

10. DELAUNAY TRIANGULATION

Motivation. Let us consider a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$. Its values are known only in the final set of points P . One way to approximate such a function is to divide the convex hull of the set P into triangles with vertices at the points of the set P and replace the function f by a function which is linear on these triangles. This means the following: Each point x of a triangle p_i, p_j, p_k is a convex combination of the vertices

$$x = t_1 p_1 + t_2 p_2 + t_3 p_3, \quad \text{where } t_1 + t_2 + t_3 = 1, \quad t_1 \geq 0, \quad t_2 \geq 0, \quad t_3 \geq 0,$$

and hence we put

$$f(x) = t_1 f(p_1) + t_2 f(p_2) + t_3 f(p_3).$$

The graph of the approximation of the function f is thus formed by triangles in $\mathbb{R}^2 \times \mathbb{R}$ with vertices $(p_i, f(p_i))$, $(p_j, f(p_j))$ and $(p_k, f(p_k))$.

FIGURE 10.1 Linear approximation of the function f .

This approximation depends substantially on how we do the triangulation of the convex hull of our set. It seems that a good approximation should be one that has no triangles with small angles. We will deal with such triangulations in the following text.

Angle-optimal triangulation. First, we show that for a given set of points P in a plane, all the triangulations of its convex hull have the same number of triangles. This allows us to compare different triangulations lexicographically according to the sizes of angles.

Theorem 10.1. *Let P be a set of n points in the plane. Let the convex hull of P have k edges. Then any triangulation of the convex hull of the set P has $2n - 2 - k$ triangles and $3n - 3 - k$ edges.*

Proof. Denote the number of triangles in the triangulation as m . Every triangle has three edges. k edges are the edge of only one triangle, the remaining edges are the edges of two triangles. So the number of all edges is equal to

$$h = \frac{3m + k}{2}.$$

The Euler Formula says that

$$n - h + (m + 1) = 2.$$

Substituting h from the first formula we derive that

$$m = 2n - 2 - k.$$

From here substituting into the first formula we get

$$h = 3n - 3 - k.$$

□

Let \mathcal{T} be a triangulation of the convex hull of a set P with $m = 2n - 2 - k$ triangles. The triangles of this triangulation have $3m$ angles. Let us order them into a sequence

$$\alpha(\mathcal{T}) = (\alpha_1, \alpha_2, \dots, \alpha_{3m}), \quad \alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_{3m}.$$

On these sequences we define the lexicographic order

$$\alpha(\mathcal{T}) < \alpha(\mathcal{T}'),$$

if there is $i \in \{1, 2, \dots, 3m\}$ such that

$$\alpha_j = \alpha'_j \quad \text{for } j < i \quad \text{and} \quad \alpha_i < \alpha'_i.$$

A triangulation \mathcal{T} is called *angle-optimal* if it is maximal in this ordering.

Legal edges and legal triangulations. For the purposes of our algorithm which should create a suitable triangulation we will need the notion of *legal edge* a *legal triangulation*. Before we start with a definition we repeat shortly high school geometry, especially what we know about the angles on the arc of a circle.

Consider points p and q on the circle \mathcal{K} with a center s . The points divide the circle into two arcs. Choose a point a on the first of them and a point b on the second. Denote the angles in the triangles paq and pbq at the vertices a and b α and β , respectively. We call them *arc angles* of the chord pq . Arc angles have these two important properties:

- The size of angles α and β does not depend on the position of points a and b on the given arcs.
- Their sum is $\alpha + \beta = 180^\circ$.

From here we can easily derive that every triangle pcq where the point c lies in the halfplane paq inside the circle \mathcal{K} has the angle

$$\gamma > \alpha$$

at the vertex c and that every triangle pdq where the point d lies in the halfplane paq outside the circle \mathcal{K} has the angle

$$\delta < \alpha$$

at the vertex d .

FIGURE 10.2 Arc angles

From the properties of arc angles it can be shown that we can circumscribe a circle to a quadrilateral if and only if the sum of opposite angles is equal to 180° .

Now consider a triangulation \mathcal{T} and in it an edge pq which has two adjacent triangles pqr a pqt . Such an edge is called *illegal edge of the triangulation* \mathcal{T} if the point t lies inside the circle circumscribed to the triangle pqr . It is equivalent to the fact that the sum of angles at vertices r a t in the quadrilateral $prqt$ is greater than 180° and also to the fact that the point r lies inside the circle circumscribed to the triangle pqt .

FIGURE 10.3 Illegal edge pq .

The edges of the triangulation \mathcal{T} which are not illegal are called *legal* and the triangulation without illegal edges is called *legal triangulation*. Immediate consequence of this definition is the following characterization of legal triangulations:

Lemma 10.2. *A triangulation is legal if and only if for every edge pq with adjacent triangles pqr and pqt the point t does not lie inside the circle circumscribed to the triangle pqr .*

The following assertion will play a crucial role in our algorithm.

Lemma 10.3. *Let \mathcal{T} be a triangulation with illegal edge pq . Denote the adjacent triangles pqr and pqt . From the triangulation \mathcal{T} let us make a new triangulation \mathcal{T}' in such a way that we replace the edge pq by the new edge rt . This edge is legal in the triangulation \mathcal{T}' and moreover in the arrangement defined in the previous section*

$$\alpha(\mathcal{T}) < \alpha(\mathcal{T}').$$

The exchange of the edge pq for the edge rt is called flip.

FIGURE 10.4 Flip

Proof. Since the edge pq is illegal in the triangulation \mathcal{T} , the sum of the angles at vertices r and t in the quadrilateral $prqt$ is bigger than 180° and so the sum of the angles at p and q in the same quadrilateral is less than 180° . (The sum of all angles in the quadrilateral is 360° .) That is why the edge rt in the triangulation \mathcal{T}' is legal.

Denote the angles before and after the flip according to the following figure:

FIGURE 10.5 Notation.

Since all the other triangles of both triangulations are the same, for

$$\alpha(\mathcal{T}) < \alpha(\mathcal{T}'),$$

it suffices to prove that

$$\min\{\alpha_i, i = 1, 2, \dots, 6\} < \min\{\beta_i, i = 1, 2, \dots, 6\}.$$

We show subsequently that every β_i is bigger than an angle α_j . Obviously,

$$\beta_1 > \alpha_1 \quad \text{a} \quad \beta_4 > \alpha_5.$$

Further, β_2 is an arc angle over the chord pr in the circle circumscribed to the triangle prt . Since the point q lies outside this circle, we get

$$\beta_2 > \alpha_5.$$

Similarly β_6 is an arc angle for the chord pt in the circle circumscribed to the triangle prt . Hence

$$\beta_6 > \alpha_4.$$

Analogously, using the circle circumscribed to the triangle qrt we can prove that

$$\beta_3 > \alpha_1 \quad \text{a} \quad \beta_5 > \alpha_2.$$

□

Corollary 10.4. *An angle-optimal triangulation is legal.*

Proof. If an angle-optimal triangulation had an illegal edge, using the flip of this edge we would have got a triangulation which is bigger in the defined arrangement which would have been a contradiction with the definition of the angle-optimal triangulation.

□

Legal triangulations need not be angle-optimal as the following figure shows on the example of two triangulations of the pentagon $p_1p_2p_3p_4p_5$ to which we can circumscribe a circle. Moreover, $p_1p_2p_3p_5$ is a square and $p_1p_2p_4$ is a isosceles triangle. Both triangulations \mathcal{T} and \mathcal{T}' are legal but because $\alpha(\mathcal{T}) < \alpha(\mathcal{T}')$, the triangulation \mathcal{T} is not angle-optimal.

FIGURE 10.6 The example of legal triangulation which is not angle-optimal.

Delaunay graph and triangulation. Besides angle-optimal triangulation and legal triangulation we introduce another notion of Delaunay triangulation. First we define what *Delaunay graph* is.

The Delaunay graph of the set P has P as the set of vertices. Two vertices p_i and p_j are connected by an edge if there is a circle on which these two points are lying and all the other points of the set P lie outside the circle. It means, that the centre of this circle lies on the edge of the Voronoi diagram which separates the cell determined by p_i from the cell determined by p_j . That is why Delaunay graph is dual to the Voronoi diagram.

FIGURE 10.7 The duality between the Voronoi diagram (dashed) and the Delaunay graph (red).

Lemma 10.5. *The Delaunay graph is a planar graph.*

Proof. Suppose that $p_i p_j$ and $p_k p_l$ are two edges of the Delaunay graph which intersect in the inner point. Let the quadrilateral $p_i p_k p_j p_l$ has the sum of angles at the vertices p_k and p_l at least 180° . Consider a circle which has the chord $p_i p_j$ and p_k lies outside it. Then p_l does not lie outside the circle. (Otherwise the sum of the angles at p_k and p_l would be smaller than 180° .) However this is a contradiction with $p_i p_j$ being an edge of the Delaunay graph. \square

FIGURE 10.8 Illustration of the proof of Lemma 10.5

Bounded faces of the Delaunay graph are polygons. Let points p_1, p_2, \dots, p_k form the vertices of one of these polygons. This polygon determines a vertex v in the dual graph, i.e. in the Voronoi diagram. Hence the points p_1, p_2, \dots, p_k have the same distance to the point v and lie on one circle with the center v .

FIGURE 10.9 The quadrilateral $p_1 p_2 p_3 p_4$ in the Delaunay graph (black) and the Voronoi diagram (red).

Delaunay triangulation is defined as an arbitrary triangulation of these polygons in the Delaunay graph. The Delaunay triangulation need not be determined uniquely. Nevertheless, the centres of the circles circumscribed to the triangles in the Delaunay triangulation form the set of the vertices of the Voronoi diagram.

FIGURE 10.10 Delaunay graph (black) and Delaunay triangulation (black and red).

If no four points of the set P lie on one circle then the Delaunay graph is already the Delaunay triangulation. In this case the Delaunay triangulation is determined uniquely. We say that in this case the points of the set P are in general position.

Every Delaunay triangulation can be characterized by the following property.

Lemma 10.6. *A triangulation of the convex hull of the set P is Delaunay if and only if for every its edge $p_i p_j$ holds: the circle circumscribed to an adjacent triangle $p_i p_j p_k$ does not contain any other point of P in its interior.*

Proof. To the proof we use the duality between the Delaunay graph and the Voronoi diagram.

Let a triangulation is Delaunay. Then the triangle $p_i p_j p_k$ has come into being by dividing a polygon of the Delaunay graph into triangles. The centre of the circle circumscribed to the triangle $p_i p_j p_k$ is also the centre of this polygon and the vertex in the Voronoi diagram. That is why in the interior of this circle cannot lie any other point of the set P .

On the contrary, suppose that all edges of a triangulation have the property described in the statement. Consider a triangle $p_i p_j p_k$ and the circle circumscribed to it. No remaining point of P lies in the interior of this circle. Some of them can lie on the circle. So the triangle $p_i p_j p_k$ is a part of polygon having the same circumscribed circle. It is easy to see that to every edge of this polygon there is a circle having the edge as a chord and having all the other points of P outside. So the edges of the polygon are the edges of the Delaunay graph and our triangulation come into being by the triangulation of the Delaunay graph, so it is Delaunay. \square

Using this lemma and the characterization of legal triangulations by Lemma 10.2, we get immediately that every Delaunay triangulation is legal. To prove that every legal triangulation is Delaunay is a little bit more complicated.

Theorem 10.7. *The set of legal triangulations is equal to the set of Delaunay triangulations.*

Proof. Let us prove by contradiction that every legal triangulation is Delaunay. Let \mathcal{T} be a legal triangulation which is not Delaunay. In this triangulation there is a triangle $p_i p_j p_k$ and a point p_l such that p_l lies in the interior of the circle circumscribed to $p_i p_j p_k$ and the angle $\angle p_i p_l p_j$ is maximal for all such triangles and points. Let $p_i p_j p_m$ be an adjacent triangle to the triangle $p_i p_j p_k$ in the triangulation \mathcal{T} . Since this triangulation is legal the point p_m does not lie in the interior of the circle \mathcal{K} circumscribed to the triangle $p_i p_j p_k$. The point p_l cannot lie inside the triangle $p_i p_j p_m$. Let p_l lies on the opposite side of the line $p_j p_m$ than the point p_i . From the following figure we can see that for the sizes of angles we have

$$\angle p_m p_l p_j > \angle p_i p_l p_j$$

which contradicts the choice of the triangle $p_i p_j p_k$ and the point p_l . \square

FIGURE 10.11 To the proof of Theorem 10.7

From here and from Corollary 10.4 we get:

Corollary 10.8. (1) *Every angle-optimal triangulation is Delaunay.*

(2) *If the points of a set P are in general position, there is just one Delaunay triangulation, just one legal triangulation and just one angle-optimal triangulation and these triangulations are the same.*

Basic idea of randomized incremental algorithm. One possibility how to construct algorithmically Delaunay triangulation is to use algorithm for the Voronoi diagram, to find the dual graph to it and divide its polygons into triangles.

We will present a different approach based on the fact that every legal triangulation is Delaunay and which insists in removing gradually illegal edges.

A naive version of such an algorithm is following: Make a triangulation of the convex hull of the set P and go through its edges. When you find an illegal edge, replace it with a legal edge by a flip and proceed by the same way further. This process is finite because after every flip we get a triangulation which is bigger than the previous one in described lexicographic order. The number of triangulations is finite, so this process must sometimes end.

A sophisticated version of this approach is the following randomized incremental algorithm. Its basic idea can be captured in several points:

- (1) We denote the point from the set P maximal in lexicographic order first with respect y -coordinate and then x -coordinate as p_0 . We add points p_{-1} and p_{-2} to the points of the set P so that all other points of P lie inside the triangle $p_0p_{-1}p_{-2}$.
- (2) We order these points randomly into a sequence p_1, p_2, \dots, p_n . Gradually for $r = 1, 2, \dots, n$ we add the point p_r to a legal triangulation \mathcal{T}_{r-1} of the triangle $p_0p_{-1}p_{-2}$ where the vertices of single triangles are in the set

$$p_{-2}, p_{-1}, p_0, p_1, \dots, p_{r-1}.$$

- (3) Using a simultaneously constructed search structure we find in which triangle $p_i p_j p_k$ of the triangulation \mathcal{T}_{r-1} the point p_r lies and if it lies in the interior or on the edge. Then we connect the point p_r by edges with vertices of this triangle (if it lies in the interior) or with vertices of adjacent triangles (if it lies on an edge). This newly created triangulation of the triangle $p_0p_{-1}p_{-2}$ with vertices in the set

$$\{p_{-2}, p_{-1}, p_0, p_1, \dots, p_{r-1}, p_r\}$$

can be "legalized" using flips which exchange illegal edges for legal ones. After removing all illegal edges we get a triangulation \mathcal{T}_r .

- (4) At the end we remove the points p_{-1} and p_{-2} and the edges going from these points from the legal triangulation \mathcal{T}_n . Due to a suitable choice of the points p_{-1} and p_{-2} the remaining triangulation is the legal and also Delaunay triangulation of the convex hull of the set $P = \{p_0, p_1, p_2, \dots, p_n\}$.

FIGURE 10.12 The set P in the triangle $p_0p_{-1}p_{-2}$.

Now we describe the single steps in more details. We start with (2) and (3).

Legalization. Let us consider a legal triangulation \mathcal{T}_{r-1} described in the step (2) above. If we find using a search structure that the point p_r lies in the interior of a triangle $p_i p_j p_k$, we connect it with single vertices by edges.

FIGURE 10.13 The point p_r in the interior of a triangle $p_i p_j p_k$.

This will create a new triangulation. What is important is that the newly added edges based on p_r are legal. This can be proved as follows: Let \mathcal{K} be a circle circumscribed to the triangle $p_i p_j p_k$. Since the triangulation \mathcal{T}_{r-1} is legal and hence also Delaunay, there is no point from the set $P_{r-1} = \{p_{-2}, p_{-1}, p_0, \dots, p_{r-1}\}$ in the interior of \mathcal{K} . Let \mathcal{L} be a circle homothetic to \mathcal{K} which has the cord $p_r p_i$. See Figure 10.14. In the interior of this circle there are no points from P_{r-1} . So the edge $p_r p_i$ is an edge of the Delaunay graph for the set P_r and this has to be legal. Analogously, one can prove that also the edges $p_r p_j$ and $p_r p_k$ are legal.

FIGURE 10.14 To the proof that $p_r p_i$ is legal.

It can happen that some of the edges $p_i p_j$, $p_i p_k$ a $p_j p_k$ become illegal in the new triangulation. Using a flip we can replace them by edges going from the point p_r . For example let $p_i p_j$ be illegal. Then it is the edge not only of the triangle $p_i p_j p_r$ but also of an adjacent triangle $p_i p_j p_l$. We replace it by the edge $p_r p_l$ which is already legal. Now some of the edges $p_i p_l$ and $p_j p_l$ can become illegal. That is why we have to replace it by a flip with an edge based in the point p_r .

FIGURE 10.15 The edge $p_i p_j$ is illegal.

We proceed in this way as long as there is no illegal edge. (After every flip the new triangulation is bigger in the lexicographic order than the previous one.) The procedure is captured in the following animation and pseudocode. The input is the point p_r , the edge $p_i p_j$ and a legal triangulation \mathcal{T}_{r-1} .

ANIMATION

PSEUDOCODE `Legalize($p_r, p_i p_j, \mathcal{T}_{r-1}$)` from pseudo.pdf, page 37, but use the form from the Czech version of e-learning.

If we find that the point p_r lies on an edge $p_i p_j$, then we replace the edge $p_i p_j$ by the edges $p_r p_i$ and $p_r p_j$ and we create new edges connecting the point p_r with the vertices p_k and p_l of adjacent triangles to the edge $p_i p_j$. Similarly as above one can show that the new edges $p_r p_i$, $p_r p_j$, $p_r p_k$ a $p_r p_l$ are legal. However the edges $p_i p_k$, $p_j p_k$, $p_i p_l$ a $p_j p_l$ can become illegal. This can be corrected by making gradually flips as in the above case.

FIGURE 10.16 The point p_r on the edge $p_i p_j$.

Search structure. In the course of the algorithm we have a relevant search structure for every triangulation. This is an oriented graph the leaves of which are the triangles of the actual triangulation and inner nodes are the triangles of the previous triangulations.

At the beginning the triangulation \mathcal{T}_0 has the only triangle $p_0 p_{-1} p_{-2}$ and the corresponding search structure \mathcal{D}_0 has only one node given by this triangle. For the triangulation \mathcal{T}_{r-1} we have a search structure \mathcal{D}_{r-1} . We change the triangulation \mathcal{T}_{r-1} gradually in the way described in the previous section. Simultaneously we change the search structure. To every transition in the triangulation there is a corresponding transition from one search structure to further search structure. This is illustrated in the Figure 10.17.

FIGURE 10.17 The change of the search structure corresponding to newly added edges inside a triangle.

FIGURE 10.18 The change of the search structure corresponding to a flip.

Using this procedure we create a search structure \mathcal{D}_r corresponding to the legal triangulation \mathcal{T}_r . The resulting search structure depends only on the random order of the points p_1, p_2, \dots, p_r .

Initial choice of points p_0, p_{-1}, p_{-2} . Now let us explain the steps (1) and (4) of our algorithm. We will refer to the geometric idea for better understanding. Important is that the points p_{-1} and p_{-2} will be determined by their properties, we will not determine their concrete coordinates. The decisions whether some edge is legal or illegal, if among four points, which are considered, is some of the points p_{-1}, p_{-2} , will not be made by computations but only by applying the rules which follow from the properties of p_{-1} and p_{-2} .

As we have already said for the point p_0 we choose that one from P which is maximal in the lexicographic order first with respect to the coordinate y and then with respect to the coordinate x .

We choose the point p_{-1} to the right of the set P and under it. (More exactly: its y -coordinate is smaller than the minimum of y -coordinates of the points from P and its x -coordinate is bigger than the maximum of x -coordinates of the points from P .) The choice of p_{-1} is carried out so that

- it lies outside all circles circumscribed to all triples of points from P which do not lie in one line,
- all points p_i with $i > 0$ lie under the line p_0p_{-1} .

We choose the point p_{-2} to the left of the set P and over it so that

- it lies outside all circles circumscribed to all triples of points from $P \cup \{p_{-1}\}$ which do not lie in one line,
- all points p_i with $i > 0$ lie under the line p_0p_{-2} ,
- all points p_i with $i \geq 0$ lie over the line $p_{-1}p_{-2}$.

So all the points p_i with $i > 0$ lie in the interior of the triangle $p_0p_{-1}p_{-2}$ and every quadrilateral $p_{-2}p_i p_{-1}p_j$ is not convex which implies that the point p_{-1} lies outside all circles circumscribed to the triangles $p_{-2}p_i p_j$.

FIGURE 10.19 The set P in the triangle $p_0p_{-1}p_{-2}$.

Under these assumptions every Delaunay (legal) triangulation of $P \cup \{p_{-1}, p_{-2}\}$ consists of

- (1) the edges going from the point p_{-1} to the vertices of the right hand side of the convex hull of the set P ,
- (2) the edges going from the point p_{-2} to the vertices of the left hand side of the convex hull of the set P ,
- (3) a Delaunay triangulation of the set P .

Consequently we can obtain a Delaunay triangulation of the set P in the way described in the step (4).

The rules how to work with our algorithm are based on the following lemma:

Lemma 10.9. *If the points p_0, p_{-1} and p_{-2} are chosen as described above the following assertions are equivalent:*

- p_j lies to the left of the oriented line $p_i p_{-1}$.
- p_j lies to the left of the oriented line $p_{-2} p_i$.
- $p_j > p_i$ in the lexicographic order first with respect to y and then with respect to x .

Instead of doing a proof we demonstrate the situation by a picture.

FIGURE 10.20 To Lemma 10.9. The points p_{-1} and p_{-2} are chosen in such a way that in the coloured angles there is no other point of the set P .

When finding whether an edge is legal or illegal we use the following rules:

- All edges of the triangle $p_0 p_{-1} p_{-2}$ are legal.
- The edge $p_{-2} p_j$ with adjacent vertices p_{-1} and p_k is legal.

FIGURE 10.21 Illustration of the previous statement.

- The edge $p_{-1} p_j$ with adjacent vertices p_{-2} and p_k is legal.
- Let the edge $p_i p_j$ have adjacent vertices p_k and p_l , let the quadrilateral $p_i p_k p_j p_l$ be convex and let there be just one negative number among the numbers i, j, k, l . Then the edge $p_i p_j$ is legal if and only if

$$\min(k, l) < \min(i, j).$$

FIGURE 10.22 $\min(-2, l) < \min(i, j)$, the edge $p_i p_j$ is legal.

FIGURE 10.23 $\min(k, l) > \min(-1, j)$, the edge $p_{-1} p_j$ is illegal.

Pseudocode of the algorithm and its expected time complexity. A brief pseudocode of the algorithm described in details in the previous sections can look like this:

PSEUDOCODE DelaunayTriangulation(P) from pseudo.pdf, page 36

Without proof we give the following statement about expected time complexity of the algorithm. See [Berg et al] for the proof.

Theorem 10.10. *Expected time complexity of the randomized incremental algorithm for the Delaunay triangulation of the set with $n + 1$ elements is $O(n \log n)$.*