Problem solving seminar I

1. Let T be a linear transformation of a vector space V into itself. Suppose that $v \in V$ is such that $T^m v = 0$, $T^{m-1}v \neq 0$ for some positive integer m. Show that $v, Tv, T^2 v, \dots, T^{m-1}v$ are linearly independent.

2. Let A be an $n \times n$ matrix over a field $\mathbb{K}$. Prove that
$$\operatorname{rank} A^2 - \operatorname{rank} A^3 \leq \operatorname{rank} A - \operatorname{rank} A^2.$$

3. (a) Prove that there is no continuous function from the closed interval $[0,1]$ onto the open interval $(0,1)$.
(b) Find a continuous surjective function from the open interval $(0,1)$ onto the closed interval $[0,1]$.
(c) Prove that no map from **(b)** is bijective.

4. Compute the 100th derivation of the function
$$\frac{x^2 + 1}{x^3 - x}.$$

5. Suppose that f is a continuous real function with period 1. Show that there is a real number x_0 such that
$$f(x_0 + \pi) = f(x_0).$$

Homework I. Find the limit
$$\lim_{x \to 0} \frac{\sin \tan x - \tan \sin x}{\arcsin \arctan x - \arctan \arcsin x}.$$