1. Topological spaces

Definition 1.1. Let X be a set. A *topology* on X is a collection $\mathcal{T} \subseteq \mathcal{P}(X)$ of subsets of X satisfying

- $\mathcal{T}$ contains $\emptyset$ and X,
- $\mathcal{T}$ is closed under arbitrary unions, i.e. if $U_i \in \mathcal{T}$ for $i \in I$ then $\bigcup_{i \in I} U_i \in \mathcal{T}$,
- $\mathcal{T}$ is closed under finite intersections, i.e. if $U_1, U_2 \in \mathcal{T}$ then $U_1 \cap U_2 \in \mathcal{T}$.

Remark. One may view the first condition as a special case of the other two since $\emptyset$ is a union of the empty collection and X is the intersection of the empty (hence finite) collection.

Definition 1.2. A *topological space* $(X, \mathcal{T})$ is a set X together with a topology $\mathcal{T}$ on it. The elements of $\mathcal{T}$ are called *open* subsets of X. A subset $F \subseteq X$ is called *closed* if its complement $X \setminus F$ is open. A subset N containing a point $x \in X$ is called a neighbourhood of x if there exists U open with $x \in U \subseteq N$. Thus an open neighbourhood of x is simply an open subset containg x.

Normally we denote the topological space by X instead of $(X, \mathcal{T})$.

Example 1.3. Every metric space (M, ρ) may be viewed as a topological space. Namely the topology is defined by declaring $U \subseteq M$ open if and only if with every $x \in U$ it also contains a small ball around x, i.e. if there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq U$. Two distinct metrics may define the same topology, in fact this happens precisely if the two metrics are equivalent. We say that a property of a metric space is topological if it does not depend on the metric but only on its equivalence class. Such properties may be usually described using topology.

Definition 1.4. Let $A \subseteq X$ be a subset of a topological space X. The *interior* of A is the biggest open subset contained in A. One has $\mathring{A} = \operatorname{int} A = \bigcup_{A \supseteq U \text{ open}} U$. Dually the *closure* of A is the smallest closed subset containing A. One has $\overline{A} = \operatorname{cl} A = \bigcap_{A \subseteq F \text{ closed}} F$.

Definition 1.5. A mapping $f : X \to Y$ between two topological spaces is called *continuous* if for every $U \subseteq Y$ open in Y the inverse image $f^{-1}(U)$ is open in X. We also say that f is a *map*.

Proposition 1.6. *The identity mapping is continuous. A composition of two continuous maps is continuous. Thus topological spaces and continuous maps between them form a category, the* category of topological spaces*.*

Definition 1.7. A *homeomorphism* $f : X \to Y$ is a continuous bijection whose inverse $f^{-1} : Y \to X$ is also continuous.

Remark. Unlike in algebra where the inverse of a bijective homomorphism is always a homomorphism this does not hold for topological spaces. The identity mapping "id" $: (X, \mathcal{T}) \to (X, \mathcal{T}')$ is continuous if and only if $\mathcal{T}' \subseteq \mathcal{T}$. Thus if we topologize X in such a way that this inclusion is proper the identity mapping in this direction will be continuous while its inverse (also the identity) will not. As an example for each X there are two extreme topologies, the discrete topology for which all subsets are open and the indiscrete one for which only $\emptyset$ and X are open.

2. Constructions with topological spaces

Definition 2.1. Let X be a topological space and $A \subseteq X$ its subset. The *subspace topology* on A is given by the collection

$$\{A \cap U \mid U \text{ open in } X\}$$

Thus a subset $V \subseteq A$ is open in this topology if and only if there exists an open subset $U \subseteq X$ such that $V = A \cap U$. We also call A endowed with the subspace topology a *subspace* of X.

Proposition 2.2. *Let X be a topological space and A its subspace. Then the inclusion $i : A \to X$ is continuous. Moreover if $f : Y \to A$ is a mapping then f is continuous if and only if the composition $if : Y \to X$ is continuous.*

$$\begin{tikzcd} & A \arrow[d, hook, "i"] \\ Y \arrow[ur, dashed, "f"] \arrow[r, "if"'] & X \end{tikzcd}$$

Remark. This has the following explanation. To define a continuous map into a subspace $A \subseteq X$ is the same as to define a continuous map into X whose image lies in A. This categorical viewpoint is often useful especially in the dual situation of quotients.

Definition 2.3. Let X and Y be topological spaces. The *product topology* on $X \times Y$ is given by

$$\{W \subseteq X \times Y \mid (\forall (x, y) \in W)\ (\exists U \subseteq X, V \subseteq Y \text{ open})\ U \times V \subseteq W\}$$

Remark. We say that a collection $\mathcal{B}$ of open subsets generates a topology $\mathcal{T}$ if every $U \in \mathcal{T}$ may be expressed as a union of elements of $\mathcal{B}$. We may thus rephrase the previous definition by saying that the product topology is generated by products $U \times V$ of open subsets of X and Y. Not every collection $\mathcal{B}$ generates a topology in this way though so there is something to be checked if one wants to define the product topology in this way.

Proposition 2.4. *The projections $p : X \times Y \to X$ and $q : X \times Y \to Y$ are continuous. Moreover a mapping $f : Z \to X \times Y$ is continuous if and only if the two compositions $pf : Z \to X$ and $qf : Z \to Y$ are continuous.*

$$\begin{tikzcd} & Z \arrow[dl, "pf"'] \arrow[d, dashed, "f"] \arrow[dr, "qf"] & \\ X & X \times Y \arrow[l, "p"] \arrow[r, "q"'] & Y \end{tikzcd}$$

Remark. Again this propositions makes it easy to define continuous maps *into* a product of two topological spaces. One needs only to specify its two components which have to be continuous and the continuity of the whole mapping is automatic. It is much harder to check continuity of a map *from* a product.

Before dualizing the notion of a subspace we define the coproducts (or disjoint unions). These are much easier.

Definition 2.5. Let X_i be a collection of topological spaces indexed by a set I. The disjoint union $\coprod_{i\in I} X_i$ has the following coproduct topology

$$\{U \subseteq \coprod_{i\in I} X_i \mid X_i \cap U \text{ is open in } X_i\}$$

Proposition 2.6. *The inclusions $j_i : X_i \to \coprod_{i\in I} X_i$ are continuous. Moreover a mapping $f : \coprod_{i\in I} X_i \to Y$ is continuous if and only if all compositions $fj_i : X_i \to Y$ are continuous.*

$$\begin{tikzcd}
X_i \arrow[d, hook, "j_i"'] \arrow[dr, "fj_i"] & \\
\coprod_{i\in I} X_i \arrow[r, dashed, "f"'] & Y
\end{tikzcd}$$

Definition 2.7. Let X be a topological space and R an equivalence relation on X. We define the quotient topology on X/R via the canonical projection $p : X \to X/R$ by

$$\{U \subseteq X/R \mid p^{-1}(U) \text{ open in } X\}$$

We call X/R with this topology the *quotient* of X and the projection p the *quotient map.*

Remark. This definition is much less transparent than that of a subspace topology. Some ugly quotients may easily end up having the indiscrete topology. For example define an equivalence relation as the kernel relation of the projection $\mathbb{R} \to \mathbb{R}/\mathbb{Q}$ on the quotient group. Then $\mathbb{R}/\mathbb{Q}$ has indiscrete topology.

Proposition 2.8. *The projection $p : X \to X/R$ is continuous. Moreover a map $f : X/R \to Y$ is continuous if and only if the composition $fp : X \to Y$ is continuous.*

$$\begin{tikzcd}
X \arrow[r, "fp"] \arrow[d, "p"'] & Y \\
X/R \arrow[ur, dashed, "f"'] &
\end{tikzcd}$$

Remark. This proposition makes defining maps *from* a quotient space particularly easy. One simply provides a continuous map from X that factors through X/R as a mapping of *sets*. This means that this map f has to be constant on each equivalence class. It is often very dificult to determine whether a mapping into a quotient space is continuous.

A special class of quotient spaces are the so-called pushouts. Let

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & X \\
Y &
\end{tikzcd}$$

be a diagram of topological spaces and their continuous maps. We define the pushout of this diagram, $X \cup_A Y$ to be the quotient $(X \sqcup Y)/\sim$ where the equivalence relation $\sim$ is defined as follows. Denote the two inclusion $i : X \to X \sqcup Y$ and $j : Y \to X \sqcup Y$. Then $\sim$ is generated by $if(a) \sim jg(a)$. Namely it is the smallest equivalence relation on $X \sqcup Y$ such that after passing to the quotient $(X \sqcup Y)/\sim$ the following square becomes commutative.

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\not\circlearrowright"] & X \arrow[d, "i"] \\
Y \arrow[r, "j"'] & X \sqcup Y
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\circlearrowright"] & X \arrow[d, "i'"] \\
Y \arrow[r, "j'"'] & X \cup_A Y
\end{tikzcd}
$$

Proposition 2.9. *The pushout enjoys the following universal property. A mapping $h : X \cup_A Y \to Z$ is continuous if and only if the two compositions $fi' : X \to Z$ and $fj' : Y \to Z$ are continuous.*

Remark. Thus defining continuous maps out of a pushout is again easy. One needs to supply two continuous maps $k : X \to Z$ and $l : Y \to Z$ for which $kf = lg : A \to Z$. Then there exists a unique (and continuous) map $X \cup_A Y \to Z$ which agrees with k on X and with l on Y. This construction is usually applied when one of the maps f, g is an embedding of a subspace.

Definition 2.10. Let X be a topological space and $A \subseteq X$ its subspace. Then the pushout of the diagram

$$
\begin{tikzcd}
A \arrow[r, hook] \arrow[d] & X \\
*
\end{tikzcd}
$$

where $*$ denotes the one-point space is called the *quotient* of X by A and denoted X/A.

Remark. This quotient space X/A is a special case of the quotient space X/R for the equivalence relation whose equivalence classes are $\{x\}$ for $x \notin A$ and A at least when $A \neq \emptyset$.

3. The Hausdorff property and other separation axioms

Definition 3.1. A topological space X is said to be *Hausdorff* if for any two distinct points $x, y \in X$ $(x \neq y)$ there exist two disjoint open subsets U, V $(U \cap V = \emptyset)$ such that $x \in U$ and $y \in V$.

This is an example of a separation axiom since one thinks of the open sets U, V as "separating" the two points x, y.

Example 3.2. Every metric space is Hausdorff since if $\rho(x, y) = 2k$ then $B_k(x), B_k(y)$ are such open subsets.

Lemma 3.3. *In every Hausdorff topological space the one-point subsets are closed.*

Remark. The condition of all points being closed is strictly weaker and is called the T_1 property. The Hausdorff condition is sometimes denoted as T_2. It belongs to a series of separation axioms. We will now describe the condition T_4 normally called the normality.

Definition 3.4. A Hausdorff topological space X is called *normal* if for any two disjoint closed subsets F, G there exist disjoint open subsets U, V such that $F \subseteq U$ and $G \subseteq V$.

Theorem 3.5. *The class of Hausdorff spaces is closed under taking subspaces, products and coproducts. Explicitly every subspace of a Hausdorff space is Hausdorff, products and coproducts of Hausdorff spaces are again Hausdorff.*

The quotients do not behave well with respect to Hausdorff spaces. There are however some special cases.

Proposition 3.6. *A quotient of a normal space by its closed subspace is again normal. Explicitly if X is normal and $A \subseteq X$ is a closed subspace then X/A is also normal.*

4. Connectedness and path-conectedness

Definition 4.1. A topological space X is called connected if it is non-empty and its only subsets which are both open and closed are $\emptyset$ and X. In other words (passing to the complement) it is not possible to write X as a disjoint union $X = U \sqcup V$ of its two non-empty open subsets.

Remark. Although the disjoint union $U \sqcup V$ in the definition is meant as a set-theoretical one it is also true that in such a case the topology on X is that of that disjoint union.

Theorem 4.2. *The real numbers $\mathbb{R}$ are connected as well as any non-empty interval in $\mathbb{R}$.*

Remark. This uses heavily the completeness of the reals since the rationals $\mathbb{Q}$ are not connected. Namely one might write $\mathbb{Q} = (\mathbb{Q} \cap (-\infty, \pi)) \sqcup (\mathbb{Q} \cap (\pi, \infty))$.

Theorem 4.3. *The class of connected spaces is closed under products and quotients (images in fact). Explicitly a product of connected spaces is connected and if $f : X \to Y$ is a surjective map with X connected then so is Y.*

In algebraic topology a more useful concept is that of a path-connected space.

Definition 4.4. A topological space X is called *path-connected* if it is non-empty and any two points $x_0, x_1 \in X$ may be joined by a continuous path, i.e. there exists a continuous map $\gamma : I \to X$ such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$.

Example 4.5. Every non-empty interval in $\mathbb{R}$ is path-connected. If x_0, x_1 are two points in the interval then one may define $\gamma(t) = (1 - t)x_0 + tx_1$.

Definition 4.6. A space X is called *locally path-connected* if for every point x and its open neighbourhood $U \ni x$ there exists a sub-neighbourhood $U \supseteq V \ni x$ such that V is path-connected.

Theorem 4.7. *Every path-connected topological space is connected. Every locally path-connected and connected topological space is path-connected.*

5. Compactness

Definition 5.1. A collection $\mathcal{U}$ of open subsets of a topological space X is called an *(open) cover* if its union is the whole of X, i.e. $\bigcup \mathcal{U} = \bigcup_{U \in \mathcal{U}} U = X$. A subcollection $\mathcal{U}' \subseteq \mathcal{U}$ is called a *sub-cover* if it is itself a cover.

Definition 5.2. A topological space X is called *compact* if every open cover admits a finite sub-cover.

Theorem 5.3. *The unit interval I is compact.*

Remark. Again this depends on the completness of the reals. Define the following cover of $\mathbb{Q} \cap I$

$$\mathcal{U} = \{\mathbb{Q} \cap ([0, \pi/6 - 1/n) \cup (\pi/6 + 1/n, 1]) \mid n \geq 3\}$$

Its union is $\mathbb{Q} \cap ([0, \pi/6) \cup (\pi/6, 1]) = \mathbb{Q} \cap I$ but no finite sub-cover exists.

Theorem 5.4. *The class of compact spaces is closed under closed subspaces, products, finite coproducts and quotients (images). Explicitly a closed subspaces of a compact space is compact. Products and finite coproducts of compact spaces are compact and if $f : X \to Y$ is a surjective map with X compact then so is Y.*

Proposition 5.5. *A continuous bijection $f : X \to Y$ from a compact space X to a Hausdorff space Y is a homeomorphism.*

Remark. This means that in this situation the inverse mapping $f^{-1} : Y \to X$ will be automatically continuous.

Remark. It might also be useful to note that every compact Hausdorff space is normal.

Theorem 5.6 (Lebesgue number Lemma)**.** *Let X be a compact metric space and $\mathcal{U}$ its open cover. Then there exists $\varepsilon > 0$ for which every subset $A \subseteq X$ of diameter at most ε (i.e. $a, b \in A \Rightarrow \rho(a, b) \leq \varepsilon$) is contained in some $U \in \mathcal{U}$.*

Definition 5.7. A topological space X is called *locally compact* if for every point x and its open neighbourhood $U \ni x$ there exists a sub-neighbourhood $U \supseteq V \ni x$ such that V is compact.

Remark. The usefulness of locally compact spaces lies in the following. Let $p : X \to X/R$ be a quotient map and Y locally compact Hausdorff. Then the product map $p \times \mathrm{id}_Y : X \times Y \to (X/R) \times Y$ is also a quotient map, i.e. the space $(X/R) \times Y$ is homeomorphic to a quotient $(X \times Y)/S$ (where S is easily determined from R) in such a way that under this homeomrphism the map $p \times \mathrm{id}$ becomes the quotient projection $X \times Y \to (X \times Y)/S$.

6. Compactly generated spaces

Maybe later...