Dynamika a vývoj galaxií – test. č. 1 (16. 4. 2009)
ŘEŠENÍ

(1) a) $I_{dV}(R) = I_e \, e^{-7.67\left[\left(\frac{R}{R_e}\right)^{1/4} - 1\right]} = \underbrace{I_e \cdot e^{+7.67}}\, e^{-7.67\left(\frac{R}{R_e}\right)^{1/4}}$

$\underline{\underline{I(R=0) \approx 2100 \cdot I_e}}$

b) exponenciální: $I(R) = I_0 \, e^{-\frac{R}{h}}$

c) $I_{TK}(R) = I_0 \dfrac{1}{\left(1+\frac{R^2}{a^2}\right)^{3/2}}$

• definice R_{eff}: $L(R<R_{eff}) = \frac{1}{2} L_{tot}$, tj.:

$$2\pi \int_0^{R_{eff}} I(R) R \, dR = \frac{1}{2} \cdot 2\pi \int_0^{\infty} I(R) R \, dR$$

• pro TK disk: $\displaystyle\int_0^{R_{eff}} \frac{R\,dR}{\left(1+\frac{R^2}{a^2}\right)^{3/2}} = \frac{1}{2} \int_0^{\infty} \frac{R\,dR}{\left(1+\frac{R^2}{a^2}\right)^{3/2}}$

$$\Rightarrow \left. \frac{1}{\left(1+\frac{R^2}{a^2}\right)^{1/2}} \right|_0^{R_{eff}} = \frac{1}{2} \left. \frac{1}{\left(1+\frac{R^2}{a^2}\right)^{1/2}} \right|_0^{\infty}$$

$$\Rightarrow \frac{1}{(R_{eff}^2 + a^2)^{1/2}} - \frac{1}{a} = -\frac{1}{2a} \Rightarrow \frac{1}{(R_{eff}^2 + a^2)^{1/2}} = \frac{1}{2a}$$

$$\Rightarrow R_{eff}^2 + a^2 = 4a^2 \Rightarrow R_{eff}^2 = 3a^2 \Rightarrow \underline{\underline{R_{eff} = \sqrt{3}\, a}}$$

d) • obecně $M_{tot} = 2\pi \int_0^{\infty} \Sigma(R) R \, dR$ (pro osově sym. disk)

• expon. disk: $\Sigma(R) = \Sigma_0 \, e^{-\frac{R}{h}}$

$$\Rightarrow M_{tot} = 2\pi\Sigma_0 \underbrace{\int_0^{\infty} e^{-\frac{R}{h}} R \, dR}_{\text{per partes}} = 2\pi\Sigma_0 \left\{ -\underbrace{\left[h e^{-\frac{R}{h}} \cdot R\right]_0^{\infty}}_{\phi} + h \int_0^{\infty} e^{-\frac{R}{h}} \, dR \right\} =$$

$$= 2\pi \Sigma_0 \underbrace{\left\{-h^2 e^{-\frac{R}{h}}\right\}_0^{\infty}}_{} = \underline{2\pi\Sigma_0 h^2 \;\; \& \;\; \Sigma_0 = \frac{M_{tot}}{2\pi h^2}}$$

② a) $\rho(r) = \rho_0 \left(\frac{r_0}{r}\right)^2$

1. možnost - využít Newtonovy theorémy, tj.:

$$F_r = -\frac{d\Phi}{dr} = -\frac{GM(r)}{r^2}$$

$$M(r) = 4\pi \int_0^r \rho(r') r'^2 dr' = 4\pi \rho_0 r_0^2 \int_0^r dr' = 4\pi \rho_0 r_0^2 r$$

$$F_r = -\frac{4\pi G \rho_0 r_0^2}{r}, \quad v_c^2 = r|F_r| = 4\pi G \rho_0 r_0^2 = \text{const} \equiv v_0^2$$

$$\Phi = -\int F_r dr + \text{const} = 4\pi G \rho_0 r_0^2 \ln r + \text{const}$$
$$= v_0^2 \ln r + \text{const}$$

$$\frac{v_{esc}^2(r)}{2} + \Phi(r) = \Phi_\infty \Rightarrow v_{esc} = \sqrt{2(\Phi_\infty - \Phi(r))}$$

$$v_{esc}(r) = \infty \quad \forall r$$

2. možnost - s použitím Poissonovy rovnice

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\Phi}{dr}\right) = 4\pi G \rho \quad \text{(při sférické symetrii)}$$
$$= \frac{4\pi G \rho_0 r_0^2}{r^2}$$

$$\Rightarrow \frac{d}{dr}\left(r^2 \frac{d\Phi}{dr}\right) = 4\pi G \rho_0 r_0^2 \Rightarrow r^2 \frac{d\Phi}{dr} = 4\pi G \rho_0 r_0^2 r + C$$

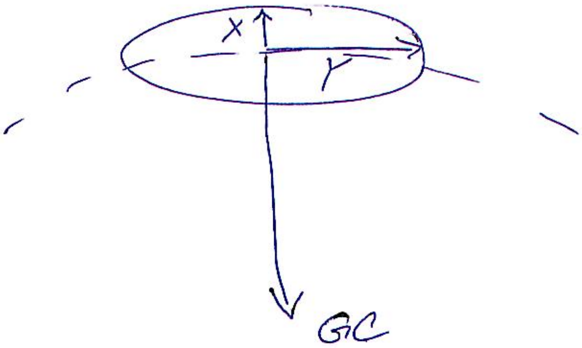
pro $r \to 0$: $0 = 0 + C \Rightarrow C = 0$

$$\frac{d\Phi}{dr} = \frac{4\pi G \rho_0 r_0^2}{r} \Rightarrow \Phi = 4\pi G \rho_0 r_0^2 \ln r + \text{const}$$

$$F_r = -\frac{d\Phi}{dr} = -\frac{4\pi G \rho_0 r_0^2}{r}$$

- dále jako u 1. možnosti

③ a) $\dfrac{X}{Y} = \dfrac{\varkappa}{2\Omega}$



$$\varkappa^2 = \frac{3}{R}\frac{\partial\Phi}{\partial R}\Big|_{z=0} + \frac{\partial^2\Phi}{\partial R^2}\Big|_{z=0}$$

$$\Omega^2 = \frac{1}{R}\frac{\partial\Phi}{\partial R}\Big|_{R=0}$$

$$\Phi_{TK} = -\frac{GM}{\sqrt{R^2 + (a+|z|)^2}}$$

$$\frac{\partial\Phi}{\partial R}\Big|_{z=0} = \frac{GM \cdot R}{(R^2+a^2)^{3/2}}$$

$$\frac{\partial^2\Phi}{\partial R^2}\Big|_{z=0} = \frac{GM}{(R^2+a^2)^{3/2}} - \frac{3GM \cdot R^2}{(R^2+a^2)^{5/2}} = \frac{GM(R^2+a^2-3R^2)}{(R^2+a^2)^{5/2}} = \frac{GM(a^2-2R^2)}{(R^2+a^2)^{5/2}}$$

$$\Omega^2 = \frac{GM}{(R^2+a^2)^{3/2}}$$

$$\varkappa^2 = \frac{3GM}{(R^2+a^2)^{3/2}} + \frac{GM(a^2-2R^2)}{(R^2+a^2)^{5/2}} = \frac{GM(3R^2+3a^2+a^2-2R^2)}{(R^2+a^2)^{5/2}} = \frac{GM(R^2+4a^2)}{(R^2+a^2)^{5/2}}$$

$$\frac{X}{Y} = \frac{\varkappa}{2\Omega} = \frac{1}{2}\sqrt{\frac{(R^2+4a^2)(R^2+a^2)^{3/2}}{(R^2+a^2)^{5/2}}} = \underline{\underline{\frac{1}{2}\sqrt{\frac{R^2+4a^2}{R^2+a^2}}}}$$

$\Rightarrow 1$ pro $r \Rightarrow 0$

$\Rightarrow \frac{1}{2}$ pro $r \Rightarrow \infty$

(3b)

$$\Sigma(R) = \frac{\Sigma_0 R_0}{R}$$

$$\rho(R,z) = \rho(R) \quad \text{pro } |z|<h$$

$$ = 0 \quad \text{pro } |z|>h$$

$$\Sigma(R) = \int_{-\infty}^{\infty} \rho(R,z)\,dz = 2\int_0^h \rho(R)\,dz = 2\rho(R)\int_0^h dz = 2\rho(R)h$$

$$\Rightarrow \rho(R) = \frac{\Sigma_0 R_0}{2Rh}$$

• Poissonova r.: $\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\Phi}{\partial R}\right) + \frac{\partial^2\Phi}{\partial z^2} = 4\pi G\rho$

$$R\frac{\partial\Phi}{\partial R} = v_c^2 = \text{const} \Rightarrow \frac{\partial^2\Phi}{\partial z^2} = 4\pi G\rho = \frac{2\pi\Sigma_0 R_0 G}{Rh}$$

$$\Rightarrow \frac{\partial\Phi}{\partial z} = \frac{2\pi G\Sigma_0 R_0}{Rh}\, z$$

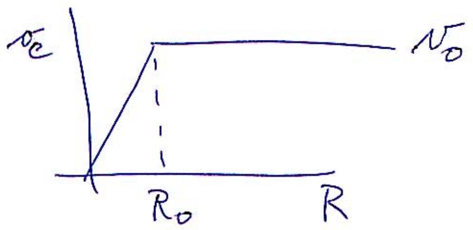
• pohyb. rov.: $\ddot{z} = -\frac{\partial\Phi}{\partial z} = -\frac{2\pi G\Sigma_0 R_0}{Rh}\, z$

- řešení: $z = A.\sin(\nu_z t + \varphi_0)$

$$\nu_z = \sqrt{\frac{2\pi G\Sigma_0 R_0}{Rh}} = \sqrt{4\pi G\rho(R)} = \sqrt{\frac{2\pi G\Sigma(R)}{h}}$$

(4) a) ILR
4/1 (UHR)
OLR

b) $v_c = \frac{v_0}{R_0} R$ pro $R < R_0$



$v_c = v_0$ pro $R > R_0$

$v_0 = 200\ \mathrm{km/s}$ $R_0 = 1\ \mathrm{kpc}$

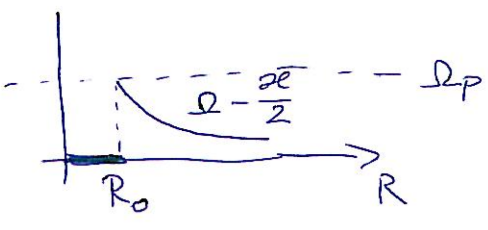
ILR: $\Omega - \frac{\kappa}{2} = \Omega_p$

1 ILR vzniká v případě, že $\Omega_p = \left(\Omega - \frac{\kappa}{2}\right)_{max}$

$\Rightarrow$ nutno najít $\left(\Omega - \frac{\kappa}{2}\right)_{max}$ (leží v R_0)

- $R < R_0$: $\kappa = 2\Omega \Rightarrow \Omega - \frac{\kappa}{2} = \Omega - \Omega = 0$
- $R > R_0$: $\kappa = \sqrt{2}\,\Omega \Rightarrow \Omega - \frac{\kappa}{2} = \Omega - \frac{\sqrt{2}}{2}\Omega = \left(1 - \frac{\sqrt{2}}{2}\right)\Omega$

$\Omega = \frac{v_0}{R} \Rightarrow \Omega - \frac{\kappa}{2} = \left(1 - \frac{\sqrt{2}}{2}\right)\frac{v_0}{R}$



$\left(\Omega - \frac{\kappa}{2}\right)_{max} = \left(1 - \frac{\sqrt{2}}{2}\right)\frac{v_0}{R_0}$

$\Rightarrow \underline{\Omega_p} = \left(1 - \frac{\sqrt{2}}{2}\right)\frac{v_0}{R_0} = \left(1 - \frac{\sqrt{2}}{2}\right)\frac{200}{1}\ \mathrm{km/s/kpc}$

$\approx 60\ \mathrm{km/s/kpc}$

c) $\left(\Omega + \frac{\kappa}{2}\right)_{OLR} = \Omega_p$ & $\left(\Omega - \frac{\kappa}{4}\right)_{UHR} = \Omega_p$

$\kappa = \sqrt{2}\,\Omega$, protože $v_c = v_0 = \mathrm{const} \Rightarrow \left(1 + \frac{\sqrt{2}}{2}\right)\Omega_{OLR} = \left(1 - \frac{\sqrt{2}}{4}\right)\Omega_{UHR}$

$\Omega_{OLR} \cdot R_{OLR} = \Omega_{UHR} \cdot R_{UHR} = v_0 \Rightarrow \frac{R_{OLR}}{R_{UHR}} = \frac{\Omega_{UHR}}{\Omega_{OLR}} = \frac{1 + \frac{\sqrt{2}}{2}}{1 - \frac{\sqrt{2}}{4}} \approx 2.6$