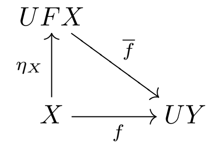
# EXERCISES IN CATEGORY THEORY 8

## 1. Adjoint functors

1.1. **Examples of adjoint functors.** A functor $U : \mathcal{A} \to \mathcal{B}$ has a left adjoint if for each $X \in \mathcal{A}$ there exists an object $FX$ and morphism $\eta_X : X \to UFX$ with the following universal property:
given $A \in \mathcal{A}$ and $f : X \to UA \in \mathcal{B}$ there exists a unique arrow $\overline{f} : FX \to A \in \mathcal{A}$ such that the triangle




(1.1)

commutes. Then $FX$ is the value of the left adjoint to $U$.

(1) Let $U : Mon \to \text{Set}$ be the forgetful functor from monoids to sets. Given a set $X$ elements of the word monoid $FX$ are lists $[x_1 \ldots x_n]$ of elements of $X$, with multiplication given by joining lists: ie. $[x, y][z] = [x, y, z]$. Show that $FX$ has the universal property of the left adjoint to $U$.
(2) The forgetful functor $U : CRing \to \text{Set}$ from the category of commutative rings to the category of sets has a left adjoint $F$. Show that the value of $F$ at the 1-element set $\{x\}$ is the commutative ring of polynomials $a_n x^n + a_1 x + \ldots a_0$ with integer coefficients $a_i \in Z$. What is $FX$ where $X$ is a finite set (or even an arbitrary set?)
(3) Consider $U : Vect \to \text{Set}$. Show that the value of the left adjoint $FX$ is the vector space with basis set $X$.
(4) Consider the forgetful functor from topological spaces $U : Top \to \text{Set}$ to sets. Show that the left adjoint to $U$ sends a set $X$ to $X$ with the *discrete* topology: all subsets are open.
(5) Given a set $X$ let $PX$ be the power set of $X$: since this is a poset we can view it as a category. Given $f : X \to Y$ we get functors $Pf : PX \to PY : U \mapsto \{fx \in Y : x \in U\}$ and $f^* : PY \to PX : U \mapsto \{x : fX \in U\}$. Show that $Pf \dashv f^*$.

1.2. **General categorical questions.**

(1) Prove that given a collection of arrows as in (1.1) that the objects $FX$ uniquely give rise to a functor $F : \mathcal{B} \to \mathcal{A}$ such that the morphisms $\eta_X : X \to UFX$ are the components of a natural transformation. In particular check that $F$ preserves composition.
(2) Prove that the left adjoint of a functor $U : \mathcal{A} \to \mathcal{B}$ is unique up to natural isomorphism.

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