Particle in constant electric a magnetic field

Non-relativistic Lorentz force

- Fields are constant:

$$\vec{E} \neq \vec{E}(\vec{r}, t) \qquad \vec{B} \neq \vec{B}(\vec{r}, t) \tag{1}$$

$$\vec{a}(t) = \frac{q}{m}\left(\vec{v}(t) \times \vec{B} + \vec{E}\right) \tag{2}$$

Initial conditions are given, without loss in generality:

$$x, y, z(t = 0) = 0 \tag{3}$$

$$v_x, v_y, v_z(t = 0) = v_x(0), v_y(0), v_z(0) \tag{4}$$

Choose a coordinate system

The annoying term is $\vec{v}(t) \times \vec{B}$, since without it we could just integrate.

- Without loss of generality, we can choose x, y, z such that:

$$\vec{B} = B\vec{z}^0 \tag{5}$$

Now the Lorentz force creates a system of ordinary differential equations:

$$\ddot{x} = \frac{qB}{m}(\dot{y} + E_x) \tag{6}$$
$$\ddot{y} = -\frac{qB}{m}(\dot{x} + E_y) \tag{7}$$
$$\ddot{z} = E_z \tag{8}$$

One of them is entirely independent and can be solved for:

$$z = z(0) + v_z(0)t + \frac{1}{2}\frac{qE_z}{m}t^2 \tag{9}$$

We are left with:

$$\ddot{x} = \frac{qB}{m}\dot{y} + \frac{q}{m}E_x \tag{10}$$

$$\ddot{y} = -\frac{qB}{m}\dot{x} + \frac{q}{m}E_y \tag{11}$$

Let's substitute away the noise:

- $\frac{qB}{m}$ is the definition of cyclotron frequency Ω
- E_x and E_y are constants, so we can "put them inside" our variable

I propose this change of variables:

$$\xi = -\Omega\dot{x} + \frac{q}{m}E_y \qquad \rightarrow \qquad \ddot{x} = \frac{\mathrm{d}}{\mathrm{d}t}\left(-\xi + \frac{q}{m\Omega}E_y\right) \tag{12}$$

$$\eta = \Omega\dot{y} + \frac{q}{m}E_x \qquad \rightarrow \qquad \ddot{y} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\eta - \frac{q}{m\Omega}E_x\right) \tag{13}$$

Then we obtain a beautiful, simple system:

$$-\dot{\xi} = \Omega\eta \tag{14}$$

$$\dot{\eta} = \Omega\xi \tag{15}$$

How to solve this?

There are many possibilities. Let's look at three ways:

- The just-substitute-it way
- The use-your-linear-algebra-skills way
- The read-the-book-first way

Just substitute it!

Take the derivative of one equation:

$$\ddot{\xi} = -\Omega\dot{\eta} \qquad \dot{\eta} = \Omega\xi \qquad \rightarrow \tag{16}$$

$$\ddot{\xi} = -\Omega^2\xi \tag{17}$$

This is a harmonic oscillator equation:

$$\xi = \alpha \mathrm{e}^{\mathrm{i}\Omega t} + \beta \mathrm{e}^{-\mathrm{i}\Omega t} \tag{18}$$

$$\eta = \gamma \mathrm{e}^{\mathrm{i}\Omega t} + \delta \mathrm{e}^{-\mathrm{i}\Omega t} \tag{19}$$

We need to integrate again to get the original variables back (I used new constants, to hide away all the ugliness)[1]:

$$x = A\cos(\Omega t) + B\sin(\Omega t) + \frac{q}{m\Omega}E_y t + K_1 \quad (20)$$

$$y = C\cos(\Omega t) + D\sin(\Omega t) - \frac{q}{m\Omega}E_x t + K_2 \quad (21)$$

[1] Using B as a constant was not a smart choice, please do not be confused.

For simplicity assume:

$$\vec{r}(0) = \vec{0} \qquad\qquad \vec{v}(0) = v_0\vec{x}^0 \tag{22}$$

$$\vec{E} = E\vec{y}^0 \qquad\qquad \vec{B} = B\vec{z}^0 \tag{23}$$

Using our initial conditions (including the inittial acceleration) we get:

$$A\Omega^2 = 0 \qquad\qquad B\Omega + \frac{q}{m\Omega}E = B\Omega + \frac{E}{B} = v_0 \tag{24}$$

$$-C\Omega^2 = -\frac{q}{m}v_0B + \frac{qE}{m} = \Omega(-v_0 + \frac{E}{B}) \qquad\qquad D = 0 \tag{25}$$

$$K1 = 0 \qquad\qquad K2 = -C \tag{26}$$

The solution is:

$$x = \frac{1}{\Omega}\left(v_0 - \frac{E}{B}\right)\sin(\Omega t) + \frac{E}{B}t \tag{27}$$

$$y = \frac{1}{\Omega}\left(v_0 - \frac{E}{B}\right)(\cos(\Omega t) - 1) \tag{28}$$

Use your linear algebra skills

$$\dot{\xi} = -\Omega\eta \tag{29}$$

$$\dot{\eta} = \Omega\xi \tag{30}$$

We think about the derivative as a linear operator and write this in matrix form:

$$\mathbf{D}\vec{w} = \Omega\mathbf{A}\vec{w}, \tag{31}$$

where $\vec{w}$ is $[\xi, \eta]$ and the matrix **A** looks like this:

$$\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{32}$$

Now it is important to realize, that we have the freedom to linearly transform at wil (any linear combination of (ξ, η) will also be a solution to the equation). The matrix will transform accordingly. So we will change our basis and find the eigenvectors of the transformation **A**. In this basis, the equations are entirely independent and we can integrate immediately:

$$\mathbf{A}\vec{a} = \lambda\vec{a} \tag{33}$$

$$\mathbf{D}\vec{a} = \Omega\mathbf{A}\vec{a} = \Omega\lambda\vec{a} \tag{34}$$

$$\vec{a}(t) = \vec{a}(0)\mathrm{e}^{\Omega\lambda t} \tag{35}$$

The derivative operator does not depend on (ξ, η) and so it does not change upon a change of basis.

The eigenvectors and corresponding eigenvalues are:

$$\vec{a}_1 = [1, -\mathrm{i}] \qquad \lambda_1 = \mathrm{i} \tag{36}$$

$$\vec{a}_2 = [1, \mathrm{i}] \qquad \lambda_2 = -\mathrm{i} \tag{37}$$

And so, the solution can be written as:

$$\xi - \mathrm{i}\eta = (\xi - \mathrm{i}\eta)(t = 0)\mathrm{e}^{\mathrm{i}\Omega t} \tag{38}$$

$$\xi + \mathrm{i}\eta = (\xi + \mathrm{i}\eta)(t = 0)\mathrm{e}^{-\mathrm{i}\Omega t} \tag{39}$$

which represents circular motion on a plane.

. . . now return to previous variables and substitute initial conditions (or vice-versa).

The read-the-book-first way

The theory tells us two things:

- particle in constant electromagnetic field will drift with a constant velocity:

$$\vec{v}_{E\times B} = \frac{\vec{E}\times\vec{B}}{B^2} \tag{40}$$

- particle in constant magnetic field will execute a circular motion in plane

We are hoping that the overall motion will be a superposition of circular motion and constant drift.

Let's start by separating the directions in the plane of the circular motion. This can be done in terms of the magnetic field direction – circular motion happens in direction perpendicular to magnetic field. We will rewrite our equations in that way:

$$\dot{\vec{v}}_\perp = \frac{q}{m}\left(\vec{v}_\perp \times \vec{B} + \vec{E}_\perp\right) \tag{41}$$

$$\dot{\vec{v}}_\parallel = \frac{q}{m}\left(\vec{0} + \vec{E}_\parallel\right) \tag{42}$$

$$\tag{43}$$

As before, we can solve for the paralllel equation immediately.

The other equation is a challenge – if it weren't for the electric field, the solution would just be circular motion. But maybe we can change our inertial frame of reference, such that the electric field (which is just a constant) will be transformed away. Now there are two ways you can do this:

- think hard, use a weird combination of vector identities and come up with a transformation that does this
- or use your physical intuition and the theory

The second will always impress your collegues more, so we will do that.

We know that the particle will drift, so let's try to change our frame of reference so that it moves with the drift:

$$\vec{w} = \vec{v}_\perp - \frac{\vec{E} \times \vec{B}}{B^2} \tag{44}$$

Subsititute into the equation of motion (derivative of a constant velocity is zero) and massage it:

$$\dot{\vec{w}} = \frac{q}{m}\left(\vec{w} \times \vec{B} + \frac{\vec{E} \times \vec{B}}{B^2} \times \vec{B} + \vec{E}_\perp\right) \tag{45}$$

$$\dot{\vec{w}} = \frac{q}{m}\left(\vec{w} \times \vec{B} + \frac{\vec{E}_\perp \times \vec{B}}{B^2} \times \vec{B} + \vec{E}_\perp\right) \tag{46}$$

$$\dot{\vec{w}} = \frac{q}{m}\left(\vec{w} \times \vec{B} + \frac{1}{B^2}\vec{B}(\vec{E}_\perp \cdot \vec{B}) - \vec{E}_\perp + \vec{E}_\perp\right) \tag{47}$$

$$\dot{\vec{w}} = \frac{q}{m}\left(\vec{w} \times \vec{B}\right) \tag{48}$$

$$\dot{\vec{w}} = \left(\vec{w} \times \vec{\Omega}\right) \tag{49}$$

We made use of the triple vector product identity.

It worked!

The equation we obtained is such circular motion, we can integrate to obtain:

$$\vec{w} = \vec{r}_{\mathrm{c}} \times \vec{\Omega} \tag{50}$$

Then we just merge all the pieces together:

$$\vec{v} = \vec{v}_{\perp} + \vec{v}_{\parallel} = \vec{r}_{\mathrm{c}} \times \vec{\Omega} + \frac{\vec{E} \times \vec{B}}{B^2} + \frac{q}{m}\vec{E}_{\parallel} t + \vec{v}_{\parallel}(t = 0) \tag{51}$$

Notice that it seems that the solution does not depend on the initial perpendicular velocity. That is not the case, it is only hidden since the cyclotron radius is:

$$r_{\mathrm{c}} = \frac{w}{\Omega} = \frac{v_{\perp}(0) + v_{E \times B}}{\Omega} \tag{52}$$

and is a constant (although **NOT** the vector!!)

The trajectory

Let's take our simplified initial conditions:

$$\vec{r}(0) = \vec{0} \qquad \vec{v}(0) = v_0 \vec{x}^0 \tag{53}$$

$$\vec{E} = E\vec{y}^0 \qquad \vec{B} = B\vec{z}^0 \tag{54}$$

for which the trajectory is:

$$x = \frac{1}{\Omega}\left(v_0 - \frac{E}{B}\right)\sin(\Omega t) + \frac{E}{B}t \tag{55}$$

$$y = \frac{1}{\Omega}\left(v_0 - \frac{E}{B}\right)(\cos(\Omega t) - 1) \tag{56}$$

The motion is entirely in **plane** (it can deceptively look 3D on the pictures, but it is not). The curve is called a trochoid. With initial velocity set to zero, it is a cycloid.

Other useful sources

- read Bittencourt
- physicstasks