Ako integrovat Maxwell-Boltzmannove (MB) rozdelenie

Integraly rozdelovacej funkcie

Makroskopicke veliciny (koncentracia, tlak, teplota,...) su definovane ako integraly z rozdelovacej funkcie.

Napr. pre ziskanie teploty z MB rozdelenia budete potrebovat vyriesit takyto integral:

$$\int_V v^2 f_{\mathrm{MB}}(\vec{v}) \mathrm{d}^3 v = \tag{1}$$

$$K \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (v_x^2 + v_y^2 + v_z^2) \mathrm{e}^{-\frac{m}{2kT}(v_x^2+v_y^2+v_z^2)} \mathrm{d}v_x \mathrm{d}v_y \mathrm{d}v_z \tag{2}$$

Integraly rozdelovacej funkcie

Po uprave budeme musiet vyriesit integraly:

$$\int_{-\infty}^{\infty} v_x^2 \mathrm{e}^{-\frac{m}{2kT}v_x^2} \mathrm{d}v_x \qquad \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{m}{2kT}v_x^2} \mathrm{d}v_x \tag{3}$$

Obecne integraly mozu byt integraly zapisane ako:

$$\int_{v} v^i \mathrm{e}^{-av^2} \mathrm{d}v \qquad i \in \{0, 1, 2, \ldots\} \tag{4}$$

Substitucia $t = v^2$ by vam pomohla len v neparnych mocninach i – tieto integraly sa bez navodu pocitaju tazko.

Symetrie

Obcas je mozne pouzit symetriu, napr.:

$$\int_{-\infty}^{\infty} v\mathrm{e}^{-av^2}\mathrm{d}v = 0, \tag{5}$$

pretoze e^{-av^2} je parna funkcia $\Rightarrow$ $v\mathrm{e}^{-av^2}$ je neparna a kladny prispevok presne vyrusi zaporny prispevok v integrale.

Alebo:

$$\int_{-\infty}^{\infty} v^2\mathrm{e}^{-av^2}\mathrm{d}v = 2\int_{0}^{\infty} v^2\mathrm{e}^{-av^2}\mathrm{d}v \tag{6}$$

pretoze funkcia je symetricka okolo $v = 0$.

Gamma funkcia

V inych pripadoch je uzitocne vediet o Gamma funkcii, ktora je definovana ako integral:

$$\Gamma(z) \equiv \int_0^\infty t^{z-1} \mathrm{e}^{-t} \mathrm{d}t \tag{7}$$

Gamma funkcia ako faktorial

$$\Gamma(z) = \int_0^\infty t^{z-1}\mathrm{e}^{-t}\mathrm{d}t = \left[-t^{z-1}\mathrm{e}^{-t}\right]_0^\infty + \int_0^\infty (z-1)t^{z-2}\mathrm{e}^{-t}\mathrm{d}t = \tag{8}$$

$$= (z-1)\int_0^\infty t^{z-2}\mathrm{e}^{-t}\mathrm{d}t \equiv (z-1)\Gamma(z-1) \tag{9}$$

$$\Gamma(1) = 1 \tag{10}$$

$$\Gamma(2) = 1 \cdot \Gamma(1) \tag{11}$$

$$\Gamma(3) = 2 \cdot \Gamma(2) = 2 \tag{12}$$

$$\Gamma(4) = 3 \cdot \Gamma(3) = 6 \tag{13}$$

Ako to suvisi s nasimi integralmi?

Substitucia je $x^2 = t$:

$$x^2 = t \qquad \Rightarrow \qquad x = \sqrt{t} \qquad\qquad (t \geq 0) \tag{14}$$

$$\Rightarrow \qquad \mathrm{d}t = 2x\mathrm{d}x = 2\sqrt{t}\mathrm{d}x \qquad \Rightarrow \qquad \mathrm{d}x = \frac{1}{2\sqrt{t}}\mathrm{d}t \tag{15}$$

$$f(y) = \int_0^\infty x^y \mathrm{e}^{-x^2}\mathrm{d}x = \int_0^\infty \sqrt{t}^y \mathrm{e}^{-t}\frac{1}{2\sqrt{t}}\mathrm{d}t = \frac{1}{2}\int_0^\infty \sqrt{t}^{y-1}\mathrm{e}^{-t}\mathrm{d}t = \tag{16}$$

$$= \frac{1}{2}\int_0^\infty t^{\frac{y-1}{2}}\mathrm{e}^{-t}\mathrm{d}t = \frac{1}{2}\int_0^\infty t^{\frac{y+1}{2}-1}\mathrm{e}^{-t}\mathrm{d}t = \frac{1}{2}\Gamma\left(\frac{y+1}{2}\right) \tag{17}$$

Príklad:

$$\int_0^\infty x^2\mathrm{e}^{-x^2}\mathrm{d}x = \frac{1}{2}\Gamma\left(\frac{2+1}{2}\right) = \frac{1}{2}\Gamma\left(\frac{3}{2}\right) \tag{18}$$

$\Gamma\left(\frac{3}{2}\right)$ je tabulovana a rovna $\frac{\sqrt{\pi}}{2}$.

Transformacia integralu

Podobnou transformaciou mozete overit (skuste si to!), ze:

$$\int_0^\infty x^y \mathrm{e}^{-ax^2} \mathrm{d}x = \frac{1}{2a^{\frac{y+1}{2}}} \Gamma\left(\frac{y+1}{2}\right) \tag{19}$$

Hodnoty Gamma funkcie

$$\Gamma(1) = 1 \qquad \Gamma(\frac{1}{2}) = \sqrt{\pi} \tag{20}$$

$$\Gamma(2) = 1 \qquad \Gamma(\frac{3}{2}) = \frac{1}{2}\sqrt{\pi} \tag{21}$$

$$\Gamma(3) = 2 \qquad \Gamma(\frac{5}{2}) = \frac{3}{4}\sqrt{\pi} \tag{22}$$

$$\Gamma(4) = 6 \qquad \Gamma(\frac{7}{2}) = \frac{15}{8}\sqrt{\pi} \tag{23}$$

Priklad - Bittencourt 6.2 a) a d)

$$f(\vec{v}) = n_0 \left(\frac{m}{2\pi k T_\perp}\right) \left(\frac{m}{2\pi k T_\parallel}\right)^{1/2} \mathrm{e}^{-\frac{m}{2k}\left(\frac{v_\perp^2}{T_\perp}+\frac{v_\parallel^2}{T_\parallel}\right)} \tag{24}$$

$$v_\parallel = v_z \qquad v_\perp^2 = v_x^2 + v_y^2 \tag{25}$$

1. Overte, ze koncentracia je n_0.
2. Overte:

$$\frac{1}{2} m \langle v_\parallel^2 \rangle = \frac{kT_\parallel}{2} \qquad \frac{1}{2} m \langle v_\perp^2 \rangle = kT_\perp \tag{26}$$

Riesenie

$$n_0 = \int\int\int_{-\infty}^{\infty} f(\vec{v})\mathrm{d}^3 v = n_0 A \int\int\int \mathrm{e}^{-\frac{m}{2k}\frac{v_x^2}{T_\perp}} \mathrm{e}^{-\frac{m}{2k}\frac{v_y^2}{T_\perp}} \mathrm{e}^{-\frac{m}{2k}\frac{v_z^2}{T_\parallel}} \mathrm{d}^3 v \tag{27}$$

$$\int_{-\infty}^{\infty} \mathrm{e}^{-\frac{m}{2k}\frac{y^2}{T_i}} \mathrm{d}y = 2\frac{1}{2}\left(\frac{m}{2kT_i}\right)^{\frac{-0-1}{2}} \Gamma(1/2) = \sqrt{\frac{2kT_i\pi}{m}} \tag{28}$$

2 kvoli medziam a parnosti funkcie v integrale.

Riesenie

$$\frac{1}{2}m\langle v_{\parallel}^2\rangle = \frac{1}{2}m\frac{1}{n_0}\int\int\int_{-\infty}^{\infty} v_z^2 f(\vec{v})\mathrm{d}^3v = \tag{29}$$

$$\frac{1}{2}mA\sqrt{\frac{2kT_{\perp}\pi}{m}}\sqrt{\frac{2kT_{\perp}\pi}{m}}2\int_0^{\infty} v_z^2 \mathrm{e}^{-\frac{m}{2k}\frac{v_z^2}{T_{\parallel}}}\mathrm{d}v_z = \tag{30}$$

$$2\frac{1}{2}\frac{1}{2}m\left(\frac{m}{2\pi kT_{\parallel}}\right)^{1/2}\left(\frac{m}{2kT_{\parallel}}\right)^{\frac{-2-1}{2}}\Gamma(3/2) = \tag{31}$$

$$m\left(\frac{m}{2kT_{\parallel}}\right)^{-1}\frac{1}{\sqrt{\pi}}\frac{1}{4}\sqrt{\pi} = \frac{1}{2}kT_{\parallel} \tag{32}$$

Riesenie

$$\frac{1}{2}m\langle v_\perp^2\rangle = \tag{33}$$

$$\frac{1}{2}m\left(\frac{m}{2\pi kT_\perp}\right)\int\int_{-\infty}^{\infty} v_x^2 \mathrm{e}^{-\frac{m}{2k}\frac{v_x^2}{T_\perp}}\mathrm{e}^{-\frac{m}{2k}\frac{v_y^2}{T_\perp}}\mathrm{d}v_x\mathrm{d}v_y + \tag{34}$$

$$\frac{1}{2}m\left(\frac{m}{2\pi kT_\perp}\right)\int\int_{-\infty}^{\infty} v_y^2 \mathrm{e}^{-\frac{m}{2k}\frac{v_x^2}{T_\perp}}\mathrm{e}^{-\frac{m}{2k}\frac{v_y^2}{T_\perp}}\mathrm{d}v_x\mathrm{d}v_y = \tag{35}$$

$$m\left(\frac{m}{2\pi kT_\perp}\right)\int\int_{-\infty}^{\infty} v_x^2 \mathrm{e}^{-\frac{m}{2k}\frac{v_x^2}{T_\perp}}\mathrm{e}^{-\frac{m}{2k}\frac{v_y^2}{T_\perp}}\mathrm{d}v_x\mathrm{d}v_y = \tag{36}$$

$$m\left(\frac{m}{2\pi kT_\perp}\right)^{1/2}\int_{-\infty}^{\infty} v_x^2 \mathrm{e}^{-\frac{m}{2k}\frac{v_x^2}{T_\perp}}\mathrm{d}v_x = \tag{37}$$

$$m\left(\frac{m}{2\pi kT_\perp}\right)^{1/2}\left(\frac{m}{2kT_\perp}\right)^{\frac{-3}{2}}\Gamma(3/2) = \tag{38}$$

$$kT_\perp \tag{39}$$