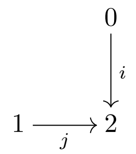
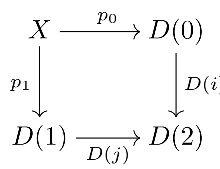
EXERCISES IN CATEGORY THEORY 6

1. Limits

(1) Let J be the category below:



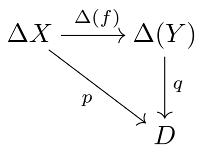


and consider a diagram $D : J \to C$. Show that a cone (X, p) on D is the same thing as a pair of arrows $X \to D(0)$ and $X \to D(1)$ making the square above commute. Show therefore that the limit of D is exactly the pullback of $D(i)$ and $D(j)$.

(2) Let $\mathcal{C}$ be a category with products. Show that each representable functor $\mathcal{C}(X, -) : \mathcal{C} \to \text{Set}$ preserves products.

(3) Let J and C be categories. Given $X \in C$ the *constant functor* $\Delta_X : J \to C$ at X is defined by $\Delta_X(j) = X$ for all $j \in J$ and sends all morphisms of J to the identity on X. Given $D : J \to C$ show that a natural transformation $p : \Delta_X \to D$ is the same thing as a cone (X, p) on D.

(4) Show that each morphism $f : X \to Y \in C$ determines a natural transformation between constant functors $\Delta_f : \Delta_X \to \Delta_Y$ and observe that a morphism of cones $(X, p) \to (Y, q)$ over D amounts to an arrow f such that the triangle of natural transformations



(5) Given a functor $F : A \to B$ and object $X \in B$ the *comma category* F/X has objects: triples $(A, p : FA \to X)$ and morphisms $f : (A, p : FA \to X) \to (B, q : FB \to X)$ are arrows $f : A \to B$ such that $q \circ Ff = p$.
Show that constant functors themselves define a functor $\Delta : C \to [J, C]$ and that $Cone(D)$ is the comma category Δ/D.

(6) Given a diagram $D : J \to \text{Set}$ show that the limit of D is given by the set of cones $Cone(1, D)$ over D with base the 1-element set.

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