Problem solving seminars IX and X

31. Let $f, g : [0,1] \to [0,\infty)$ be continuous functions safisfying

$$\sup\{f(x);\ x \in [0,1]\} = \sup\{g(x);\ x \in [0,1]\}.$$

Prove that there exists $t \in [0,1]$ with

$$f^2(t) + 3f(t) = g^2(t) + 3g(t).$$

32. Let R be the ring of matrices of the form

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}.$$

What are all 2-sided ideals in R?

33. Let A be the 3×3 matrix

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

Determine all real numbers a for which

$$\lim_{n \to \infty} a^n A^n$$

exists and is nonzero as a matrix.

34. Let E be the set of all continuous functions $u : [0,1] \to \mathbb{R}$ satisfying

$$|u(x) - u(y)| \le |x - y| \text{ for } x, y \in [0,1], \quad u(0) = 0.$$

Let $\varphi : E \to \mathbb{R}$ be defined by

$$\varphi(u) = \int_0^1 (u(x)^2 - u(x))dx.$$

Show that φ achieves its maximum value on some element of E.

35. Let $M_{n\times n}(\mathbb{F})$ be the ring of $n \times n$ matrices over a field $\mathbb{F}$. Prove that it has no 2-sided ideals except $M_{n\times n}(\mathbb{F})$ and $\{0\}$.

36. Find all left ideals of the ring $M_{n\times n}(\mathbb{F})$.