Problem solving seminar II

6. Let the real valued function f be defined in an open interval about the point a on the real line and be differentiable at a. Prove that if (x_n) is an increasing sequence and (y_n) is a decreasing sequence in the domain of f, and both sequences converge to a, then

$$\lim_{n\to\infty} \frac{f(y_n) - f(x_n)}{y_n - x_n} = f'(a).$$

7. Let $\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_n$ be distinct real numbers. Show that the n exponential functions $e^{\alpha_1 t},\ e^{\alpha_2 t},\ \ldots,\ e^{\alpha_n t}$ are linearly independent over the real numbers.

8. Let A and B be $n \times n$ complex unitary matrices. Prove that

$$|\det(A + B)| \leq 2^n.$$

9. Suppose that f is continuous real valued function. Show that

$$\int_0^1 f(x)x^2 dx = \frac{1}{3} f(\xi)$$

for some $\xi \in [0, 1]$.

Homework II. Does every positive polynomial in two variables take its minimum in the plane?