2.1. Vlastnosti základních funkcí

$f(x) = \sin(x)$

1, $D(f) = (-\infty, \infty)$

2, $H(f) = \langle -1; 1 \rangle$

3, spojitá v celém $D(f)$

4, ohraničená shora i zdola

5, je periodická

6, funkce je lichá

7, funkce je rostoucí

$\left(-\frac{\pi}{2} + 2k\pi; \frac{\pi}{2} + 2k\pi\right)$

funkce je klesající

$\left(\frac{\pi}{2} + 2k\pi, \frac{3}{2}\pi + 2k\pi\right)$

8, $\lim\limits_{x \to \pm\infty} f(x) =$ neexistuje

$f(x) = x^3$

1, $D(f) = (-\infty, \infty)$

2, $H(f) = (-\infty, \infty)$

3, spojitá v celém $D(f)$

4, není ohraničená

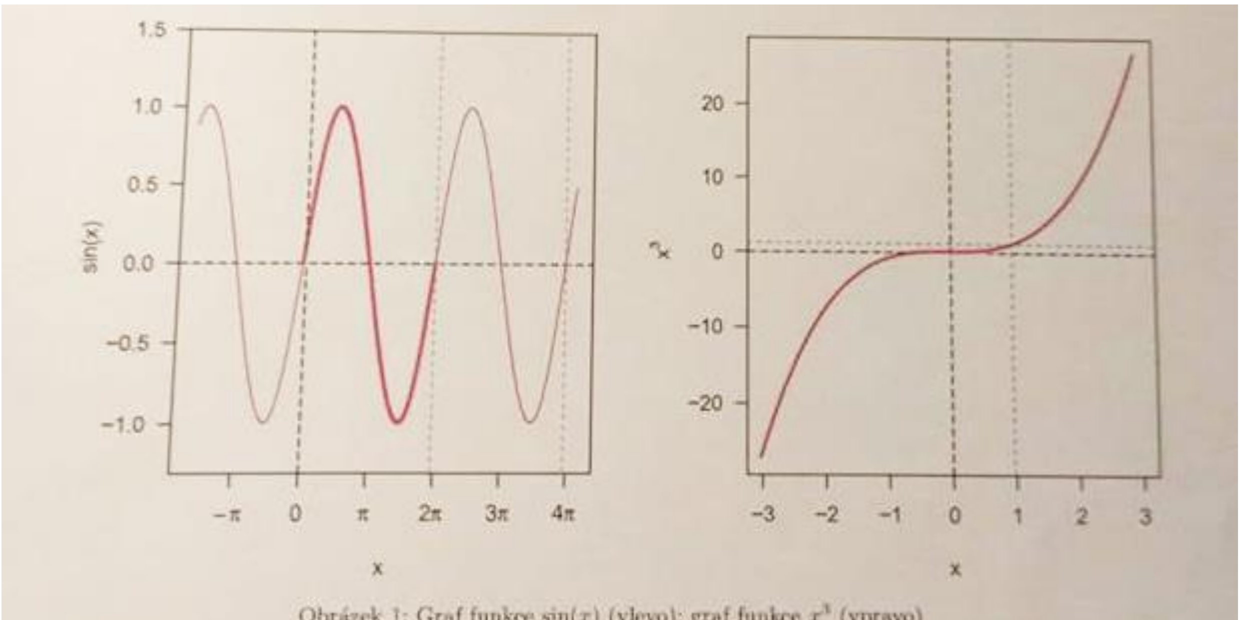
5, není periodická

6, funkce je lichá

7, funkce je rostoucí

8, $\lim\limits_{x \to \infty} f(x) = \infty$ $\quad \lim\limits_{x \to -\infty} f(x) = -\infty$

$\lim\limits_{x \to 0} f(x) = 0$



Obrázek 1: Graf funkce $\sin(x)$ (vlevo); graf funkce x^3 (vpravo)

2.2 Výpočty limit

① $x^2-3x+2=(x\mp\ \ )(x\mp\ \ )$ možné kořeny $\pm1, \pm2$

	1	-3	+2
1	1	-2	0

$\Rightarrow x=1 \quad (x-1)$

$(x-1)(x-2)$

② x^3-3x^2-6x+8

	1	-3	-6	+8
1	1	-2	-8	0
-1	1	-3	-5	
2	1	0	-8	
-2	1	-4	0	

Row 1: $\Rightarrow x=1 \quad (x-1)(x^2-2x-8)$

Row −2: $\Rightarrow x=-2 \quad (x+2)(x-4)(x-1)$

2.4 Limity funkcí ve vlastním bodě

① $\lim\limits_{x\to2} x^3+x-5 = 2^3+2-5 = 5$

③ $\lim\limits_{x\to4} \dfrac{x^3-4x^2-x+4}{\underbrace{x^2-2x-8}_{0\ \Rightarrow\ \text{musím si to rozložit}}} = \lim\limits_{x\to4} \dfrac{(x+1)(x-1)\cancel{(x-4)}}{(x+2)\cancel{(x-4)}} = \dfrac{(4+1)(4-1)}{(4+2)} = \dfrac{15}{6} = \dfrac{5}{2}$

x^3-4x^2-x+4

	1	-4	-1	4
+1	1	-3	-4	0
-1	1	-4	0	

Row +1: $\Rightarrow x=1 \quad (x-1)(x^2-3x-4)$

Row −1: $\Rightarrow x=-1 \quad (x+1)(x-4)$

$(x-4)(x+1)(x-1)$

x^2-2x-8

	1	-2	-8
1	1	-1	-9
-1	1	-3	-5
2	1	0	-8
-2	1	-4	0

Row −2: $\Rightarrow x=-2$

$(x+2)(x-4)$

② $\lim\limits_{x\to1} \dfrac{5^x+3^x}{2^x}-1 = \dfrac{5^1+3^1}{2^1}-1 = \dfrac{8}{2}-1 = 4-1 = 3$

④ $\lim\limits_{x\to-3} \dfrac{2x^2+x+3}{3x+5} = \dfrac{2\cdot(-3)^2\overbrace{-3+3}^{0}}{3\cdot(-3)+5} \quad \dfrac{2\cdot9}{-9+5} = \dfrac{18}{-4} = -\dfrac{9}{2}$

2.5 Limity funkcí v nevlastním bodě

① $\lim\limits_{x\to\infty} \frac{3x^2+2}{4} = \frac{3\infty^2+2}{4} = \underline{\underline{\infty}}$

② $\lim\limits_{x\to-\infty} \frac{5x^2-x^4-6x^6+x^3}{x^2-x^3+4} = \lim\limits_{x\to-\infty} \frac{-6x^6-x^4+x^3+5x^2}{-x^3+x^2+4} =$

$= \lim\limits_{x\to-\infty} \frac{x^{6}\left(-6-\frac{1}{x^2}+\frac{1}{x^3}+\frac{5}{x^4}\right)}{x^3\left(-1+\frac{1}{x}+\frac{4}{x^3}\right)} = \frac{-\infty\left(-6-\frac{1}{\infty}+\frac{1}{\infty}+\frac{5}{\infty}\right)}{-1+\frac{1}{\infty}+\frac{4}{\infty}} =$

$= \frac{-\infty(-6-0+0+0)}{-1+0+0} = \frac{-\infty}{1} = \underline{\underline{-\infty}}$

③ $\lim\limits_{x\to-\infty} \frac{4^x+6^x}{4^x} = \left(\frac{4}{4}\right)^x + \left(\frac{6}{4}\right)^x = 1+0 = \underline{\underline{1}}$

④ $\lim\limits_{x\to\infty} \frac{x^2-x^3+4}{5x^2-5x^4+x+3} = \lim\limits_{x\to\infty} \frac{-x^3+x^2+4}{-5x^4+5x^2+x+3} = \lim\limits_{x\to\infty} \frac{x^3\left(-1+\frac{1}{x^2}+\frac{4}{x^3}\right)}{x^4\left(-1+\frac{5}{x^2}+\frac{1}{x^3}+\frac{3}{x^4}\right)} =$

$= \frac{-1+\frac{1}{\infty}+\frac{4}{\infty}}{\infty\left(-1+\frac{5}{\infty}+\frac{1}{\infty}+\frac{3}{\infty}\right)} = \frac{-1+0+0}{\infty(-1+0+0+0)} = \frac{1}{\infty} = \underline{\underline{0}}$

⑤ $\lim\limits_{x\to-\infty} \frac{4^x-5}{3^x-2} = \frac{4^{-\infty}-5}{3^{-\infty}-2} = \frac{0-5}{0-2} = \underline{\underline{\frac{5}{2}}}$

⑥ $\lim\limits_{x\to\infty} \frac{3^x+5^x}{7^x} = \frac{\infty+\infty}{\infty} = \frac{\infty}{\infty}$ – neurčitý výraz

$\to \lim\limits_{x\to\infty} \frac{3^x}{7^x} + \frac{5^x}{7^x} = \left(\frac{3}{7}\right)^x + \left(\frac{5}{7}\right)^x = 0+0 = \underline{\underline{0}}$

⑦ $\lim\limits_{x\to-\infty} \frac{3x^6-1+4x^2}{3x^3-x^2+4x^4-x^6} = \frac{3x^6+4x^2-1}{-x^6+4x^4+3x^3-x^2} = \frac{x^6}{x^6}\left(\frac{3+\frac{4}{x^4}-\frac{1}{x^6}}{-1+\frac{4}{x^2}+\frac{3}{x^3}-\frac{1}{x^4}}\right) =$

$= \frac{3+\frac{4}{-\infty}-\frac{1}{-\infty}}{-1+\frac{4}{-\infty}+\frac{3}{-\infty}-\frac{1}{-\infty}} = \frac{3+0+0}{-1-0-0+0} = \underline{\underline{-3}}$

⑧ $\lim\limits_{x\to\infty} \frac{4^x-8^x}{5^x} = \frac{4^\infty-8^\infty}{5^\infty} = \frac{\infty-\infty}{\infty}$ neurčitý výraz

$\lim\limits_{x\to\infty} \frac{4^x}{5^x} - \frac{8^x}{5^x} = \left(\frac{4}{5}\right)^x - \left(\frac{8}{5}\right)^x = 0-\infty = \underline{\underline{-\infty}}$

2.6

1. $\left(\frac{x^2-x+1}{\cos x}\right)' = \frac{(2x-1)\cos x-(x^2-x+1)(-\sin x)}{\cos^2 x} = \frac{(2x-1)\cos x+(x^2-x+1)\sin x}{\cos^2 x}$

2. $(x^6 - x^{-6} - x^0 + \cos x - \ln x)' = 6x^5+6x^{-7}-0-\sin x-\frac{1}{x} = 6x^5+6x^{-7}-\sin x-\frac{1}{x}$

3. $\left(\frac{e^{-x}}{1-x}\right)' = \frac{-e^{-x}(1-x)-e^{-x}(-1)}{(1-x)^2} = \frac{-e^{-x}(1-x-1)}{(1-x)^2} = \frac{xe^{-x}}{(1-x)^2}$

4. $(3x\,tg x + (3x-x^4)e^x)' = 3tg x+3x\cdot\frac{1}{\cos^2 x}+(3-4x^3)e^x+(3x-x^4)e^x$

5. $(2\cos x\sin x - e^x tg x)' = 2(-\sin x)\sin x+2\cos x\cdot\cos x - e^x tg x - e^x\frac{1}{\cos^2 x}$

$= -2\sin^2 x + 2\cos^2 x - e^x\left(tg x+\frac{1}{\cos^2 x}\right)$

6. $\left(\frac{x^2-2x+1}{4x-2}\right)' = \frac{(2x-2)(4x-2)-(x^2-2x+1)(4)}{(4x-2)^2} = \frac{8x^2-12x+4-4x^2+8x-4}{(4x-2)^2}$

$= \frac{4x^2-4x}{(4x-2)^2} = \frac{4(x^2-x)}{(2\cdot(2x-1))^2} = \frac{4(x^2-x)}{4(2x-1)^2} = \frac{x^2-x}{(2x-1)^2}$

7. $(x^7+3x^5-2x^2+x+7)' = x^6+15x^4-4x+1$

8. $(3\cos^2 x - 4\cos(x^2))' = -6\cos x\sin x + 4\sin(x^2)\cdot 2x = 8x\sin(x^2) - 6\sin x\cos x$

2.7

1. $((x^5-x)e^x)'' = ((5x^4-1)e^x+(x^5-x)e^x)' = 20x^3e^x+(5x^4-1)e^x +$
$+(5x^4-1)e^x+(x^5-x)e^x = \underline{e^x(x^5+10x^4+20x^3-x-2)}$

2. $(2\cos x\, e^x)'' = (-2\sin x\, e^x + 2\cos x\, e^x)' = -2\cos x\, e^x - 2\sin x\, e^x - 2\sin x\, e^x + 2\cos x\, e^x$
$= \underline{-4\sin x\, e^x}$

3. $(\ln(\cos x)+\ln(\ln x))'' = \left(\frac{1}{\cos x}\cdot(-\sin x) + \frac{1}{\ln x}\cdot\frac{1}{x}\right)' = \left(-\frac{\sin x}{\cos x}+\frac{1}{x\ln x}\right)'$
$= \left(-tg x + \frac{1}{x\ln x}\right)' = -\frac{1}{\cos^2 x}+\frac{0\cdot x\ln x - 1\,(x\ln x)'}{x^2\ln^2 x}$
$= -\frac{1}{\cos^2 x}+\frac{-1\cdot\left(\ln x+\frac{x}{x}\right)}{x^2\ln^2 x} = \underline{-\frac{1}{\cos^2 x}-\frac{\ln x+1}{x^2\ln^2 x}}$

4. $(x^7+3x^5-2x^2+x+7)'' = (7x^6+15x^4-4x+1)' = \underline{42x^5+60x^3-4}$

2.8

1. $\lim_{x\to4}\frac{x^3-4x^2-x+4}{x^2-2x-8} = \left|\frac{0}{0}\right| \overset{L'H}{=} \lim_{x\to4}\frac{3x^2-8x-1}{2x-2} = \left|\frac{3\cdot4^2-8\cdot4-1}{2\cdot4-2}\right| = \frac{48-32-1}{8-2} = \frac{15}{6}$
$= \underline{\frac{5}{2}}$

2. $\lim_{x\to4}\frac{x^3-4x^2+x+4}{x^2-2x-8} = \left|\frac{8}{0}\right|$ nie je možné použiť L'Hospitalovo pravidlo

3. $\lim_{x\to3}\frac{2x^3-5x^2-4x+3}{x^2-9} = \left|\frac{0}{0}\right| \overset{L'H}{=} \lim_{x\to3}\frac{6x^2-10x-4}{2x} = \left|\frac{6\cdot9-10\cdot3-4}{2\cdot3}\right| = \frac{20}{6} = \underline{\frac{10}{3}}$