B

3.1. PARCIÁLNÍ DERIVACE 1. ŘÁDU

1. $2xy + \ln y - \cos x$

$$\frac{\partial}{\partial x} \cdot f(x;y) = 2y + \sin x$$

$$\frac{\partial}{\partial y} \cdot f(x;y) = 2x + \frac{1}{y}$$

2. $\sin(x-y)$

$$\frac{\partial}{\partial x} \cdot f(x,y) = +\cos(x-y)$$

$$\frac{\partial}{\partial y} f(x,y) = -\cos(x-y)$$

3. $\frac{1}{xy} + \ln x$

$$\frac{\partial}{\partial x} \cdot f(x,y) = 1 \cdot x^{-1} y^{-1} + \frac{1}{x} = -1x^{-2}y^{-1} + \frac{1}{x} = \frac{-1}{x^2y} + \frac{1}{x} = \frac{-1+xy}{x^2y} = \frac{xy-1}{x^2y}$$

$$\frac{\partial}{\partial y} f(x,y) = 1 \cdot x^{-1} \cdot y^{-1} = -1x^{-2}y^{-1} = \frac{-1}{x^2y}$$

4. $\ln(x^2-y^2)$

$$\frac{\partial}{\partial x} f(x,y) = \ln(2x^1-y^2) = \frac{2x}{x^2-y^2}$$

$$\frac{\partial}{\partial y} f(x,y) = \ln(x^2-y^2) = -2y(x^2-y^2)^{-1} = \frac{-2y}{x^2-y^2}$$

5. $\frac{x^2}{y^2}$

$$\frac{\partial}{\partial x} f(x,y) = \frac{2x}{y^2}$$

$$\frac{\partial}{\partial y} f(x,y) = x^2 \cdot y^{-2} = x^2 \cdot (-2) y^{-2-1} = x^2 \cdot (-2) y^{-3} = -\frac{2x^2}{y^3}$$

6. $\cos xy$

$$\frac{\partial}{\partial x} f(x,y) = -\sin x\,(xy)$$

$$\frac{\partial}{\partial y} f(x,y) = -\sin y\,(xy)$$

7. $e^{(1-x^2)/y}$

$$\frac{\partial}{\partial x} \cdot f(x,y) = -2xy\,e^{y-yx^2}$$

$$\frac{\partial}{\partial y} f(x,y) = +(1-x^2)\cdot 1\,e^{y-yx^2} = -(x^2-1)\,e^{y-yx^2}$$

3.2. PARCIÁLNÍ DERIVACE 2. ŘÁDU

1. $2xy + \ln y - \cos x$

$$\frac{\partial}{\partial x} f(x,y) = 2y + 0 + \sin$$

$$\frac{\partial^2}{\partial x^2} f(x,y) = \cos x$$

$$\frac{\partial^2}{\partial x \partial y} f(x,y) = 2$$

$$\frac{\partial}{\partial y} f(x,y) = 2x + \frac{1}{y}$$

$$\frac{\partial^2}{\partial y^2} \cdot f(x,y) = -\frac{1}{y^2}$$

$$\frac{\partial^2}{\partial y \partial x} f(x,y) = 2$$

2. $\sin(x-y) = \sin x - \sin y$

$$\frac{\partial}{\partial x} \cdot f(x,y) = \cos(x-y)$$

$$\frac{\partial^2}{\partial x^2} f(x,y) = -\sin(x-y)$$

$$\frac{\partial^2}{\partial x \partial y} f(x,y) = -\sin(x-y)\cdot(-1) = \sin(x-y)$$

$$\frac{\partial}{\partial y} = \cos(x-y)\cdot(-1) = -\cos(x-1)$$

$$\frac{\partial^2}{\partial y^2} = \sin(x-y)\cdot(-1) = -\sin(x-y)$$

$$\frac{\partial^2}{\partial y \partial x} = \sin(x-y)$$

3. $\frac{1}{xy} + \ln x = x^{-1}y^{-1} + \ln x$

$$\frac{\partial}{\partial x} f(x,y) = -\frac{1}{y\cdot x^2} + \frac{1}{x} = -y^{-1}x^{-2} + x^{-1}$$

$$= \frac{xy-1}{x^2 y}$$

$$\frac{\partial^2}{\partial x^2} f(x,y) = 2x^{-3}\cdot y^{-1} - x^{-2} = \frac{2}{x^3 y} - \frac{1}{x^2} = \frac{2-xy}{x^3 y}$$

$$\frac{\partial^2}{\partial x \partial y} f(x,y) = y^{-2}x^{-2} = \frac{1}{x^2 y^2}$$

$$\frac{\partial}{\partial y} = -\frac{1}{xy^2} = -x^{-1}y^{-2}$$

$$\frac{\partial^2}{\partial y^2} f(x,y) = 2x^{-1}y^{-3} = \frac{2}{xy^3}$$

$$\frac{\partial^2}{\partial y \partial x} f(x,y) = x^{-2}y^{-2} = \frac{1}{x^2 y^2}$$

4. $\ln(x^2-y^2)$

$$\frac{\partial}{\partial x} f(x,y) = \frac{1}{x^2-y^2} \cdot 2x = \frac{2x}{x^2-y^2}$$

$$\frac{\partial}{\partial y} f(x,y) = \frac{1}{x^2-y^2} \cdot (-2y) = -\frac{2y}{x^2-y^2}$$

$$\frac{\partial^2}{\partial x^2} f(x,y) = \frac{2\cdot(x^2-y^2) - 2x\cdot(2x)}{(x^2-y^2)^2} = \frac{2x^2-2y^2-4x^2}{(x^2-y^2)^2} = \frac{-2\cdot(x^2+y^2)}{(x^2-y^2)^2}$$

$$\frac{\partial^2}{\partial x \partial y} f(x,y) = \frac{0\cdot(x^2-y^2)-2x\cdot(-2y)}{(x^2-y^2)^2} = \frac{+4xy}{(x^2-y^2)^2}$$

$$\frac{\partial^2}{\partial y^2} f(x,y) = \frac{2\cdot(x^2-y^2)-2y\cdot(-2y)}{(x^2-y^2)^2} = -\frac{2x^2-2y^2+4y^2}{(x^2-y^2)^2} = \frac{-2(x^2+y^2)}{(x^2-y^2)^2}$$

$$\frac{\partial^2}{\partial y \partial x} f(x,y) = -\frac{0\cdot(x^2-y^2)-2y\cdot(2x)}{(x^2-y^2)^2} = \frac{4xy}{(x^2-y^2)^2}$$

5. $\frac{x^2}{y^2} = x^2 \cdot y^{-2}$

$$\frac{\partial}{\partial x} f(x,y) = 2x\cdot y^{-2} = \frac{2x}{y^2}$$

$$\frac{\partial^2}{\partial x^2} f(x,y) = 2y^{-2} = \frac{2}{y^2}$$

$$\frac{\partial^2}{\partial x \partial y} f(x,y) = 2x\cdot(-2)\cdot y^{-3} = \frac{-4x}{y^3}$$

$$\frac{\partial}{\partial y} f(x,y) = x^2(-2)y^{-3} = -2\frac{x^2}{y^3}$$

$$\frac{\partial^2}{\partial y^2} f(x,y) = x^2\cdot 6y^{-4} = \frac{6x^2}{y^4}$$

$$\frac{\partial^2}{\partial x \partial y} f(x,y) = 2x\cdot(-2)y^{-3} = \frac{-4x}{y^3}$$

6. $\cos xy$

$$\frac{\partial}{\partial x} f(x,y) = -\sin xy \cdot y = -y\cdot\sin xy$$

$$\frac{\partial^2}{\partial x^2} f(x,y) = 0 - y\cdot\cos xy\cdot y = -y^2\cdot\cos xy$$

$$\frac{\partial^2}{\partial x \partial y} f(x,y) = -1\sin xy - y\cdot\cos xy\cdot x = -\sin xy - xy\cdot\cos xy$$

$$\frac{\partial}{\partial y} f(x,y) = (-\sin xy)\cdot x = -x\cdot\sin xy$$

$$\frac{\partial^2}{\partial y^2} f(x,y) = 0 + (-x)\cdot\cos xy\cdot x = -x^2\cos xy$$

$$\frac{\partial^2}{\partial y \partial x} f(x,y) = -1\sin xy - x\cos xy\cdot y = -\sin xy - xy\cdot\cos xy$$

7. $e^{\overbrace{(1-x^2)\cdot y}^{y-yx^2}}$

$$\frac{\partial}{\partial x} f(x,y) = e^{(1-x^2)y} \cdot (0-2yx) = -2xy \cdot e^{y-yx^2}$$

$$\frac{\partial}{\partial y} f(x,y) = e^{y-yx^2} \cdot (1-x^2) = -1 \cdot (x^2-1) \cdot e^{y-x^2y}$$

$$\frac{\partial^2}{\partial x^2} f(x,y) = -2y \cdot e^{y-yx^2} + (-2xy) \cdot e^{y-yx^2} \cdot (0-2xy) = e^{y-yx^2} \cdot (4x^2y^2-2y) =$$
$$2y \cdot e^{y-yx^2} \cdot (2x^2y-1)$$

$$\frac{\partial^2}{\partial x \partial y} f(x,y) = -2x \cdot e^{y-yx^2} + (-2xy) \cdot e^{y-yx^2} \cdot (1-x^2) = e^{y-yx^2} \cdot (-2x-2xy+2x^3y) =$$
$$= 2xe^{y-yx^2} \cdot (x^2y-y-1)$$

$$\frac{\partial^2}{\partial y^2} f(x,y) = e^{y-yx^2} \cdot (1-x^2) \cdot (1-x^2) + e^{y-yx^2} \cdot 0 = (1-x^2)^2 \cdot e^{y-yx^2} = (x^2-1)^2 \cdot e^{y-yx^2}$$

$(1-x^2)^2 = 1-2x^2+x^4$
$(x^2-1)^2 = x^4-2x^2+1$

$$\frac{\partial^2}{\partial y \partial x} f(x,y) = e^{y-yx^2} \cdot \overbrace{(0-2xy) \cdot (1-x^2)}^{-2xy+2x^3y = 2x(x^2y-y)} + e^{y-yx^2} \cdot (-2x) =$$
$$= e^{y-yx^2} [2x(x^2y-y)-2x)] = 2xe^{y-yx^2} \cdot (x^2y-y-1)$$

3.3. LOKÁLNÍ EXTRÉMY FCE DVOU PROMĚNNÝCH

1. $f(x,y) = 2xy - 2x - 4y$

$\frac{\partial}{\partial x} f(x,y) = 2y - 2 = y - 1 \qquad y = 1$

STACIONÁRNÍ BOD [2; 1]

$\frac{\partial}{\partial y} f(x,y) = 2x - 4 = x - 2 \qquad x = 2$

$\frac{\partial^2}{\partial x^2} f(x,y) = \frac{\partial}{\partial x}\, 2y - 2 = 0$

$\frac{\partial^2}{\partial x \partial y} f(x,y) = \frac{\partial}{\partial y}\, 2y - 2 = 2$

$\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \qquad |H| = 0 \cdot 0 - 2 \cdot 2 = -4$

$\frac{\partial^2}{\partial y^2} f(x,y) = \frac{\partial}{\partial y} f(x,y) = 2x - 4 = 0$

POUZE STACIONÁRNÍ BOD

$\frac{\partial^2}{\partial y \partial x} f(x,y) = \frac{\partial}{\partial x} f(x,y)\; 2x - 4 = 2$

2. $f(x,y) = 2x^3 + 3y^2 - 6xy$

$\frac{\partial}{\partial x} = 2 \cdot 3x^2 - 6y = 6x^2 - 6y \qquad x_1 = 0, \quad x_2 = 1$

$\frac{\partial}{\partial y} f(x,y) = 6y - 6x = 0 \qquad y_1 = 0, \quad y_2 = 1$

STACIONÁRNÍ BODY [0; 0]
[1; 1]

$\frac{\partial^2}{\partial x^2} f(x,y) = 12x$

$\frac{\partial}{\partial x \partial y} = -6$

$\begin{pmatrix} 12x & -6 \\ -6 & 0 \end{pmatrix} \qquad |H| = 12 \cdot 0 - (-6 \cdot (-6)) = -36$

$|H| = 12 \cdot 1 - (-6 \cdot (-6)) = 0$

$\frac{\partial^2}{\partial y^2} f(x,y) = 6 - 6 = 0$

$\frac{\partial}{\partial y \partial x} f(x,y) = -6$

FCE MÁ POUZE STACIONÁRNÍ BODY [0; 0]
[+1; 1]

3. $f(x,y) = x^2 + 4xy + 6y^2 - 2x + 8y - 5$

$\frac{\partial}{\partial x} f(x,y) = 2x - 4y - 2 = 2y + 1 = 1$

$x = -1$

$\frac{\partial}{\partial y} f(x,y) = -4x + 12y - 8$

$12y = 4x - 6$

$12y = 4(2y + 1) - 8$

$12y = 8y + 4 - 8$

$4y = -4$

$y = -1$

STACIONÁRNÍ BOD $[-1, -1]$

$\frac{\partial^2}{\partial x^2} = 2$

$\frac{\partial^2}{\partial x \partial y} = \frac{\partial}{\partial y} 2x - 4 - 2 = -4$

$\frac{\partial^2}{\partial y^2} = 12$

$\frac{\partial^2}{\partial y \partial x} = -4$

$\begin{pmatrix} 2 & -4 \\ -4 & 12 \end{pmatrix}$

$|H| = 2 \cdot 12 - (-4 \cdot (-4)) = 24 - 16 = \underline{8}$

FCE MÁ EXTRÉM V MINIMU V BODĚ $[-1, -1]$

4. $f(x,y) = 5 + 6x - 4x^2 - 3y^2$

$\frac{\partial}{\partial x} = 6 - 8x$

$8x = 6$

$x = \frac{3}{4}$

STACIONÁRNÍ BOD $[\frac{3}{4}; 0]$

$\frac{\partial}{\partial y} = -6y$

$y = 0$

$\frac{\partial^2}{\partial x^2} = 8$

$\frac{\partial^2}{\partial x \partial y} = 0$

$\frac{\partial^2}{\partial y^2} = -6$

$\frac{\partial^2}{\partial y \partial x} = 0$

$\begin{pmatrix} -8 & 0 \\ 0 & -6 \end{pmatrix}$

$|H| = -8 \cdot (-6) - 0 \cdot 0 = 48$

FCE MÁ EXTRÉM V MAXIMU V BODĚ $[\frac{3}{4}; 0]$

5. $f(x,y) = 8x^3 + y^3 - 6xy + 4$

$\frac{\partial}{\partial x} = 8 \cdot 3x^2 - 6y = 24x^2 - 6y$

$6y = 24x^2$

$y = 4x^2$

$y_1 = 1 \qquad y_2 = 0$

$\frac{\partial}{\partial y} = 3y^2 - 6x$

$3(4x^2) - 6x = 0$

$12x^2 - 6x = 0$

$2x^2 - x = 0$

$x(2x - 1) = 0$

$x_1 = \frac{1}{2} \qquad x_2 = 0$

STACIONÁRNÍ BODY $[\frac{1}{2}; 1]$
$[0; 0]$

$\frac{\partial^2}{\partial x^2} = 48x$

$\frac{\partial^2}{\partial x \partial y} = -6$

$\frac{\partial^2}{\partial y^2} = 6y$

$\frac{\partial^2}{\partial y \partial x} = -6$

$\begin{pmatrix} 48x & -6 \\ -6 & 6y \end{pmatrix} = |H| = 48 \cdot \frac{1}{2} \cdot 6 \cdot 1 - (-6 \cdot (-6)) = 144 - 36 = 108$

$|H| = 48 \cdot 0 \cdot 0 - (-6 \cdot (-6)) = -36$

FCE MÁ EXTRÉM V MINIMUM V BODĚ $[\frac{1}{2}; 1]$
BOD $[0;0]$ JE POUZE STACIONÁRNÍ

6. $f(x,y) = x(x-6) + y(y-9) + xy$

$f(x,y) = x^2 - 6x + y^2 - 9y + xy$

STACIONÁRNÍ BOD $[1; 4]$

$\frac{\partial}{\partial x} = 2x - 6 + y$

$y = 6 - 2x$

$y = 6 - 2 \cdot 1$

$y = 4$

$\frac{\partial}{\partial y} = 2y + 9 + x$

$2 \cdot (6 - 2x)$

$2 - 4x - 9 + x = 0$

$12 - 9 = 3x$

$x = 1$

$\frac{\partial^2}{\partial x^2} = 2$

$\frac{\partial^2}{\partial x \partial y} = 1 \qquad \frac{\partial^2}{\partial y \partial x} = 1$

$\frac{\partial^2}{\partial y^2} = 2$

$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = |H| = 2 \cdot 2 - (1 \cdot 1) = 3$

FCE MÁ EXTRÉM V MINIMU V BODĚ $[1; 4]$

7. $f(x,y) = x^3 + xy^2 + 6xy$

$\frac{\partial}{\partial x} f(x,y) = 3x^2 + y^2 + 6y$

$3x^2 + y^2 + 6y = 0$

$3x^2 + \left(-\frac{3x}{x}\right)^2 + 6\left(-\frac{3x}{x}\right) = 0$

$3x^2 + \frac{9x^2}{x^2} - 6\frac{3x}{x} = 0$

STACIONÁRNÍ BODY $[0;0]$ $[0;-6]$

?

$3x^2 + 9 - 6 \cdot 3 = 0$

$3x^2 + 9 - 18 = 0$

$3x^2 - 9 = 0$

$3x^2 = 9$

$x = \pm\sqrt{3}$ $\quad x_1 = +\sqrt{3}$

$x_2 = -\sqrt{3}$

$\frac{\partial}{\partial y} f(x,y) = 2yx + 6x$

$2yx + 6x = 0$

$yx + 3x = 0$

$y = -\frac{3x}{x}$

$y_1 = -\frac{3 \cdot \sqrt{3}}{\sqrt{3}}$ $\quad y_2 = -\frac{3(-\sqrt{3})}{-\sqrt{3}}$

$y_1 = -3$ $\quad y_2 = -3$

$\frac{\partial^2}{\partial x^2} = 6x$ $\quad \begin{pmatrix} 6x & 2y+6 \\ 2y+6 & 2x \end{pmatrix}$

$\frac{\partial}{\partial x \partial y} = 2y + 6$

$\frac{\partial^2}{\partial y^2} = 2x$

pro: $x_1 = +\sqrt{3}$ a $y_1 = -3$

kladné

$\begin{pmatrix} 6x & 2y+6 \\ 2y+6 & 2x \end{pmatrix} = |H| = 6 \cdot \sqrt{3} \cdot 2\sqrt{3} - (2 \cdot (-3) + 6) \cdot (2 \cdot (-3) + 6) = \underline{\underline{36}}$

$\frac{\partial}{\partial y \partial x} = 2y + 6$

pro $x_1 = -\sqrt{3}$ $\quad y = -3$

záporné

$\begin{pmatrix} 6x & 2y+6 \\ 2y+6 & 2x \end{pmatrix} = |H| = 6 \cdot (-\sqrt{3}) \cdot 2(-\sqrt{3}) - (2 \cdot (-3) + 6) \cdot (2 \cdot (-3) + 6) = \underline{\underline{36}}$

FCE MÁ EXTRÉM V MINIMU V BODĚ $[\sqrt{3}; -3]$

MÁ V MAXIMU V BODĚ $[-\sqrt{3}; -3]$