot \mathbf{v} + \tfrac{1}{2}a_3 v^2) \tag{3.22}$$

This equation is identical to (7.1.9), which we solved in Chapter 7 to obtain the Maxwell-Boltzmann distribution function. Hence, it leads in identical fashion to the equilibrium distribution function

$$f = n\left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left(-\frac{m\mathbf{c}^2}{2kT}\right) \tag{3.23}$$

with $\mathbf{c} = \mathbf{v} - \mathbf{u}$.

The Maxwellian distribution function, besides being the *equilibrium solution* of the Boltzmann equation is, therefore, also the *most probable* distribution consistent with the specified macroscopic parameters n, $\mathbf{u}$ and T of the system.

3.4 Mixture of Various Particle Species

For the case of a mixture containing different species of particles, *each* species having a given number density n_α, average velocity $\mathbf{u}_\alpha$ and temperature T_α, we can still perform a variational calculation to determine the *most probable* distribution subjected to the constraints provided by the set of macroscopic parameters n_α, $\mathbf{u}_\alpha$ and T_α, for each species. Note that this is *not* an equilibrium situation unless the temperatures and mean velocities of all species are equal.

In order to determine the most probable distribution function for this nonequilibrium gas mixture (*each* species having their *own* number density, mean velocity and temperature), we independently minimize each H_α,

$$H_\alpha = \int_v f_\alpha \ln f_\alpha \, d^3v \tag{3.24}$$

This also minimizes H for the mixture, since

$$H = \sum_\alpha H_\alpha \tag{3.25}$$

For the species of type α, when H_α is at its minimum, we must have $\delta H_\alpha = 0$ for a small variation δf_α in f_α. The macroscopic parameters n_α, $\mathbf{u}_\alpha$ and T_α must all remain fixed when f_α is varied. The problem is completely analogous to the one we solved in the previous subsection for a one-component gas and leads, in identical fashion, to equation (3.23) for each species. Therefore, each particle species has a Maxwellian distribution function, but with its own number density, mean velocity and

temperature. Although this it not an equilibrium situation for the *whole* gas (unless the mean velocities and temperatures of all species are the same), it is nevertheless the most probable distrbution function for this system under the specified constraints.

4. BOLTZMANN COLLISION TERM FOR A WEAKLY IONIZED PLASMA

In this section we shall derive, from the Boltzmann equation, an approximate expression for the collision term for a weakly ionized plasma, in which the collisions between electrons and neutral particles play a dominant role. The distribution function for the neutral particles is assumed to be homogeneous and isotropic. The external force acting on the electrons is assumed to be small, so that the electrons are not very far from the equilibrium state. Consequently, the spatial inhomogeneity and the anisotropy of the nonequilibrium distribution function for the electrons are very small, since the nonequilibrium state is only slightly perturbed from the equilibrium state. Under equilibrium conditions the electrons are assumed to have no drift velocity and their distribution function is isotropic and homogeneous.

4.1 Spherical Harmonic Expansion of the Distribution Function

Let (v, θ, ϕ) denote spherical coordinates in velocity space, as shown in Fig. 4. Since the anisotropy of the nonequilibrium distribution function is very small, the dependence of $f(\mathbf{r}, \mathbf{v}, t)$ on θ and ϕ is very small. Hence, it is appropriate to expand $f(\mathbf{r}, \mathbf{v}, t)$ in terms of the velocity space angular variables θ and ϕ, and retain only the first few terms of this expansion. Since ϕ varies between 0 and 2π, we can expand $f(\mathbf{r}, \mathbf{v}, t)$ in a *Fourier series* in ϕ. Furthermore, θ varies between 0 and π, and consequently $\cos\theta$ varies between $+1$ and -1, which means that we can expand $f(\mathbf{r}, \mathbf{v}, t)$ in a series of *Legendre polynomials* in $\cos\theta$. Therefore, we can make a *spherical harmonic* expansion of the distribution function, as follows,

$$f(\mathbf{r}.\mathbf{v}, t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} P_n^m(\cos\theta) \cdot$$

$$\cdot \left[f_{mn}(\mathbf{r}, v, t) \cos(m\phi) + g_{mn}(\mathbf{r}, v, t) \sin(m\phi) \right] \tag{4.1}$$

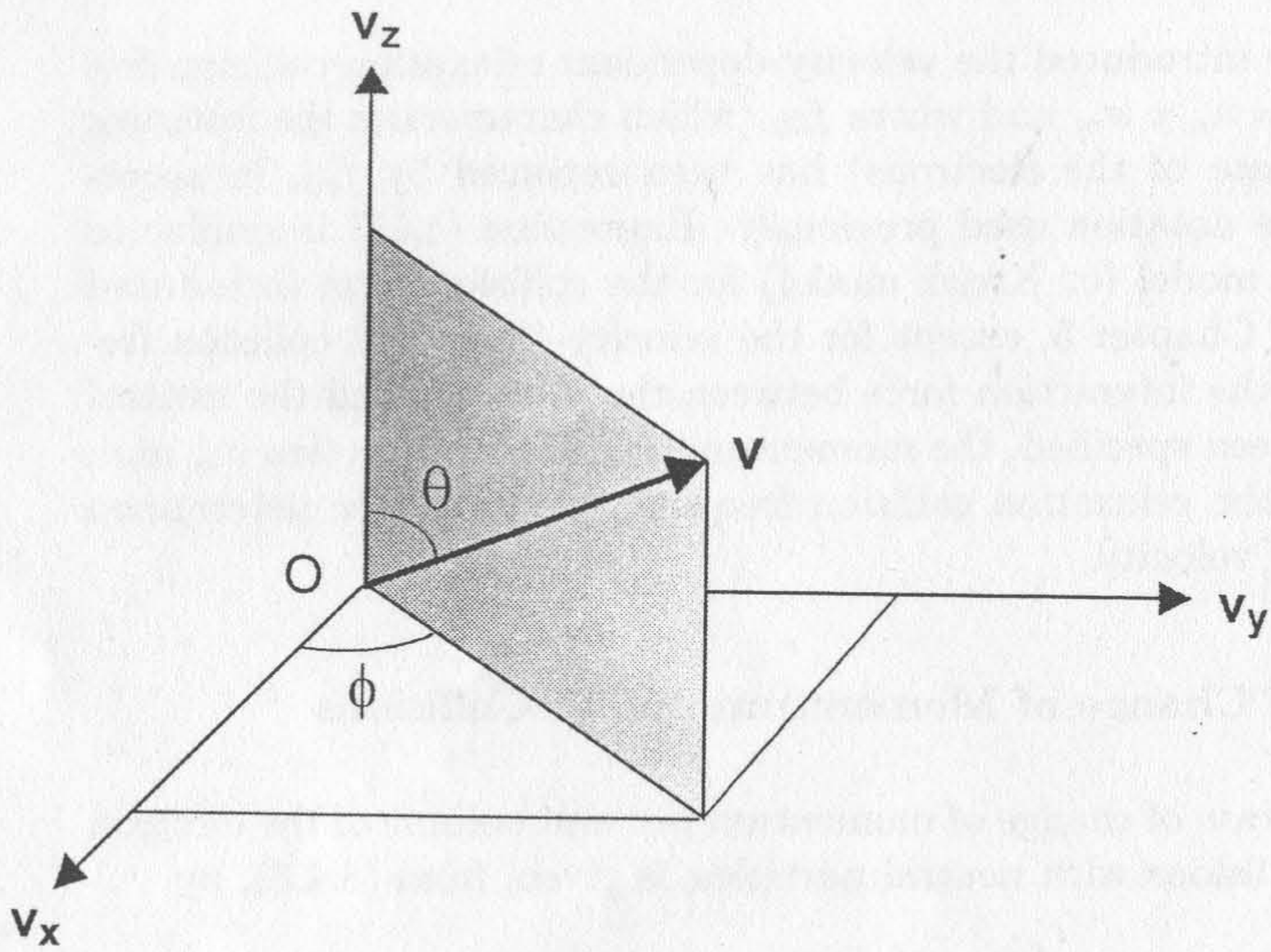

Fig. 4 *Spherical coordinates* (v, θ, ϕ) *in velocity space.*

where the functions $P_n^m(\cos\theta)$ represent the *associated Legendre polynomials* and the functions f_{mn} and g_{mn} can be considered as coefficients of the expansion.

The first term in the expansion (4.1) corresponds to $m = 0$ and $n = 0$, and since $P_0^0(\cos\theta) = 1$, it follows that it is given by $f_{00}(\mathbf{r}, v, t)$. This leading term is the isotropic distribution function corresponding to the equilibrium state of the electrons. The term corresponding to $m = 1$ and $n = 0$ vanishes, since $P_0^1(\cos\theta) = 0$. The next highest order term in (4.1) corresponds to $m = 0$ and $n = 1$, and since $P_1^0(\cos\theta) = \cos\theta$, it is given by $f_{01}(\mathbf{r}, v, t)\cos\theta$. Therefore, retaining only the first two nonzero terms of the spherical harmonic expansion (4.1), in view of the fact that the anisotropy is assumed to be small, we obtain

$$f(\mathbf{r}, \mathbf{v}, t) = f_{00}(\mathbf{r}, v, t) + \frac{\mathbf{v} \cdot \widehat{\mathbf{v}}_z}{v} f_{01}(\mathbf{r}, v, t) \tag{4.2}$$

where we have replaced $\cos\theta$ by $(\mathbf{v} \cdot \widehat{\mathbf{v}}_z)/v$ (see Fig. 4). The second term in the right-hand side of (4.2) corresponds to the small anisotropy due to the spatial inhomogeneity and the external forces on the electrons.

4.2 Approximate Expression for the Boltzmann Collision Term

The Boltzmann collision integral, given in (2.12), can be written for the case of binary electron-neutral collisions as

$$\left(\frac{\delta f_e}{\delta t}\right)_{coll} = \int_b \int_\epsilon \int_{v_1} (f'_e f'_{n1} - f_e f_{n1})\, g\, b\, db\, d\epsilon\, d^3 v_1 \tag{4.3}$$

where we have replaced $\sigma(\Omega)\, d\Omega$ by $b\, db\, d\epsilon$. Here f_e represents the nonequilibrium distribution function for the electrons and f_n is the isotropic equilibrium function for the neutral particles.

In a first approximation we may assume the neutral particles to be stationary and not affected by collisions with the electrons, since the mass of a neutral particle is much larger than that of an electron. Hence, we assume that

$$\mathbf{v}_1 = \mathbf{v}'_1 = 0 \tag{4.4}$$

$$f_{n1} = f'_{n1} \tag{4.5}$$

Therefore, (4.3) becomes

$$\left(\frac{\delta f_e}{\delta t}\right)_{coll} = \int_{v_1} f_{n1}\, d^3 v_1 \int_0^{2\pi} d\epsilon \int_0^\infty (f'_e - f_e)\, g\, b\, db \tag{4.6}$$

Since the number density of the neutral particles is given by

$$n_n = \int_{v_1} f_{n1}\, d^3 v_1 \tag{4.7}$$

we can write (4.6) as

$$\left(\frac{\delta f_e}{\delta t}\right)_{coll} = n_n \int_0^{2\pi} d\epsilon \int_0^\infty (f'_e - f_e)\, g\, b\, db \tag{4.8}$$

Further, from (4.2) the distribution function for the electrons, *before collision*, is given by

$$f_e = f_e(\mathbf{r}, \mathbf{v}, t) = f_{00}(\mathbf{r}, v, t) + \frac{\mathbf{v} \cdot \widehat{\mathbf{v}}_z}{v}\, f_{01}(\mathbf{r}, v, t) \tag{4.9}$$

and, *after collision*, by

$$f'_e = f_e(\mathbf{r}, \mathbf{v}', t) = f_{00}(\mathbf{r}, v', t) + \frac{\mathbf{v}' \cdot \widehat{\mathbf{v}}_z}{v'}\, f_{01}(\mathbf{r}, v', t) =$$

$$= f_{00}(\mathbf{r}, v, t) + \frac{\mathbf{v}' \cdot \widehat{\mathbf{v}}_z}{v} f_{01}(\mathbf{r}, v, t) \tag{4.10}$$

In this last equation we have considered $v' = v$, in view of the fact that the electrons do not loose energy on collisions, since the neutrals are much more massive and considered at rest in a first approximation. This means that $\mathbf{v} = \mathbf{g}$ and $\mathbf{v}' = \mathbf{g}'$ [see equation (4.4)], and since $g = g'$ [see equation (20.2.16)] we have $v = v'$. Note, however, that $\mathbf{v} \neq \mathbf{v}'$. Therefore, from (4.9) and (4.10), we have

$$f_e' - f_e = \frac{(\mathbf{v}' - \mathbf{v}) \cdot \widehat{\mathbf{v}}_z}{v} f_{01}(\mathbf{r}, v, t) \tag{4.11}$$

Without any loss of generality we can choose the v_z axis as being parallel to the initial relative velocity $\mathbf{g}$ of the electron. Therefore,

$$(\mathbf{v}' - \mathbf{v}) \cdot \widehat{\mathbf{v}}_z = (\mathbf{g}' - \mathbf{g}) \cdot \widehat{\mathbf{v}}_z = g(\cos\chi - 1) = v(\cos\chi - 1) \tag{4.12}$$

where χ denotes the scattering angle (the angle between $\mathbf{g}$ and $\mathbf{g}'$, as indicated in Fig. 3 of Chapter 20). Substituting (4.12) into (4.11), we obtain

$$f_e' - f_e = -(1 - \cos\chi)\, f_{01}(\mathbf{r}, v, t) \tag{4.13}$$

Plugging this result into (4.8), yields

$$\left(\frac{\delta f_e}{\delta t}\right)_{coll} = -n_n\, g\, f_{01}(\mathbf{r}, v, t) \int_0^{2\pi} d\epsilon \int_0^{\infty} (1 - \cos\chi)\, b\, db \tag{4.14}$$

Since the momentum transfer cross section σ_m for collisions between electrons and neutral particles is defined by [see (20.5.10)]

$$\sigma_m = \int_\Omega (1 - \cos\chi)\, \sigma(\Omega)\, d\Omega = \int_0^{2\pi} d\epsilon \int_0^{\infty} (1 - \cos\chi)\, b\, db \tag{4.15}$$

we can write (4.14) as

$$\left(\frac{\delta f_e}{\delta t}\right)_{coll} = -n_n\, g\, \sigma_m\, f_{01}(\mathbf{r}, v, t) \tag{4.16}$$

If we substitute $f_{01}(\mathbf{r}, v, t)$ in (4.16), using (4.9), and noting that for this case $(\mathbf{v} \cdot \widehat{\mathbf{v}}_z)/v = (\mathbf{g} \cdot \widehat{\mathbf{v}}_z)/g = 1$, we obtain

$$\left(\frac{\delta f_e}{\delta t}\right)_{coll} = -n_n\, v\, \sigma_m\, (f_e - f_{e0}) = -\nu_r(v)\, (f_e - f_{e0}) \tag{4.17}$$

where we have introduced the velocity-dependent relaxation collision frequency $\nu_r(v) = n_n\ v\ \sigma_m$ and where f_{00} (which characterizes the isotropic equilibrium state of the electrons) has been replaced by f_{e0}, in accordance with the notation used previously. Expression (4.17) is similar to the relaxation model (or Krook model) for the collision term introduced in section 6 of Chapter 5, except for the velocity-dependent collision frequency. Once the interaction force between the electrons and the neutral particles has been specified, the momentum transfer cross section σ_m and, consequently, the relaxation collision frequency $\nu_r(v)$ can be determined as functions of velocity.

4.3 Rate of Change of Momemtum due to Collisions

The time rate of change of momentum per unit volume of the electron gas, due to collisions with neutral particles, is given, from (8.4.3), by

$$\mathbf{A}_e = \left[\frac{\delta(\rho_{me}\mathbf{u}_e)}{\delta t}\right]_{coll} = m_e \int_v \mathbf{v} \left(\frac{\delta f_e}{\delta t}\right)_{coll} d^3v \tag{4.18}$$

Substituting (4.17) into (4.18), we obtain

$$\mathbf{A}_e = -m_e \int_v \nu_r(v)\ \mathbf{v}\ f_e\ d^3v + m_e \int_v \nu_r(v)\ \mathbf{v}\ f_{e0}\ d^3v \tag{4.19}$$

If we assume that the relaxation collision frequency ν_r does not depend on velocity and if we consider that the electron gas has no drift velocity in the equilibrium state, that is,

$$\mathbf{u}_{e0} = \frac{1}{n_e} \int_v \mathbf{v}\ f_{e0}\ d^3v = 0 \tag{4.20}$$

then (4.19) becomes

$$\mathbf{A}_e = -n_e\ m_e\ \nu_r\ \mathbf{u}_e = -\rho_{me}\ \nu_r\ \mathbf{u}_e \tag{4.21}$$

where $\mathbf{u}_e$ is the average velocity of the electrons in the nonequilibrium state. Equation (4.21) corresponds to the expression used in the Langevin equation for the time rate of change of momentum per unit volume, as a result of collisions, in which a constant collisions frequency ν_c was introduced phenomenologically.

5. THE FOKKER-PLANCK EQUATION

In this section we shall present a derivation of the Fokker-Planck equation, in which the collision term takes into account the simultaneous Coulomb interactions between the charged particles. For this purpose we assume that the large-angle deflection of a charged particle, in a multiple Coulomb interaction, can be considered as a series of consecutive weak binary collisions (or *grazing collisions*), that is, as a succession of small-angle scatterings. Therefore, the Fokker-Planck collision term can be derived directly from the Boltzmann collision integral, which is valid for binary collisions, under the assumption that a series of consecutive weak (small-angle deflection) binary collisions is a valid representation for the multiple Coulomb interaction. In the derivation that follows, we will consider collisions between species of particles represented by the indices α and β.

5.1 Derivation of the Fokker-Planck Collision Term

If $\chi(\mathbf{v})$ is some arbitrary function of velocity associated with the particles of type α, then, according to (2.27) and (2.29), the time rate of change of the quantity $\chi(\mathbf{v})$ per unit volume, as a result of collision between particles of type α and those of type β, can be expressed as

$$\int_v \chi(\mathbf{v})\Big(\frac{\delta f_\alpha}{\delta t}\Big)_{coll} d^3v = \int_\Omega \int_{v_1} \int_v (f'_\alpha f'_{\beta 1} - f_\alpha f_{\beta 1})\, \chi\, g\, \sigma(\Omega)\, d\Omega\, d^3v_1\, d^3v =$$

$$= \int_\Omega \int_{v_1} \int_v f_\alpha f_{\beta 1}\, (\chi' - \chi)\, g\, \sigma(\Omega)\, d\Omega\, d^3v_1\, d^3v \qquad (5.1)$$

where χ' denotes $\chi(\mathbf{v}')$. In this last expression only the quantity χ' is a function of the after-collision velocity $\mathbf{v}'$.

For weak binary collisions (or grazing collisions), we can write

$$\mathbf{v}' = \mathbf{v} + \Delta\mathbf{v} \qquad (5.2)$$

where the change $\Delta\mathbf{v}$, due to collision, is assumed to be small. Since

$$\chi' = \chi(\mathbf{v}') = \chi(\mathbf{v} + \Delta\mathbf{v}) \qquad (5.3)$$

we can expand χ' in a Taylor series about the velocity $\mathbf{v}$, as

$$\chi(\mathbf{v} + \Delta\mathbf{v}) = \chi(\mathbf{v}) + \sum_i \frac{\partial\chi}{\partial v_i}\, \Delta v_i + \tfrac{1}{2} \sum_{ij} \frac{\partial^2\chi}{\partial v_i\, \partial v_j}\, \Delta v_i\, \Delta v_j + \ldots \qquad (5.4)$$

Substituting (5.4) into (5.1) we obtain

$$\int_v \chi \left(\frac{\delta f_\alpha}{\delta t}\right)_{coll} d^3v = \int_\Omega \int_{v_1} \int_v f_\alpha f_{\beta 1} \left(\sum_i \frac{\partial \chi}{\partial v_i} \Delta v_i + \right.$$

$$\left. + \tfrac{1}{2} \sum_{ij} \frac{\partial^2 \chi}{\partial v_i \, \partial v_j} \Delta v_i \, \Delta v_j \right) g \, \sigma(\Omega) \, d\Omega \, d^3v_1 \, d^3v \tag{5.5}$$

where the higher order terms have been neglected.

The next step is to factor out the arbitrary function $\chi(\mathbf{v})$ from (5.5). This can be accomplished by integrating the first group of integrals involving $\partial\chi/\partial v_i$ by parts once and the second group of integrals involving $\partial^2\chi/(\partial v_i \, \partial v_j)$ by parts twice. For example, for the x component of the first group of integrals involving $\partial\chi/\partial v_i$, we have

$$\int_\Omega \int_{v_1} \int_v \frac{\partial \chi(\mathbf{v})}{\partial v_x} \Delta v_x \, f_\alpha(\mathbf{v}) \, f_\beta(\mathbf{v}_1) \, g \, \sigma(\Omega) \, d\Omega \, d^3v_1 \, d^3v =$$

$$= \int_\Omega \int_{v_1} \left[\int_v dv_y \, dv_z \, \frac{\partial \chi(\mathbf{v})}{\partial v_x} \, dv_x \, (v'_x - v_x) \cdot \right.$$

$$\left. \cdot \, f_\alpha(\mathbf{v}) \, g \, \sigma(\Omega) \, d\Omega \right] f_\beta(\mathbf{v}_1) \, d^3v_1 \tag{5.6}$$

For the term within brackets we can take

$$dV = \frac{\partial \chi(\mathbf{v})}{\partial v_x} \, dv_x \tag{5.7}$$

and

$$U = (v'_x - v_x) \, f_\alpha(\mathbf{v}) \, g \, \sigma(\Omega) \, d\Omega \tag{5.8}$$

and perform the integral over v_x by parts to obtain

$$\int_{v_x} \frac{\partial \chi(\mathbf{v})}{\partial v_x} \, dv_x \, (v'_x - v_x) \, f_\alpha(\mathbf{v}) \, g \, \sigma(\Omega) \, d\Omega =$$

$$= -\int_{v_x} \chi(\mathbf{v}) \, \frac{\partial}{\partial v_x} \left[(v'_x - v_x) \, f_\alpha(\mathbf{v}) \, g \, \sigma(\Omega) \, d\Omega \right] dv_x \tag{5.9}$$

where the integrated term is equal to zero, since f must vanish at $\pm\infty$. Therefore, we find for the integral in (5.6),

$$\int_\Omega \int_{v_1} \int_v \frac{\partial \chi(\mathbf{v})}{\partial v_x} \, \Delta v_x \, f_\alpha(\mathbf{v}) \, f_\beta(\mathbf{v}_1) \, g \, \sigma(\Omega) \, d\Omega \, d^3v_1 \, d^3v \; =$$

$$= -\int_\Omega \int_{v_1} \int_v \chi(\mathbf{v}) \, \frac{\partial}{\partial v_x} \Big[\Delta v_x \, f_\alpha(\mathbf{v}) \, g \, \sigma(\Omega) \, d\Omega \Big] \, f_\beta(\mathbf{v}_1) \, d^3v_1 \, d^3v \quad (5.10)$$

Performing the other integrals in (5.5) by parts, in a similar way, we obtain the following expression for the collision term

$$\int_v \chi \Big(\frac{\delta f_\alpha}{\delta t} \Big)_{coll} \, d^3v \; =$$

$$= -\int_\Omega \int_{v_1} \int_v \chi \sum_i \frac{\partial}{\partial v_i} \, [\, \Delta v_i \, f_\alpha \, g \, \sigma(\Omega) \, d\Omega] \, f_{\beta 1} \, d^3v_1 \, d^3v \, +$$

$$+ \int_\Omega \int_{v_1} \int_v \tfrac{1}{2} \chi \sum_{ij} \frac{\partial^2}{\partial v_i \, \partial v_j} \, [\, \Delta v_i \, \Delta v_j \, f_\alpha \, g \, \sigma(\Omega) \, d\Omega] \, f_{\beta 1} \, d^3v_1 \, d^3v \; =$$

$$= -\int_v \chi \, \Big[\sum_i \frac{\partial}{\partial v_i} \Big(f_\alpha \int_\Omega \int_{v_1} \Delta v_i \, g \, \sigma(\Omega) \, d\Omega \, f_{\beta 1} \, d^3v_1 \Big) \Big] \, d^3v \, +$$

$$+ \int_v \chi \, \Big[\tfrac{1}{2} \sum_{ij} \frac{\partial^2}{\partial v_i \, \partial v_j} \Big(f_\alpha \int_\Omega \int_{v_1} \Delta v_i \, \Delta v_j \, g \, \sigma(\Omega) \, d\Omega \, f_{\beta 1} \, d^3v_1 \Big) \Big] \, d^3v \quad (5.11)$$

We now define the quantities

$$< \Delta v_i >_{av} = \int_\Omega \int_{v_1} \Delta v_i \, g \, \sigma(\Omega) \, d\Omega \, f_{\beta 1} \, d^3v_1 \quad (5.12)$$

and

$$< \Delta v_i \, \Delta v_j >_{av} = \int_\Omega \int_{v_1} \Delta v_i \, \Delta v_j \, g \, \sigma(\Omega) \, d\Omega \, f_{\beta 1} \, d^3v_1 \quad (5.13)$$

which are modified averages over the scattering angle and the velocity distribution function of the scatterers. Using this notation, (5.11) becomes

$$\int_v \chi \Big(\frac{\delta f_\alpha}{\delta t} \Big)_{coll} \, d^3v = -\int_v \chi \, \Big[\sum_i \frac{\partial}{\partial v_i} \, (f_\alpha \; < \Delta v_i >_{av}) \, +$$

$$+ \tfrac{1}{2} \sum_{ij} \frac{\partial^2}{\partial v_i \, \partial v_j} \left(f_\alpha \; < \Delta v_i \; \Delta v_j >_{av} \right) \Big] \, d^3v \tag{5.14}$$

Since this equation holds for any arbitrary function of velocity $\chi(\mathbf{v})$, it follows that for $\chi = 1$ we must have

$$\left(\frac{\delta f_\alpha}{\delta t} \right)_{coll} = - \sum_i \frac{\partial}{\partial v_i} \left(f_\alpha \; < \Delta v_i >_{av} \right) +$$

$$+ \tfrac{1}{2} \sum_{ij} \frac{\partial^2}{\partial v_i \, \partial v_j} \left(f_\alpha \; < \Delta v_i \; \Delta v_j >_{av} \right) \tag{5.15}$$

This is the collision term of the *Fokker-Planck equation.* The average quantitites $< \Delta v_i >_{av}$ and $< \Delta v_i \; \Delta v_j >_{av}$ are known as the Fokker-Planck coefficients of *dynamical friction* and of *diffusion in velocity space*, respectively. They give the mean rate at which Δv_i and $\Delta v_i \; \Delta v_j$ are changed due to many consecutive weak Coulomb collisions.

Note that the Fokker-Planck collision term (5.15) includes two terms having opposite sign, which may result in no net change in f_α as a result of collisions. A dimensional analysis of (5.12) shows that the Fokker-Planck coefficient $< \Delta v_i >_{av}$ has dimensions of force per unit mass and tends to accelerate or decelerate the particles until they reach the average equilibrium velocity. This process is usually called *dynamical friction.* On the other hand, the Fokker-Planck coefficient $< \Delta v_i \; \Delta v_j >_{av}$ represents *diffusion in velocity space*, until equilibrium is reached. Under equilibrium conditions, diffusion in velocity space is balanced by dynamical friction and there is no net change in f_α as a result of collisions, so that the collision term in (5.15) vanishes. Fig. 5 illustrates schematically the effect of these processes on the distribution function.

In principle, the expansion procedure used to obtain the Fokker-Planck collision term can be extended to any number of terms. However, in practice only the first two terms of the expansion shown in (5.15) are ever used, so that (5.15) can be considered as a reasonable approximation to the collision term when $\Delta \mathbf{v} = (\mathbf{v}' - \mathbf{v})$ is small for most collisions. This is generally considered to be the case for the long-range Coulomb force. Recall that, in section 8 of Chapter 20, we have shown that the large number of particles interacting mildly with the target particle (resulting in small-angle deflections) are much more important than the small number of particles interacting strongly with the target particle (resulting in large-angle deflections) and have a dominant contribution to the total scattering cross section.

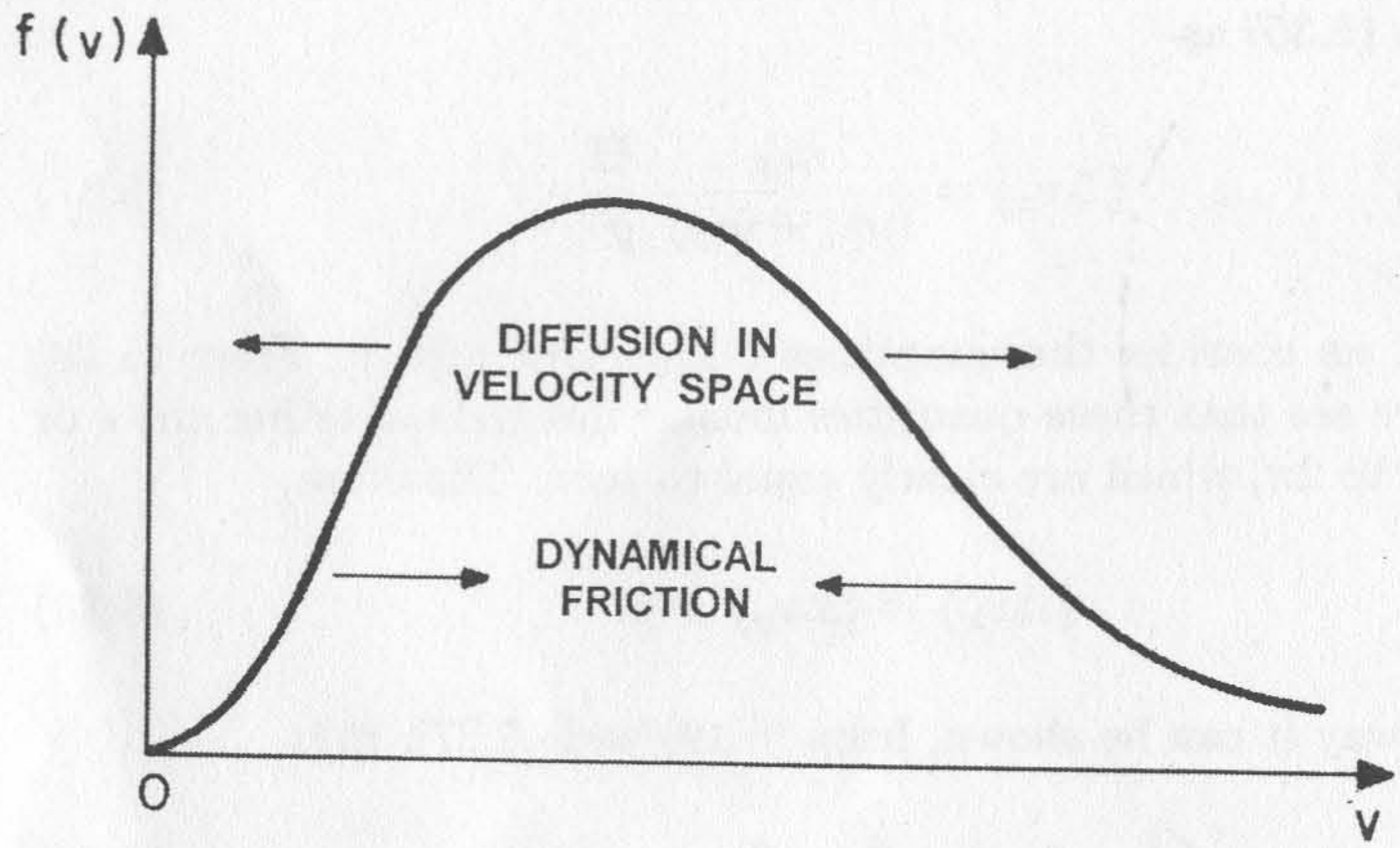

Fig. 5 *Schematic illustration of the processes of dynamical friction and of diffusion in velocity space, associated with the Fokker-Planck coefficients.*

5.2 The Fokker-Planck Coefficients for Coulomb Interactions

We shall evaluate now the coefficients of dynamical friction $< \Delta v_i >_{av}$ and of diffusion in velocity space $< \Delta v_i \ \Delta v_j >_{av}$, which appear in the Fokker-Planck collision term (5.15). It is convenient to perform first the integral over the solid angle Ω, since it does not require a knowlegde of the distribution function $f_\beta(\mathbf{v}_1)$. For this purpose let us write

$$< \Delta v_i >_{av} = \int_{v_1} \{\Delta v_i\} \ f_{\beta 1} \ d^3 v_1 \tag{5.16}$$

and

$$< \Delta v_i \ \Delta v_j >_{av} = \int_{v_1} \{\Delta v_i \ \Delta v_j\} \ f_{\beta 1} \ d^3 v_1 \tag{5.17}$$

where the curly bracket notation has been introduced to represent the following integrals over solid angle

$$\{\Delta v_i\} = \int_\Omega \Delta v_i \ g \ \sigma(\Omega) \ d\Omega \tag{5.18}$$

and

$$\{\Delta v_i \, \Delta v_j\} = \int_\Omega \Delta v_i \, \Delta v_j \; g \; \sigma(\Omega) \; d\Omega \tag{5.19}$$

In order to calculate these integrals over the solid angle, recall first that in the center of mass coordinate system we have, from (20.2.5) and (20.2.8),

$$\mathbf{v} = \mathbf{c}_0 - \frac{m_\beta}{(m_\alpha + m_\beta)} \, \mathbf{g} \tag{5.20}$$

$$\mathbf{v}' = \mathbf{c}_0 - \frac{m_\beta}{(m_\alpha + m_\beta)} \, \mathbf{g}' \tag{5.21}$$

so that

$$\Delta \mathbf{v} = \mathbf{v}' - \mathbf{v} = \frac{m_\beta}{(m_\alpha + m_\beta)} \, (\mathbf{g} - \mathbf{g}') \tag{5.22}$$

In a Cartesian coordinate system in which the vector $\mathbf{g}$ is along the z axis (as shown in Fig. 3 of Chapter 20), we have

$$g_z = g \quad ; \quad g_x = g_y = 0 \tag{5.23}$$

and

$$g'_x = g \sin \chi \cos \epsilon \tag{5.24}$$

$$g'_y = g \sin \chi \sin \epsilon \tag{5.25}$$

$$g'_z = g \cos \chi \tag{5.26}$$

Plugging equations (5.23) to (5.26), in (5.22), gives

$$\Delta \mathbf{v} = \frac{m_\beta}{(m_\alpha + m_\beta)} \, g \, [(1 - \cos \chi) \, \widehat{\mathbf{z}} - \sin \chi \, (\cos \epsilon \, \widehat{\mathbf{x}} + \sin \epsilon \, \widehat{\mathbf{y}})] \tag{5.27}$$

The differential scattering cross section for the Coulomb potential was calculated in section 7 of Chapter 20 and was found to be given by

$$\sigma(\chi) = \frac{b_0^2}{4 \sin^4(\chi/2)} = \frac{b_0^2}{(1 - \cos \chi)^2} \tag{5.28}$$

where b_0 is defined in (20.4.9).

Proceeding in the evaluation of $\{\Delta v_i\}$ for $i = x, \; y, \; z$, let us first calculate $\{\Delta v_z\}$. From (5.18), (5.27) and (5.28), we have

$$\{\Delta v_z\} = \frac{m_\beta}{(m_\alpha + m_\beta)} \, g^2 b_0^2 \int_0^{2\pi} d\epsilon \int_{\chi_{min}}^{\pi} \frac{\sin \chi}{(1 - \cos \chi)} \, d\chi \tag{5.29}$$

where the lower limit of the integral in χ was taken to be χ_{min}, in order to avoid the divergence of the integral that would result if we take zero as the lower limit. As we have seen, the charged particles in the plasma that are separated by distances greater than the Debye length λ_D are effectively shielded from one another. Therefore, in order to avoid an infinite result for the integral in (5.29), we take the lower limit χ_{min} to be the value of the scattering angle corresponding to an impact paramter b equal to λ_D.

With reference to (20.4.13) let us introduce the new variable

$$u = \frac{b}{b_0} = \cot\left(\tfrac{1}{2}\chi\right) \tag{5.30}$$

from which we obtain

$$du = -\frac{d\chi}{(1 - \cos\chi)} \tag{5.31}$$

and

$$\sin\chi = \frac{2u}{(1 + u^2)} \tag{5.32}$$

With this change of variable and introducing the cut-off value for the impact paramter at λ_D, that is, at

$$u_c = \frac{\lambda_D}{b_0} = \Lambda \tag{5.33}$$

we obtain for (5.29),

$$\{\Delta v_z\} = 2\pi \frac{m_\beta}{(m_\alpha + m_\beta)} g^2 b_0^2 \int_\Lambda^0 \frac{2u}{(1 + u^2)} (-du) =$$

$$= 2\pi \frac{m_\beta}{(m_\alpha + m_\beta)} g^2 b_0^2 \ln(1 + \Lambda^2) \tag{5.34}$$

In general $\Lambda \gg 1$, so that $\ln(1 + \Lambda^2) \simeq 2 \ln \Lambda$ and (5.34) simplifies to

$$\{\Delta v_z\} = \frac{m_\beta}{(m_\alpha + m_\beta)} \frac{Z^2 e^4 \ln \Lambda}{4 \pi \epsilon_0^2 \mu^2 g^2} \tag{5.35}$$

where we have substituted b_0 by the expression given in (20.4.9). Introducing the notation

$$\Theta = \frac{Z^2 e^4 \ln \Lambda}{4 \pi \epsilon_0^2 \mu^2} \tag{5.36}$$

we can write (5.35) as

$$\{\Delta v_z\} = \frac{m_\beta}{(m_\alpha + m_\beta)} \frac{\Theta}{g^2} \tag{5.37}$$

Next let us consider the quantities $\{\Delta v_x\}$ and $\{\Delta v_y\}$. From (5.18) and (5.27) we see that these quantities involve integrals of either $cos\ \epsilon$ or $sin\ \epsilon$ from 0 to 2π, which are clearly equal to zero. Therefore,

$$\{\Delta v_x\} = \{\Delta v_y\} = 0 \tag{5.38}$$

In a similar way it can be shown, from (5.19) and (5.27), that

$$\{\Delta v_i\ \Delta v_j\} = 0 \qquad for \qquad i \neq j \tag{5.39}$$

since the integrals over ϵ from 0 to 2π vanish.

To evaluate $\{\Delta v_z\ \Delta v_z\} = \{\Delta v_z^2\}$, we use (5.19), (5.27) and (5.28), which yield

$$\{\Delta v_z^2\} = 2\pi \frac{m_\beta^2}{(m_\alpha + m_\beta)^2} g^3 b_0^2 \int_{\chi_{min}}^{\pi} \sin\chi\ d\chi \tag{5.40}$$

Changing variables according to (5.30), we obtain

$$\{\Delta v_z^2\} = 2\pi \frac{m_\beta^2}{(m_\alpha + m_\beta)^2} g^3 b_0^2 \int_0^{\Lambda} \frac{4u}{(1+u^2)^2}\ du =$$

$$= 4\pi \frac{m_\beta^2}{(m_\alpha + m_\beta)^2} g^3\ b_0^2 \frac{\Lambda^2}{(1+\Lambda^2)} \tag{5.41}$$

Since $\Lambda \gg 1$, (5.41) simplifies to

$$\{\Delta v_z^2\} = 4\pi \frac{m_\beta^2}{(m_\alpha + m_\beta)^2} g^3\ b_0^2 = \frac{m_\beta^2}{(m_\alpha + m_\beta)^2} \frac{Z^2\ e^4}{4\ \pi\ \epsilon_0^2\ \mu^2\ g} \tag{5.42}$$

or, using the notation introduced in (5.36),

$$\{\Delta v_z^2\} = \frac{m_\beta^2}{(m_\alpha + m_\beta)^2} \frac{\Theta}{g \ln \Lambda} \tag{5.43}$$

In a similar way, we can calculate $\{\Delta v_x^2\}$ and $\{\Delta v_y^2\}$ from (5.19), (5.27) and (5.28), which give

$$\{\Delta v_x^2\} = \frac{m_\beta^2}{(m_\alpha + m_\beta)^2}\, g^3 b_0^2 \int_0^{2\pi} \cos^2\epsilon\, d\epsilon \int_{\chi_{min}}^{\pi} \frac{\sin^3\chi}{(1-\cos\chi)^2}\, d\chi \quad (5.44)$$

$$\{\Delta v_y^2\} = \frac{m_\beta^2}{(m_\alpha + m_\beta)^2}\, g^3 b_0^2 \int_0^{2\pi} \sin^2\epsilon\, d\epsilon \int_{\chi_{min}}^{\pi} \frac{\sin^3\chi}{(1-\cos\chi)^2}\, d\chi \quad (5.45)$$

Therefore, evaluating the integral over ϵ, we find

$$\{\Delta v_x^2\} = \{\Delta v_y^2\} = \pi\, \frac{m_\beta^2}{(m_\alpha + m_\beta)^2}\, g^3 b_0^2 \int_{\chi_{min}}^{\pi} \frac{\sin^3\chi}{(1-\cos\chi)^2}\, d\chi \quad (5.46)$$

If we change variables according to (5.30), we readily find

$$\{\Delta v_x^2\} = \{\Delta v_y^2\} = 4\pi\, \frac{m_\beta^2}{(m_\alpha + m_\beta)^2}\, g^3 b_0^2 \int_0^{\Lambda} \frac{u^3}{(1+u^2)^2}\, du =$$

$$= 2\pi\, \frac{m_\beta^2}{(m_\alpha + m_\beta)^2} g^3 b_0^2 \left[\ln(1+\Lambda^2) - \frac{\Lambda^2}{(1+\Lambda^2)} \right] \quad (5.47)$$

Since $\Lambda \gg 1$ and replacing b_0 using (20.4.9), we obtain finally

$$\{\Delta v_x^2\} = \{\Delta v_y^2\} = \frac{m_\beta^2}{(m_\alpha + m_\beta)^2}\, \frac{Z^2\, e^4 \ln\Lambda}{4\,\pi\,\epsilon_0^2\,\mu^2\, g} = \frac{m_\beta^2}{(m_\alpha + m_\beta)^2}\, \frac{\Theta}{g} \quad (5.48)$$

The next step in the evaluation of the Fokker-Planck coefficients consists in integrating the expressions for $\{\Delta v_i\}$ and $\{\Delta v_i\ \Delta v_j\}$, for $i,\ j = x,\ y,\ z$, over the distribution function of the particles which constitute the scattering centers. Thus, using the results we have just calculated, we find

$$< \Delta v_z >_{av} = \frac{m_\beta}{(m_\alpha + m_\beta)} \int_{v_1} \frac{\Theta}{g^2}\, f_{\beta 1}\, d^3 v_1 \quad (5.49)$$

$$< \Delta v_z^2 >_{av} = \frac{m_\beta^2}{(m_\alpha + m_\beta)^2} \int_{v_1} \frac{\Theta}{g \ln\Lambda}\, f_{\beta 1}\, d^3 v_1 \quad (5.50)$$

$$< \Delta v_x^2 >_{av} = < \Delta v_y^2 >_{av} = \frac{m_\beta^2}{(m_\alpha + m_\beta)^2} \int_{v_1} \frac{\Theta}{g}\, f_{\beta 1}\, d^3 v_1 \quad (5.51)$$

All other coefficients vanish.

5.3 Application to Electron-Ion Collisions

Let us calculate the Fokker-Planck coefficients for the case of electron-ion collisions. For simplicity we shall assume that the electron is colliding with a field of heavy stationary positive ions. This assumption is reasonable, since on the average the electron velocities are much larger than the ion velocities ($< v_i^2 > = 3kT_i/m_i$ while $< v_e^2 > = 3kT_e/m_e$ and generally $T_e/m_e \gg T_i/m_i$). Thus, assuming that the positive ions are motionless, we can represent their velocity distribution function by the Dirac delta function

$$f_{\beta 1} = n_0 \, \delta(v_{1x}) \, \delta(v_{1y}) \, \delta(v_{1z}) \tag{5.52}$$

In addition, because $m_i \gg m_e$, we can take $(m_e + m_i) \simeq m_i$ and $\mu \simeq m_e$. Substituting (5.52) into equations (5.49) through (5.51), we obtain at once

$$< \Delta v_z >_{av} = \frac{n_0 \, \Theta}{g^2} \tag{5.53}$$

$$< \Delta v_z^2 >_{av} = \frac{n_0 \, \Theta}{g \ln \Lambda} \tag{5.54}$$

$$< \Delta v_x^2 >_{av} =< \Delta v_y^2 >_{av} = \frac{n_0 \, \Theta}{g} \tag{5.55}$$

where Θ is defined according to (5.36). Note that $\mathbf{v}_z$ is along the direction of the initial relative velocities, whereas $\mathbf{v}_x$ and $\mathbf{v}_y$ are in the plane perpendicular to the initial relative velocities.

PROBLEMS

21.1 Consider a system consisting of a mixture of two types of particles having masses m and M, and subjected to an external force $\mathbf{F}$. Denote the corresponding distribution functions by f and g, respectively. Write down the set of coupled Boltzmann transport equations for the system.

21.2 Consider a plasma in which the electrons and the ions are characterized, respectively, by the following distribution functions

$$f_e = n_0 \left(\frac{m_e}{2\pi k T_e}\right)^{3/2} \exp\left[-\frac{m_e(\mathbf{v} - \mathbf{u}_e)^2}{2kT_e}\right]$$

$$f_i = n_0 \left(\frac{m_i}{2\pi k T_i}\right)^{3/2} \exp\left[-\frac{m_i(\mathbf{v} - \mathbf{u}_i)^2}{2kT_i}\right]$$

(a) Calculate the difference $(f_e' f_{i1}' - f_e f_{i1})$.

(b) Show that this plasma of electrons and ions will be in the equilibrium state, that is, the difference $(f'_e f'_{i1} - f_e f_{i1})$ will vanish if and only if $\mathbf{u}_e = \mathbf{u}_i$ and $T_e = T_i$.

21.3 Use a Lagrange multiplier technique to show that for a system characterized by the following modified Maxwell-Boltzmann distribution function

$$f(\mathbf{r}, v) = n(\mathbf{r}) \left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left(-\frac{mv^2}{2kT}\right)$$

where T is constant, the entropy S, defined by

$$S = -k \int_r \int_v f \ln(f)\, d^3v\, d^3r$$

is a *maximum* when the density n is constant, independent of $\mathbf{r}$. Consider that the system has a total of N particles in a fixed volume V at a temperature T.

21.4 Consider the case of *Maxwell molecules*, for which the interparticle force is of the form

$$\mathbf{F}(\mathbf{r}) = \frac{K}{r^5}\, \hat{\mathbf{r}}$$

where K is a constant.
(a) Without specifying the form of the distribution functions $f_\alpha(\mathbf{v})$ and $f_\beta(\mathbf{v}_1)$ for the particles of type α and β, show that the time rate of change of momentum for the particles of type α per unit volume, due to collisions, is given by

$$\left(\frac{\delta \mathcal{P}_\alpha}{\delta t}\right)_{coll} = \int_v m_\alpha \mathbf{c}_\alpha \left(\frac{\delta f_\alpha}{\delta t}\right)_{coll} d^3v = \sum_\beta n_\alpha m_\alpha \nu_{\alpha\beta}(\mathbf{u}_\beta - \mathbf{u}_\alpha)$$

where $\nu_{\alpha\beta}$ is the collision frequency for momentum transfer, given explicity by

$$\nu_{\alpha\beta} = 2\pi \frac{(K\mu)^{1/2}}{m_\alpha}\, n_\beta\, A_1(5)$$

where $A_1(5)$ is a dimensionless number (of order unity) defined by (with $p = 5$ and $\ell = 1$)

$$A_\ell(p) = \int_0^\infty (1 - \cos^\ell \chi)\, v_0\, dv_0$$

with

$$v_0 = b\left(\frac{\mu g^2}{K}\right)^{1/(p-1)}$$

Also,

$$n_\alpha = \int_v f_\alpha \, d^3v \quad ; \quad \mathbf{u}_\alpha = \frac{1}{n_\alpha}\int_v \mathbf{v}\, f_\alpha \, d^3v$$

$$n_\beta = \int_{v_1} f_{\beta 1} \, d^3v_1 \quad ; \quad \mathbf{u}_\beta = \frac{1}{n_\beta}\int_{v_1} \mathbf{v}_1\, f_{\beta 1} \, d^3v_1$$

(b) For the same case, show that the time rate of change of energy for the particles of type α per unit volume, due to collsions, is given by

$$\left(\frac{\delta \mathcal{E}_\alpha}{\delta t}\right)_{coll} = \tfrac{1}{2}\int_v m_\alpha c_\alpha^2 \left(\frac{\delta f_\alpha}{\delta t}\right)_{coll} d^3v \; =$$

$$= \frac{n_\alpha m_\alpha \nu_{\alpha\beta}}{(m_\alpha + m_\beta)}\left[3k(T_\beta - T_\alpha) + m_\beta(\mathbf{u}_\beta - \mathbf{u}_\alpha)^2\right]$$

where

$$T_\alpha = \frac{m_\alpha}{3k} < c_\alpha^2 > = \frac{m_\alpha}{3kn_\alpha}\int_v c_\alpha^2 \, f_\alpha \, d^3v$$

$$T_\beta = \frac{m_\beta}{3k} < c_\beta^2 > = \frac{m_\beta}{3kn_\beta}\int_{v_1} c_{\beta 1}^2 \, f_{\beta 1} \, d^3v_1$$

21.5 Consider a gas mixture of two types of particles ($\alpha = 1,\ 2$), each one characterized by a Maxwellian distribution function

$$f_\alpha(\mathbf{v}) = n_\alpha\left(\frac{m_\alpha}{2\pi k T_\alpha}\right)^{3/2} \exp\left(-\frac{m_\alpha v^2}{2kT_\alpha}\right)$$

with its own mass, density and temperature.
(a) Make the following transformation of velocity variables

$$\mathbf{v}_1 = \mathbf{v}_c' + \overline{M}_2 \mathbf{g}$$

$$\mathbf{v}_2 = \mathbf{v}_c' - \overline{M}_1 \mathbf{g}$$

where $\mathbf{v}_c'$ is a velocity similar to the center of mass velocity, $\mathbf{g}$ is the relative velocity between the two species ($\mathbf{g} = \mathbf{v}_1 - \mathbf{v}_2$) and

$$\overline{M}_1 = \frac{m_1/T_1}{m_1/T_1 + m_2/T_2}$$

$$\overline{M}_2 = \frac{m_2/T_2}{m_1/T_1 + m_2/T_2}$$

Show that the Jacobian of this transformation,

$$J = \frac{\partial(\mathbf{v}'_c,\ \mathbf{g})}{\partial(\mathbf{v}_1,\ \mathbf{v}_2)}$$

satisfies $|J| = 1$, so that $d^3v'_c\ d^3g = d^3v_1\ d^3v_2$.
(b) The relative speed between the two species, $g = |\mathbf{v}_1 - \mathbf{v}_2|$, when averaged over both their velocity distribution functions, is given by

$$< g > = \frac{1}{n_1 n_2} \int_{v_1} \int_{v_2} g\ f_1(\mathbf{v}_1)\ f_2(\mathbf{v}_2)\ d^3v_1\ d^3v_2$$

Transform the variables of integration $\mathbf{v}_1$ and $\mathbf{v}_2$ to $\mathbf{v}'_c$ and $\mathbf{g}$, and perform the integrals over $\mathbf{v}'_c$ and $\mathbf{g}$ to show that

$$< g > = \left(\frac{8k}{\pi}\right)^{1/2} \left(\frac{T_1}{m_1} + \frac{T_2}{m_2}\right)^{1/2}$$

(c) If only one kind of particles is present, so that $m_1 = m_2 = m$, $T_1 = T_2 = T$ and $n_1 = n_2 = n$, show that

$$< g > = \sqrt{2} < v > = \left(\frac{8kT}{\pi\mu}\right)^{1/2}$$

where $< v > = (8kT/\pi m)^{1/2}$ is the average speed and $\mu = m/2$ is the reduced mass. If the mutual scattering cross section is σ, show that the collision frequency in a homogeneous Maxwellian gas is given by

$$\nu = n\sigma < g > = 4n\sigma \left(\frac{kT}{\pi m}\right)^{1/2}$$

21. 6 Consider the following expressions which define the Fokker-Planck coefficients of dynamical friction and of diffusion in velocity space,

$$< \Delta v_i >_{av} = \int_{\Omega} \int_{v_1} \Delta v_i\ g\ \sigma(\Omega)\ d\Omega\ f_{\beta 1}\ d^3v_1$$

$$< \Delta v_i\ \Delta v_j >_{av} = \int_{\Omega} \int_{v_1} \Delta v_i\ \Delta v_j\ g\ \sigma(\Omega)\ d\Omega\ f_{\beta 1}\ d^3v_1$$

(a) With reference to Fig. 6, verify that

$$\Delta v_x = -\frac{m_\beta}{(m_\alpha + m_\beta)}\, g \sin\chi \cos\epsilon$$

$$\Delta v_y = -\frac{m_\beta}{(m_\alpha + m_\beta)}\, g \sin\chi \sin\epsilon$$

$$\Delta v_z = \frac{m_\beta}{(m_\alpha + m_\beta)}\, g\,(1 - \cos\chi)$$

For a general inverse-power interparticle force of the form

$$\mathbf{F}(\mathbf{r}) = \frac{K}{r^p}\,\widehat{\mathbf{r}}$$

where K is a constant and p is a positive integer number, show that [see (5.18)]

$$\{\Delta v_x\} = \{\Delta v_y\} = 0$$

$$\{\Delta v_z\} = \frac{\mu g^2 \sigma_m}{m_\alpha}$$

where μ is the reduced mass and σ_m is the momentum transfer cross section given by

$$\sigma_m = 2\pi\left(\frac{K}{\mu g^2}\right)^{2/(p-1)} A_1(p)$$

where

$$A_\ell(p) = \int_0^\infty (1 - \cos^\ell\chi)\, v_0\, dv_0$$

$$v_0 = b\left(\frac{\mu g^2}{K}\right)^{1/(p-1)}$$

Verify also that [see (5.19)]

$$\{\Delta v_i\, \Delta v_j\} = 0 \qquad for \qquad i \neq j$$

$$\{\Delta v_x^2\} = \{\Delta v_y^2\} = \pi\frac{\mu^2 g^3}{m_\alpha^2}\left(\frac{K}{\mu g^2}\right)^{2/(p-1)} A_2(p)$$

$$\{\Delta v_z^2\} = 2\pi\frac{\mu^2 g^3}{m_\alpha^2}\left(\frac{K}{\mu g^2}\right)^{2/(p-1)} [2A_1(p) - A_2(p)]$$

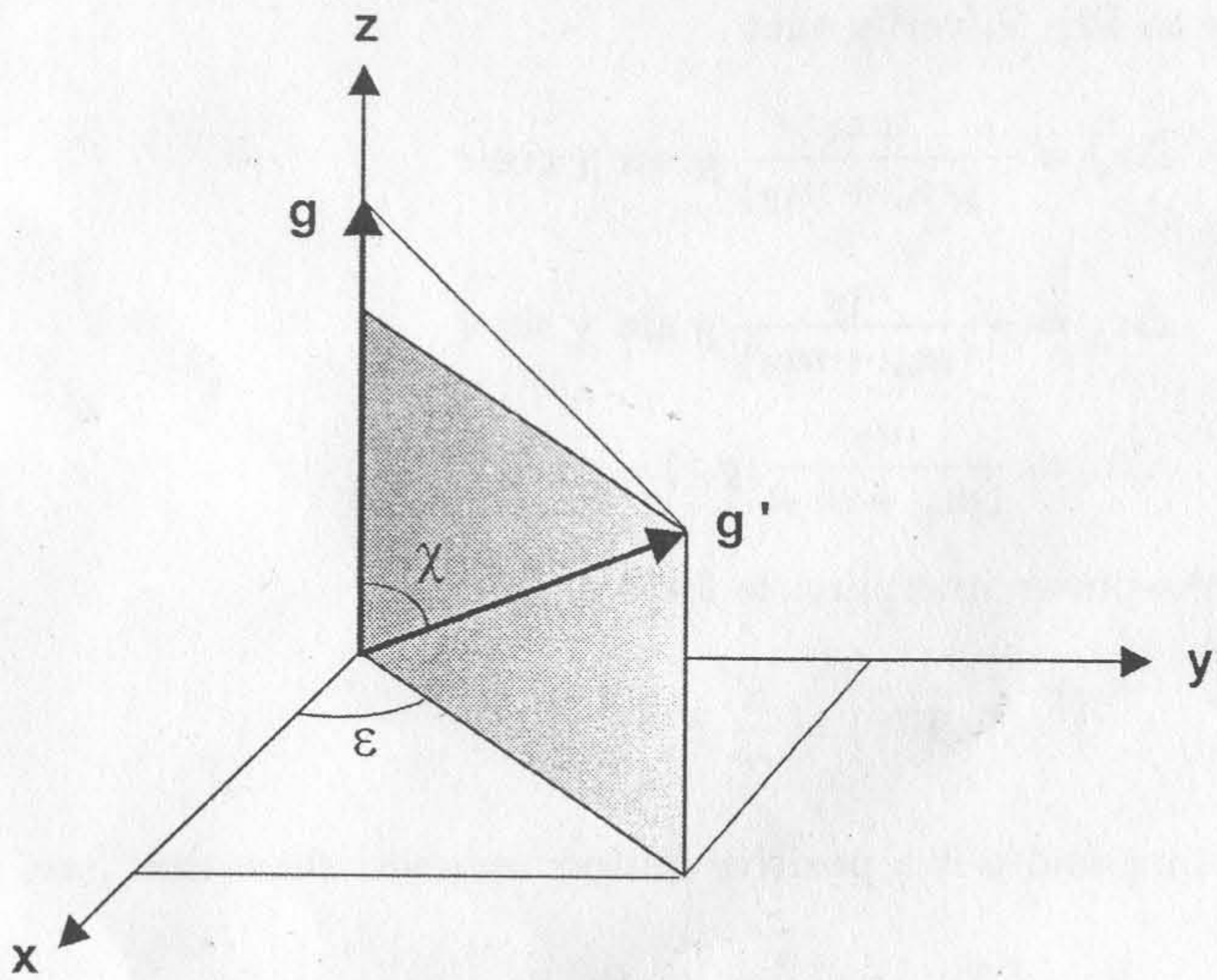

Fig. 6 *Coordinate system showing the relative orientations of the velocity vectors* g *and* g′.

(b) For the case of *Maxwell molecules* ($p = 5$), where the results are independent of $f_{\beta 1}$, show that the Fokker-Planck coefficients are given by

$$< \Delta v_x >_{av} =< \Delta v_y >_{av} = 0$$

$$< \Delta v_z >_{av} = \nu_{\alpha\beta} < g >_\beta$$

$$< \Delta v_i \, \Delta v_j >_{av} = 0 \qquad for \qquad i \neq j$$

$$< \Delta v_x^2 >_{av} =< \Delta v_y^2 >_{av} = \frac{\mu}{2m_\alpha} \frac{A_2(5)}{A_1(5)} \, \nu_{\alpha\beta} < g^2 >_\beta$$

$$< \Delta v_z^2 >_{av} = \frac{\mu}{m_\alpha} \left[2 - \frac{A_2(5)}{A_1(5)} \right] \nu_{\alpha\beta} < g^2 >_\beta$$

where

$$\nu_{\alpha\beta} = 2\pi \frac{(K\mu)^{1/2}}{m_\alpha} \, n_\beta \, A_1(5)$$

$$< g^i >_\beta = \frac{1}{n_\beta} \int_{v_1} g^i \, f_{\beta 1} \, d^3 v_1 \qquad ; \qquad (i = 1, \, 2)$$

(c) Calculate the Fokker-Planck coefficients for the case of the Coulomb interaction ($p = 2$) using the results of part (a) and of Problem 20.6, in terms of integrals over $f_{\beta 1}$, and compare with the results derived in subsection 5.2.

(d) Calculate the Fokker-Planck coefficients for electron-electron interactions, when $f_{\beta 1}$ is the Maxwellian distribution function. Refer to (5.49), (5.50) and (5.51).