Problem solving seminar VI

21. Let $U \subseteq \mathbb{R}^n$ be an open set. Suppose that the map $h : U \to \mathbb{R}^n$ is a homeomorphism from U onto $\mathbb{R}^n$, which is uniformly continuous. Prove that $U = \mathbb{R}^n$.

22. Suppose that f maps the compact interval I into itself and that

$$|f(x) - f(y)| < |x - y|$$

for all $x, y \in I$, $x \neq y$. Can one conclude that there is some constant $M < 1$ such that

$$|f(x) - f(y)| < M|x - y|?$$

23. Let V be a finite dimensional vector space and A and B two linear transformations of V into itself such that $A^2 = B^2 = 0$ and $AB + BA = \mathrm{id}$.
(a) Prove that $\ker A = A(\ker B)$, $\ker B = B(\ker A)$ and $V = \ker A \oplus \ker B$.
(b) Prove that the dimension of V is even.

24. Prove that the group $G = \mathbb{Q}/\mathbb{Z}$ has no proper subgroup of finite index.

Homework VI.
(a) Let M be a compact metric space and let $f : M \to \mathbb{R}$ be an upper semicontinuous function. Prove that f is bounded from above and that it takes its maximum in a point of M.
(b) Let I and J be two metric spaces and let $g : I \times J \to \mathbb{R}$ be continuous and bounded from below. Prove that the function $f : I \to \mathbb{R}$ defined

$$f(x) = \inf\{g(x, y); y \in J\}$$

is upper semicontinuous.