Equation (7.1) states that the values of X at all times prior to $n-1$ have no effect whatsoever on the conditional probability distribution of X_n given X_{n-1}. Thus a Markov process has memory of its past values, but only to a limited extent.

The collection of quantities

$$\Pr\{X_n = s_{k_n} | X_{n-1} = s_{k_{n-1}}\}$$

for various n, s_{k_n} and $s_{k_{n-1}}$, is called the set of one-time-step **transition probabilities**. It will be seen later (Section 8.4) that these provide a **complete description** of the Markov process, for with them the joint distribution function of $(X_n, X_{n-1}, \ldots, X_1, X_0)$, or any subset thereof, can be found for any n. Furthermore, one only has to know the initial value of the process (in conjunction with its transition probabilities) to determine the probabilities that it will take on its various possible values at all future times. This situation may be compared with initial-value problems in differential equations, except that here probabilities are determined by the initial conditions.

All the random processes we will study in the remainder of this book are Markov processes. In the present chapter we study simple random walks which are Markov processes in discrete time and with a discrete state space. Such processes are examples of **Markov chains** which will be discussed more generally in the next chapter.

One note concerning terminology. We often talk of the **value of a process** at time t, say, which really refers to the value of a single random variable ($X(t)$), even though a process is a collection of several random variables.

7.2 UNRESTRICTED SIMPLE RANDOM WALK

Suppose a particle is initially at the point $x=0$ on the x-axis. At each subsequent time unit it moves a unit distance to the right, with probability p, or a unit distance to the left, with probability q, where $p+q=1$.

At time unit n let the position of the particle be X_n. The above assumptions yield

$$X_0 = 0, \qquad \text{with probability one,}$$

and in general,

$$X_n = X_{n-1} + Z_n, \qquad n = 1, 2, \ldots,$$

where the Z_n are identically distributed with

$$\Pr\{Z_1 = +1\} = p$$
$$\Pr\{Z_1 = -1\} = q.$$

It is further assumed that the steps taken by the particle are mutually independent random variables.

Definition. The collection of random variables $X = \{X_0, X_1, X_2, \ldots\}$ is called a simple random walk in one dimension. It is 'simple' because the steps take only the values ± 1, in distinction to cases where, for example, the Z_n are continuous random variables.

The simple random walk is a random process indexed by a discrete time parameter ($n = 0, 1, 2, \ldots$) and has a discrete state space because its possible values are $\{0, \pm 1, \pm 2, \ldots\}$. Furthermore, because there are no bounds on the possible values of X, the random walk is said to be **unrestricted**.

Sample paths

Two possible beginnings of sequences of values of X are

$$\{0, +1, +2, +1, 0, -1, 0, +1, +2, +3, \ldots\}$$
$$\{0, -1, 0, -1, -2, -3, -4, -3, -4, -5, \ldots\}$$

The corresponding sample paths are sketched in Fig. 7.2.

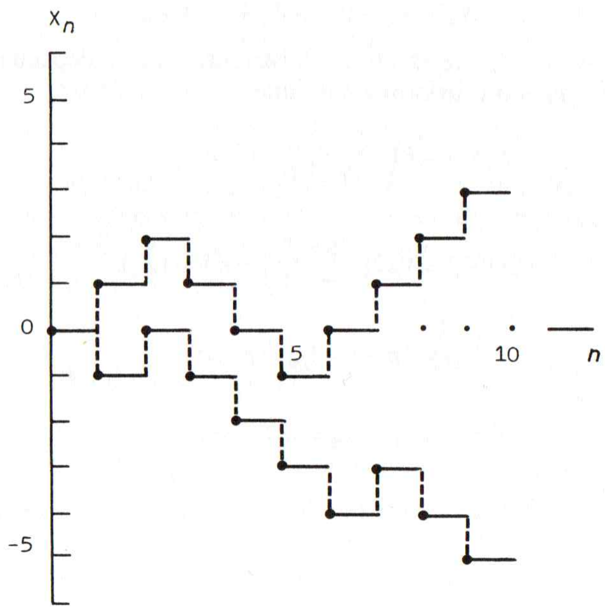

Figure 7.2 Two possible sample paths of the simple random walk.

Markov property

A simple random walk is clearly a Markov process. For example,

$$\Pr\{X_4=2|X_3=3,X_2=2,X_1=1,X_0=0\} = \Pr\{X_4=2|X_3=3\} = \Pr\{Z_4=+1\}=q.$$

That is, the probability is q that X_4 has the value 2 given that $X_3=3$, regardless of the values of the process at epochs 0, 1, 2.

The one-time-step **transition probabilities** are

$$p_{jk}=\Pr\{X_n=k|X_{n-1}=j\}=\begin{cases}p, & \text{if } k=j+1\\ q, & \text{if } k=j-1\\ 0, & \text{otherwise}\end{cases}$$

and in this case these do not depend on n.

Mean and variance

We first observe that

$$\begin{aligned}X_1&=X_0+Z_1\\ X_2&=X_1+Z_2=X_0+Z_1+Z_2\\ &\vdots\\ X_n&=X_0+Z_1+Z_2+\cdots+Z_n.\end{aligned}$$

Then, because the Z_n are identically distributed and independent random variables and $X_0=0$ with probability one,

$$E(X_n)=E\left(\sum_{k=1}^{n} Z_k\right)=nE(Z_1)$$

and

$$\operatorname{Var}(X_n)=\operatorname{Var}\left(\sum_{k=1}^{n} Z_k\right)=n\operatorname{Var}(Z_1).$$

Now,

$$E(Z_1)=1p+(-1)q=p-q$$

and

$$E(Z_1^2)=1p+1q=p+q=1.$$

Thus

$$\begin{aligned}\operatorname{Var}(Z_1)&=E(Z_1^2)-E^2(Z_1)\\ &=1-(p-q)^2\\ &=1-(p^2+q^2-2pq)\\ &=1-(p^2+q^2+2pq)+4pq\\ &=4pq,\end{aligned}$$

since $p^2+q^2+2pq=(p+q)^2=1$. Hence we arrive at the following expressions for the mean and variance of the process at epoch n:

$$E(X_n)=n(p-q) \tag{7.2}$$

$$\operatorname{Var}(X_n)=4npq \tag{7.3}$$

We see that the mean and variance grow linearly with time.

The probability distribution of X_n

Let us derive an expression for the probability distribution of the random variable X_n, the value of the process (or x-coordinate of the particle) at time $n\geqslant 1$. That is, we seek

$$p(k,n)=\Pr\{X_n=k\},$$

where k is an integer.

We first note that $p(k,n)=0$ if $n<|k|$ because the process cannot get to level k in less than $|k|$ steps. Henceforth, therefore, $n\geqslant|k|$.

Of the n steps let the number of magnitude $+1$ be N_n^+ and the number of magnitude -1 be N_n^-, where N_n^+ and N_n^- are random variables. We must have

$$X_n=N_n^+-N_n^-$$

and

$$n=N_n^++N_n^-.$$

Adding these two equations to eliminate N_n^- yields

$$N_n^+=\tfrac{1}{2}(n+X_n). \tag{7.4}$$

Thus $X_n=k$ if and only if $N_n^+=\frac{1}{2}(n+k)$. We note that N_n^+ is a binomial random variable with parameters n and p. Also, since from (7.4), $2N_n^+=n+X_n$ is necessarily even, X_n must be even if n is even and X_n must be odd if n is odd. Thus we arrive at

$$p(k,n)=\binom{n}{(k+n)/2}p^{(k+n)/2}q^{(n-k)/2};$$

$n\geqslant|k|$, k and n either both even or both odd.

For example, the probability that the particle is at $k=-2$ after $n=4$ steps is

$$p(-2,4)=\binom{4}{1}pq^3=4pq^3. \tag{7.5}$$

This will be verified graphically in Exercise 3.

Approximate probability distribution

If $X_0 = 0$, then

$$X_n = \sum_{k=1}^{n} Z_k,$$

where the Z_k are i.i.d. random variables with finite means and variances. Hence, by the central limit theorem (Section 6.4),

$$\frac{X_n - E(X_n)}{\sigma(X_n)} \xrightarrow{d} N(0,1)$$

as $n \to \infty$. Since $E(X_n)$ and $\sigma(X_n)$ are known from (7.2) and (7.3), we have

$$\frac{X_n - n(p-q)}{\sqrt{4npq}} \xrightarrow{d} N(0,1).$$

Thus for example,

$$\Pr\{n(p-q) - 1.96\sqrt{4npq} < X_n < n(p-q) + 1.96\sqrt{4npq}\} \simeq 0.95.$$

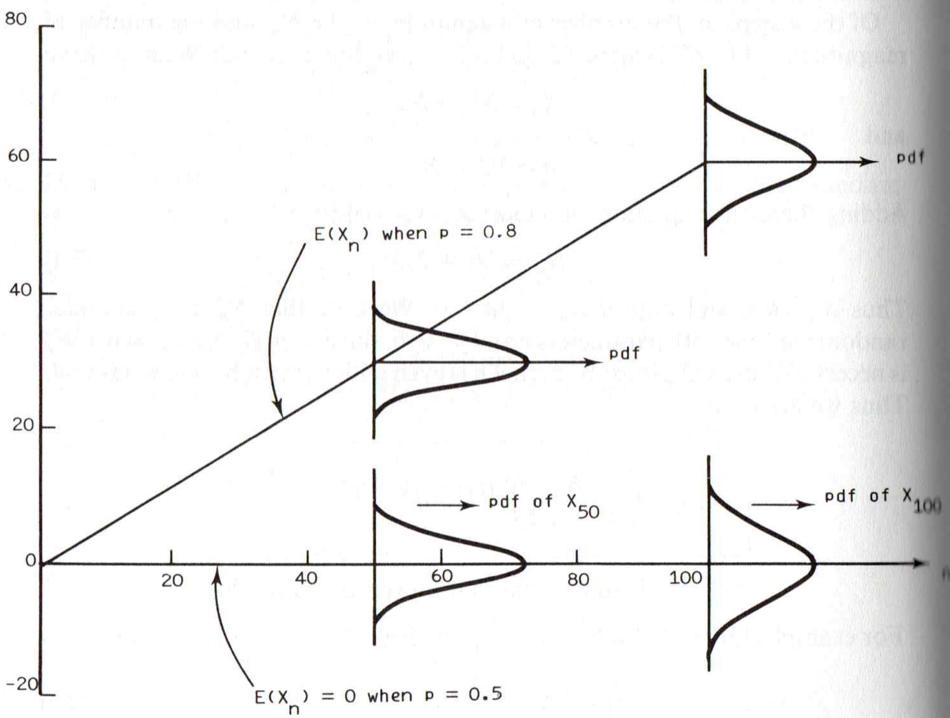

Figure 7.3 Mean of the random walk versus n for $p = 0.5$ and $p = 0.8$ and normal density approximations for the probability distributions of the process at epochs $n = 50$ and $n = 100$.

After $n = 10\,000$ steps with $p = 0.6$, $E(X_n) = 2000$ and

$$\Pr\{1808 < X_{10\,000} < 2192\} \simeq 0.95,$$

whereas when $p = 0.5$ the mean is 0 and

$$\Pr\{-196 < X_{10\,000} < 196\} \simeq 0.95.$$

Figure 7.3 shows the growth of the mean with increasing n and the approximating normal densities at $n = 50$ and $n = 100$ for various p.

7.3 RANDOM WALK WITH ABSORBING STATES

The paths of the process considered in the previous section increase or decrease at random, indefinitely. In many important applications this is not the case as particular values have special significance. This is illustrated in the following classical example.

A simple gambling game

Let two gamblers, A and B, initially have \$$a$ and \$$b$, respectively, where a and b are positive integers. Suppose that at each round of their game, player A wins \$1 from B with probability p and loses \$1 to B with probability $q = 1 - p$. The total capital of the two players at all times is

$$c = a + b.$$

Let X_n be player A's capital at round n where $n = 0, 1, 2, \ldots$ and $X_0 = a$. Let Z_n be the amount A wins on trial n. The Z_n are assumed to be independent.

It is clear that as long as both players have money left,

$$X_n = X_{n-1} + Z_n, \qquad n = 1, 2, \ldots,$$

where the Z_n are i.i.d. as in the previous section. Thus $\{X_n, n = 0, 1, 2, \ldots\}$ is a simple random walk but there are now some restrictions or boundary conditions on the values it takes.

Absorbing states

Let us assume that A and B play until one of them has no money left; i.e., has 'gone broke'. This may occur in two ways. A's capital may reach zero or A's capital may reach c, in which case B has gone broke. The process $X = \{X_0, X_1, X_2, \ldots\}$ is thus restricted to the set of integers $\{0, 1, 2, \ldots, c\}$ and it terminates when either the value 0 or c is attained. The values 0 and c are called absorbing states, or we say there are **absorbing barriers** at 0 and c. Figure 7.4 shows plots of A's capital X_n versus trial number for two possible

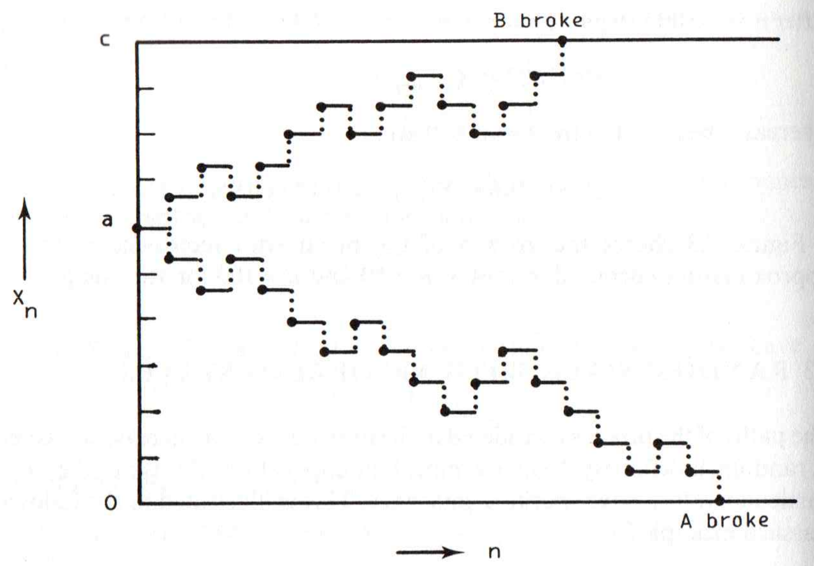

Figure 7.4 Two sample paths of a simple random walk with absorbing barriers at 0 and c. The upper path results in absorption at c (corresponding to player A winning all the money) and the lower one in absorption at 0 (player A broke).

games. One of these sample paths leads to absorption of X at 0 and the other to absorption at c.

7.4 THE PROBABILITIES OF ABSORPTION AT 0

Let P_a, $a = 0, 1, 2, \ldots, c$ denote the probabilities that player A goes broke when his initial capital is \$$a$. Equivalently P_a is the probability that X is absorbed at 0 when $X_0 = a$. The calculation of P_a is referred to as a **gambler's ruin problem**. We will obtain a difference equation for P_a.

First, however, we observe that the following boundary conditions must apply:

$$\boxed{\begin{aligned} P_0 &= 1 \\ P_c &= 0 \end{aligned}}$$

since if $a = 0$ the probability of absorption at 0 is one whereas if $a = c$, absorption at c has already occurred and absorption at 0 is impossible.

Now, when a is not equal to either 0 or c, all games can be divided into two mutually exclusive categories:

(i) A wins the first round;
(ii) A loses the first round.

Thus the event $\{A$ goes broke from $a\}$ is the union of two mutually exclusive events:

$$\begin{aligned} &\{A \text{ goes broke from } a\} = \\ &\{A \text{ wins the first round and goes broke from } a+1\} \\ &\quad \cup \{A \text{ loses the first round and goes broke from } a-1\}. \end{aligned} \tag{7.6}$$

Also, since going broke after winning the first round and winning the first round are independent,

$$\begin{aligned} &\Pr\{A \text{ wins the first round and goes broke from } a+1\} \\ &= \Pr\{A \text{ wins the first round}\} \Pr\{A \text{ goes broke from } a+1\} \\ &= pP_{a+1}. \end{aligned} \tag{7.7}$$

Similarly,

$$\begin{aligned} &\Pr\{A \text{ loses the first round and goes broke from } a-1\} \\ &= qP_{a-1}. \end{aligned} \tag{7.8}$$

Since the probability of the union of two mutually exclusive events is the sum of their individual probabilities, we obtain from (7.6)–(7.8), the key relation

$$\boxed{P_a = pP_{a+1} + qP_{a-1}}, \qquad a = 1, 2, \ldots, c-1. \tag{7.9}$$

This is a difference equation for P_a which we will solve subject to the above boundary conditions.

Solution of the difference equation (7.9)

There are three main steps in solving (7.9).

(i) *The first step is to rearrange the equation*

Since $p + q = 1$, we have

$$(p+q)P_a = pP_{a+1} + qP_{a-1},$$

or

$$p(P_{a+1} - P_a) = q(P_a - P_{a-1}).$$

Dividing by p and letting

$$r = \frac{q}{p}$$

gives

$$P_{a+1} - P_a = r(P_a - P_{a-1}).$$

It is seen that when $p = q$ and $c = 10$ and both players in the gambling game start with the same capital, the expected duration of the game is 25 rounds. If the total capital is $c = 1000$ and is equally shared by the two players to start with, then the average duration of their game is 250 000 rounds!

Finally we note that when $c = \infty$, the expected times to absorption are

$$\boxed{D_a = \begin{cases} \dfrac{a}{q-p}, & p < q \\ \infty, & p \geq q \end{cases}} \tag{7.24}$$

as will be proved in Exercise 13.

7.8 SMOOTHING THE RANDOM WALK – THE WIENER PROCESS AND BROWNIAN MOTION

In Fig. 7.8a are shown portions of two possible sample paths of a simple unrestricted random walk with steps up or down of equal magnitudes. The illustrations in Fig. 7.8b–f were obtained by successive reductions of Fig. 7.8a. In (a), the 'steps' are discernible, but after several reductions the paths become smooth in appearance. In terms of the position and time scales in (a), the steps in (f) are very small and so is the time between them. The point of this is to illustrate that paths may be discontinuous but appear quite smooth when viewed from a distance.

Consider the time interval $(0, t]$. Subdivide this into subintervals of length Δt so that there are $t/\Delta t$ such subintervals. We now suppose that a particle, initially at $x = 0$, makes a step (in one space dimension) at the times $\Delta t, 2\Delta t, \ldots$, and that the size of the step is either $+\Delta x$ or $-\Delta x$, the probability being 1/2 that the move is to the left or the right. Thus the position of the particle, $X(t)$, at time t, is a random walk which has executed $t/\Delta t$ steps. Since the position will depend on the choice of Δt and Δx, we write the position as $X(t; \Delta t, \Delta x)$.

We may write

$$X(t; \Delta t, \Delta x) = \sum_{i=1}^{t/\Delta t} Z_i, \tag{7.25}$$

where the Z_i are independent and identically distributed with

$$\Pr[Z_i = +\Delta x] = \Pr[Z_i = -\Delta x] = 1/2, \qquad i = 1, 2, \ldots.$$

For the Z_i we have,

$$E[Z_i] = 0,$$

and

$$\mathrm{Var}[Z_i] = E[Z_i^2] = (\Delta x^2).$$

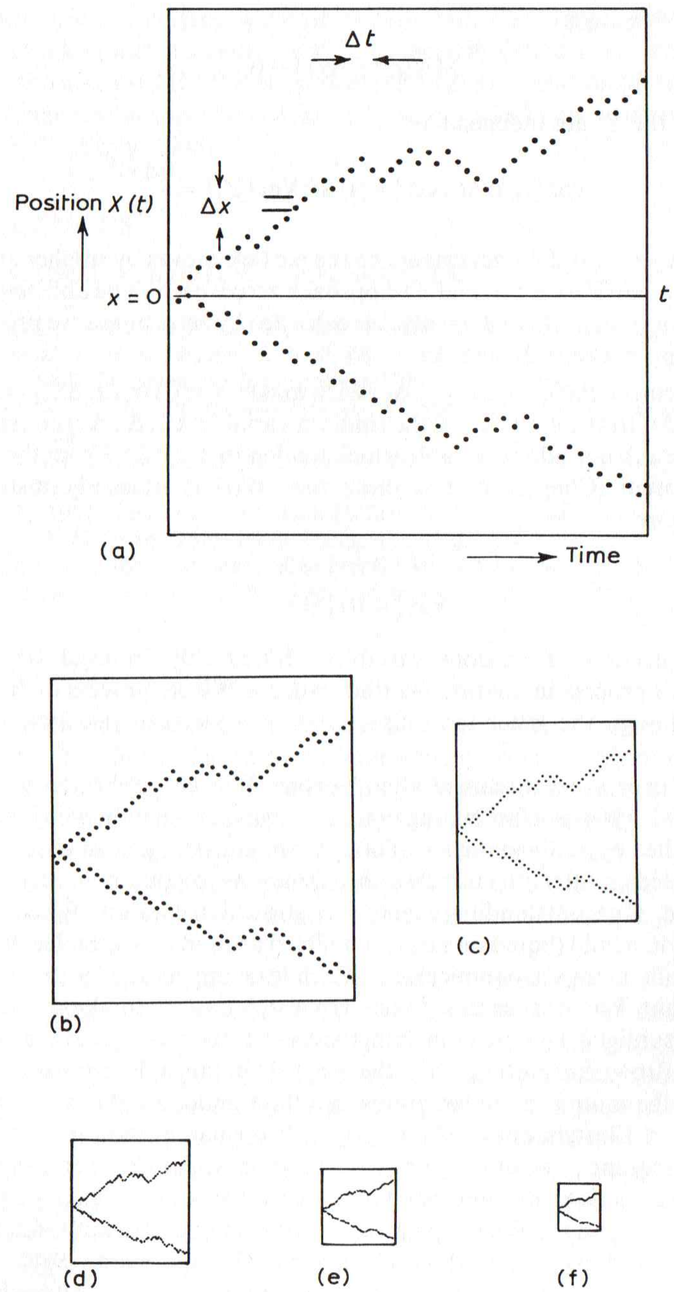

Figure 7.8 In (a) are shown two sample paths of a random walk, (b) to (f) were obtained by successive reductions of (a).

From (7.25) we get

$$E[X(t;\Delta t,\Delta x)]=0,$$

and since the Z_i are independent,

$$\mathrm{Var}[X(t;\Delta t,\Delta x)]=(t/\Delta t)\,\mathrm{Var}[Z_i]=\frac{t(\Delta x)^2}{\Delta t}.$$

Now we let Δt and Δx get smaller so the particle moves by smaller amounts but more often. If we let Δt and Δx approach zero we won't be able to find the limiting variance as this will involve zero divided by zero, unless we prescribe a relationship between Δt and Δx.

A convenient choice is $\Delta x=\sqrt{\Delta t}$ which makes $\mathrm{Var}[X(t;\Delta t,\Delta x)]=t$ for all values of Δt. In the limit $\Delta t\to 0$ the random variable $X(t;\Delta t,\Delta x)$ converges in distribution to a random variable which we denote by $W(t)$. From the central limit theorem (Chapter 6) it is clear that $W(t)$ is normally distributed. Furthermore,

$$E[W(t)]=0$$
$$\mathrm{Var}[W(t)]=t.$$

The collection of random variables $\{W(t),t\geqslant 0\}$, indexed by t, is a continuous process in continuous time called a **Wiener process** or **Brownian motion**, though the latter term also refers to a physical phenomenon (see below).

The Wiener process (named after Norbert Wiener, celebrated mathematician, 1894–1964) is a fascinating mathematical construction which has been much studied by mathematicians. Though it might seem just an abstraction, it has provided useful mathematical approximations to random processes in the real world. One outstanding example is Brownian motion. When a small particle is in a fluid (liquid or gas) it is buffeted around by the molecules of the fluid, usually at an astronomical rate. Each little impact moves the particle a tiny amount. You can see this if you ever watch dust or smoke particles in a stream of sunlight. This phenomenon, the erratic motion of a particle in a fluid, is called Brownian motion after the English botanist Robert Brown who observed the motion of pollen grains in a fluid under a light microscope. In 1905, Albert Einstein obtained a theory of Brownian motion using the same kind of reasoning as we did in going from random walk to Wiener process. The theory was subsequently confirmed by the experimental results of Perrin. For further reading on the Wiener process see, for example, Parzen (1962), and for more advanced aspects, Karlin and Taylor (1975) and Hida (1980).

Random walks have also been employed to represent the voltage in nerve cells (neurons). A step up in the voltage is called **excitation** and a step down is called **inhibition**. Also, there is a critical level (threshold) of excitation of which

the cell emits a travelling wave of voltage called an **action potential**. The random walk model of a neuron was introduced by Gerstein and Mandelbrot (1964), who also used the Wiener process as an approximation for the voltage. Many other neural models have since been proposed and analysed (see, for example, Tuckwell, 1988).

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EXERCISES

1. Given physical examples of the four kinds of random process ((a)–(d) in Section 7.1). State in each case whether the process is a Markov process.
2. Let $X=\{X_0,X_1,X_2,\dots\}$ be a random process in discrete time and with a discrete state space. Given that successive **increments** X_1-X_0, $X_2-X_1,\dots$ are independent, show that X is a Markov process.
3. For a simple random walk enumerate all possible sample paths that lead to the value $X_4=-2$ after 4 steps. Hence verify formula (7.5) for $\Pr(X_4=-2)$.
4. Let $X_n=X_{n-1}+Z_n$, $n=1,2,\dots$, describe a random walk in which the Z_n are independent normal random variables each with mean μ and variance σ^2. Find the exact probability law of X_n if $X_0=x_0$ with probability one.
5. In certain gambling situations (e.g. horse racing, dogs) the following is an approximate description. At each trial a gambler bets \$$m$, assumed fixed. With probability q he loses all the \$$m$ and with probability $p=1-q$ he wins back his \$$m$ plus a profit on each dollar which is a random variable with mean μ and variance σ^2. Let X_n be the gambler's fortune after n bets. Deduce that $\{X_0,X_1,X_2,\dots\}$ is a random walk with $X_0=x_0$, the gambler's initial capital, and

$$X_n=X_{n-1}+Z_n,\qquad n=1,2,\dots,$$
$$Z_n=m[I_nY_n+(1-I_n)],$$