EXERCISES IN CATEGORY THEORY 4

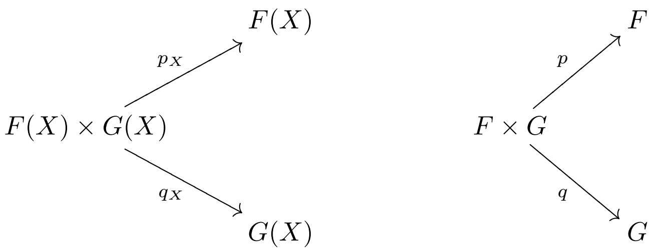
1. The Yoneda Lemma

In Set elements of a set X correspond to maps $1 \to X$ where 1 is the set with 1 element. From one perspective, the Yoneda lemma says that representable functors have a similar behaviour: maps $C(X, -) \to F$ in $[C, \mathrm{Set}]$ correspond to elements of the set FX.

(1) Use the Yoneda lemma to show that a morphism $\theta : F \to G \in [C, \mathrm{Set}]$ is mono if and only if each of its components $\theta_X : FX \to GX$ is mono is Set: an injective function. *Note: one direction is straightforward and does not use the Yoneda lemma.*

(2) Given $F, G \in [C, \mathrm{Set}]$ we want to work out what the product functor $F \times G$ looks like. Use the Yoneda lemma and the universal property of products to show that we must have $(F \times G)(X) \cong F(X) \times G(X)$.

(3) Set $F \times G(X) = F(X) \times G(X)$. Use the universal property of the product projections in Set



to define $F \times G$ on morphisms and to construct a product diagram in $[C, \mathrm{Set}]$ as above right.

(4) To each object X of C we have assigned a functor $C(X, -) : C \to \mathrm{Set}$. For each $f : X \to Y$ describe a natural transformation $C(f, -) : C(Y, -) \to C(X, -)$.

(5) Prove that these assignments define a functor $Y : C^{op} \to [C, \mathrm{Set}]$. *This is called the Yoneda embedding.*

(6) A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be fully faithful if given $f : FX \to FY \in D$ there exists a unique $g : X \to Y$ such that $F(g) = f$. Use the Yoneda Lemma to prove that the Yoneda embedding $C^{op} \to [C, \mathrm{Set}]$ is fully faithful.

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