

Problem solving seminar VII

25. Let $f : [0, \infty) \rightarrow \mathbb{R}$ be a function satisfying $f(1) = 1$ and

$$f'(x) = \frac{1}{x^2 + f(x)}.$$

Prove that

$$\lim_{x \rightarrow \infty} f(x)$$

exists and is less than $1 + \frac{\pi}{4}$.

26. Let G be the abelian group defined by generators x, y and z , and relations

$$15x + 3y = 0$$

$$3x + 7y + 4z = 0$$

$$18x + 14y + 8z = 0$$

- (1) Express G as a direct product of two cyclic groups. (A direct product is the same as a direct sum.)
- (2) Express G as a direct product of cyclic groups of prime power.
- (3) How many elements of G have order two?

27. Let X be a metric space and let V be a finite-dimensional subspace of the vector space of continuous real valued functions on X . Prove that there is a basis $\{f_1, f_2, \dots, f_n\}$ for V and points x_1, x_2, \dots, x_n in X such that

$$f_i(x_j) = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases}$$