C7790 Introduction to Molecular Modelling
TSM Modelling Molecular Structures

Lesson 6
Thermodynamics & Modelling
Statistical Thermodynamics

PS/2020 Distant Form of Teaching: Rev1

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Fundamental relation

$$\Delta G_r^{0} = -RT \ln K$$

$$\Delta G_r^0 = c\Delta G_{f,C}^0 + d\Delta G_{f,D}^0 - \left(a\Delta G_{f,A}^0 + b\Delta G_{f,B}^0\right)$$

$$K = \frac{[C]_r^c [D]_r^d}{[A]_r^a [B]_r^b}$$

What do we need to know?

A

B

C

D

We only need to know the **properties of individual components** involved in the reaction at standard conditions (or at different conditions, which are well defined).

solution at equilibrium

We need to know the **composition of solution** at equilibrium.

Thermodynamics & Modelling

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easier for modelling

A C B D D C C D D B

solution at equilibrium

We need to know the **composition of solution** at equilibrium.

It is hard or impossible to model.

Overview

macroworld **microworld**

states

(thermodynamic properties, G, T,…)

phenomenological thermodynamics

equilibrium (equilibrium constant)
kinetics (rate constant)

free energy
(Gibbs/Helmholtz)

partition function

statistical thermodynamics

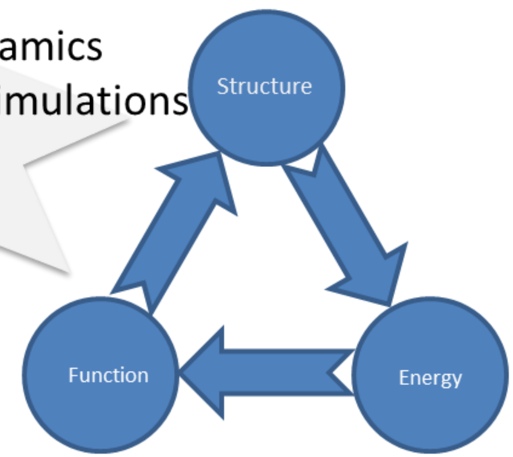
microstates
(mechanical properties, E)

Description levels (model chemistry):

- quantum mechanics
  - semiempirical methods
  - ab initio methods
  - post-HF methods
  - DFT methods
- molecular mechanics
- coarse-grained mechanics

Simulations:

- molecular dynamics
- Monte Carlo simulations
- docking
- …



microstate ≠ microworld

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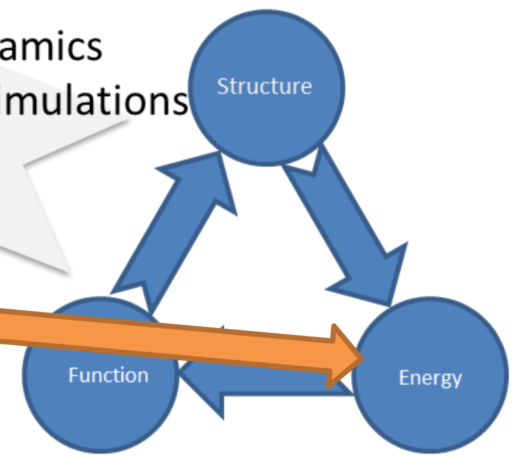

microstate ≠ microworld

Statistical Thermodynamics

Or what you should already know….

Two approaches - I

1. Phenomenological approach:

Thermodynamics examines the interrelationships between quantities that characterize the macroscopic state of the system and changes in these quantities in physical processes. **Many of the features of the system can be clarified without a thorough knowledge of its internal structure.** It is based on several axiomatically pronounced (and experimentally confirmed) laws, which, in connection with the known properties of the system, served to derive other properties and relationships. The state of the system is described using state functions and equations, which determine the relationships between individual state functions.

Level of description:

- state functions
- state equations
- thermodynamic theorems

wikipedia.cz, simplified

Two approaches - II

2. Statistical approach:

Statistical physics (statistical mechanics) relates two levels of description of physical reality, namely the macroscopic and microscopic levels. In a more traditional sense, it deals with the study of the properties of macroscopic systems or systems, considering the microscopic structure of these systems (**statistical thermodynamics**). The founders were Ludwig Boltzmann and Josiah Willard Gibbs.

Level of description:

- particles and interactions between them
- equations of motions

wikipedia.cz, simplified

System properties

The observable value ($\overline{M}$) of the property M can be determined by two approaches:

Time average:

snapshot of the system at time *t* is called a microstate

$$\overline{M} = \frac{1}{t_{tot}} \int_{o}^{t_{tot}} M(t)dt$$

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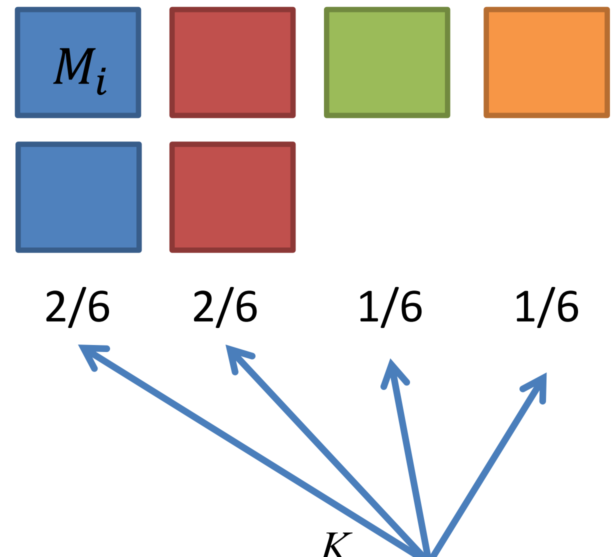
See later: We can run **molecular dynamics simulations** to get value of property by molecular modelling.

Ensemble average:



$$\overline{M} = \sum_{i=1}^{K} p_i M_i$$

See later: We can run **Monte Carlo simulations** to get value of property by molecular modelling.

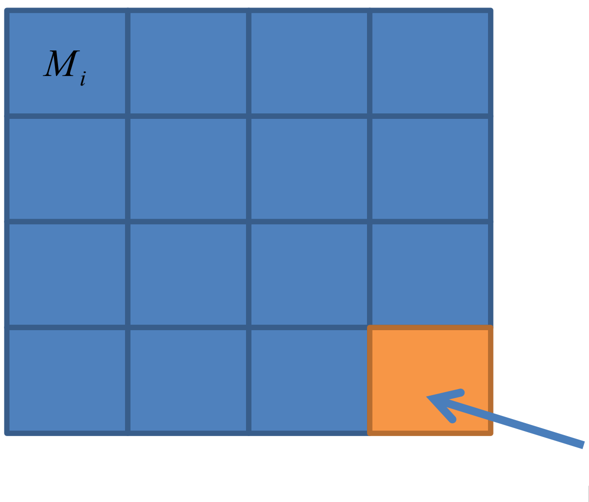
Statistical ensemble

Statistical ensemble (Gibbs ensemble) is **thought construction**, in which the ensemble is formed by **large number of copies of the system (prototype)**, whose thermodynamic properties we want to determine.

Each replica of the prototype is located in exactly one microstate.

Interactions between prototype replicas are very weak (it is practically possible to neglect them), however, sufficient for the ansemble to be located in **thermodynamic equilibrium**.

Statistical view:



- L number of copies of the prototype
- K number of microstates that the prototype can acquire
- n_i number of prototypes in microstate *i*
- p_i the probability of prototype occurrence in the given microstate *i*

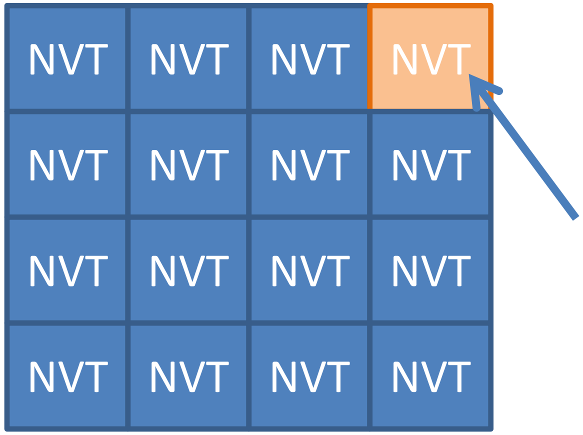
$$\overline{M} = \sum_{i=1}^{K} p_i M_i \qquad p_i = \frac{n_i^*}{L} = ?$$

Types of statistical ensembles

The most common types of statistical ensembles include:

- microcanonical ensemble (NVE) - the prototype contains a constant number of particles, has a constant volume, and energy
- grand canonical ensemble (μVT) - the prototype has a constant chemical potential, volume, and temperature
- **canonical ensemble** (NVT) - the prototype contains a constant number of particles, has a constant volume, and temperature

Canonical ensemble



Consider the system (**prototype**), which has a constant number of molecules, a constant volume, and temperature.

prototype

$$U_e = LU$$
(internal energy)

$$S_e = LS$$
(entropy)

$$F_e = LF$$
(Helmholtz energy)

Two constrains apply:

1. The ensemble total energy is equal to the sum of the energies of the subsystems (the copies do not interact with each other) - energy conservation:

$$E = \sum_{i=1}^{L} E_i = \sum_{i=1}^{K} n_i E_i$$

2. The sum of number of prototypes in given microstate must be constant and equal to L (total number of prototype copies):

$$L = \sum_{i=1}^{K} n_i$$

Entropy of canonical ensemble

It can be shown that the entropy of the statistical ensemble is related to the statistical weight *W*.

$$\boxed{S_e = k_B \ln W}$$

k_B - Boltzmann constant

k_B - is not 1.0 because the **definition of absolute temperature**

The statistical weight *W* determines number of possible ensemble implementations.

Statistical weight

number of all combinations

$$\underbrace{W(n_1,\ldots,n_K)}_{\text{number of microstates}} = \frac{L!}{\prod_{i=1}^{K} n_i!}$$

L - number of copies of the prototype
K - number of microstates
n_i - number of prototypes of the given microstate *i*

correction for indistinguishability of individual microstates

Entropy of canonical ensemble, II

Because L is a large number, there is an implementation for which the statistical weight of the distribution W^* dominates over others.

$$W^*(n_1^*, \dots, n_K^*) >> W_{others}$$

Then, we search for the ensemble composition, in which its entropy reaches maximum value and all imposed constraints are fulfilled.

$$S_e = k_b \ln W\,(n_1^*, \dots, n_K^*) = k_b \ln \frac{L!}{\prod_{i=1}^{K} n_i!} \rightarrow \mathrm{max!}$$

Two constrains apply:

$$E = \sum_{i=1}^{L} E_i = \sum_{i=1}^{K} n_i E_i \qquad L = \sum_{i=1}^{K} n_i$$

Canonical ensemble - solution

The final result:

$$p_i^* = \frac{e^{-\beta E_i}}{\sum_{j=1}^{K} e^{-\beta E_j}} = \frac{e^{-\beta E_i}}{Q}$$

$$\beta = \frac{1}{k_B T}$$

k_B - Boltzmann constant
T - absolute temperature

Canonical partition function:

it is a normalization value

$$\boxed{Q = \sum_{j=1}^{K} e^{-\beta E_j}}$$

Partition function determines various thermodynamic properties.

Value of observable property:

$$\overline{M} = \sum_{i=1}^{K} p_i^* M_i$$

Thermodynamic properties

Canonical ensemble

Internal energy:

$$U = k_B T \left(\frac{\partial \ln Q}{\partial \ln T} \right)_{N,V}$$

$$U = \sum_{i=1}^{K} E_i p_i^* = \frac{\sum_{i=1}^{K} E_i e^{-\beta E_i}}{\sum_{j=1}^{K} e^{-\beta E_j}} = \frac{\sum_{i=1}^{K} E_i e^{-\beta E_i}}{Q}$$

Entropy:

$$S = \frac{U}{T} + k_B \ln Q$$

$$F = U - TS$$

Helmholtz energy F:

$$F = -k_B T \ln Q$$

Canonical partition function:

$$Q = \sum_{j=1}^{K} e^{-\beta E_j}$$

Partition function and modelling

Canonical partition function:

$$Q = \sum_{j=1}^{K} e^{-\beta E_j}$$

Helmholtz energy F:

$$F = -k_B T \ln Q$$

Evaluation of all (or important) microstates

$$E_1, E_2, E_3, \ldots$$

Molecular dynamics (classical continuous system)

For example:

- ideal gas model with contributions from
  - electronic microstates
  - vibration microstates
  - rotation microstates
  - translation microstates
- Monte Carlo simulations

$$Q = \frac{1}{h^3} \iint_{\Omega} e^{-\beta H(\boldsymbol{x},\boldsymbol{p})} \, d\boldsymbol{x} d\boldsymbol{p}$$

h - Planck constant

H - Hamiltonian (energy of the system)

We need to find discrete energies of microstates.

We need to solve describe energy evolution in time.

Partition function and modelling

Canonical partition function:

$$\boxed{Q = \sum_{j=1}^{K} e^{-\beta E_j}}$$

Helmholtz energy F:

$$\boxed{F = -k_B T \ln Q}$$

approximation

Consider only the most important microstate

$$E_1, \cancel{E_2, E_3, \ldots}$$

The most important microstate is the microstate with the lowest energy.

$$F = E_1$$

Very often used for qualitative consideration or when computationally demanding methods are employed (typically quantum chemical calculations).

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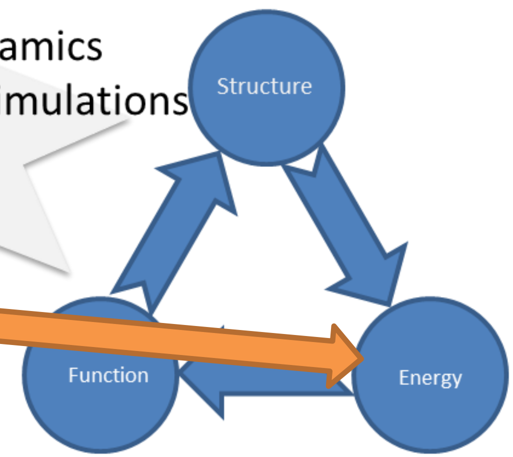

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