Téma 5: Integrální počet, skupina C

5.1. Neurčité integrály

1) $\int \sqrt{x}\,dx = \int x^{\frac{1}{2}}\,dx = \int \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\,dx = \int \frac{x^{\frac{3}{2}}}{\frac{3}{2}}\,dx = \int \frac{x^{\frac{3}{2}}}{1}\cdot\frac{2}{3} = \frac{2x^{\frac{3}{2}}}{3} = \frac{2}{3}\sqrt{x^3} + c$

2) $\int \frac{3}{4}\,dx = \frac{3}{4}x + c$

3) $\int 4x^{-3}\,dx = 4\int \frac{x^{-3+1}}{-3+1}\,dx = \cancel{4}^{2}\int \frac{x^{-2}}{-\cancel{2}_{1}}\,dx = 2\cdot x^{-2} = -2x^{-2} + c$

4) $\int 3\sqrt{x}\,dx = \int 3x^{\frac{1}{2}}\,dx = 3\int \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\,dx = 3\int \frac{x^{\frac{3}{2}}}{\frac{3}{2}}\,dx = \cancel{3}\int \frac{x^{\frac{3}{2}}}{1}\cdot\frac{2}{\cancel{3}}\,dx = 2x^{\frac{3}{2}} + c$

5) $\int e^x\left(1+\frac{e^x}{3}\right)dx = \int e^x + \frac{e^{2x}}{3} = e^x + \frac{1}{3}\int e^{2x} = e^x + \frac{e^{2x}}{3\cdot 2} + c$

6) $\int\left(\frac{(2\sqrt{x}+1)^2}{x^2} + \cos^{-2}x\right)dx = \int \frac{4x+2\sqrt{x}+2\sqrt{x}+1}{x^2} + \frac{1}{\cos^2 x}\,dx = \int\left(\frac{4x}{x^2} + \frac{4\sqrt{2}}{x^2} + \frac{1}{x^2} + \frac{1}{\cos^2 x}\right)dx =$

$= \int\left(4\frac{1}{x} + 4x^{-\frac{3}{2}} + x^{-2} + tg(x)\right)dx = 4\ln(x) + 4\frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} + \frac{x^{-1}}{-1} + tg(x) = 4\ln(x) + 4\cdot\left(-\frac{2}{1}\right)\left(\frac{1}{\sqrt{x}}\right) + (-1)\cdot\frac{1}{x} + tg\,x = 4\ln(x) - \frac{8}{\sqrt{x}}$

$- \frac{1}{x} + tg(x) + c$

7) $\int(\sqrt{x}+1)(x-\sqrt{x}+1)\,dx = \int(x\sqrt{x} - x + \sqrt{x} + x - \sqrt{x} + 1)\,dx = \int(x\sqrt{x}+1)\,dx =$

$= \int\left(x^{\frac{2}{2}}\cdot x^{\frac{1}{2}} + 1\right)dx = \int\left(x^{\frac{3}{2}}+1\right)dx = \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + x = \frac{2x^{\frac{5}{2}}}{5} + x + c$

8) $\int(4x^5 + x^3 - 5)\,dx = \frac{4x^6}{6} + \frac{x^4}{4} - 5x + c = \frac{16x^6 + 6x^4 - 120x}{24} + c = \frac{\cancel{2}(8x^6+3x^4-60x)}{\cancel{24}\,12} + c =$

$= \frac{2}{3}x^6 + \frac{1}{4}x^4 - 5x + c$

9) $\int \frac{x^4 - 10x^2 + 5}{x^2}\,dx = \int\left(x^2 - 10 + \frac{5}{x^2}\right)dx = \frac{x^3}{3} - 10x + \frac{5x^{-1}}{-1} = \frac{x^3}{3} - 10x - \frac{5}{x} + c$

10) $\int \frac{\sqrt{x}}{x^2}\,dx = \int(\sqrt{x}\cdot x^{-2})\,dx = \int\left(x^{\frac{1}{2}}\cdot x^{-\frac{4}{2}}\right)dx = \int\left(x^{-\frac{3}{2}}\right)dx = \frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} = -2x^{-\frac{1}{2}} = -\frac{2}{\sqrt{x}} + c$

$-\frac{2}{7} + \frac{7}{7} = \frac{-2+7}{7} = \frac{5}{7}$

11) $\int \frac{5}{x^{\frac{2}{7}}}\,dx = \int\left(5\cdot x^{-\frac{2}{7}}\right)dx = 5\cdot\frac{x^{\frac{5}{7}}}{\frac{5}{7}} = \frac{5}{1}\cdot\frac{7x^{\frac{5}{7}}}{5} = 7x^{\frac{5}{7}}$

12) $\int \frac{x^3 - 2x + 1}{x^3}\,dx = \int\left(1 - \frac{2}{x^2} + \frac{1}{x^3}\right)dx = x - 2\frac{x^{-1}}{-1} + 1\frac{x^{-2}}{-2} + c = x - 2\cdot(-1)\cdot\frac{1}{x} + 1\cdot\left(-\frac{1}{2}\right)\cdot\frac{1}{x^2} + c =$

$= x + \frac{2}{x} - \frac{1}{2x^2} + c$

13) $\int\left(\frac{3}{x^4} + \frac{1}{\sqrt{x}}\right)dx = \int\left(3x^{-4} + x^{-\frac{1}{2}}\right)dx = \frac{3}{1}\cdot\frac{x^{-3}}{-3} + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c = -\frac{1}{x^3} + 2\sqrt{2} + c$

5.2. Substituční metoda

1) $$\int \sin(2x-5)\,dx = \left|\begin{matrix} 2x-5=t \\ 2dx = dt \end{matrix}\right| = \frac{1}{2}\int \sin t\,dt = \frac{1}{2}(-\cos t)+c = -\frac{1}{2}\cos(2x-5))+c =$$
$$= -\frac{\cos(2x-5)}{2}+c$$

2) $$\int \frac{3\ln^2 x}{x}\,dx = \left|\begin{matrix} t=x \\ dt=1 \end{matrix}\right| \; 3\int \frac{\ln^2 t}{t}\cdot 1 = 3\int \ln^2 t\cdot t^{-1}\,1 = \frac{3}{1}\cdot\frac{\ln^3 t}{3} = \ln^3 t + c = \ln^3 x + c$$

3) $$\int \frac{1}{\sqrt{5-4x}}\,dx = \left|\begin{matrix} 5-4x=t \\ 0-4dx=dt \\ dx=-\frac{1}{4}dt \end{matrix}\right| = \int \frac{1}{\sqrt{t}}\cdot\left(-\frac{1}{4}dt\right) = -\frac{1}{4}\int t^{-\frac{1}{2}}\,dt = -\frac{1}{4}\cdot\frac{t^{\frac{1}{2}}}{\frac{1}{2}}\,dt =$$
$$= -\frac{1}{4}\cdot\frac{2\sqrt{5-4x}}{1}+c = -\frac{\sqrt{5-4x}}{2}+c$$

4) $$\int \frac{e^{2x}-1}{e^x}\,dx = \int e^{-x}(e^{2x}-1)\,dx = \int e^{2x-x} - \int e^{-x}\,dx = \int e^x - \int e^{-x}\,dx \;\Big|\begin{matrix} t=-x \\ dt=-1dx \end{matrix} \Rightarrow$$
$$e^x - \int e^t\cdot(-1) = e^x - e^t\cdot(-1) = e^x + e^{-x} = e^x + \frac{1}{e^x}+c$$

5) $$\int x e^{-x^2}\,dx \;\Big|\begin{matrix} t=-x^2 \\ dt=-2x \end{matrix} \Rightarrow dx = \frac{dt}{-2x} = \int x e^t\cdot\frac{dt}{-2x} = \int -\frac{1}{2}e^t\cdot dt = -\frac{1}{2}e^t + c = -\frac{1}{2}e^{-x^2}+c$$

6) $$\int \frac{1}{6}\left(1-\frac{x}{6}\right)^{-2}dx \;\Big|\begin{matrix} t=1-\frac{x}{6} \\ dt=-\frac{1}{6}dx \end{matrix} \Rightarrow \int \frac{1}{6\cdot t^2}\,dt = \frac{1}{6}\cdot\int\frac{1}{t^2}\,dt = \frac{1}{6}\cdot\int\frac{1}{t^2} = \frac{1}{6}\cdot\frac{1\cdot 1}{t^1} = \frac{1}{6}\cdot\frac{1}{1-\frac{x}{6}} =$$
$$= \frac{6}{6-x}$$

7) $$\int \frac{1}{\cos^2(1-x)}\,dx = \left|\begin{matrix} 1-x=t \\ 0-1dx=dt \\ dx=-dt \end{matrix}\right| = -\int\frac{1}{\cos^2 t}\,dt = -\int(\cos^{-2}t\,dt) = -\left(\frac{\cos^{-1}t}{-1}\cdot(-\sin t)\right)+c =$$
$$= -\left(\frac{\sin(1-x)}{\cos(1-x)}\right)+c = \frac{\sin(x-1)}{\cos(x-1)}+c = \tan(x-1)+c$$

8) $$\int 6x^2 e^{-2x^3}\,dx \;\Big|\begin{matrix} t=-2x^3 \\ dt=-6x^2 \\ -dt=6x^2 \end{matrix} \quad dx = -\frac{1}{6x^2} \Rightarrow \int 6x^2 e^t - \frac{1}{6x^2} = \int -e^t\,dt = -\int e^t\,dt = -e^{-2x^3}$$

9) $$\int \frac{\sin x}{2\cdot\sqrt{\cos^3 x}}\,dx = \frac{1}{2}\cdot\int\frac{\sin x}{\sqrt{\cos^3 x}}\,dx = \frac{1}{2}\int \sin x\cdot\cos^{-\frac{3}{2}}x\,dx = \left|\begin{matrix} \cos x=t \\ -\sin x\,dx=dt \\ \sin x\,dx=-dt \end{matrix}\right| = \frac{1}{2}\int t^{-\frac{3}{2}}\cdot(-dt) =$$
$$= -\frac{1}{2}\int t^{-\frac{3}{2}}\,dt = -\frac{1}{2}\cdot\frac{t^{-\frac{1}{2}}}{-\frac{1}{2}}+c = t^{-\frac{1}{2}}+c = \cos^{-\frac{1}{2}}x+c = \frac{1}{\sqrt{\cos x}}+c$$

10) $$\int \frac{4\cos x}{\sqrt[3]{1+2\sin x}}\,dx = \int\frac{4\cos x}{(1+2\sin x)^{\frac{1}{3}}} = \int 4\cos x\cdot(1+2\sin x)^{-\frac{1}{3}}\,dx = 4\int\cos x\cdot(1+2\sin x)^{-\frac{1}{3}}\,dx =$$
$$= \left|\begin{matrix} (1+2\sin x)=t \\ 2\cos x\,dx=dt \\ \cos x\,dx=\frac{dt}{2} \end{matrix}\right| = 4\int t^{-\frac{1}{3}}\cdot\frac{dt}{2} = \frac{1}{2}\cdot 4\cdot\frac{t^{\frac{2}{3}}}{\frac{2}{3}}+c = 2t^{\frac{2}{3}}\cdot\frac{3}{2}+c = 3t^{\frac{2}{3}}+c = 3(1+2\sin x)^{\frac{2}{3}}+c$$

11) $\int \sqrt{1+2x}\,dx = \int (1+2x)^{\frac{1}{2}}\,dx = \left| \begin{array}{l} (1+2x) = t \\ 2\,dx = dt \\ dx = \frac{dt}{2} \end{array} \right| = \int t^{\frac{1}{2}} \cdot \frac{dt}{2} = \frac{1}{2} \int t^{\frac{1}{2}}\,dt = \frac{1}{2} \cdot \frac{t^{\frac{3}{2}}}{\frac{3}{2}} + c =$

$= \frac{1}{2} \cdot t^{\frac{3}{2}} \cdot \frac{2}{3} + c = \frac{2}{6} \cdot t^{\frac{3}{2}} + c = \frac{1}{3} \cdot t^{\frac{3}{2}} + c = \frac{1}{3} \cdot (1+2x)^{\frac{3}{2}} + c = \frac{(1+2x)^{\frac{3}{2}}}{3} + c$

5.3. Určité integrály

1) $\int_1^4 3\sqrt{x}\,dx = 3\int_1^4 \sqrt{x}\,dx = 3\cdot\int_1^4 x^{\frac{1}{2}}\,dx = 3\left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right)_1^4 = 3\cdot\left(x^{\frac{3}{2}}\cdot\frac{2}{3}\right)_1^4 = 3\left(\sqrt{x^3}\cdot\frac{2}{3}\right)_1^4 =$

$= 3\cdot\left(\frac{2\cdot\sqrt{x^3}}{3}\right)_1^4 = 3\cdot\left[\left(\frac{2\cdot\sqrt{4^3}}{3}\right)-\left(\frac{2\cdot\sqrt{1^3}}{3}\right)\right] = 3\cdot\left[\frac{16}{3}-\frac{2}{3}\right] = 3\cdot\frac{14}{3} = \mathbf{14}$

2) $\int_2^5 \frac{4}{x}\,dx = \int_2^5 4\cdot\frac{1}{x}\,dx = 4\int_2^5 \cdot\frac{1}{x}\,dx = 4(\ln x)_2^5 = 4[(\ln 5)-(\ln 2)] = \mathbf{4\cdot\ln\frac{5}{2}}$

$(\to \ln x - \ln y) = \ln\frac{x}{y})$

3) $\int_0^{\pi} 5\sin 4x\,dx = 5\int_0^{\pi} \sin 4x = \left[\frac{5-\cos 4x}{4}\right]_0^{\pi} = \left(\frac{5-\cos 4\pi}{4}\right) - \left(\frac{5-\cos 0}{4}\right) = -\frac{5}{4}+\frac{5}{4} = \mathbf{0}$

4) $\int_0^{\pi/2} \sin x\,dx = [-\cos x]_0^{\pi/2} = (-\cos\frac{\pi}{2})-(-\cos 0) = 0-(-1) = \mathbf{1}$

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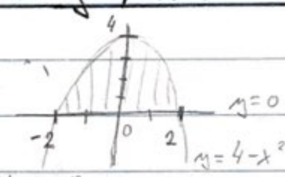
5.4. Aplikace určitého integrálu - výpočet plochy pod křivkou

1) $y = 4-x^2$; $y = 0$



$0 = 4-x^2$

$x = 2, -2$

$24-8 \qquad -24+8$

$\int_{-2}^{2} 4-x^2-0 = \left[\frac{12x-x^3}{3}\right]_{-2}^{2} = \left(\frac{12\cdot 2-2^3}{3}\right) - \left(\frac{12(-2)-(-2)^3}{3}\right) = \mathbf{\frac{32}{3}}$

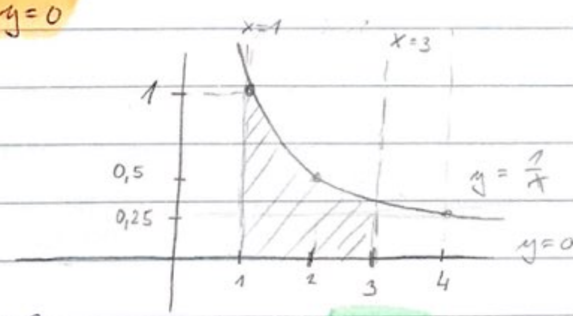
2) $yx = 1$ $\quad x = 1$ $\quad x = 3$ $\quad y = 0$

$y = \frac{1}{x}$

$\frac{1}{1} = 1$

$\frac{1}{2} = 0{,}5$

$\frac{1}{4} = 0{,}25$



$\int_1^3 \frac{1}{x} - 0\,dx = [\ln x]_1^3 = \ln 3 - \underset{0}{\underset{\parallel}{\ln 1}} = \mathbf{\ln 3}$

* 5) $\int_1^2 \frac{2(1+\ln x)}{x}\,dx$