MASARYK UNIVERSITY
FACULTY OF SCIENCE
DEPARTMENT OF MATHEMATICS AND STATISTICS

Mathematics I

Šárka Uchytilová

Brno 2014

Contents

Introduction

This textbook, Mathematics I, was written to meet the needs of the foreign students taking M1010 Mathematics I and M1020 Mathematics I - seminar at Faculty of Science at Masaryk University.

Moreover, we hope the textbook helps Czech students to get acquainted with English Mathematical terminology as well as enables their vocabulary development, further facilitating their studies abroad.

This textbook builds on the Czech textbook called *Matematika pro chemiky, 1. díl* written by prof. RNDr. Zuzana Došlá, DSc and is supported by the project *MUNI/FR/0134/2014*.

Chapter 1

Linear Algebra

1.1 Matrices

A **matrix** is a rectangular array of numbers arranged in rows and columns. An $m \times n$ matrix has m rows and n columns and $m \times n$ specifies the **order** of a matrix.

An example of matrix which has order 2×3 is

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}.$$

In general, matrix A which has order $m \times n$ can be written

$$\mathbf{A} = \mathbf{a_{ij}}$$

where $i = 1, 2, 3, \ldots, m$; $j = 1, 2, 3, \ldots, n$ and a_{ij} is the element in the ith row and jth column.

- Two matrices *are equal* if they have the same order and the elements in corresponding positions are equal.
- A matrix consisting of only one row is called **row matrix** or **row vector**.
- A matrix consisting of only one column is called **column matrix** or **column vector**.
- A matrix in which all elements are zero is called **zero matrix**.
- A $n \times n$ square matrix in which elements on the main diagonal are equal to one and are zeros elsewhere is called **identity matrix** or **unit matrix**. It is denoted by $\mathbf{E}$.

If $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the value $ad - bc$ is called the **determinant** of $\mathbf{A}$, denoted $|\mathbf{A}|$ or $\det\mathbf{A}$.

- We can only find the determinants of square matrices.
- If $|\mathbf{A}| = 0$ then we say that $\mathbf{A}$ is **singular**.

- If $|\mathbf{A}| \neq 0$ then we say that $\mathbf{A}$ is **non-singular** or **invertible**.

If a matrix $\mathbf{C}$ exists such that $\mathbf{CA}$=$\mathbf{E}$ then $\mathbf{C}$ is said to be the **inverse matrix** of $\mathbf{A}$, and we denote $\mathbf{C}$ by $\mathbf{A}^{-1}$.

- The multiplicative inverse of $\mathbf{A}$, denoted $\mathbf{A}^{-1}$, satisfies $\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{E}$.
- If $|\mathbf{A}| = 0$ then $\mathbf{A}^{-1}$ does not exist.

Matrix addition

To add two matrices $\mathbf{A}$ and $\mathbf{B}$ they must be of the same order and then we add corresponding elements. The result of addition is another matrix $\mathbf{C}$ of the same order in which $c_{ij} = a_{ij} + b_{ij}$.

Example 1.1. Find $\mathbf{A} + \mathbf{B}$, where

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \ \mathbf{B} = \begin{pmatrix} 7 & 8 & 9 \\ 5 & 3 & 2 \end{pmatrix}$$

Solution.

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} + \begin{pmatrix} 7 & 8 & 9 \\ 5 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1+7 & 2+8 & 3+9 \\ 4+5 & 5+3 & 6+2 \end{pmatrix} = \begin{pmatrix} 8 & 10 & 12 \\ 9 & 8 & 8 \end{pmatrix}$$

□

Multiples of matrices

If $\mathbf{A} = (a_{ij})$ is a matrix and k is a scalar, then $k\mathbf{A} = (k \cdot a_{ij})$. That means that we multiply each element in matrix $\mathbf{A}$ by k.

Example 1.2. Find $3 \cdot \mathbf{A}$, where

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

Solution.

$$3 \cdot \mathbf{A} = 3 \cdot \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 3 \cdot 1 & 3 \cdot 2 & 3 \cdot 3 \\ 3 \cdot 4 & 3 \cdot 5 & 3 \cdot 6 \end{pmatrix} = \begin{pmatrix} 3 & 6 & 9 \\ 12 & 15 & 18 \end{pmatrix}$$

□

Matrix multiplication

The product of an $m \times n$ matrix $\mathbf{A}$ with an $n \times p$ matrix $\mathbf{B}$ is the $m \times p$ matrix $\mathbf{C}$ in which the element in the ith row and jth column is the sum of the products of the elements in the ith row of $\mathbf{A}$ with the corresponding elements in the jth column of $\mathbf{B}$. That is

$$\mathbf{C} = \mathbf{A} \cdot \mathbf{B} \qquad \text{and} \qquad (c_{ij}) = \sum_{k=1}^{n} a_{ik} b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{in}b_{nj}$$

The product exists only if the number of columns of $\mathbf{A}$ equals the number of rows of $\mathbf{B}$. In general, we cannot reverse the order in which we multiply matrices ($\mathbf{A} \cdot \mathbf{B} \neq \mathbf{B} \cdot \mathbf{A}$).

Example 1.3. Find $\mathbf{A} \cdot \mathbf{B}$, where

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}, \ \mathbf{B} = \begin{pmatrix} 7 & 8 & 9 \\ 5 & 3 & 2 \end{pmatrix}$$

Solution.

$$\mathbf{A} \cdot \mathbf{B} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} + \begin{pmatrix} 7 & 8 & 9 \\ 5 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1+7 & 2+8 & 3+9 \\ 4+5 & 5+3 & 6+2 \end{pmatrix} = \begin{pmatrix} 8 & 10 & 12 \\ 9 & 8 & 8 \end{pmatrix}$$

□

1.2 Solving systems using row operations

The system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n &= b_k \end{aligned} \tag{1.1}$$

where $a_{ij}, \ b_i \ (i = 1, \ \ldots, k; \ j = 1, \ \ldots, n)$ are real numbers, is called a **linear system of k equations in n variables** (or $k \times n$ **system**).

- If $(b_1, b_2, \ldots, b_k) = \vec{0}$ then we say that system (1.1) is **homogeneous**.
- If $(b_1, b_2, \ldots, b_k) \neq \vec{0}$ then we say that system (1.1) is **non-homogeneous**.

A solution of this system is any ordered n-tuple $(t_1, \ t_2, \ \ldots, \ t_n)$ such numbers that satisfy the system (1.1).

In matrix form $\mathbf{AX} = \mathbf{B}$ the system (1.1) is

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \end{pmatrix},$$

where $\mathbf{A}$ is a **coefficient matrix**.

The matrix

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} & b_k, \end{pmatrix}$$

is the **system's augmented matrix form**. To solve such a system, we need to reduce the augmented matrix to the echelon form using elementary row operations.
This method is called the **Gaussian-elimination algorithm** and is based on the fact that we do not change a solution if we

a) interchange rows,

b) multiply any row by a non-zero number,

c) add a multiple of any row to another row.

Number of solutions

1. Linear system of equations has a solution if and only if a rank of coefficient matrix is equal to rank of augmented matrix.

2. Linear system of k equations in n unknowns has only one solution if a rank of coefficient matrix is equal to rank of augmented matrix and it is equal to number of unknowns n, i.e. $r = n$.

3. Linear system of k equations in n unknowns has infinitely many solutions if a rank of coefficient matrix is equal to rank of augmented matrix and it is smaller than number of unknowns n, i.e. $r < n$.

Example 1.4. Solve the system:
$$\begin{aligned} x - 2y + z &= 1 \\ -x + 3y + 2z &= 0 \\ 2x - y + 5z &= 5 \end{aligned}$$

Solution.

In augmented form, the system is

$$\left(\begin{array}{rrr|r} 1 & -2 & 1 & 1 \\ -1 & 3 & 2 & 0 \\ 2 & -1 & 5 & 5 \end{array}\right).$$

The first row stays the same. We add the first row to the second one and subtract double first row from the third one

$$\left(\begin{array}{rrr|r} 1 & -2 & 1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 3 & 3 & 3 \end{array}\right).$$

Now, the first and the second rows stay the same and we subtract triple second row from the third row

$$\left(\begin{array}{rrr|r} 1 & -2 & 1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -6 & 0 \end{array}\right).$$

The last row corresponds to equation $-6z = 0$ and therefore $z = 0$.

Substituting $z = 0$ into the second equation $y + 3z = 1$ we get $y = 1$.

Using the first row and $z = 0$ and $y = 1$ we have $x = 3$.

Thus we have a unique solution $(x, y, z) = (3, 1, 0)$.

$\square$

Chapter 2

Functions of One Variable

2.1 Basic properties of functions

Let $D \subset \mathbb{R}$ and $H \subset \mathbb{R}$ be given sets. A **function** f is a rule that assigns to each element x in set D exactly one element, called $f(x)$, in set H.

- The set D is called the **domain** of the function and is denoted $D(f)$. It gives us all permissible values that x may attain. Symbol x is called an **independent variable**.
- The set H is called the **range** of the function and is denoted $H(f)$. The range of f is the set of all possible values of $f(x)$ and symbol $f(x)$ is called a **dependent variable**.

The graph of function $f: \ D(f) \to \mathbb{R}$ is the set of ordered pairs $\{(x, f(x)) \in \mathbb{R}^2 : x \in D(f)\}$.

Properties of functions

- Function f is said to be **bounded** if there exists a constant $K \in \mathbb{R}$, $K < 0$ for which $|f(x)| \leq K$ for every $x \in D(f)$. For example, $y = \sin x$ is a bounded function.
- We say that the function f is an **even function** if it satisfies $f(-x) = f(x)$ for every $x \in D(f)$. For example, $y = \cos x$ is an even function.
- We say that the function f is an **odd function** if it satisfies $f(-x) = -f(x)$ for every $x \in D(f)$. For example, function $y = \sin x$ is an odd function.
- Function f is called **one-to-one function** if $f(x_1) \neq f(x_2)$ for any pair $x_1, \ x_2 \in D(f)$ where $x_1 \neq x_2$. For example, $y = x$ is a one-to-one function.
- Function f is called **periodic function** with period $p \in \mathbb{R}, \ p > 0$ if for every $x \in D(f)$ also $x + p \in D(f)$ and $f(x + p) = f(x - p) = f(x)$. If p is a period of function f, then also $2p, \ 3p, \ 4p, \ldots$ are its periods. For example, trigonometric functions $(y = \sin x, \ y = \cos x, \ y = \tan x, \ y = \text{cotg } x)$ are periodic.

Definition 2.1. Let f be a function $f: D(f) \to \mathbb{R}$ and $I \subseteq D(f)$. Function f is said to be

a) **increasing on the set I** if $f(x_1) < f(x_2)$ for all $x_1, \ x_2 \in I$ such that $x_1 < x_2$;
b) **decreasing on the set I** if $f(x_1) > f(x_2)$ for all $x_1, \ x_2 \in I$ such that $x_1 < x_2$.

Functions that are increasing or decreasing are called **strictly monotonic**.

Given two functions $u : A \to B$ and $f : B \to \mathbb{R}$, then the function $F : A \to \mathbb{R}$ given by $y = f(u(x))$ is called a **composite function**. Function u is called the **inner component** and function f is called the **outer component**.

Example 2.1. For the functions below find its inner and outer component.

a) $f\colon y = \sin(x^2)$, b) $g\colon y = \sqrt{\frac{x-1}{x+1}}$.

Solution.

a) Here is $y = \sin u$, $u = x^2$. That means that the outer component is the sinus function and the inner component is quadratic function.

b) The outer component of given composite function is $y = \sqrt{u}$ and the inner component is
$$u = \frac{x-1}{x+1},$$
where u must be nonnegative and therefore
$$\frac{x-1}{x+1} \geq 0 \qquad \to \qquad x \in (-\infty, -1) \cup [1, \infty).$$

□

The inverse function of a one-to-one function f is a function f^{-1} for which

1. $D(f^{-1}) = H(f)$, $H(f^{-1}) = D(f)$
2. For every $y \in D(f^{-1})$ it holds $f^{-1}(y) = x$ if and only if $f(x) = y$.

Note:

An inverse function can be defined only for one-to-one function.
The symbol f^{-1} does not mean $\frac{1}{f}$.
The graph of function f^{-1} is a reflection of the graph of function f in the line $y = x$.

Example 2.2. Find the inverses of the given functions

a) $f\colon y = \frac{2x-1}{3x+5}$, b) $g\colon y = \ln(5 - 2x)$, c) $h\colon y = \mathrm{e}^{x-3}$.

Solution.

a) We will describe the procedure for finding the inverse function of a given function. We first interchange x and y in equation $y = f(x)$ and try to find y as a function of variable x using equivalent operations. In this case, we get
$$x = \frac{2y-1}{3y+5} \quad \Leftrightarrow \quad 3yx + 5x = 2y - 1 \quad \Leftrightarrow \quad y(3x-2) = -1 - 5x \quad \Leftrightarrow \quad y = -\frac{1+5x}{3x-2}.$$
Thus the inverse function has a formula $\qquad f^{-1}\colon y = -\dfrac{1+5x}{3x-2}.$

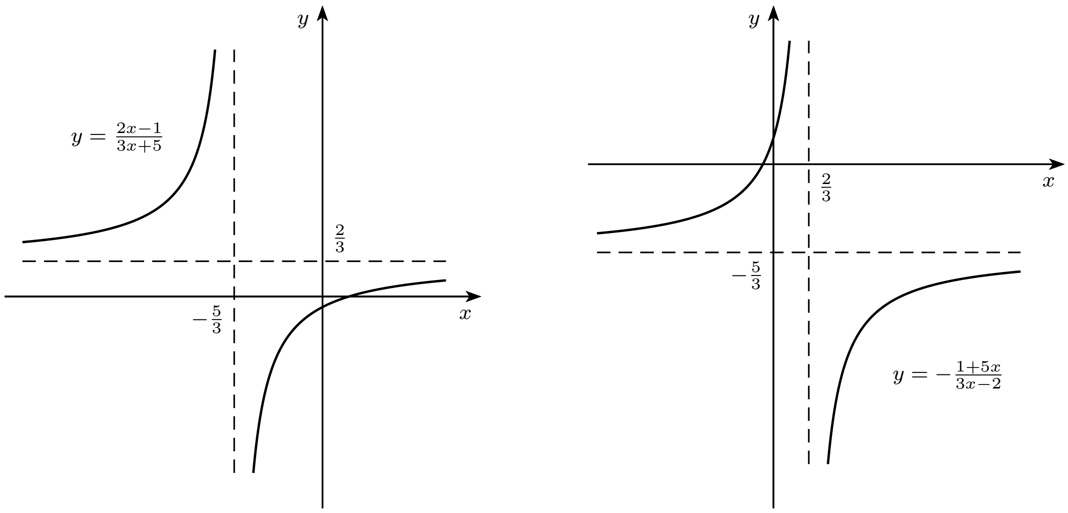


Figure 2.1: Graphs of functions f and f^{-1}

b) By interchanging x and y, we attain

$$x = \ln(5-2y) \quad \Leftrightarrow \quad e^x = e^{\ln(5-2y)} \Leftrightarrow \quad e^x = 5-2y \quad \Leftrightarrow \quad y = \frac{5-e^x}{2}.$$

The inverse of the given function g is therefore $g^{-1}\colon y = \dfrac{5-e^x}{2}$.

c) After interchange x and y, we have

$$x = e^{y-3} \quad \Leftrightarrow \quad \ln x = \ln e^{y-3} \quad \Leftrightarrow \quad \ln x = y-3 \quad \Leftrightarrow \quad y = \ln x + 3.$$

Thus the inverse function of h is the function $h^{-1}\colon y = \ln x + 3$.

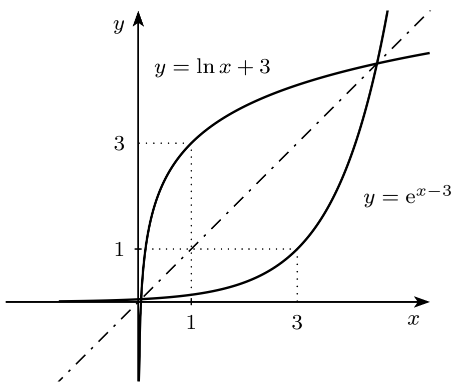


Figure 2.2: Graph of function h and its inverse function

□

Note:
Figures 2.1 and 2.2 illustrate an important property of inverse functions: Function f is increasing on interval I if and only if its inverse function f^{-1} is increasing on interval J, where $J = f(I)$. Similarly for decreasing functions.

2.2 Polynomials

Definition 2.2. Function $P : \mathbb{R} \to \mathbb{R}$ given by $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, where $a_0,\ a_1, \ldots, a_n \in \mathbb{R}$ is called **polynomial**. Numbers a_i are called **coefficients** of the polynomial.

If $a_n \neq 0$, then the **degree** of the polynomial is number n.

Number $\alpha \in \mathbb{C}$ is called **root** of polynomial P if $P(\alpha) = 0$.
Number $\alpha \in \mathbb{C}$ is called **k-tuple root** of polynomial P if there exists polynomial Q such that $P(x) = (x-\alpha)^k Q(x)$ and α is not a root of polynomial Q (that means $Q(\alpha) \neq 0$). Number k is the **multiplicity** of root α.

Examples of polynomials:

a) linear function $y = ax + b$, where $a \neq 0$ (polynomial of degree 1)
b) quadratic function $y = ax^2 + bx + c$, where $a \neq 0$ (polynomial of degree 2)

Theorem 2.1.

a) Any polynomial P of degree n (where $n \geq 0$) has n complex roots (counted with the multiplicity).

b) If complex number α is a k-tuple root of polynomial P, then the complex conjugate $\overline{\alpha}$ is also k-tuple root of polynomial P.

c) Let $\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_r$ be all real roots of polynomial P with multiplicities $k_1,\ k_2,\ \ldots,\ k_r$ and $(c_1 \pm id_1),\ \ldots,\ (c_s \pm id_s)$ are all complex roots with multiplicities $r_1,\ \ldots,\ r_s$, then

$$P(x) = a_n(x-\alpha_1)^{k_1} \cdot \cdots \cdot (x-\alpha_r)^{k_r} \cdot [(x-c_1)^2 + d_1^2]^{r_1} \cdot \cdots \cdot [(x-c_s)^2 + d_s^2]^{r_s}.$$

d) Polynomials $P,\ Q$ of degree n are equal if and only if all coefficients of corresponding terms are equal.

Sign of Polynomials

When investigating the behavior of function, we need to find intervals on which a given function assumes positive values (its graph is above the x-axis) and on which it assumes negative values (its graph is below the x-axis).

This can be solved by factorizing the polynomial P in real domain. If $x_1 < x_2 < \cdots < x_m$ are all different real roots of P with odd multiplicities then polynomial P keeps the same positive or negative sign on each interval $(-\infty, x_1), (x_1, x_2), \ldots, (x_m, \infty)$. If we choose two neighbouring intervals, P changes its sign from negative to positive or vice versa. Passing over the root of even multiplicity polynomial P does not change its sign.

Example 2.3. Determine the sign of the polynomial $P(x) = (x^2 - x)(x-2)^2$ and sketch the graph.

Solution. We factorize the polynomial $P(x) = x(x-1)(x-2)^2$. Polynomial P has two simple real roots (i.e. with odd multiplicities) $x_1 = 0$ and $x_2 = 1$ and one double root (i.e. even multiplicity) $x_{3,4} = 2$. We mark numbers $x_1 = 0$ and $x_2 = 1$ on the number line. Polynomial is positive for high values of x and it changes its sign when passing over the roots 0 a 1. For $x = 2$, the polynomial is equal to zero.
Thus $P(x) > 0$ for $x \in (-\infty, 0) \cup (1, 2) \cup (2, \infty)$ (see fig. 2.3).

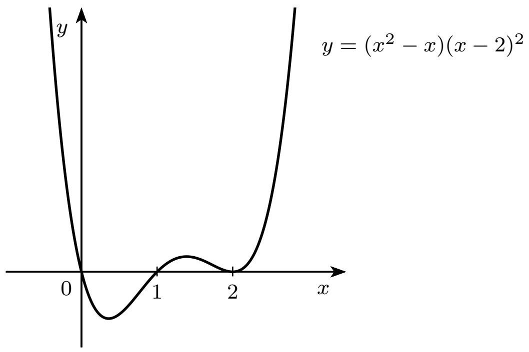


Figure 2.3: Graph of polynomial $P(x) = (x^2 - x)(x - 2)^2$

2.3 Rational functions

Let $P(x),\ Q(x)$ be polynomials. Function

$$R(x) = \frac{P(x)}{Q(x)}$$

is called **rational function**.

a) If degree of *P(x)* is smaller than degree of $Q(x)$, we call function $R(x)$ **proper rational function**.
b) If degree of *P(x)* is greater than or equal to degree of $Q(x)$, we call function $R(x)$ **improper rational function**.

Examples of rational functions:

a) $y = \frac{k}{x}$, where $x \neq 0,\ k \neq 0$,

b) $y = \frac{ax+b}{cx+d}$, where $x \neq -\frac{d}{c},\ c \neq 0,\ ad \neq bc$.

Graph of both of these rational functions is a curve called **hyperbola**.

Sign of Rational Functions

The procedure of finding the sign of rational function is similar to finding the sign of polynomial. A rational function $R(x) = \frac{P(x)}{Q(x)}$ where P and Q are polynomials, can change its sign only at points, where P or Q change it. However, we have to remember that rational functions are not defined at real roots of the denominator.

Consider $R(x)$ to be a rational function whose numerator and denominator have no common roots. Let $x_1 < \cdots < x_p$ be all different real roots of numerator and denominator with odd multiplicities. Then P keeps the same positive or negative sign on each interval $(-\infty, x_1), (x_1, x_2), \dots, (x_m, \infty)$ (where R is defined). If we choose two neighbouring intervals, P changes its sign from negative to positive or vice versa.

Example 2.4. Determine the sign of $R(x)$.

a) $R(x) = \frac{(x-2)(x-5)}{(x-3)(x+2)(x+5)}$, b) $R(x) = \frac{(x^2-1)(x^2-2x+1)(x-2)^2}{(x^2+x-2)^2(x+2)}$.

Solution.

a) We first find all real roots of numerator and denominator. These are $-5, -2, 2, 3$ and 5. We choose any one number that is not the root and determine the sign of R. For example, for $x = 0$ is $R(0) < 0$. Hence $R(x)$ has the negative sign on the interval involving 0, that is the interval $(-2, 2)$, and passing over the roots $-5, -2, 2, 3, 5$ function R changes its sign from negative to positive or vice versa (see table below).

x	$(-\infty,-5)$	$(-5,-2)$	$(-2,2)$	$(2,3)$	$(3,5)$	$(5,\infty)$
$R(x)$	$-$	$+$	$-$	$+$	$-$	$+$

b) Numerator and denominator are not fully factorized. Therefore, we have to start with finding all real roots

$$x^2 - 1 = (x+1)(x-1), \qquad x^2 - 2x + 1 = (x-1)^2, \qquad x^2 + x - 2 = (x-1)(x+2)$$

and the complete factorization is

$$R(x) = \frac{(x+1)(x-1)(x-1)^2(x-2)^2}{(x-1)^2(x+2)^2(x+2)} = \frac{(x-1)^3(x+1)(x-2)^2}{(x-1)^2(x+2)^3}.$$

After cancellation we obtain

$$R(x) = \frac{(x-1)(x+1)(x-2)^2}{(x+2)^3}.$$

Real roots with odd multiplicities are -2, -1 and 1. These points separate the domain of R into intervals. We calculate the value of R at an interior point of some interval, for example $x = 0$, where $R(0) < 0$. The sign changes when passing over the roots -2, -1 and 1.

x	$(-\infty,-2)$	$(-2,-1)$	$(-1,1)$	$(1,\infty)$
$R(x)$	$-$	$+$	$-$	$+$

2.4 Partial Fractions

Let $R(x) = \frac{P(x)}{Q(x)}$ be a proper rational function. We can decompose any of proper rational functions into *partial fractions*:

a) If number α is a simple real root of polynomial Q, then function R can be written as a partial fraction

$$\frac{A}{(x-\alpha)}.$$

b) If number α is a k-tuple real root of Q, then R can be written as a sum of k partial fractions

$$\frac{A}{(x-\alpha)} + \frac{B}{(x-\alpha)^2} + \cdots + \frac{M}{(x-\alpha)^k}.$$

c) If $\alpha \pm i\beta$ are simple complex conjugate roots of Q, then R can be written as a partial fraction

$$\frac{Ax+B}{ax^2+bx+c},$$

where ax^2+bx+c has roots $\alpha \pm i\beta$.

d) If $\alpha \pm i\beta$ are double complex conjugate roots of Q, then R can be written as a sum of two partial fractions

$$\frac{Ax+B}{ax^2+bx+c} + \frac{Cx+D}{(ax^2+bx+c)^2}.$$

Analogously, to l-tuple pair of complex conjugate roots of the denominator there corresponds a sum of l partial fractions.

To find numbers $A,\ B,\ C,\ D$ we use a *method of undetermined coefficients*:

1. We decompose the denominator into the product of factors irreducible in the real domain.

2. We write the assumed form of the decomposition into the sum of partial fractions with undetermined coefficients.

3. We write an equation where the original proper rational function is equal to the sum of the assumed form of partial fractions.

4. We multiply the equation by the denominator of the original rational function and get the equality of two polynomials.

5. Two polynomials are equal if and only if coefficients of corresponding powers of x on the two sides of the equation are equal. Thus we expand the polynomials on both sides and equate coefficients of corresponding terms. That gives us a system of linear equations.

6. We solve the system of equation and obtain the only solution.

7. We decompose the original rational function.

Example 2.5. Express

$$R(x) = \frac{x^2 + 4x}{x^4 - 16}$$

as a sum of partial fractions.

Solution. First of all, we have to factorize the denominator

$$x^4 - 16 = (x^2)^2 - (2^2)^2 = (x^2 - 4)(x^2 + 4) = (x - 2)(x + 2)(x^2 + 4).$$

Therefore, the assumed form of the sum of partial fractions is

$$\frac{x^2 + 4x}{x^4 - 16} = \frac{A}{x - 2} + \frac{B}{x + 2} + \frac{Cx + D}{x^2 + 4},$$

and after multiplying by $(x - 2)(x + 2)(x^2 + 4)$ we obtain

$$x^2 + 4x = A(x + 2)(x^2 + 4) + B(x - 2)(x^2 + 4) + (Cx + D)(x^2 - 4).$$

Equating the coefficients of corresponding powers of x we get a system

$$\begin{aligned}
x^3 &: 0 = A + B + C \\
x^2 &: 1 = 2A - 2B + D \\
x^1 &: 4 = 4A + 4B - 4C \\
x^0 &: 0 = 8A - 8B - 4D.
\end{aligned}$$

The solution of this system is $A = \frac{3}{8}$, $B = \frac{1}{8}$, $C = -\frac{1}{2}$, $D = \frac{1}{2}$ and the decomposition of $R(x)$ is

$$R(x) = \frac{x^2 + 4x}{x^4 - 16} = \frac{3}{8(x - 2)} + \frac{1}{8(x + 2)} + \frac{1 - x}{2(x^2 + 4)}.$$

□

Chapter 3

Limits, derivatives, minima and maxima

3.1 Limits

A limit describes the behaviour of function in a neighbourhood of given point (apart from the point).
Function $y = f(x)$ has a **limit $\boldsymbol{L}$** at point x_0 if values of function $f(x)$ approaches number L as x approaches x_0. We write

$$\lim_{x \to x_0} f(x) = L.$$

Note:

Point x_0 may be any real number but also ∞ or $-\infty$.

If L is a real number, we say that function $f(x)$ has a **finite limit**.

If L is ∞ or $-\infty$, we say that function $f(x)$ has a **infinite limit**.

Function f has at most one limit at any point.
Limits, as x approaches a number x_0 from left and right sides, may differ. We call the limit of function f as x approaches x_0 from the right the **right-hand limit** of f at point x_0 and the limit of function f as x approaches x_0 from the left the **left-hand limit** of f at point x_0.

A function $f(x)$ has a limit L at point x_0 if and only if the right-hand limit and the left-hand limit at x_0 exist and they are equal, i.e.

$$\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = L = \lim_{x \to x_0} f(x).$$

Example 3.1. a) Let $f(x)$ be a function $y = x^2$ and $x_0 = 0$. Then

$$\lim_{x \to 0^-} x^2 = \lim_{x \to 0^+} x^2 = 0.$$

And therefore

$$\lim_{x \to 0} x^2 = 0.$$

b) Let $f(x)$ be a function $y = \frac{1}{x}$ and $x_0 = 0$. Then

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty \qquad \text{and} \qquad \lim_{x \to 0^+} \frac{1}{x} = \infty.$$

Therefore,

$$\lim_{x \to 0} \frac{1}{x} \quad \text{does not exist.}$$

Rules for limits

If $\lim_{x\to x_0} f(x) = L_1$ and $\lim_{x\to x_0} g(x) = L_2$, where $L_1,\ L_2 \in \mathbb{R}$, then

1. $\lim_{x\to x_0}(f(x) \pm g(x)) = L_1 \pm L_2$,
2. $\lim_{x\to x_0}(f(x) \cdot g(x)) = L_1 \cdot L_2$,
3. $\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{L_1}{L_2}$ if $L_2 \neq 0$,
4. $\lim_{x\to x_0} |f(x)| = |\lim_{x\to x_0} f(x)|$.

Note:
When calculating the limit of quotient

$$\lim_{x \to x_0} \frac{f(x)}{g(x)},$$

where

$$\lim_{x \to x_0} f(x) = 1, \quad \lim_{x \to x_0} g(x) = \pm\infty \ \text{ or } \ \lim_{x \to x_0} g(x) = 0,$$

symbolic expressions

$$\frac{1}{\pm\infty} = 0, \quad \frac{1}{+0} = +\infty, \quad \frac{1}{-0} = -\infty \tag{3.1}$$

holds.

If $\lim_{x\to x_0} f(x) = c$ and $\lim_{x\to x_0} g(x) = 0$, then we have to consider sign of number c.

In case of limits in the form

$$\infty - \infty, \quad \frac{\infty}{\infty}, \quad \frac{0}{0}$$

the operations are not defined in $\mathbb{R}$. These cases are therefore called **indeterminate forms**. We will show how to deal with such limits in section 3.4.

Definition 3.1. a) A function $f(x)$ is **continuous** at point x_0 if

$$\lim_{x \to x_0} f(x) = f(x_0).$$

b) If the function $f(x)$ is defined and continuous at every point of the closed interval $[a, b]$, we call the function **continuous on interval** $[a, b]$.

Theorem 3.1. *If the function $f(x)$ is continuous on interval $I = [a, b]$, then $f(x)$ is bounded and attains its minimal and maximal value on I.*

If the function $f(x)$ is continuous on interval $I = [a, b]$, then $f(x)$ attains all the values between its maximum and minimum on the interval I.

Corollary 3.1. *If the function $f(x)$ is continuous on $I = [a, b]$ and $f(a) \cdot f(b) < 0$, then there is at least one point $c \in (a, b)$ where $f(c) = 0$.*

3.2 Derivatives

Definition 3.2. Let f be a function of x and $x_0 \in D(f)$. If the limit

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists and is finite, we call this limit the **derivative of f at point** x_0 and denote $f'(x_0)$. Similarly,

$$\lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0}$$

is called right derivative and denoted $f'_+(x_0)$;

$$\lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0}$$

is called left derivative and denoted $f'_-(x_0)$.

The derivative $f'(x_0)$ equals to the *slope of tangent line* to the graph of $f(x)$ at point x_0. Therefore,

$$y = f(x_0) + f'(x_0)(x - x_0)$$

is the equation of **tangent line** at point $(x_0, f(x_0))$.

A **normal** to the graph of $f(x)$ at point x_0 is a line which is perpendicular to the tangent line at point x_0. Such a line has an equation

a) $y = f(x_0) - \frac{1}{f'(x_0)}(x - x_0)$ if $f'(x_0) \neq 0$,

b) $x = x_0$ if $f'(x_0) = 0$.

Theorem 3.2 (Rules for derivatives)**.** *Let functions $f(x)$, $g(x)$ have derivative on set M. Then*

1. $(cf(x))' = cf'(x)$,
2. $(f(x) + g(x))' = f'(x) + g'(x)$,

3. $(f(x) \cdot g(x))' = f'(x)g(x) + f(x)g'(x)$.

4. *If* $g(x) \neq 0$, *then* $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x)-f(x)g'(x)}{g^2(x)}$.

The **second derivative** of function $f(x)$ is a function $f'' = (f')'$. In general, ***n*-th order derivative of function** $f(x)$ is a function $f^{(n)} = (f^{(n-1)})'$.

Theorem 3.3. *Let function f has a derivative on an open interval I.*

a) If $f'(x) > 0$ for all $x \in I$, then f is increasing on I.
b) If $f'(x) < 0$ for all $x \in I$, then f is decreasing on I.
c) If $f'(x) = 0$ for all $x \in I$, then f is constant on I.

Derivatives in physics

Consider a function $s(t)$ to be a **displacement function**. Then the first derivative $s'(t)$ is a **velocity function** $v(t)$ and the second derivative $s''(t)$ (which is also the first derivative of velocity function) is an **instantaneous acceleration** at time t.

3.3 Maxima and Minima

There are two different types of minima and maxima. The first one is a local minimum (or maximum) and the second type is a global minimum (or maximum).

Definition 3.3. A neighborhood $\mathcal{O}(x_0)$ of $x_0 \in \mathbb{R}$ is an open interval $(x_0 - \delta,\ x_0 + \delta)$, where $\delta > 0$.

Definition 3.4. A function f has at point x_0

- a **local minimum** if there exists a neighborhood $\mathcal{O}(x_0)$ of x_0 such that $f(x) \geq f(x_0)$ for all $x \in \mathcal{O}(x_0)$,
- a **local maximum** if there exists a neighborhood $\mathcal{O}(x_0)$ of x_0 such that $f(x) \leq f(x_0)$ for all $x \in \mathcal{O}(x_0)$,
- a **strict local minimum** if there exists a neighborhood $\mathcal{O}(x_0)$ such that $f(x) > f(x_0)$ for all $x \in \mathcal{O}(x_0) \setminus \{x_0\}$,
- a **strict local maximum** if there exists a neighborhood $\mathcal{O}(x_0)$ such that $f(x) < f(x_0)$ for all $x \in \mathcal{O}(x_0) \setminus \{x_0\}$.

Theorem 3.4. *Suppose that a function $f(x)$ has a local minimum or maximum at point x_0 and that $f'(x)$ is defined at x_0. Then*

$$f'(x_0) = 0. \tag{3.2}$$

The point satisfying (3.2) is called a **stationary point**.

The points where $f' = 0$ or f' fails to exist are commonly called the **critical points** of f. None of these points is necessarily the location of a local maximum or minimum, but these points are only points where local extreme may exist .

Theorem 3.5 (Sufficient Condition)**.**

a) If the first derivative f' of the function f changes its sign from positive to negative at critical point x_0, then the function f has a local minimum at point x_0.

b) If the first derivative f' of the function f changes its sign from negative to positive at critical point x_0, then the function f has a local maximum at point x_0.

Theorem 3.6 (Second Derivative Test for Local Maxima and Minima)**.**

a) If $f'(x_0) = 0$ and $f''(x_0) < 0$, then function f has a strict local maximum at point x_0.

b) If $f'(x_0) = 0$ and $f''(x_0) > 0$, then function f has a strict local minimum at point x_0.

Example 3.2. Find the local minimum and maximum values, if any, of the function

$$f(x) = x^3 - 12x - 6.$$

Solution. To find all the stationary points we have to calculate the first derivative

$$f'(x) = (x^3 - 12x - 6)' = 3x^2 - 12.$$

The derivative is defined at every point of the function's domain and is zero at $x = 2$ or $x = -2$. The sign diagram of f' is

interval	$(-\infty, -2)$	$(-2, 2)$	$(2, +\infty)$
f'	$+$	$-$	$+$
f	$\nearrow$	$\searrow$	$\nearrow$

Since the derivative f' changes its sign at both points, it has a local maximum at point $x = -2$ and its value is 10. Local minimum value is - 22 and is located at $x = 2$.

$\square$

Definition 3.5. Let f be a function on set M. If $x_0 \in M$ and

- $f(x) \geq f(x_0)$ for all $x \in M$, we say that the function f has a **global minimum** at point x_0.
- $f(x) \leq f(x_0)$ for all $x \in M$, we say that the function f has a **global maximum** at point x_0.

The only points where we can find a global minimum or maximum are critical points or endpoints of the function's domain (set M).

Example 3.3. Find the absolute minimum and maximum of

$$f(x) = x - x^2, \quad x \in [0, 1].$$

Solution. Function $f(x)$ is continuous and bounded on closed interval and therefore has absolute minimum and maximum. The derivative is

$$f'(x) = 1 - 2x$$

and we find stationary points where

$$f'(x) = 0.$$

Therefore, there is a stationary point (which is also the only critical point) at $x = \frac{1}{2}$. Finally, we have to compare the values at all critical points and endpoints of the interval $[0, 1]$

$$f\left(\frac{1}{2}\right) = \frac{1}{4}, \quad f(0) = 0, \quad f(1) = 0.$$

Thus, function f has its absolute maximum at point $x = \frac{1}{2}$ and absolute minimum at $x = 0$ and $x = 1$.

□

3.4 L'Hospital's Rule

Theorem 3.7. *Suppose $x_0 \in \mathbf{R} \cup \{-\infty, \infty\}$. If*

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0 \qquad \textit{or} \qquad \lim_{x \to x_0} |f(x)| = \lim_{x \to x_0} |g(x)| = \infty$$

and

$$\lim_{x \to x_0} \frac{f'(x)}{g'(x)} \quad \textit{exists, then} \quad \lim_{x \to x_0} \frac{f(x)}{g(x)} \quad \textit{exists as well}$$

and

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}.$$

Example 3.4. Find the limit

$$\lim_{x \to 0} \frac{x\mathrm{e}^x}{4 - 4\mathrm{e}^x}.$$

Solution. Substituting $x = 0$, we obtain „$\frac{0}{0}$". Therefore, we can use a L'Hospital Rule:

$$\lim_{x \to 0} \frac{x\mathrm{e}^x}{4 - 4\mathrm{e}^x} = \lim_{x \to 0} \frac{\mathrm{e}^x + x\mathrm{e}^x}{-4\mathrm{e}^x} = -\frac{1}{4}.$$

□

Chapter 4

Behaviour of a function, Approximation of a function

4.1 Concavity and Points of Inflection

Definition 4.1. a) We say that function f is **convex on interval I** if the graph of function f is located above the tangent at any point on I, i.e.

$$f(x) \geq f(x_0) + f'(x_0)(x - x_0) \qquad \text{where } x,\ x_0 \in I.$$

b) We say that function f is **concave on interval I** if the graph of function f is located under the tangent at any point on I, i.e.

$$f(x) \leq f(x_0) + f'(x_0)(x - x_0) \qquad \text{where } x,\ x_0 \in I.$$

c) A point on the curve where the concavity changes is called a **point of inflection**.

Theorem 4.1. *Let I be an open interval and let f have a second derivative f'' on I.*

a) If $f''(x) > 0$ for all $x \in I$, then f is ***convex****.*

b) If $f''(x) < 0$ for all $x \in I$, then f is ***concave****.*

c) If $f''(x_0) = 0$ and the sign of $f''(x_0)$ changes on either side of x_0, then there is a point of inflection at $x = x_0$.

Note:
Convex function is sometimes called **concave up** and concave function may be also called **concave down**. (See e.g. [3] and [7]).

4.2 Asymptotes

If the distance between the graph of a function and some line approaches zero as a point on the graph moves increasingly far from the origin, we say that the line is an asymptote of the graph. For instance, the x-axis and y-axis are asymptotes of the curves $y = \frac{1}{x}$.

Definition 4.2. A line $x = x_0$ is called a **vertical asymptote** of the graph of a function f if

$$\lim_{x \to x_0^+} f(x) = \pm\infty \qquad \text{or} \qquad \lim_{x \to x_0^-} f(x) = \pm\infty.$$

A function f can have a vertical asymptote only at point $x = x_0$ where f is not defined.

Definition 4.3. A line $y = ax + b$, $a, b \in \mathbb{R}$, is called an **asymptote** of f if

$$\lim_{x \to -\infty} (f(x) - (ax + b)) = 0 \qquad \text{or} \qquad \lim_{x \to +\infty} (f(x) - (ax + b)) = 0.$$

If $a = 0$, the asymptote is called **horizontal**, otherwise it is called **oblique**.

Theorem 4.2. *A line $y = ax + b$ is an asymptote of function f for $x \to +\infty$ if*

$$a = \lim_{x \to +\infty} \frac{f(x)}{x}, \qquad b = \lim_{x \to +\infty} (f(x) - ax)$$

(where both limits are proper limits). Analogically for $x \to -\infty$.

Example 4.1. Find all asymptotes of the graph of functions

$$\text{a)} \quad f(x) = \frac{x-2}{x+1}, \qquad\qquad \text{b)} \quad f(x) = \frac{x^3}{x^2 - 1}.$$

Solution.

a) Since $D(f) = \mathbb{R} \setminus \{-1\}$ and f is continuous on domain $D(f)$, a vertical asymptote can only occur at the point $x = -1$. We have

$$\lim_{x \to -1^-} \frac{x-2}{x+1} = \infty.$$

Limit is improper thus there is a vertical asymptote at point $x = -1$.
Now, we find oblique and horizontal asymptotes. That means that we have to find limits

$$a = \lim_{x \to +\infty} \frac{f(x)}{x} = \lim_{x \to +\infty} \frac{x-2}{x(x+1)} = 0$$

$$b = \lim_{x \to +\infty} (f(x) - ax) = \frac{x-2}{x+1} = 1.$$

Asymptote for $x \to \infty$ is a line $y = 1$. Similarly for $x \to -\infty$ and therefore line $y = 1$ is an asymptote also for $x \to -\infty$.

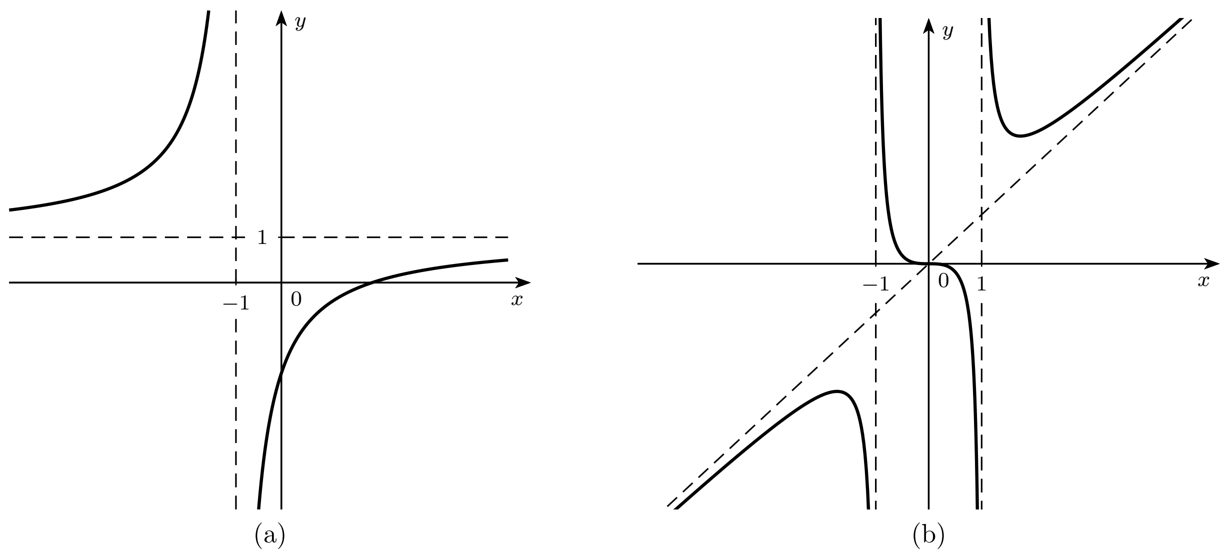


Figure 4.1: Graphs of functions and asymptotes

b) Function is not defined at points $x = -1$ and $x = 1$. We evaluate limits

$$\lim_{x\to -1^-} \frac{x^3}{x^2-1} = \infty \qquad \text{and} \qquad \lim_{x\to 1^+} \frac{x^3}{x^2-1} = \infty.$$

Both limits are improper. Therefore lines $x = -1$ a $x = 1$ are vertical asymptotes. To find oblique and horizontal asymptotes we have to evaluate limits

$$\lim_{x\to\infty} \frac{x^3}{x^3-x} = 1$$

$$\lim_{x\to\infty} \left(\frac{x^3}{x^2-1} - x\right) = \lim_{x\to\infty} \frac{x}{x^2-1} = 0.$$

Both limits are proper hence the graph of f has an oblique asymptote $y = x$ as $x \to \infty$. Since all previous calculations are valid also when $x \to -\infty$, the line $y = x$ is an oblique asymptote as $x \to -\infty$.

□

4.3 Behaviour of Function

When studying a behaviour of a function, we go through the following steps:

1. Identify the domain $D(f)$. Find (if possible) roots of f and intervals where function f is positive and where it is negative (sign diagram). Determine if f is even, odd or periodic.

2. Calculate f' and find

a) intervals where f is increasing (i.e. $f' > 0$),
b) intervals where f is decreasing (i.e. $f' < 0$),
c) local minima and maxima.

3. Calculate f'' and find
   a) points of inflection,
   b) intervals where f is convex (i.e. $f'' > 0$),
   c) intervals where f is concave (i.e. $f'' < 0$).
4. Find all asymptotes of the graph of function f (vertical, oblique, horizontal).
5. Sketch the graph of function f.

Example 4.2. Investigate the behaviour of function

$$f\colon y = x^3 - 3x^2 + 4.$$

Solution. 1. The function f is a polynomial so the domain $D(f) = \mathbb{R}$. Since $f(-x) = (-x)^3 - 3(-x)^2 + 4 = -x^3 - 3x^2 + 4$, function f is neither even nor odd. Function is not periodic (nonconstant polynomials are not periodic).
Roots of f are $x = -1$ and $x = 2$. Therefore, a sign diagram is

interval	$(-\infty,-1)$	$(-1,2)$	$(2,+\infty)$
f	$-$	$+$	$+$

2. We evaluate the first derivative

$$y' = (x^3 - 3x^2 + 4)' = 3x^2 - 6x.$$

Roots of polynomial $3x^2 - 6x = 3x(x-2)$ are $x = 0$ and $x = 2$. Those roots correspond to stationary points.
The sign diagram for y' is

interval	$(-\infty,0)$	$(0,2)$	$(2,+\infty)$
f'	$+$	$-$	$+$
f	$\nearrow$	$\searrow$	$\nearrow$

Thus function f has a local maximum at $x = 0$ and a local minimum $x = 2$.

3. The second derivative is

$$y'' = (3x^2 - 6x)' = 6x - 6.$$

We equate it with zero and get $x = 1$ which is the only potential point of inflection.
Sign diagram for y'':

interval	$(-\infty,1)$	$(1,\infty)$
f''	$+$	$-$
f	$\cup$	$\cap$

The function has a point of inflection at $x = 1$.

4. We find any asymptotes.
   As f is continuous on $\mathbb{R}$ there is no vertical asymptote.
   We identify an oblique or horizontal asymptote:

$$a = \lim_{x \to \pm\infty} \frac{f(x)}{x} = \lim_{x \to \pm\infty} \frac{x^3 - 6x^2 + 4}{x} = \infty.$$

   The limit is improper, therefore, there are no such asymptotes.

5. When calculating the values at several significant points we obtain: $f(0) = 4$, $f(1) = 2$, $f(2) = 0$. Now, we can sketch the graph (see fig. 4.2).

$\square$

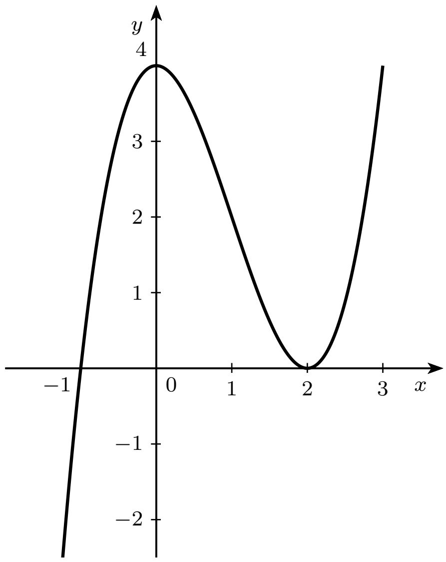


Figure 4.2: Graph of function $y = x^3 - 3x^2 + 4$

Chapter 5

Indefinite Integral

Chapter 3 deals with an important term *derivative of function*. In this chapter we will investigate a converse problem. We are looking for a function F whose derivative $f = F'$ is known. To find such functions F is part of **integral calculus**. To integrate function means to find all the functions that have a given function as a derivative.

5.1 Basic Integrals

Definition 5.1. Let f and F be functions defined on interval I. Function F is called an **antiderivative of function** f (or a **primitive function of** f) on interval I if

$$F'(x) = f(x) \text{ for all } x \in I.$$

The set of all antiderivatives of f is called an **indefinite integral of function** f and is denoted

$$\int f(x)\,\mathrm{d}x.$$

Note:

a) In case that interval I is not open, the derivatives at the end points are one-sided derivatives.

b) Symbol $\int$ originate from stretching letter S, which is the first letter of the word sum.

c) The function f is the **integrand** of the integral. The term $\mathrm{d}x$ is the **differential of variable** x. It tells us that the variable of integration is x.

Theorem 5.1. *If F is an antiderivative of f on interval I, then all other antiderivatives of f on I has the form $F + c$, where $c \in \mathbb{R}$.*
We indicate this by writing

$$\int f(x)\,\mathrm{d}x = F(x) + c, \qquad \textit{where } c \in \mathbb{R}.$$

Number c is the **constant of integration**.

Example 5.1. Find antiderivatives of following functions

$$\text{a) } y = 1, \qquad \text{b) } y = e^x, \qquad \text{c) } y = e^{-x}.$$

Solution. a) Antiderivative of function $f(x) = 1$ is $F(x) = x$ because $F'(x) = x' = 1$.

b) Antiderivative of function $f(x) = e^x$ is $F(x) = e^x$ because when differentiating function e^x it stays the same.

c) Similarly, $(e^{-x})' = -e^{-x} = -f(x)$ and thus the antiderivative of f is function $F(x) = -e^{-x}$.

Using indefinite integrals we can write

$$\int \mathrm{d}x = x + c, \qquad \int e^x \,\mathrm{d}x = e^x + c, \qquad \int e^{-x} \,\mathrm{d}x = -e^{-x} + c.$$

□

Example 5.2. Find an indefinite integral of

$$\text{a) } \quad y = \cos 2x, \qquad \text{b) } \quad y = \frac{1}{x}.$$

Solution. a) We know that

$$(\sin x)' = \cos x, \quad \text{therefore} \quad \int \cos x \,\mathrm{d}x = \sin x.$$

Similarly,

$$(\sin 2x)' = 2\cos 2x, \quad \text{thus} \quad \int \cos 2x \,\mathrm{d}x = \frac{1}{2}\sin 2x.$$

b) The antiderivative of function $f(x) = \frac{1}{x}$ is $F(x) = \ln x$ on interval $(0, \infty)$ and $F(x) = \ln(-x)$ on interval $(-\infty, 0)$. Therefore function $F(x) = \ln|x|$ is a primitive function on $(-\infty, 0)$ and $(0, \infty)$ and we get

$$\int \frac{1}{x} \,\mathrm{d}x = \ln|x| + c. \tag{5.1}$$

□

Theorem 5.2. *If a function f is continuous on the interval I, then it has an antiderivative on I.*

Theorem 5.3. *Let integrals $\int f(x)\,\mathrm{d}x$ and $\int g(x)\,\mathrm{d}x$ exist on an interval I and let α be a constant. Then the integrals $\int (f(x) + g(x))\,\mathrm{d}x$ and $\int \alpha f(x)\,\mathrm{d}x$ also exist on I and the following equalities hold:*

$$\int (f(x) + g(x))\,\mathrm{d}x = \int f(x)\,\mathrm{d}x + \int g(x)\,\mathrm{d}x, \tag{5.2}$$

$$\int \alpha f(x)\,\mathrm{d}x = \alpha \int f(x)\mathrm{d}x. \tag{5.3}$$

The following formulas give the indefinite integrals of several basic functions:

(1) $\int 1\,\mathrm{d}x = x + c,$

(2) $\int x^n\,\mathrm{d}x = \frac{x^{n+1}}{n+1} + c, \quad n \neq -1,$

(3) $\int \frac{1}{x}\,\mathrm{d}x = \ln|x| + c,$

(4) $\int \mathrm{e}^x\,\mathrm{d}x = \mathrm{e}^x + c,$

(5) $\int a^x\,\mathrm{d}x = \frac{a^x}{\ln a} + c, \quad a > 0,\ a \neq 1,$

(6) $\int \sin x\,\mathrm{d}x = -\cos x + c,$

(7) $\int \cos x\,\mathrm{d}x = \sin x + c,$

(8) $\int \frac{1}{x^2+1}\,\mathrm{d}x = \operatorname{arctg} x + c$

(9) $\int \frac{1}{(x-x_0)^2+a^2}\,\mathrm{d}x = \frac{1}{a}\operatorname{arctg}\left(\frac{x-x_0}{a}\right) + c,$

(10) $\int \frac{1}{\sqrt{a^2-x^2}}\,\mathrm{d}x = \arcsin\frac{x}{a} + c,$

(11) $\int \frac{1}{\sqrt{x^2+a}}\,\mathrm{d}x = \ln|x + \sqrt{x^2+a}| + c,$

(12) $\int \frac{1}{\cos^2 x}\,\mathrm{d}x = \tan x + c,$

(13) $\int \frac{1}{\sin^2 x}\,\mathrm{d}x = -\operatorname{cotg} x + c,$

(14) $\int \frac{f'(x)}{f(x)}\,\mathrm{d}x = \ln|f(x)| + c.$

In the above table a means any non-zero number, apart from Formula (5). Number $c \in \mathbb{R}$ is the integration constant. The formulas are valid on intervals on which both sides are defined and x is an independent variable.

Example 5.3. Evaluate the indefinite integrals

a) $\int x^3\,\mathrm{d}x,$

b) $\int \frac{1}{x^2}\,\mathrm{d}x,$

c) $\int \left(\sqrt{x} + \frac{1}{x-1} + 2\right)\,\mathrm{d}x,$

d) $\int \frac{1}{2x-5}\,\mathrm{d}x,$

e) $\int \tan x\,\mathrm{d}x,$

f) $\int \tan^2 x\,\mathrm{d}x,$

g) $\int \frac{x+1}{x^2+2x+9}\,\mathrm{d}x,$

h) $\int \frac{1}{x^2+9}\,\mathrm{d}x,$

i) $\int \frac{1}{\sqrt{9-x^2}}\,\mathrm{d}x,$

j) $\int \frac{x^4}{x^2+9}\,\mathrm{d}x.$

Solution. a) Using formula (2) from the table above we obtain

$$\int x^3\,\mathrm{d}x = \frac{x^4}{4} + c.$$

b) Rewriting the integral and using formula (2) we have

$$\int \frac{1}{x^2}\,\mathrm{d}x = \int x^{-2}\,\mathrm{d}x = \frac{x^{-1}}{-1} + c = -\frac{1}{x} + c.$$

c) Dividing the integral by the Theorem 5.3 and using the formulas (2), (14) and (1) we get

$$\begin{aligned}\int\left(\sqrt{x}+\frac{1}{x-1}+2\right)\,\mathrm{d}x &= \int x^{\frac{1}{2}}\,\mathrm{d}x+\int\frac{1}{x-1}\,\mathrm{d}x+2\int\,\mathrm{d}x=\\ &= \frac{2}{3}x^{\frac{3}{2}}+\ln|x-1|+2x+c.\end{aligned}$$

d) We rewrite the integral and then use the formula (14)

$$\int\frac{1}{2x-5}\,\mathrm{d}x=\frac{1}{2}\int\frac{2}{2x-5}\,\mathrm{d}x=\frac{1}{2}\ln|2x-5|+c.$$

e) We apply the trigonometric identity $\tan x=\frac{\sin x}{\cos x}$ to be able to use formula (14)

$$\int\tan x\,\mathrm{d}x=\int\frac{\sin x}{\cos x}\,\mathrm{d}x=-\int\frac{-\sin x}{\cos x}\,\mathrm{d}x=-\ln|\cos x|+c.$$

f) Using trigonometric identities and formulas (1) and (12) we have

$$\begin{aligned}\int\tan^2 x\,\mathrm{d}x &= \int\frac{\sin^2 x}{\cos^2 x}\,\mathrm{d}x=\int\frac{1-\cos^2 x}{\cos^2 x}\,\mathrm{d}x=\\ &= \int\frac{1}{\cos^2 x}\,\mathrm{d}x-\int\,\mathrm{d}x=\tan x-x+c.\end{aligned}$$

g) Since $(x^2+2x+9)'=2x+2$ we multiply the integral by $\frac{1}{2}$ and rewrite the integrand. Afterwards we use the formula (14)

$$\int\frac{x+1}{x^2+2x+9}\,\mathrm{d}x=\frac{1}{2}\int\frac{2x+2}{x^2+2x+9}\,\mathrm{d}x=\frac{1}{2}\ln|x^2+2x+9|+c\,.$$

h) We can use the formula (9) for $x_0=0$ and $a=3$

$$\int\frac{1}{x^2+9}\,\mathrm{d}x=\frac{1}{3}\operatorname{arctg}\frac{x}{3}+c\,.$$

i) We use the formula (10) for $a=3$

$$\int\frac{1}{\sqrt{9-x^2}}\,\mathrm{d}x=\arcsin\frac{x}{3}+c\,.$$

j) We rewrite the given function as a sum of polynomial and proper rational function and use formulas (1), (2) and (9)

$$\int\frac{x^4}{x^2+9}\,\mathrm{d}x=\int\left(x^2-9+\frac{81}{x^2+9}\right)=\frac{x^3}{3}-9x+27\operatorname{arctg}\frac{x}{3}+c\,.$$

□

Example 5.4. Decide whether the following functions are the antiderivatives of the same function.

$$f\colon y = \sin^2 x, \qquad g\colon y = -\frac{1}{2}\cos 2x.$$

Solution. We show two different methods of solution.

a) Using the definition, i.e. we calculate the derivatives

$$(\sin^2 x)' = 2\sin x\cos x = \sin 2x, \quad \left(-\frac{1}{2}\cos 2x\right)' = -\frac{1}{2}(-\sin 2x)2 = \sin 2x.$$

That means that given functions are antiderivatives of the same function.

b) We will show that given functions differ only in constant.

$$\sin^2 x - \left(-\frac{1}{2}\cos 2x\right) = \sin^2 x + \frac{1}{2}(\cos^2 x - \sin^2 x) = \frac{1}{2}.$$

□

5.2 Basic Integration Methods

Two principal integration methods will be introduced in this chapter – integration by parts and the substitution method.

Integration by Parts

Integration by parts is a technique for replacing difficult integrals by ones that are usually easier to integrate. The formula for integration by parts comes from the product rule,

$$(uv)' = u'v + uv'.$$

By integrating we get

$$uv = \int u'v + \int uv'.$$

Theorem 5.4. *Let $u(x)$ and $v(x)$ be functions differentiable on the interval I. Then the equality*

$$\int u(x)v'(x)\,\mathrm{d}x = u(x)v(x) - \int u'(x)v(x)\,\mathrm{d}x. \tag{5.4}$$

holds if at least one of the integrals involved exists.

Choosing u and v'

In general, we choose for u something that becomes simpler when differentiated and for v' something whose integral is simple.

Most of the functions that are suitable for integration by parts can be divided into two groups.

1. In the first group a *polynomial is differentiated* and the second factor is integrated. The new integral is the product of the polynomial of degree smaller by one and of the second function, which is of a type similar to that of the original integral (exponential function e^{ax} is kept, sine and cosine are exchanged).
   Examples:
$$\int P(x)e^{ax}\,\mathrm{d}x, \quad \int P(x)\sin ax\,\mathrm{d}x, \quad \int P(x)\cos ax\,\mathrm{d}x.$$
2. In the second group a *polynomial is integrated* and the second factor is differentiated. The reverse choice would not be possible, because we do not yet know how to integrate a logarithm, inverse sine, etc.
   Examples:
$$\int P(x)\ln x\,\mathrm{d}x, \quad \int P(x)\ \mathrm{arctg}\,ax\,\mathrm{d}x, \quad \int P(x)\arcsin ax\,\mathrm{d}x.$$

Example 5.5. Evaluate the indefinite integrals

a) $\int x\cos x\,\mathrm{d}x$, b) $\int x^2\mathrm{e}^x\,\mathrm{d}x$,

c) $\int \mathrm{arctg}\,x\,\mathrm{d}x$, d) $\int \mathrm{e}^x\sin x\,\mathrm{d}x$.

Solution. a) The integral is of the first type, thus $v' = \cos x$ will be integrated and the polynomial function $u = x$ will be differentiated. Then $u' = 1$, $v = \sin x$ and by substituting we get
$$\int x\cos x\,\mathrm{d}x = x\sin x - \int \sin x\,\mathrm{d}x = x\sin x + \cos x + c.$$

b) Sometimes we have to use integration by parts more than once to obtain an answer. Here is an example.
We choose $u = x^2$, $v' = \mathrm{e}^x$. Then $u' = 2x$, $v = \mathrm{e}^x$ and we have
$$\int x^2\mathrm{e}^x\,\mathrm{d}x = x^2\mathrm{e}^x - \int 2x\mathrm{e}^x\,\mathrm{d}x.$$
The integral on the right side now requires another integration by parts: $u = 2x$, $v' = \mathrm{e}^x$. We obtain $u' = 2$, $v = \mathrm{e}^x$ and
$$\begin{aligned}\int x^2\mathrm{e}^x\,\mathrm{d}x &= x^2\mathrm{e}^x - \int 2x\mathrm{e}^x\,\mathrm{d}x = x^2\mathrm{e}^x - 2x\mathrm{e}^x + 2\int \mathrm{e}^x\,\mathrm{d}x \\ &= x^2\mathrm{e}^x - 2x\mathrm{e}^x + 2\mathrm{e}^x + c = (x^2 - 2x + 2)\mathrm{e}^x + c.\end{aligned}$$

c) It seems that there is no product in this integral. However, we can rewrite it as
$$\mathrm{arctg}\,x = 1\cdot\mathrm{arctg}\,x.$$

Therefore $u = \operatorname{arctg} x$, $v' = 1$ and we get $u' = \frac{1}{1+x^2}$, $v = x$. Thus

$$\begin{aligned} \int \operatorname{arctg} x \, \mathrm{d}x &= x \operatorname{arctg} x - \int \frac{x}{1+x^2} \, \mathrm{d}x \\ &= x \operatorname{arctg} x - \frac{1}{2} \int \frac{2x}{1+x^2} \, \mathrm{d}x \\ &= x \operatorname{arctg} x - \frac{1}{2} \ln(1+x^2) + c. \end{aligned}$$

d) After using the integration-by-parts method, we can sometimes attain an expression that again contains the original unknown integral. In this case, we can understand the whole expression as an equation from which we find the unknown integral.
We choose $u = \sin x$, $v' = \mathrm{e}^x$. Then $u' = \cos x$, $v = \mathrm{e}^x$ and

$$\int \mathrm{e}^x \sin x \, \mathrm{d}x = \mathrm{e}^x \sin x - \int \mathrm{e}^x \cos x \, \mathrm{d}x.$$

Similarly $u = \cos x$ and $v' = \mathrm{e}^x$. Then $u' = -\sin x$, $v = \mathrm{e}^x$ and

$$\int \mathrm{e}^x \sin x \, \mathrm{d}x = \mathrm{e}^x \sin x - \mathrm{e}^x \cos x - \int \mathrm{e}^x \sin x \, \mathrm{d}x$$

from which we find

$$\int \mathrm{e}^x \sin x \, \mathrm{d}x = \frac{1}{2}(\mathrm{e}^x \sin x - \mathrm{e}^x \cos x) + c.$$

□

The Substitution Method

The substitution method of integration is derived from the derivative chain rule.

Theorem 5.5. *Let function f have an antiderivative F on interval J. Let function $t = \varphi(x)$ be differentiable on I and $\varphi(x) \in J$ for $x \in I$. Then composite function $f(\varphi(x))\varphi'(x)$ has an antiderivative on interval I and the equality*

$$\int f(\varphi(x))\varphi'(x) \, \mathrm{d}x = F(\varphi(x)) + c$$

holds.

To evaluate the integral $\int f(\varphi(x))\varphi'(x) \, \mathrm{d}x$ we carry out the following steps:

1. Substitute $t = \varphi(x)$ and $\mathrm{d}t = \varphi'(x) \, \mathrm{d}x$ to obtain the integral $\int f(t) \, \mathrm{d}t$.

2. Integrate with respect to t.

3. Replace t by $\varphi(x)$ in the result.

That means

$$\int f(\varphi(x))\varphi'(x)\,\mathrm{d}x = \int f(t)\,\mathrm{d}t = F(t) + c = F(\varphi(x)) + c. \tag{5.5}$$

Example 5.6. Evaluate

a) $\int (3x-4)^7\,\mathrm{d}x$,

b) $\int \frac{x}{x^2+1}\,\mathrm{d}x$,

c) $\int 10x(x^2+13)^{12}\,\mathrm{d}x$,

d) $\int \frac{\ln^2 x}{x}\,\mathrm{d}x$,

e) $\int \cos^3 x\,\mathrm{d}x$,

f) $\int x\sqrt{1-x^2}\,\mathrm{d}x$.

Solution. a) We substitute $t = 3x-4$. To find $\mathrm{d}x$ we have to differentiate the equation for substitution and we attain

$$\mathrm{d}t = 3\,\mathrm{d}x.$$

Therefore,

$$\mathrm{d}x = \frac{1}{3}\,\mathrm{d}t.$$

Then

$$\int (3x-4)^7\,\mathrm{d}x = \int t^7\,\frac{\mathrm{d}t}{3} = \frac{t^8}{24} + c = \frac{(3x-4)^8}{24} + c.$$

b) We use substitution $t = x^2+1$. Then $\mathrm{d}t = 2x\,\mathrm{d}x$, $\mathrm{d}x = \frac{\mathrm{d}t}{2x}$, thus

$$\int \frac{x}{x^2+1}\,\mathrm{d}x = \int \frac{x}{t}\,\frac{\mathrm{d}t}{2x} = \frac{1}{2}\int \frac{1}{t}\,\mathrm{d}t = \frac{1}{2}\ln|t| + c = \frac{1}{2}\ln|x^2+1| + c.$$

c) We substitute $t = x^2+13$. Then $\mathrm{d}t = 2x\,\mathrm{d}x$ and $\mathrm{d}x = \frac{\mathrm{d}t}{2x}$. Therefore,

$$\int 10x(x^2+13)^{12}\,\mathrm{d}x = \int 10x\cdot t^{12}\cdot\frac{\mathrm{d}t}{2x} = 5\int t^{12}\,\mathrm{d}t = \frac{5}{13}t^{13} + c = \frac{5}{13}(x^2+13)^{13} + c.$$

d) We choose the substitution $t = \ln x$. Then $\mathrm{d}t = \frac{1}{x}\,\mathrm{d}x$, $\mathrm{d}x = x\,\mathrm{d}t$ and

$$\int \frac{\ln^2 x}{x}\,\mathrm{d}x = \int \frac{t^2}{x}x\,\mathrm{d}t = \int t^2\,\mathrm{d}t = \frac{1}{3}t^3 + c = \frac{1}{3}\ln^3 x + c.$$

e) Substituting $t = \sin x$ we get $\mathrm{d}t = \cos x\,\mathrm{d}x$ and $\mathrm{d}x = \frac{\mathrm{d}t}{\cos x}$. We use the trigonometric identity $\cos^2 x = 1 - \sin^2 x$ and obtain

$$\begin{aligned}\int \cos^3 x\,\mathrm{d}x &= \int \cos^3 x\,\frac{\mathrm{d}t}{\cos x} = \int \cos^2 x\,\mathrm{d}t = \int (1-\sin^2 x)\,\mathrm{d}t = \\ &= \int (1-t^2)\,\mathrm{d}t = t - \frac{t^3}{3} + c = \sin x - \frac{\sin^3 x}{3} + c.\end{aligned}$$

f) We substitute $t = 1 - x^2$. Then $\mathrm{d}t = -2x\,\mathrm{d}x$ and $\mathrm{d}x = -\frac{\mathrm{d}t}{2x}$. Thus

$$\int x\sqrt{1-x^2}\,\mathrm{d}x = \int x\sqrt{t}\,\frac{\mathrm{d}t}{-2x} = -\frac{1}{2}\int t^{\frac{1}{2}}\,\mathrm{d}t = -\frac{1}{3}t^{\frac{3}{2}} + c = -\frac{1}{3}(1-x^2)^{\frac{3}{2}} + c.$$

$\square$

Note:
We can also use a "converse" substitution, i.e. $x = \psi(t)$. However, this substitution is not so common. We will show the method for integral

$$\int \sqrt{a^2 - x^2}\,dx.$$

We use the substitution $x = a\sin t$. This integral is used when calculating the surface of a circle.

Example 5.7. Evaluate $\int \sqrt{1-x^2}\,\mathrm{d}x$.

Solution. First we substitute $x = \sin t$, $\mathrm{d}x = \cos t\,\mathrm{d}t$

$$\int \sqrt{1-x^2}\,\mathrm{d}x = \int \sqrt{1-\sin^2 t}\cdot\cos t\,\mathrm{d}t = \int \sqrt{\cos^2 t}\cos t\mathrm{d}t = \int \cos^2 t\,\mathrm{d}t.$$

Then we apply the trigonometric identity $\cos^2 t = \frac{1+\cos 2t}{2}$ and get

$$\begin{aligned}\int \cos^2 t\,\mathrm{d}t &= \int \frac{1+\cos 2t}{2}\,\mathrm{d}t = \frac{1}{2}\left(t + \frac{1}{2}\sin 2t\right) + c = \frac{1}{2}(t + \sin t\cos t) + c = \\ &= \frac{1}{2}(t + \sin t\sqrt{1-\sin^2 t}) + c = \frac{1}{2}\left(\arcsin x + x\sqrt{1-x^2}\right) + c.\end{aligned}$$

$\square$

5.3 Integration of Rational Functions

Rational function is a function in the form

$$R(x) = \frac{P(x)}{Q(x)},$$

where P, Q are polynomials.

The basic method for preparing rational functions for integration is the method of partial fractions (see chapter 2.3).
In general,

- if $R(x)$ is an improper rational function, we divide the polynomial $P(x)$ by $Q(x)$ and rewrite it as a sum of polynomial and proper rational function.
  To integrate a polynomial we use formulas (1) and (2) from the table on page 28.

- if $R(x)$ is a proper rational function, we decompose it into partial fractions, which might be of two types:

$$\frac{M}{(x-\alpha)^k}, \qquad \text{or} \qquad \frac{Ax+B}{(ax^2+bx+c)^k}$$

  depending on the type of roots of polynomial Q.

Integration of Partial Fractions with Real Roots in the Denominator

If number α is a simple real root of polynomial Q, then the rational function R can be written in the form

$$\frac{A}{x-\alpha}. \tag{5.6}$$

We use the formula (14) to integrate the fraction (5.6) and obtain

$$\int \frac{A}{x-\alpha}\,\mathrm{d}x = A\ln|x-\alpha| + c.$$

If number α is a k-tuple real root of Q, the rational function R can be written as a sum of k partial fractions

$$\frac{A}{x-\alpha} + \frac{B}{(x-\alpha)^2} + \cdots + \frac{K}{(x-\alpha)^k}.$$

To integrate these fractions we use (5.6) and for $k \geq 2$ formula (2) on page 28, by which

$$\int \frac{M}{(x-\alpha)^k}\,\mathrm{d}x = M\int (x-\alpha)^{-k}\,\mathrm{d}x = M\frac{(x-\alpha)^{-k+1}}{-k+1} + c = \frac{M}{(1-k)(x-\alpha)^{k-1}} + c.$$

Integration of Partial Fractions with Complex Roots in the Denominator

If numbers $\alpha \pm i\beta$ are complex conjugate simple roots of polynomial Q then the partial fraction is

$$\frac{Ax+B}{ax^2+bx+c},$$

where ax^2+bx+c has roots $\alpha \pm i\beta$. We use formulas (9) and (14) on page 28 for integration.

If numbers $\alpha \pm i\beta$ are complex conjugate double roots of Q, then R contains two partial fractions

$$\frac{Ax+B}{ax^2+bx+c} + \frac{Cx+D}{(ax^2+bx+c)^2}.$$

Example 5.8. Evaluate the integrals:

a) $\int \frac{2x^2+6x-2}{x(x+2)(x-1)}\,\mathrm{d}x$, b) $\int \frac{x^2+2x+6}{(x-1)^3}\,\mathrm{d}x$,

c) $\int \frac{x-1}{x^4+3x^2+2}\,\mathrm{d}x$, d) $\int \frac{1}{x^3+1}\,\mathrm{d}x$.

Solution. a) We decompose the function into partial fractions and we use the formula $\int \frac{f'(x)}{f(x)}\,\mathrm{d}x = \ln|f(x)| + c$:

$$\begin{aligned} \int \frac{2x^2+6x-2}{x(x+2)(x-1)}\,\mathrm{d}x &= \int \left(\frac{1}{x} + \frac{2}{x-1} - \frac{1}{x+2}\right)\mathrm{d}x = \\ &= \ln|x| + 2\ln|x-1| - \ln|x+2| + c. \end{aligned}$$

b) We decompose the function and substitute $t = x - 1$. Then $\mathrm{d}x = \mathrm{d}t$ and

$$\begin{aligned} \int \frac{x^2+2x+6}{(x-1)^3}\,\mathrm{d}x &= \int \left(\frac{9}{(x-1)^3} - \frac{4}{(x-1)^2} + \frac{1}{x-1}\right)\mathrm{d}x = \int \left(\frac{9}{t^3} - \frac{5}{t^2} + \frac{6}{t}\right)\mathrm{d}t = \\ &= -\frac{9}{2t^2} + \frac{5}{t} + 6\ln|t| + c = -\frac{9}{2(x-1)^2} + \frac{5}{x-1} + 6\ln|x-1| + c. \end{aligned}$$

c) We decompose the given function into partial fractions

$$\int \frac{x-1}{x^4+3x^2+2}\,\mathrm{d}x = \int \frac{x-1}{x^2+1}\,\mathrm{d}x - \int \frac{x-1}{x^2+2}\,\mathrm{d}x.$$

We evaluate each of the integrals separately. We first multiply each integrand by a constant so that the numerator would be equal to the derivative of denominator

$$\int \frac{x-1}{x^2+1}\,\mathrm{d}x = \frac{1}{2}\int \frac{2x-2}{x^2+1}\,\mathrm{d}x, \qquad \int \frac{x-1}{x^2+2}\,\mathrm{d}x = \frac{1}{2}\int \frac{2x-2}{x^2+2}\,\mathrm{d}x$$

and divide into two fractions

$$\begin{aligned} \int \frac{x-1}{x^2+1}\,\mathrm{d}x &= \frac{1}{2}\left(\int \frac{2x}{x^2+1}\,\mathrm{d}x - 2\int \frac{1}{x^2+1}\,\mathrm{d}x\right), \\ \int \frac{x-1}{x^2+2}\,\mathrm{d}x &= \frac{1}{2}\left(\int \frac{2x}{x^2+2}\,\mathrm{d}x - 2\int \frac{1}{x^2+2}\,\mathrm{d}x\right). \end{aligned}$$

We use formula $\int \frac{f'(x)}{f(x)}\,\mathrm{d}x = \ln|f(x)| + c$ for the first integrals and formula (9) for the second ones, i.e. $\int \frac{\mathrm{d}x}{(x-x_0)^2+a^2} = \frac{1}{a}\operatorname{arctg}\frac{x-x_0}{a} + c$:

$$\begin{aligned} \int \frac{x-1}{x^2+1}\,\mathrm{d}x &= \frac{1}{2}\ln|x^2+1| - \operatorname{arctg} x, \\ \int \frac{x-1}{x^2+2}\,\mathrm{d}x &= \frac{1}{2}\ln|x^2+2| + \frac{1}{\sqrt{2}}\operatorname{arctg}\frac{x}{\sqrt{2}}. \end{aligned}$$

Altogether

$$\int \frac{x-1}{x^4+3x^2+2}\,\mathrm{d}x = \frac{1}{2}\ln|x^2+1| - \operatorname{arctg} x - \frac{1}{2}\ln|x^2+2| - \frac{1}{\sqrt{2}}\operatorname{arctg}\frac{x}{\sqrt{2}} + c.$$

d) We decompose integrand into partial fractions

$$\int \frac{1}{x^3+1}\,\mathrm{d}x = \frac{1}{3}\int \frac{1}{x+1}\,\mathrm{d}x - \frac{1}{3}\int \frac{x-2}{x^2-x+1}\,\mathrm{d}x.$$

And then

$$\frac{1}{3}\int \frac{1}{x+1}\,\mathrm{d}x = \frac{1}{3}\ln|x+1|$$

$$\begin{aligned}
\int \frac{x-2}{x^2-x+1}\,\mathrm{d}x &= \frac{1}{2}\int \frac{2x-4}{x^2-x+1}\,\mathrm{d}x = \frac{1}{2}\int \frac{2x-1-3}{x^2-x+1}\,\mathrm{d}x = \\
&= \frac{1}{2}\int \frac{2x-1}{x^2-x+1}\,\mathrm{d}x - \frac{3}{2}\int \frac{1}{x^2-x+1}\,\mathrm{d}x = \\
&= \frac{1}{2}\int \frac{2x-1}{x^2-x+1}\,\mathrm{d}x - \frac{3}{2}\int \frac{1}{(x-\frac{1}{2})^2+\frac{3}{4}}\,\mathrm{d}x = \\
&= \frac{1}{2}\ln|x^2-x+1| + \frac{3}{\sqrt{3}}\operatorname{arctg}\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\,.
\end{aligned}$$

Altogether

$$\int \frac{1}{x^3+1}\,\mathrm{d}x = \frac{1}{3}\ln|x+1| - \frac{1}{6}\ln|x^2-x+1| - \frac{\sqrt{3}}{3}\operatorname{arctg}\left(\frac{2}{\sqrt{3}}x - \frac{1}{\sqrt{3}}\right) + c\,.$$

$\square$

Chapter 6

Definite Integral

The previous chapter deals with the concept of the indefinite integral, which assign a set of functions (varying in constant) to a function. The definite integral, on the contrary, assigns a number to a function.

The definition of a definite integral is based on the requirement of the meaning of the area of the region between the graph of a nonnegative continuous function $y = f(x)$ and an interval $a \leq x \leq b$.

6.1 Definition and Basic Properties of Definite Integral

When we define the definite integral, we can use either antiderivative (*Newton-Leibniz formula*) or summation definition (*Riemann definite integral*). Since both definitions coincide for continuous functions, we will show only the Newton-Leibniz formula.

Definition 6.1. Let function f be continuous on the interval $[a; b]$. The **definite integral** of function f over $[a; b]$ is a number

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a), \tag{6.1}$$

where F is the antiderivative of function f on interval $[a, b]$.

Note:

i) We often write $\big[F(x)\big]_a^b$ instead of $F(b) - F(a)$, i.e.

$$\int_a^b f(x)\,\mathrm{d}x = \Big[F(x)\Big]_a^b.$$

ii) The existence of the antiderivative F is assured for continuous function by Theorem 5.2. The number $\int_a^b f(x)\,\mathrm{d}x$ is unique which can be proved by using Theorem 5.1. The theorem says that any two functions that have f as a derivative on $[a; b]$ must differ by some fixed constant throughout $[a; b]$. Therefore,

$$\int_a^b f(x)\,\mathrm{d}x = F(b) + c - (F(a) + c) = F(b) - F(a).$$

iii) The number a is a **lower limit**, the number b is a **upper limit**, the interval $[a;b]$ is an **integration domain**, and the function f is an **integrand**. The common name for lower and upper limits is **limits of integration**.

Geometrical meaning
Let a function f be nonnegative and continuous function on interval $[a;b]$. The geometrical interpretation of definite integral is the area P of the region beneath the graph of f from a to b (see fig. 6.1.). This region is called a **subgraph** of function f.

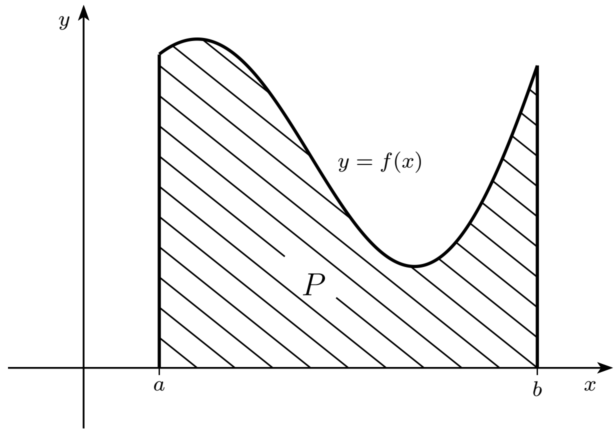


Figure 6.1: Geometrical meaning of definite integral

The definition is based on approximation of the region with inscribed rectangles.
The technique is:

a) We divide the interval $[a, b]$ into n intervals $[x_{i-1}, x_i]$ (called **intervals of partition**) of the same length so that $x_0 = a$ and $x_n = b$. The norm of partition is

$$x_i - x_{i-1} = (b-a)/n.$$

b) We approximate each interval with an inscribed rectangle with sides of the length $(x_i - x_{i-1})$ and $f(c_i)$ where c_i lies inside the interval of partition. The area of the rectangle, its height times its base length, is

$$f(c_i)(x_i - x_{i-1}).$$

The sum of all these areas provides the estimate of the area P, i.e.

$$P \approx \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}) = \sum_{i=1}^{n} f(c_i)\frac{b-a}{n}.$$

The higher number n is, the more accurate the area value we get.

c) When $n \to \infty$, we get an exact value of the area

$$P = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\frac{b-a}{n}. \tag{6.2}$$

This number is called a definite integral and is denoted $\int_a^b f(x)\,\mathrm{d}x$.

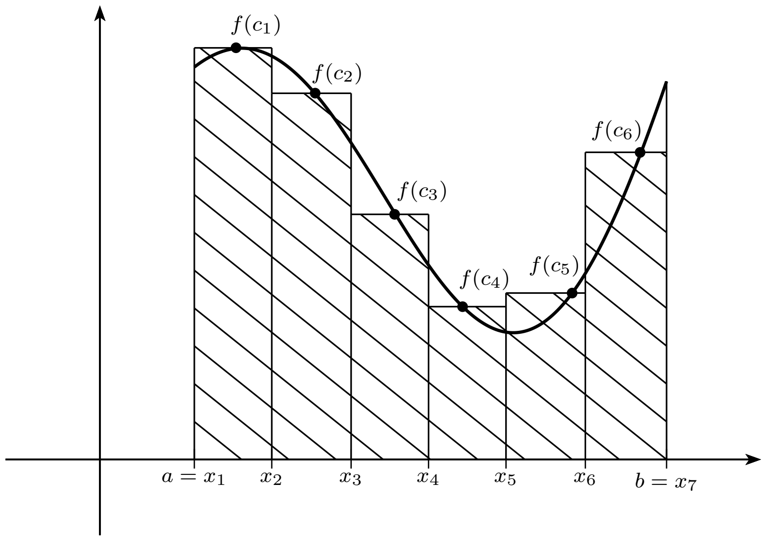


Figure 6.2: Approximating the area with rectangles

Example 6.1. Find the area bounded by the parabola $y = x^2$, x-axis and line $x = a$. What are odds of $S_1 : S_2$ (see fig. 6.3)?

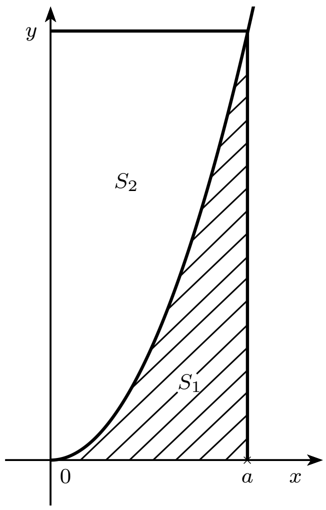


Figure 6.3: Parabola and rectangular

Solution. To find the area S_1 under the parabola, we calculate the definite integral

$$S_1 = \int_0^a x^2 \mathrm{d}x = \left[\frac{x^3}{3}\right]_0^a = \frac{a^3}{3}.$$

The area of rectangle is $a \cdot a^2 = a^3$ and therefore the area above the parabola is $S_2 = \frac{2a^3}{3}$. The odds are $S_1 : S_2 = 1 : 2$.

□

Example 6.2. Evaluate the definite integrals

$$\text{a)} \quad \int_0^{\pi} \sin x \, \mathrm{d}x, \qquad\qquad \text{b)} \quad \int_0^{1} \frac{1}{x^2+1}.$$

Solution. From (6.1) we attain

$$\int_0^{\pi} \sin x \, \mathrm{d}x = \big[-\cos x\big]_0^{\pi} = -\cos\pi - (\cos 0) = -(-1) + 1 = 2.$$

Similarly,

$$\int_{-1}^{1} \frac{1}{x^2+1} \, \mathrm{d}x = \big[\operatorname{arctg} x\big]_{-1}^{1} = \operatorname{arctg} 1 - \operatorname{arctg}(-1) = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}.$$

□

Theorem 6.1 (Properties of the Definite Integral)**.** *If f and g are continuous functions on the interval $[a,b]$, then*

a) $\int_a^b [f(x)+g(x)] \, \mathrm{d}x = \int_a^b f(x) \, \mathrm{d}x + \int_a^b g(x) \, \mathrm{d}x$;

b) $\int_a^b cf(x) \, \mathrm{d}x = c \int_a^b f(x) \, \mathrm{d}x$;

c) $\int_a^b f(x) \, \mathrm{d}x = \int_a^c f(x) \, \mathrm{d}x + \int_c^b f(x) \, \mathrm{d}x$, *where* $a < c < b$;

d) $\int_a^b f(x) \, \mathrm{d}x \geq 0$, *if* $f(x) \geq 0$ *on interval* $[a,b]$;

e) $\int_a^b f(x) \, \mathrm{d}x \geq \int_a^b g(x) \, \mathrm{d}x$, *if* $f(x) \geq g(x)$ *on interval* $[a,b]$.

Note:

i) Formula c) can be used in case when a function f is continuous on $[a,c]$ and $[c,b]$ and is not continuous at point c. For example, function $\operatorname{sgn} x$ is not continuous at $x=0$, however, we can define the definite integral

$$\int_{-2}^{1} \operatorname{sgn} x \, \mathrm{d}x = \int_{-2}^{0} \operatorname{sgn} x \, \mathrm{d}x + \int_0^{1} \operatorname{sgn} x \, \mathrm{d}x = -2 + 1 = -1.$$

ii) For $a > b$ we define

$$\int_a^b f(x) \, \mathrm{d}x = -\int_b^a f(x) \, \mathrm{d}x,$$

and integral $\int_a^a f(x) \, \mathrm{d}x$ is

$$\int_a^a f(x) \, \mathrm{d}x = 0.$$

6.2 Integration Methods for the Definite Integral

Integration by Parts

Theorem 6.2 (Integration by Parts for the Definite Integrals)**.** *Let $u(x)$ and $v(x)$ have continuous derivatives on interval $[a, b]$. Then*

$$\int_a^b u(x)v'(x)\,\mathrm{d}x = \big[u(x)v(x)\big]_a^b - \int_a^b u'(x)v(x)\,\mathrm{d}x.$$

Example 6.3. Evaluate the integral

$$\int_1^{\mathrm{e}} x^3 \ln x \,\mathrm{d}x.$$

Solution. We integrate by parts and choose $u = \ln x$, $v' = x^3$. Then

$$u' = \frac{1}{x}, \qquad v = \frac{x^4}{4}$$

and we attain

$$\int_1^{\mathrm{e}} x^3 \ln x \,\mathrm{d}x = \left[\frac{x^4}{4}\ln x\right]_1^{\mathrm{e}} - \frac{1}{4}\int_1^{\mathrm{e}} x^3 \,\mathrm{d}x = \frac{\mathrm{e}^4}{4} - \frac{1}{4}\left[\frac{x^4}{4}\right]_1^{\mathrm{e}} = \frac{3\mathrm{e}^4 + 1}{16}.$$

□

Substitution Method

Theorem 6.3 (Substitution Method for the Definite Integral)**.** *Let function $f(t)$ be continuous on $[a, b]$. Let function $\varphi(x)$ have the continuous derivative on the interval $[\alpha, \beta]$ and $\varphi(x)$ maps the interval $[\alpha, \beta]$ into interval $[a, b]$. Then*

$$\int_\alpha^\beta f(\varphi(x))\varphi'(x)\,\mathrm{d}x = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(t)\,\mathrm{d}t.$$

Note: When using a substitution method for the definite integral, we have to find new limits of integration.

Example 6.4. Evaluate

$$\int_0^5 \frac{x}{\sqrt{1+3x}}\,\mathrm{d}x.$$

Solution.

We substitute

$$t = \sqrt{1+3x}.$$

Then $t^2 = 1 + 3x$. Hence $x = \frac{1}{3}(t^2 - 1)$ and $\mathrm{d}x = \frac{2}{3}t\,\mathrm{d}t$. To obtain new limits for variable t, we substitute $x = 0$ and $x = 5$ into the equation $t = \sqrt{1+3x}$. Thus we get $0 \rightsquigarrow 1$, $5 \rightsquigarrow 4$. Finally, we evaluate the definite integral with new limits of integration

$$\int_0^5 \frac{x}{\sqrt{1+3x}}\mathrm{d}x = \frac{2}{9}\int_1^4 (t^2 - 1)\mathrm{d}t = \frac{2}{9}\left[\frac{t^3}{3} - t\right]_1^4 = 4.$$

□

6.3 Applications of the Definite Integral

In this part we present several applications of the definite integral in geometry.

Area of a Plane Figure

Let function f be a continuous and nonnegative on the interval $[a, b]$. Then the area of subgraph of function f (see fig. 6.1) is given by the formula

$$P = \int_a^b f(x)\,\mathrm{d}x. \tag{6.3}$$

Example 6.5. Evaluate the area of disc with radius r.

Solution. The area of disc P is equal to the area of one quarter of the disc multiplied by four. That is

$$P = 4\int_0^r \sqrt{r^2 - x^2}\,dx.$$

We substitute $x = r\sin t$. Then $\mathrm{d}x = r\cos t\,\mathrm{d}t$ and new limits of integration are: $0 \rightsquigarrow 0$, $r \rightsquigarrow \pi/2$. We obtain

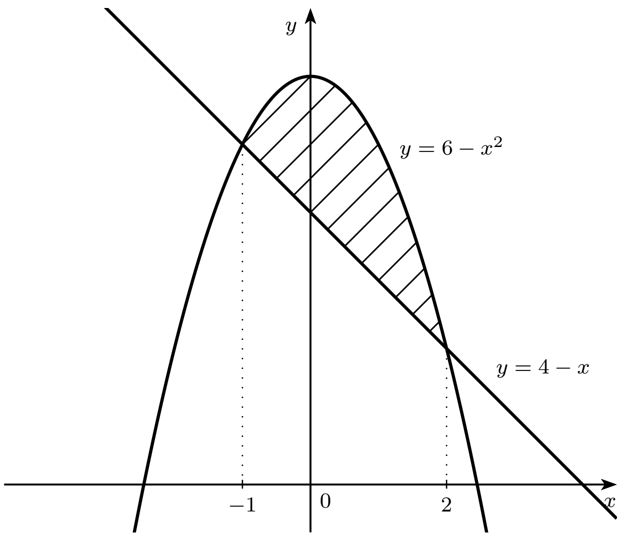
$$P = 4\int_0^r \sqrt{r^2 - x^2}\,\mathrm{d}x = 4r^2\int_0^{\frac{\pi}{2}} \sqrt{1 - \sin^2 t}\cos t\,\mathrm{d}t = 4r^2\int_0^{\frac{\pi}{2}} \sqrt{\cos^2 t}\cos t\,\mathrm{d}t = 4r^2\int_0^{\frac{\pi}{2}} \cos^2 t\,\mathrm{d}t.$$

Applying the formula $\cos^2 t = \frac{1+\cos 2t}{2}$ we get

$$P = 4r^2\int_0^{\frac{\pi}{2}} \frac{1+\cos 2t}{2}\,\mathrm{d}t = \frac{4r^2}{2}\left[t + \frac{1}{2}\sin 2t\right]_0^{\frac{\pi}{2}} = \pi r^2.$$

□

Example 6.6. Find the area of the region bounded by $y = 6 - x^2$ and $x + y - 4 = 0$.



Solution. We first find limits of integration, i.e. x-coordinates of points of intersection. We solve the equation

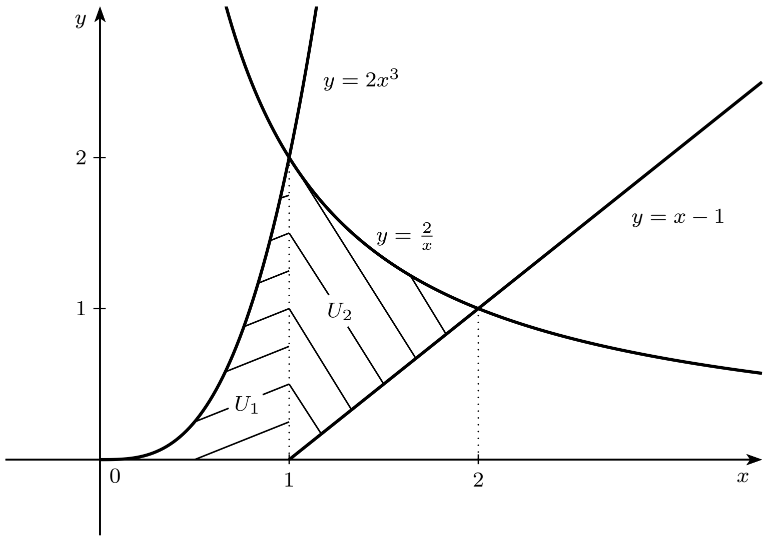
$$6 - x^2 = 4 - x,$$

and therefore $x = -1$ and $x = 2$. Thus the area of the region is

$$\int_{-1}^{2} [6 - x^2 - (4 - x)]\,\mathrm{d}x = \int_{-1}^{2} (2 + x - x^2)\,\mathrm{d}x = \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{2} = \frac{9}{2}.$$

□

Example 6.7. Evaluate the area of the region bounded by the graphs of the functions $y = 2x^3$, $y = \frac{2}{x}$ and $y = x - 1$ for $x \geq 0$.



Solution. We have to divide the region U into two parts U_1 and U_2, where $U_1 \cup U_2 = U$ (see figure above). We find points of intersection of boundary curves

$$2x^3 = \frac{2}{x} \quad \Rightarrow \quad x = 1 \qquad (x = -1 \text{ does not satisfy geometrically})$$

and

$$x - 1 = \frac{2}{x} \quad \Rightarrow \quad x = 2.$$

Therefore, the area of the given region $S(U)$ is

$$\begin{aligned} S(U) = S(U_1) + S(U_2) &= \int_0^1 2x^3\,\mathrm{d}x + \int_1^2 \left[\frac{2}{x} - (x - 1)\right]\mathrm{d}x = \\ &= \left[\frac{x^4}{2}\right]_0^1 + \left[2\ln x - \frac{x^2}{2} + x\right]_1^2 = \\ &= \frac{1}{2} + 2\ln 2 - 2 + 2 + \frac{1}{2} - 1 = 2\ln 2. \end{aligned}$$

□

Volume of a Solid of Revolution

Let function $y = f(x)$ be continuous nonnegative on interval $[a, b]$. The volume of a solid made by revolving a subgraph of function f

$$P = \{(x, y) : \ a \le x \le b, \ 0 \le y \le f(x)\}$$

about an x-axis is given by the formula

$$V = \pi \int_a^b f^2(x)\,\mathrm{d}x. \tag{6.4}$$

Example 6.8. Evaluate the volume of the cone with height v and a base radius r.

Solution. We get a cone by revolving a triangle bounded by the lines

$$y = 0, \quad y = kx + q \quad \text{and} \quad x = v$$

about the x-axis. Where $y = kx + q$ is equation for a line passing points $[0, 0]$ and $[v, r]$. Therefore the gradient of the line is $\frac{r}{v}$, $q = 0$ and the volume of the cone is

$$\begin{aligned} V &= \pi \int_0^v \left(\frac{r}{v}x\right)^2 \mathrm{d}x = \pi \frac{r^2}{v^2} \int_0^v x^2\,\mathrm{d}x = \\ &= \pi \frac{r^2}{v^2} \left[\frac{x^3}{3}\right]_0^v = \frac{1}{3}\pi r^2 v. \end{aligned}$$

□

Example 6.9. (*Revolving of catenary curve*) Find the volume generated by revolving the subgraph of function $y = \frac{1}{2}(\mathrm{e}^x + \mathrm{e}^{-x})$ about the x-axis for $x \in [0, a]$.

Solution. The volume of the solid is given by the formula (6.4) and therefore

$$\begin{aligned} V &= \pi \int_0^a \frac{1}{4}(\mathrm{e}^x + \mathrm{e}^{-x})^2\,\mathrm{d}x = \frac{\pi}{4} \int_0^a (\mathrm{e}^{2x} + 2 + \mathrm{e}^{-2x})\,\mathrm{d}x = \\ &= \frac{\pi}{4} \left[\frac{\mathrm{e}^{2x}}{2} + 2x - \frac{\mathrm{e}^{-2x}}{2}\right]_0^a = \frac{\pi}{4} \left(\frac{\mathrm{e}^{2a}}{2} + 2a - \frac{\mathrm{e}^{-2a}}{2}\right). \end{aligned}$$

□

Example 6.10. (*Volume of Torus*) Find the volume of solid generated by revolving the circle

$$x^2 + (y - 3)^2 = 1$$

about the x-axis.

Solution. We first divide the circle into two functions $f_1\colon y = 3 + \sqrt{1-x^2}$ and $f_2\colon y = 3 - \sqrt{1-x^2}$. The volume of generated solid is equal to the difference in volumes of solids generated by revolving the subgraphs of f_1 and f_2. Furthermore, both functions are symmetrical about the y-axis. Therefore,

$$\begin{aligned} V &= V_1 - V_2 = 2\pi \int_0^1 (3+\sqrt{1-x^2})^2 \,\mathrm{d}x - 2\pi \int_0^1 (3-\sqrt{1-x^2})^2 \,\mathrm{d}x = \\ &= 2\pi \int_0^1 (3+\sqrt{1-x^2})^2 - (3-\sqrt{1-x^2})^2 \,\mathrm{d}x = \\ &= 2\pi \int_0^1 [9+6\sqrt{1-x^2}+1-x^2-(9-6\sqrt{1-x^2})+1-x^2] \,\mathrm{d}x = \\ &= 24\pi \int_0^1 \sqrt{1-x^2}\,\mathrm{d}x = 6\pi^2. \end{aligned}$$

□

Length of a Curve

Let function f have a continuous derivative on interval $[a, b]$. The length of its graph is given by

$$L = \int_a^b \sqrt{1+(f'(x))^2}\,\mathrm{d}x. \tag{6.5}$$

Example 6.11. Evaluate the length of the curve $y = \ln x$ on interval $[\sqrt{3}, \sqrt{8}]$.

Solution. Since $(\ln x)' = \frac{1}{x}$ and the length of the curve is given by (6.5), we have

$$\int_{\sqrt{3}}^{\sqrt{8}} \sqrt{1+\frac{1}{x^2}}\,\mathrm{d}x = \int_{\sqrt{3}}^{\sqrt{8}} \frac{\sqrt{x^2+1}}{x}\,\mathrm{d}x.$$

To solve the obtained integral, we substitute $t^2 = x^2+1$. Then $2x\,\mathrm{d}x = 2t\,\mathrm{d}t$ and therefore $\mathrm{d}x = \frac{t}{x}\,\mathrm{d}t$. New limits of integration are $\sqrt{3} \rightsquigarrow 2$ and $\sqrt{8} \rightsquigarrow 3$. We attain

$$\int_2^3 \frac{\sqrt{t^2}}{x}\frac{t}{x}\,\mathrm{d}t = \int_2^3 \frac{t^2}{t^2-1}\,\mathrm{d}t.$$

Finally, we decompose the integrand into partial fractions and evaluate the integral:

$$\int_2^3 \left(1 + \frac{1}{2}\frac{1}{t-1} - \frac{1}{2}\frac{1}{t+1}\right)\mathrm{d}t = \left[t + \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|\right]_2^3 = 1 + \frac{1}{2}\ln\frac{3}{2}.$$

□

6.4 Improper Integrals

In the definition of the infinite integrals that we dealt with in the previous chapter, functions to be integrated had to satisfy two restrictions:

- the integration domain was a bounded closed interval,
- the function was bounded on this interval.

This section deals with improper integrals, whose integration domain is a closed interval unbounded from one side or whose integrand is an unbounded function.

Improper Integral on Unbounded Intervals

Definition 6.2. Let function f be continuous on interval $[a, \infty)$. If there exists a proper limit

$$\lim_{b\to\infty} \int_a^b f(x)\,\mathrm{d}x, \tag{6.6}$$

we say that **improper integral**

$$\int_a^\infty f(x)\,\mathrm{d}x$$

converges and assumes the value

$$\int_a^\infty f(x)\,\mathrm{d}x = \lim_{b\to\infty} \int_a^b f(x)\,\mathrm{d}x. \tag{6.7}$$

On the contrary, if the limit 6.6 is infinite or does not exist, the **improper integral diverges**.

Example 6.12. Evaluate the improper integrals

a) $\int_0^\infty \frac{1}{x^2+1}\,\mathrm{d}x$,	b) $\int_0^\infty e^{-x}\,\mathrm{d}x$,
c) $\int_1^\infty \frac{1}{x}\,\mathrm{d}x$,	d) $\int_1^\infty \frac{1}{x^3}\,\mathrm{d}x$,
e) $\int_1^\infty \frac{1}{\sqrt[3]{x}}\,\mathrm{d}x$,	f) $\int_0^\infty \sin x\,\mathrm{d}x$.

Solution.

a) First we evaluate

$$F(b) = \int_0^b \frac{1}{x^2+1}\,\mathrm{d}x = [\operatorname{arctg} x]_0^b = \operatorname{arctg} b.$$

Then

$$\lim_{b\to\infty} F(b) = \lim_{b\to\infty} \operatorname{arctg} b = \frac{\pi}{2}.$$

Therefore, the improper integral converges and its value is

$$\int_0^\infty \frac{1}{x^2+1}\,\mathrm{d}x = \frac{\pi}{2}.$$

b) Similarly, we evaluate

$$\lim_{b\to\infty}\int_0^b \mathrm{e}^{-x}\,\mathrm{d}x = \lim_{b\to\infty}\left[-\mathrm{e}^{-x}\right]_0^b = \lim_{b\to\infty}(-\mathrm{e}^{-b}+1) = 1.$$

Thus the integral converges and assumes the value

$$\int_0^\infty e^{-x}\,\mathrm{d}x = 1.$$

c) To find the value of improper integral, we find the limit

$$\lim_{b\to\infty}\int_1^b \frac{1}{x}\,\mathrm{d}x = \lim_{b\to\infty}[\ln x]_1^b = \lim_{b\to\infty}\ln b = \infty.$$

Therefore, the integral diverges.

d) This time we have

$$\lim_{b\to\infty}\int_1^b \frac{1}{x^3}\,\mathrm{d}x = \lim_{b\to\infty}\left[-\frac{1}{2x^2}\right]_1^b = \lim_{b\to\infty}\left(-\frac{1}{2b^2}+\frac{1}{2}\right) = \frac{1}{2}.$$

e) We evaluate

$$\lim_{b\to\infty}\int_1^b \frac{1}{\sqrt[3]{x}}\,\mathrm{d}x = \lim_{b\to\infty}\left[\frac{3}{2}x^{\frac{2}{3}}\right]_1^b = \frac{3}{2}\lim_{b\to\infty}\left(b^{\frac{2}{3}}-1\right) = \infty.$$

f) We first find

$$F(b) = \int_0^b \sin x\,\mathrm{d}x = \left[-\cos x\right]_0^b = -\cos b + 1.$$

Then

$$\lim_{b\to\infty} F(b) = 1 - \lim_{b\to\infty}\cos b$$

does not exist and the given integral diverges.

$\square$

Note:
In general, integral

$$\int_1^\infty \frac{1}{x^k}\,\mathrm{d}x$$

converges to $\frac{1}{k-1}$ for $k>1$ and diverges for $k\le 1$.

Improper Integral of Unbounded Functions

Definition 6.3. Let function f be continuous and unbounded on interval $(a, b]$. Then the point a is called a **singular point of function** f and the **improper integral** is defined by

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{c\to a+} \int_c^b f(x)\,\mathrm{d}x. \tag{6.8}$$

If the limit (6.8) is a proper limit, we say that the integral **converges**.
If the limit is infinite or does not exist, the improper integral **diverges**.

Example 6.13. Find the singular points and evaluate the improper integrals

a) $\int_0^1 \frac{1}{x}\,\mathrm{d}x$,

b) $\int_0^2 \frac{1}{2-x}\,\mathrm{d}x$,

c) $\int_0^1 \frac{1}{\sqrt{1-x}}\,\mathrm{d}x$,

d) $\int_0^1 \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x$.

Solution.

a) As we approach zero from the right side, function $\frac{1}{x}$ is increasing to infinity and $x = 0$ is a singular point. Using (6.8) we get

$$\int_0^1 \frac{1}{x}\,\mathrm{d}x = \lim_{c\to 0^+} \int_c^1 \frac{1}{x}\,\mathrm{d}x = \lim_{c\to 0^+} [\ln x]_c^1 = \lim_{c\to 0^+} (\ln 1 - \ln c) = \infty,$$

That means that integral diverges.

b) Singular is $x = 2$. From (6.8) we obtain

$$\int_0^2 \frac{1}{2-x}\,\mathrm{d}x = \lim_{c\to 2^-} \int_0^c \frac{1}{2-x}\,\mathrm{d}x = \lim_{c\to 2^-} [-\ln(2-x)]_0^c = \lim_{c\to 2^-} [-\ln(2-c) + \ln 2] = \infty,$$

Integral diverges.

c) Since the singular point is $x = 1$, we evaluate the limit

$$\lim_{c\to 1^-} \int_0^c \frac{1}{\sqrt{1-x}}\,\mathrm{d}x = \lim_{c\to 1^-} \left[-2\sqrt{1-x}\right]_0^c = \lim_{c\to 1^-} (-2\sqrt{1-c} + 2\sqrt{1}) = 2.$$

Therefore $\int_0^1 \frac{1}{\sqrt{1-x}}\,\mathrm{d}x = 2$.

d) Singular point is $x = 1$. We have

$$\lim_{c\to 1^-} \int_0^c \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \lim_{c\to 1^-} [\arcsin x]_0^c = \lim_{c\to 1^-} (\arcsin c - \arcsin 0) = \frac{\pi}{2}.$$

Thus $\int_0^1 \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \frac{\pi}{2}$.

□

Note:
Generally, the integral

$$\int_0^1 \frac{1}{x^k}\,\mathrm{d}x$$

diverges when $k \geq 1$ and converges to $\frac{1}{1-k}$ when $k < 1$.

Example 6.14. Evaluate

$$\text{a)} \quad \int_0^1 x \ln x \,\mathrm{d}x, \qquad \text{b)} \quad \int_{-1}^1 \frac{1}{x^2-1}\,\mathrm{d}x.$$

Solution. a) Singular point is $x = 0$. First we evaluate the definite integral

$$F(b) = \int_b^1 x \ln x \,\mathrm{d}x$$

using integration by parts, where $u = \ln x$ and $v' = x$. Therefore $u' = \frac{1}{x}$, $v = \frac{x^2}{2}$ and we get

$$\begin{aligned} \int_b^1 x \ln x \,\mathrm{d}x &= \left[\frac{x^2}{2}\ln x\right]_b^1 - \int_b^1 \frac{x}{2}\,\mathrm{d}x = \left[\frac{x^2}{2}\ln x\right]_b^1 - \left[\frac{x^2}{4}\right]_b^1 = \\ &= \frac{b^2}{4} - \frac{1}{4} - \frac{1}{2}b^2 \ln b. \end{aligned}$$

Then we evaluate the limit

$$\lim_{b\to 0^+} F(b) = \left(\lim_{b\to 0^+} \frac{b^2}{4} - \frac{1}{4} - \frac{1}{2}b^2 \ln b\right) = -\frac{1}{4} - \frac{1}{2}\lim_{b\to 0^+} b^2 \ln b.$$

Applying the L'Hospital's rule we attain

$$\lim_{b\to 0^+} b^2 \ln b = \lim_{b\to 0^+} \frac{\ln b}{\frac{1}{b^2}} = \lim_{b\to 0^+} \frac{\frac{1}{b}}{\frac{-2}{b^3}} = \lim_{b\to 0^+} (\frac{1}{-2}b^2) = 0.$$

Altogether

$$\int_0^1 x \ln x \,\mathrm{d}x = -\frac{1}{4}.$$

b) Since both limits of integration are singular points, we divide the given integral into two improper integrals whose one limit of integration is not a singular point:

$$\int_{-1}^1 \frac{1}{x^2-1}\,\mathrm{d}x = \int_{-1}^0 \frac{1}{x^2-1}\,\mathrm{d}x + \int_0^1 \frac{1}{x^2-1}\,\mathrm{d}x.$$

We evaluate the integrals separately and start with the second one. Using partial fractions we attain

$$\lim_{b\to 1^-}\int_0^b \frac{1}{x^2-1}\,\mathrm{d}x = \lim_{b\to 1^-}\frac{1}{2}\int_0^b \left(\frac{1}{x-1}-\frac{1}{x+1}\right)\mathrm{d}x = \frac{1}{2}\lim_{b\to 1^-}\left[\ln|x-1|-\ln|x+1|\right]_0^b =$$
$$= \frac{1}{2}\lim_{b\to 1^-}\left(\ln\left|\frac{b-1}{b+1}\right| - \ln|-1|\right) = \frac{1}{2}\lim_{b\to 1^-}\ln\left|\frac{b-1}{b+1}\right| = -\infty.$$

As this particular integral diverges, the given integral diverges as well and the behaviour of the first integral is not important.

□

Bibliography

[1] Z. Došlá. *Matematika pro chemiky, 1. díl*. 1st ed. Masaryk university, 2012. 116 p. ISBN 978-80-210-5263-5.

[2] Z. Došlá, P. Liška. *Matematika pro nematematické obory s aplikacemi v přírodních a technických vědách*. Grada Publishing, a.s., 2014. 304 p. ISBN 978-80-247-5322-5.

[3] G. B. Thomas Jr., R. L. Finney. *Calculus and Analytic Geometry*. 7th ed. Addison-Wesley Publishing Company, 1990. 1136 p. ISBN 0-201-16320-9.

[4] J. Kuben. 2011 *Anglická matematická terminologie / English mathematical terminology*. [online] Brno: Univerzita obrany. Avaible from: https://moodle.unob.cz/mod/resource/view.php?id=14910 [Accessed 20 December 2014]

[5] J. Kuben. 2012 *Differential Calculus for Functions of a Single Variable*. [online] Brno: Univerzita obrany. Avaible from: https://moodle.unob.cz/mod/resource/view.php?id=14911 [Accessed 20 December 2014]

[6] J. Kuben. 2011 *Integral Calculus for Functions of a Single Variable*. [online] Brno: Univerzita obrany. Avaible from: https://moodle.unob.cz/mod/resource/view.php?id=14912 [Accessed 20 December 2014]

[7] P. Urban, D. Martin, R. Haese, S. Haese, M. Haese, M. Humphries. *Mathematics for the international student, Mathematics HL (Core)*. 2nd ed. Haese & Harris Publications, 2008. 935 p. ISBN 978-1-876543-11-2.