Debye shielding using Vlasov equation

We examine the shielding effect of plasma we consider a charged particle isolated for observation in plasma with positive charge Q. We will use the steady-state Vlasov equations for electrons and ions (with charge $+e$) and only consider the electric field. The equations then are:

$$\vec{v} \cdot \nabla f_{\mathrm{e,i}} \pm \frac{e}{m_{\mathrm{e,i}}} (\nabla \phi) \cdot \nabla_v f_{\mathrm{e,i}} = 0 \tag{1}$$

We can express the charge density through the particle densities:

$$\rho(\vec{r}) = \sum_\alpha q_\alpha n_\alpha, \tag{2}$$

and the densities as the zeroth moment of the distribution function:

$$n_\alpha(\vec{r}) = \int f_\alpha(\vec{r}, \vec{v}) \mathrm{d}^3 v \tag{3}$$

To get the total charge density we need to include the "additional" particle and we choose the frame of reference such that the particle is at the origin.

$$\rho(\vec{r}) = Q\delta(\vec{r}) + \int e\left(f_{\mathrm{i}} - f_{\mathrm{e}}\right) \mathrm{d}^3 v \tag{4}$$

To obtain the electric potential, we have to solve the Poisson equation:

$$\nabla^2 \phi = \int \frac{e}{\varepsilon_0} \left(f_{\mathrm{e}} - f_{\mathrm{i}}\right) \mathrm{d}^3 v - \frac{Q}{\varepsilon_0} \delta(\vec{r}) \tag{5}$$

Now we have three equations (2 $\times$ Vlasov + Poisson) to solve for 2 distributions and 1 potential

We assume solution of the distributions in the form of:

$$f_\alpha(\vec{r}, \vec{v}) = f_{0,\alpha}(v) \mathrm{e}^{-\frac{q_\alpha \phi(\vec{r})}{kT}}, \tag{6}$$

where $f_{0\alpha}$ is the Maxwell-Boltzmann distribution, which assumes steady-state and spherical symmetry. Integrating the Maxwell-Boltzmann distribution should give the density at equilibrium:

$$\int f_{0\alpha} \mathrm{d}^3 v = n_{0\alpha} \equiv n_0 \tag{7}$$

Substituting into Poisson equation:

$$\nabla^2 \phi = \int \frac{e}{\varepsilon_0} \left(f_{0,\alpha}(v) \mathrm{e}^{\frac{e\phi(\vec{r})}{kT}} - f_{0,\alpha}(v) \mathrm{e}^{-\frac{e\phi(\vec{r})}{kT}} \right) \mathrm{d}^3 v - \frac{Q}{\varepsilon_0} \delta(\vec{r}) = \frac{n_0 e}{\varepsilon_0} \left(\mathrm{e}^{\frac{e\phi(\vec{r})}{kT}} - \mathrm{e}^{-\frac{e\phi(\vec{r})}{kT}} \right) - \frac{Q}{\varepsilon_0} \delta(\vec{r}) \tag{8}$$

To proceed analytically further we assume:

$$e^{\pm\frac{e\phi(\vec{r})}{kT}} \approx 1 \pm \frac{e\phi(\vec{r})}{kT} \tag{9}$$

which gives us:

$$\nabla^2\phi = 2\frac{n_0 e^2}{\varepsilon_0 kT}\phi(\vec{r}) - \frac{Q}{\varepsilon_0}\delta(\vec{r}) \tag{10}$$

The differential equation can be solved by using Green's functions. [1]

Green's function is defined by:

$$\mathcal{L}G(\vec{r},\vec{r}') = -\delta(\vec{r}-\vec{r}'), \tag{11}$$

where $\mathcal{L}$ is a linear differential operator. It has the nice property:

$$\int G(\vec{r},\vec{r}')f(\vec{r}')\mathrm{d}^3r' = \phi(\vec{r}), \tag{12}$$

where $f(\vec{r})$ is the RHS of the differential equation (source term).

For our problem:

$$\mathcal{L} = \nabla^2 - \lambda^2 \qquad f(\vec{r}) = \delta(\vec{r}) \tag{13}$$

Using Fourier transform:

$$G(\vec{r}) = \frac{1}{(2\pi)^3}\int G(\vec{k})e^{i\vec{k}\cdot\vec{r}}\mathrm{d}^3k \tag{14}$$

the definition of Green's function is:

$$\mathcal{F}\mathcal{L}\left(G(\vec{r})\right) = \left(i^2k^2 - \lambda^2\right)G(\vec{k}) = -\mathcal{F}\left(\delta(\vec{r})\right) = -1 \qquad \Rightarrow \tag{15}$$

$$G(\vec{k}) = \frac{1}{k^2+\lambda^2} \tag{16}$$

Doing the inverse FT:

$$G(\vec{r}) = \frac{1}{(2\pi)^3}\int \frac{e^{i\vec{k}\cdot\vec{r}}}{k^2+\lambda^2}\mathrm{d}^3k \tag{17}$$

Using spherical coordinates:

$$G(\vec{r}) = \frac{1}{(2\pi)^3}\int k^2\sin\theta\frac{e^{i\vec{k}\cdot\vec{r}}}{k^2+\lambda^2}\mathrm{d}^3k = \tag{18}$$

$$\frac{1}{(2\pi)^3}\int k_r^2\sin\theta\frac{e^{ik_r r\cos\theta}}{k_r^2+\lambda^2}\mathrm{d}^3k = \frac{2\pi}{(2\pi)^3}\int \mathrm{d}k_r\frac{k_r^2}{k_r^2+\lambda^2}\int_0^\pi \sin\theta e^{ik_r r\cos\theta}\mathrm{d}\theta = \tag{19}$$

$$-\frac{2\pi}{(2\pi)^3}\int \mathrm{d}k_r\frac{k_r^2}{k_r^2+\lambda^2}\int_1^{-1} e^{ik_r rt}\mathrm{d}t = -\frac{1}{(2\pi)^2}\int \mathrm{d}k_r\frac{k_r}{k_r^2+\lambda^2}\frac{e^{-ik_r r}-e^{ik_r r}}{ir} = \tag{20}$$

$$\frac{1}{r(2\pi)^2}\int_0^\infty \mathrm{d}k_r\frac{k_r}{k_r^2+\lambda^2}2\sin(k_r r) = \dots \tag{21}$$

$$\dots = \frac{1}{4r\pi}e^{-|\lambda r|} \qquad \Rightarrow \qquad G(\vec{r},\vec{r}') = \frac{e^{-\lambda|\vec{r}-\vec{r}'|}}{4|\vec{r}-\vec{r}'|\pi} \tag{22}$$

For our problem:

$$\phi(\vec{r}) = \int f(\vec{r}')G(\vec{r},\vec{r}')\mathrm{d}^3r' = \tag{23}$$

$$\frac{Q}{\varepsilon_0}\int\delta(\vec{r}')\frac{e^{-\lambda|\vec{r}-\vec{r}'|}}{4|\vec{r}-\vec{r}'|\pi}\mathrm{d}^3r' = \frac{Q}{\varepsilon_0}\frac{e^{-\lambda|\vec{r}-0|}}{4|\vec{r}-0|\pi} = \frac{Q}{\varepsilon_0}\frac{e^{-\lambda|r|}}{4r\pi} \tag{24}$$

[1] Bittencourt: seek solution in form of $\phi(r) = \phi_{\mathrm{Coulomb}}F(r)$, $\lim_{r\to 0}F(r) = 1$ $\lim_{r\to\infty}F(r) = 0$.

Definite integral:

$$\int_0^\infty \frac{x \sin(a\,x)}{x^2+b^2}\,dx = \frac{1}{2}\,\pi\,\mathrm{sgn}(a)\,e^{-|a\,b|}$$

Fig. 1: Solution from wolframalpha.com

with:

$$\lambda^2 = 2\frac{n_0 e^2}{\varepsilon_0 kT} = \frac{2}{\lambda_\mathrm{D}^2} \tag{25}$$

$$\phi(\vec{r}) = \frac{Q}{\varepsilon_0}\frac{\mathrm{e}^{-\frac{\sqrt{2}}{\lambda_\mathrm{D}}r}}{4r\pi} \tag{26}$$

The charge density is then given by:

$$\rho(\vec{r}) = -\frac{2n_0 e^2}{\varepsilon_0 kT} Q \frac{\mathrm{e}^{-\frac{\sqrt{2}}{\lambda_\mathrm{D}}r}}{4r\pi} + Q\delta(\vec{r}) = -\frac{Q}{2\pi r\lambda_\mathrm{D}^2}\mathrm{e}^{-\frac{\sqrt{2}}{\lambda_\mathrm{D}}r} + Q\delta(\vec{r}) = \tag{27}$$

The total charge is given by integral over space:

$$q_\mathrm{total} = -4\pi\int_0^\infty \mathrm{d}r r^2 \frac{Q}{2\pi r\lambda_\mathrm{D}^2}\mathrm{e}^{-\frac{\sqrt{2}}{\lambda_\mathrm{D}}r} + \int \mathrm{d}^3x Q\delta(\vec{r}) = -2\int_0^\infty \mathrm{d}r r\frac{Q}{\lambda_\mathrm{D}^2}\mathrm{e}^{-\frac{\sqrt{2}}{\lambda_\mathrm{D}}r} + Q = \tag{28}$$

$$-2\frac{Q}{\lambda_\mathrm{D}^2}\frac{\lambda_\mathrm{D}^2}{2}\mathrm{e}^{-\frac{\sqrt{2}}{\lambda_\mathrm{D}}0} + Q = Q - Q = 0 \tag{29}$$

- Since $n_{\mathrm{e,i}} = n_0 \mathrm{e}^{\pm\frac{e\phi(r)}{kT}}$, around Q the charge density of electrons is larger.
- Decays much faster than the Coulomb potential.
- Shielding takes place over the distance of order of Debye length – plasma dimensions must be greter than Debye length.

To test the validity of approximation:

$$\frac{e\phi(\vec{r})}{kT} = \frac{eQ}{\varepsilon_0 kT}\frac{\mathrm{e}^{-\frac{\sqrt{2}}{\lambda_\mathrm{D}}r}}{4r\pi}\frac{\frac{1}{3}n_0 eQ}{4r\pi\frac{1}{3}n_0\varepsilon_0 kT}\mathrm{e}^{-\frac{\sqrt{2}}{\lambda_\mathrm{D}}r} = \frac{\frac{1}{3}Q}{4er\pi\lambda_\mathrm{D}^2\frac{1}{3}n_0}\mathrm{e}^{-\frac{\sqrt{2}}{\lambda_\mathrm{D}}r} = \frac{\lambda_\mathrm{D}Q}{3eN_\mathrm{D}r}\mathrm{e}^{-\frac{\sqrt{2}}{\lambda_\mathrm{D}}r}, \tag{30}$$

therefore the approximation is justified when:

$$r \gg \frac{Q\lambda_\mathrm{D}}{eN_\mathrm{D}} \tag{31}$$