# Additional study material for the course of global analysis

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Autumn semester 2019 / 2020

**Exercise.** $\mathbb{RP}^n$ *is orientable* $\iff$ *n is odd.*

*Solution.*

- Consider $\mathbb{R}^{n+1}$ with its standard orientation. Then the linear map $A\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ given by $x \mapsto A(x) = -x$ is orientation preserving $\iff \det A = (-1)^{n+1} > 0 \iff n$ is odd.
- Now consider $S^n \subset \mathbb{R}^{n+1}$. Note that $\nu(x) = x$ defines a global unit normal vector field for $S^n \subset \mathbb{R}^{n+1}$, which in turn determines an orientation on $S^n$ as follows: a basis $(\zeta_1, \zeta_2, \ldots, \zeta_n)$ of $\mathrm{T}_x S^n$ is positively oriented $\iff (\nu(x), \zeta_1, \zeta_2, \ldots, \zeta_n)$ is positively oriented basis of $\mathrm{T}_x \mathbb{R}^{n+1} \cong \mathbb{R}^{n+1}$. Let us fix this orientation on $S^n$.
- $A|_{S^n}\colon S^n \to S^n$ is orientation preserving $\iff n$ is odd.
- Consider the projection

$$\pi\colon S^n \to S^n/_\sim \cong \mathbb{RP}^n$$

under which antipodal points on $S^n$ get identified, i.e. $x \sim A(x) = -x$. Then one can check that $\pi$ is a local diffeomorphism, which implies the isomorphism of tangent spaces

$$\mathrm{T}_x \pi\colon\ \mathrm{T}_x S^n \xrightarrow{\cong} \mathrm{T}_{\pi(x)} \mathbb{RP}^n \quad \forall x \in S^n \tag{1}$$

Moreover, note that $\pi \circ A = \pi$ which implies $\mathrm{T}(\pi \circ A) = T\pi$. Thus we can try to define orientation on $\mathbb{RP}^n$ by requiring (1) to be orientation preserving $\forall x \in S^n$. This will lead to a well-defined orientation $\iff$ $A$ is orientation preserving.

**Exercise.** *Suppose* $(M, \nabla)$ *is a smooth manifold equipped with an affine connection. Show that*

1. *the torsion* $\mathrm{T}$ *and curvature* $\mathrm{R}$ *of* $\nabla$ *are* $(1,2)$ *and* $(1,3)$ *tensor fields on* $M$
2. *if* $\nabla$ *is torsion-free, then for all* $\xi, \eta, \zeta \in \Gamma(TM)$ *the Bianchi identity holds:*

$$\mathrm{R}(\xi,\eta)\zeta + \mathrm{R}(\eta,\zeta)\xi + \mathrm{R}(\zeta,\xi)\eta = 0 \tag{2}$$

Recall that a connection $\nabla$ on a smooth manifold $M$ is a bilinear operator on the space of vector fields $\nabla\colon \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM), (\xi, \eta) \mapsto \nabla_\xi \eta$ satisfying

- $\nabla_{f\xi} \eta = f \nabla_\xi \eta$ for all $f \in C^\infty(M)$
- $\nabla_\xi (f\eta) = \xi(f)\eta + f \nabla_\xi \eta$

*Solution.* 1. Torsion T of $\nabla$ is given by

$$\mathrm{T}(\xi, \eta) := \nabla_\xi \eta - \nabla_\eta \xi - [\xi, \eta] \ , \tag{3}$$

where $[-,-]$ is the usual bracket on vector fields. Since $\mathrm{T}(\xi, \eta)$ is a vector field, it is automatically a $(1, 0)$ tensor field on $M$. Then showing that T is bilinear with respect to the ring of smooth functions $C^\infty(M)$ suffices to show that it is a $(1, 2)$ tensor field. Consider arbitrary $f \in C^\infty(M)$ and $\xi, \eta, \zeta \in \Gamma(TM)$ and compute

$$\begin{aligned} T(f\xi + \eta, \zeta) &= \nabla_{f\xi+\eta} \zeta - \nabla_\zeta (f\xi + \eta) - [f\xi + \eta, \zeta] \\ &= f \nabla_\xi \zeta + \nabla_\eta \zeta - \nabla_\zeta (f\xi) - \nabla_\zeta \eta + [\zeta, f\xi] - [\eta, \zeta] \\ &= f \nabla_\xi \zeta + \nabla_\eta \zeta - \zeta(f)\xi - f \nabla_\zeta \xi - \nabla_\zeta \eta + \zeta(f)\xi + f[\zeta, \xi] - [\eta, \zeta] \\ &= f(\nabla_\xi \zeta - \nabla_\zeta \xi - [\xi, \zeta]) + \nabla_\eta \zeta - \nabla_\zeta \eta - [\eta, \zeta] \\ &= fT(\xi, \zeta) + T(\eta, \zeta) \end{aligned}$$

Similarly for the second argument.

For arbitrary $\xi, \eta, \zeta \in \Gamma(TM)$, the curvature R of $\nabla$ is given by

$$\mathrm{R}(\xi, \eta)\zeta := \nabla_\xi \nabla_\eta \zeta - \nabla_\eta \nabla_\xi \zeta - \nabla_{[\xi,\eta]} \zeta \tag{4}$$

Observe that $\mathrm{R}(\xi, \eta)\zeta$ is a vector field and hence a $(1, 0)$ tensor field. We pick arbitrary $f \in C^\infty(M)$ and $\xi, \eta, \zeta, \phi \in \Gamma(TM)$. It is easy to check that

$$\mathrm{R}(\xi + \phi, \eta)\zeta = \mathrm{R}(\xi, \eta)\zeta + \mathrm{R}(\phi, \eta)\zeta$$

and similarly for other arguments. Now we want to show that $\mathrm{R}(f\xi, \eta)\zeta = \mathrm{R}(\xi, f\eta)\zeta = \mathrm{R}(\xi, \eta)(f\zeta) = f\,\mathrm{R}(\xi, \eta)\zeta$. We start with the first argument and then, since the computation for the second argument is similar, we proceed with the third argument.

$$\begin{aligned} \mathrm{R}(f\xi, \eta)\zeta &= \nabla_{f\xi} \nabla_\eta \zeta - \nabla_\eta \nabla_{f\xi} \zeta - \nabla_{[f\xi,\eta]} \zeta \\ &= f \nabla_\xi \nabla_\eta \zeta - \nabla_\eta f \nabla_\xi \zeta - \nabla_{-[\eta,f\xi]} \zeta \\ &= f \nabla_\xi \nabla_\eta \zeta - (\eta(f) \nabla_\xi \zeta + f \nabla_\eta \nabla_\xi \zeta) - \nabla_{-\eta(f)\xi - f[\eta,\xi]} \\ &= \nabla_\xi \nabla_\eta \zeta - \eta(f) \nabla_\xi \zeta - f \nabla_\eta \nabla_\xi \zeta + \eta(f) \nabla_\xi \zeta - f \nabla_{[\xi,\eta]} \zeta \\ &= f(\nabla_\xi \nabla_\eta \zeta - \nabla_\eta \nabla_\xi \zeta - \nabla_{[\xi,\eta]} \zeta) \\ &= f\,\mathrm{R}(\xi, \eta)\zeta \end{aligned}$$

and the computation for the third argument

$$\begin{aligned}
\mathrm{R}(\xi,\eta)(f\zeta) &= \nabla_\xi \nabla_\eta (f\zeta) - \nabla_\eta \nabla_\xi (f\zeta) - \nabla_{[\xi,\eta]}(f\zeta) \\
&= \nabla_\xi(\eta(f)\zeta + f \nabla_\eta \zeta) - \nabla_\eta(\xi(f)\zeta + f \nabla_\xi \zeta) - [\xi,\eta](f)\zeta - f \nabla_{[\xi,\eta]} \zeta \\
&= \xi(\eta(f))\zeta + \eta(f) \nabla_\xi \zeta + \xi(f) \nabla_\eta \zeta + f \nabla_\xi \nabla_\eta \zeta \\
&\quad - (\eta(\xi(f))\zeta + \xi(f) \nabla_\eta \zeta + \eta(f) \nabla_\xi \zeta + f \nabla_\eta \nabla_\xi \zeta) \\
&\quad - (\xi(\eta(f))\zeta - \eta(\xi(f))\zeta - f \nabla_{[\xi,\eta]} \zeta \\
&= f(\nabla_\xi \nabla_\eta \zeta - \nabla_\eta \nabla_\xi \zeta - \nabla_{[\xi,\eta]} \zeta) \\
&= f\, \mathrm{R}(\xi,\eta)\zeta
\end{aligned}$$

*Solution.* 2. Let us express the left-hand side of (2) using $\nabla$

$$\begin{aligned}
\mathrm{R}(\xi,\eta)\zeta + \mathrm{R}(\eta,\zeta)\xi + \mathrm{R}(\zeta,\xi)\eta = (&\nabla_\xi \nabla_\eta - \nabla_\eta \nabla_\xi - \nabla_{[\xi,\eta]})\zeta \\
&+ (\nabla_\eta \nabla_\zeta - \nabla_\zeta \nabla_\eta - \nabla_{[\eta,\zeta]})\xi \\
&+ (\nabla_\zeta \nabla_\xi - \nabla_\xi \nabla_\zeta - \nabla_{[\zeta,\xi]})\eta
\end{aligned}$$

The righ-hand side of the last equality is a sum of the following terms

$$\nabla_\xi(\nabla_\eta \zeta - \nabla_\zeta \eta) - \nabla_{[\eta,\zeta]} \xi \tag{5}$$

$$\nabla_\eta(\nabla_\zeta \xi - \nabla_\xi \zeta) - \nabla_{[\zeta,\xi]} \eta \tag{6}$$

$$\nabla_\zeta(\nabla_\xi \eta - \nabla_\eta \xi) - \nabla_{[\xi,\eta]} \zeta \tag{7}$$

For a torsion free connection we have $T(\xi,\eta) = 0$ which is by (3) equivalent to

$$[\xi,\eta] = \nabla_\xi \eta - \nabla_\eta \xi\ . \tag{8}$$

If we apply (8) on (5), (6) and (7) and sum up we get

$$\nabla_\xi[\eta,\zeta] - \nabla_{[\eta,\zeta]} \xi + \nabla_\eta[\zeta,\xi] - \nabla_{[\zeta,\xi]} \eta + \nabla_\zeta[\xi,\eta] - \nabla_{[\xi,\eta]} \zeta$$

Using (8) again we obtain

$$\mathrm{R}(\xi,\eta)\zeta + \mathrm{R}(\eta,\zeta)\xi + \mathrm{R}(\zeta,\xi)\eta = [\xi,[\eta,\zeta]] + [\eta,[\zeta,\xi]] + [\zeta,[\xi,\eta]]$$

Since Lie bracket satisfies Jacobi identity, the right-hand side of the last equality is zero. We conclude

$$\mathrm{R}(\xi,\eta)\zeta + \mathrm{R}(\eta,\zeta)\xi + \mathrm{R}(\zeta,\xi)\eta = 0\ .$$